On the number of pairs of positive integers $x, y \leq H$ such that $x^2 + y^2 + 1$, $x^2 + y^2 + 2$ are square-free

by

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Dedicated to Professor Doychin Tolev on the occasion of his 60th birthday

1. Notations. Let $H$ be a sufficiently large positive number. We denote by $\varepsilon$ an arbitrary small positive number, not the same in all appearances. The letters $d, h, k, l, q, r, x, y$ with or without subscript will denote positive integers. By the letters $D_1, D_2, H_0$ and $t$ we denote real numbers and by $m, n$ integers. As usual $\mu(n)$ is the Möbius function and $\tau(n)$ denotes the number of positive divisors of $n$. Further $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of $t$. Instead of $m \equiv n \, (\text{mod } d)$ we write for simplicity $m \equiv n \,(d)$. Moreover $(l, m)$ is the greatest common divisor of $l$ and $m$, and $(l, m, n)$ is the greatest common divisor of $l$, $m$ and $n$. The letter $p$ will always denote a prime number. We put $\psi(t) = \{t\} - 1/2$. As usual $e(t) = \exp(2\pi it)$. For any odd $q$ we denote by $(\frac{\cdot}{q})$ the Jacobi symbol. For any $n$ and $q$ such that $(n, q) = 1$ we denote by $(\bar{n})_q$ the inverse of $n$ modulo $q$. If the value of the modulus is understood from the context then we simply write $\bar{n}$.

We shall consider the Gauss sums

\begin{equation}
G(q, m, n) = \sum_{x=1}^{q} e \left( \frac{mx^2 + nx}{q} \right), \quad G(q, m) = G(q, m, 0),
\end{equation}

and the Kloosterman sums

\begin{equation}
K(q, m, n) = \sum_{\substack{x=1 \\
(x,q)=1}}^{q} e \left( \frac{mx + nx}{q} \right).
\end{equation}

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2. Introduction and statement of the result. The problem of consecutive square-free numbers arose in 1932 when Carlitz [1] proved that

\[
\sum_{n \leq H} \mu^2(n)\mu^2(n + 1) = \prod_p \left(1 - \frac{2}{p^2}\right) H + O(H^{\theta + \varepsilon}),
\]

where \(\theta = 2/3\). Formula (2.1) was subsequently improved by Heath-Brown [6] to \(\theta = 7/11\) and by Reuss [9] to \(\theta = (26 + \sqrt{433})/81\).

In 2018 the author [2] showed that for any fixed \(1 < c < 22/13\) there exist infinitely many consecutive square-free numbers of the form \([n^c], [n^c] + 1\).

Recently the author [3] proved that there exist infinitely many consecutive square-free numbers of the form \([\alpha n], [\alpha n] + 1\), where \(n\) is natural and \(\alpha > 1\) is an irrational number with bounded partial quotient or irrational algebraic number.

Also recently the author [4] showed that there exist infinitely many consecutive square-free numbers of the form \([\alpha p], [\alpha p] + 1\), where \(p\) is prime and \(\alpha > 0\) is an irrational algebraic number.

On the other hand, in 2012 Tolev [11] proved ingeniously that there exist infinitely many square-free numbers of the form \(x^2 + y^2 + 1\). More precisely he established the asymptotic formula

\[
\sum_{1 \leq x,y \leq H} \mu^2(x^2 + y^2 + 1) = cH^2 + O(H^{4/3 + \varepsilon}),
\]

where

\[
c = \prod_p \left(1 - \frac{\lambda(p^2)}{p^4}\right) \quad \text{and} \quad \lambda(q) = \sum_{1 \leq x,y \leq q} 1,
\]

Define

\[
\Gamma(H) = \sum_{1 \leq x,y \leq H} \mu^2(x^2 + y^2 + 1) \mu^2(x^2 + y^2 + 2)
\]

and

\[
\lambda(q_1, q_2, m, n) = \sum_{x,y:\ (2.4)} e\left(\frac{mx + ny}{q_1q_2}\right),
\]

where the summation is taken over the integers \(x, y\) satisfying the conditions

\[
1 \leq x, y \leq q_1q_2,
\]

(2.4) \[x^2 + y^2 + 1 \equiv 0 \ (q_1),\]

\[x^2 + y^2 + 2 \equiv 0 \ (q_2).\]

We also define

\[
\lambda(q_1, q_2) = \lambda(q_1, q_2, 0, 0).
\]
Consecutive square-free $x^2 + y^2 + 1$, $x^2 + y^2 + 2$

Motivated by these results and following the method of Tolev [11] we shall prove the following theorem.

**Theorem 2.1.** For the sum $\Gamma(H)$ defined by (2.2) we have the asymptotic formula

\[
\Gamma(H) = \sigma H^2 + O(H^{8/5+\varepsilon}),
\]

where

\[
\sigma = \prod_p \left( 1 - \frac{\lambda(p^2, 1) + \lambda(1, p^2)}{p^4} \right).
\]

From Theorem 2.1 it follows that there exist infinitely many consecutive square-free numbers of the form $x^2 + y^2 + 1$, $x^2 + y^2 + 2$.

### 3. Lemmas.

The first lemma we need gives us the basic properties of the Gauss sum.

**Lemma 3.1 ([5] [7]).**

(i) If $(q_1, q_2) = 1$ then

\[
G(q_1 q_2, m_1 q_2 + m_2 q_1, n) = G(q_1, m_1 q_2^2, n) G(q_2, m_2 q_1^2, n).
\]

(ii) If $(q, m) = d$ then

\[
G(q, m, n) = \begin{cases} 
   dG(q/d, m/d, n/d) & \text{if } d \mid n, \\
   0 & \text{if } d \nmid n.
\end{cases}
\]

(iii) If $(q, 2m) = 1$ then

\[
G(q, m, n) = e \left( -\frac{4m n^2}{q} \right) \left( \frac{m}{q} \right) G(q, 1).
\]

(iv) If $(q, 2) = 1$ then

\[
G^2(q, 1) = (-1)^{(q-1)/2} q.
\]

The next lemma gives us A. Weil’s estimate for the Kloosterman sum.

**Lemma 3.2 ([8]).**

\[
|K(q, m, n)| \leq \tau(q) q^{1/2} (q, m, n)^{1/2}.
\]

The next lemma is the central moment in the proof of Theorem 2.1. Right here we apply the properties of the Gauss sum and A. Weil’s estimate for the Kloosterman sum.

**Lemma 3.3.** Let $8 \nmid q_1$, $8 \nmid q_2$ and $(q_1, q_2) = 1$. Then for the function $\lambda$ defined by (2.3) we have

\[
|\lambda(q_1, q_2, m, n)| \leq 16 \tau^2(q_1 q_2) (q_1 q_2)^{1/2} (q_1 q_2, m, n)^{1/2}.
\]
In particular,
\[(3.2) \quad \lambda(q_1, q_2) \ll (q_1 q_2)^{1+\varepsilon}.
\]

Remark 3.4. An estimate of type (3.1) is valid for any positive integers \(q_1, q_2\). We introduce the restrictions \(8 \nmid q_1, 8 \nmid q_2\) since in this case the proof is slightly simpler and only such \(q_1, q_2\) appear in our work.

Proof. We consider three cases.

Case 1: \(2 \nmid q_1 q_2\). Using (1.1), (2.3), (2.4) and Lemma 3.1 we get
\[(3.3) \quad \lambda(q_1, q_2, m, n) = \frac{1}{q_1 q_2} \sum_{1 \leq x, y \leq q_1 q_2} e\left(\frac{mx + ny}{q_1 q_2}\right) \sum_{1 \leq h_1 \leq q_1} e\left(\frac{h_1(x^2 + y^2 + 1)}{q_1}\right) \times \sum_{1 \leq h_2 \leq q_2} e\left(\frac{h_2(x^2 + y^2 + 2)}{q_2}\right) = \frac{1}{q_1 q_2} \sum_{1 \leq h_1 \leq q_1} e\left(\frac{h_1}{q_1}\right) \sum_{1 \leq h_2 \leq q_2} e\left(\frac{2h_2}{q_2}\right) G(q_1 q_2, h_1 q_2 + h_2 q_1, m) \times G(q_1 q_2, h_1 q_2 + h_2 q_1, n) = \frac{1}{q_1 q_2} \sum_{1 \leq h_1 \leq q_1} e\left(\frac{h_1}{q_1}\right) G(q_1, h_1 q_2^2, m) G(q_1, h_1 q_2^2, n) \times \sum_{1 \leq h_2 \leq q_2} e\left(\frac{2h_2}{q_2}\right) G(q_2, h_2 q_1^2, m) G(q_2, h_2 q_1^2, n) = \frac{1}{q_1 q_2} \sum_{l_1|q_1} \sum_{1 \leq h_1 \leq q_1 \atop (h_1, q_1) = q_1/l_1} e\left(\frac{h_1}{q_1}\right) G(q_1, h_1 q_2^2, m) G(q_1, h_1 q_2^2, n) \times \sum_{l_2|q_2} \sum_{1 \leq h_2 \leq q_2 \atop (h_2, q_2) = q_2/l_2} e\left(\frac{2h_2}{q_2}\right) G(q_2, h_2 q_1^2, m) G(q_2, h_2 q_1^2, n).
\]

Bearing in mind (1.2), (3.3), \(2 \nmid q_1 q_2\) and Lemma 3.1 we obtain
\[(3.4) \quad \lambda(q_1, q_2, m, n) = q_1 q_2 \sum_{l_1|q_1} \frac{1}{l_1^2} \sum_{1 \leq r_1 \leq l_1 \atop (r_1, l_1) = 1} e\left(\frac{r_1}{l_1}\right) G(l_1, r_1 q_2^2, m l_1 q_1^{-1}) G(l_1, r_1 q_2^2, n l_1 q_1^{-1}).
\]
From (3.4) and Lemma 3.2 it follows that

\[ \lambda(q_1q_2) = q_1q_2 \sum_{l_1 \mid |q_1, l_1 \mid (m,n)} \frac{\tau(l_1)}{l_1^{1/2}} \sum_{l_2 \mid q_2, l_2 \mid (l_2, l_2) = 1} \frac{\tau(l_2)}{l_2^{1/2}} \]

\[ \times \sum_{l_1 \mid |q_1, l_1 \mid (m,n)} \frac{G^2(l_1, 1)}{l_1^{2}} \sum_{1 \leq r_1 \leq l_1} e \left( \frac{2r_1}{l_1} \right) G(l_2, r_2q_1^2, ml_2q_2^{-1}) G(l_2, r_2q_1^2, nl_2q_2^{-1}) \]

\[ \times \sum_{l_2 \mid q_2, l_2 \mid (l_2, l_2) = 1} e \left( \frac{2r_2}{l_2} \right) \frac{G^2(l_2, 1)}{l_2^{2}} \sum_{1 \leq r_2 \leq l_2} \frac{r_1 - 4r_2q_1^2(m^2 + n^2)l_1q_1^{-2}}{l_1^{2}} \]

\[ = q_1q_2 \sum_{l_1 \mid |q_1, l_1 \mid (m,n)} \frac{(-1)^{(l_1-1)/2}}{l_1} K(l_1, 1, 4q_2^2(m^2 + n^2)l_1q_1^{-2}) \]

\[ \times \sum_{l_2 \mid q_2, l_2 \mid (l_2, l_2) = 1} \frac{(-1)^{(l_2-1)/2}}{l_2} K(l_2, 2, 4q_2^2(m^2 + n^2)l_2q_2^{-2}). \]

From (3.4) and Lemma 3.2 it follows that

\[ |\lambda(q_1, q_2, m, n)| \leq q_1q_2 \sum_{l_1 \mid |q_1, l_1 \mid (m,n)} \frac{\tau(l_1)}{l_1^{1/2}} \sum_{l_2 \mid q_2, l_2 \mid (l_2, l_2) = 1} \frac{\tau(l_2)}{l_2^{1/2}} \]

\[ \times \sum_{r_1 \mid (q_1, m, n)} q_1^{-1/2} \sum_{r_2 \mid (q_2, m, n)} q_2^{-1/2} \]

\[ \leq \tau^2(q_1q_2)^{1/2}(q_1q_2, m, n)^{1/2}. \]

**Case 2:** $q_1 = 2^h q'_1$, where $2 \nmid q'_1$ and $h \leq 2$, and $2 \nmid q_2$. The function $\lambda(q_1, q_2, m, n)$ defined by (2.3) is such that if

\[ (q'_1q''_1, q'_2q''_2) = (q'_1, q''_1) = (q'_2, q''_2) = 1 \]

then

\[ \lambda(q'_1q''_1, q'_2q''_2, m, n) = \lambda(q'_1, q''_1, m(q''_1q''_2q'_1q''_2, n(q''_1q''_2q'_1q''_2)) \times \lambda(q''_1, q''_2, m(q''_1q''_2q'_1q''_2, n(q''_1q''_2q'_1q''_2))) \]

(Since the proof is elementary we skip the details.)

Using (3.5), (3.6) and the trivial estimate $|\lambda(2^h, 1, m, n)| \leq 4^h$ we get

\[ |\lambda(2^h q'_1, q_2, m, n)| = |\lambda(2^h, 1, m(q'_1q''_2)_{2^h}, n(q''_1q''_2)_{2^h}) \times \lambda(q'_1, q_2, m(2^h)q'_1q_2, n(2^h)q'_1q_2)| \]

\[ \leq 16\tau^2(q'_1q_2)(q'_1q_2)^{1/2}(q'_1q_2, m, n)^{1/2} \]

\[ \leq 16\tau^2(q_1q_2)(q_1q_2)^{1/2}(q_1q_2, m, n)^{1/2}. \]
Case 3: $2 \nmid q_1$ and $q_2 = 2^h q_2'$, where $2 \nmid q_2'$ and $h \leq 2$. By (3.5), (3.6) and the trivial estimate $|\lambda(1, 2^h, m, n)| \leq 4^h$ we get

$$|\lambda(q_1, 2^h q_2', m, n)| = |\lambda(1, 2^h, m(q_1 q_2')_2 h, n(q_1 q_2')_2 h) \times \lambda(q_1, q_2', m(2^h q_1 q_2'), n(2^h q_1 q_2'))|$$

$$\leq 16\tau^2(q_1 q_2')(q_1 q_2')^{1/2}(q_1 q_2', m, n)^{1/2}$$

$$\leq 16\tau^2(q_1 q_2')(q_1 q_2')^{1/2}(q_1 q_2, m, n)^{1/2}.$$ 

Now (3.1) follows from (3.5), (3.7) and (3.8). As a byproduct of (3.1) we obtain (3.2).

**Lemma 3.5.** Assume that $8 \nmid q_1$, $8 \nmid q_2$, $(q_1, q_2) = 1$ and $H_0 \geq 2$. Then for the sums

$$A_1 = \sum_{1 \leq m \leq H_0} \frac{|\lambda(q_1, q_2, m, 0)|}{m}, \quad A_2 = \sum_{1 \leq m, n \leq H_0} \frac{|\lambda(q_1, q_2, m, n)|}{mn}$$

we have

$$A_1 \ll (q_1 q_2)^{1/2+\varepsilon} H_0^\varepsilon, \quad A_2 \ll (q_1 q_2)^{1/2+\varepsilon} H_0^\varepsilon.$$

**Proof.** Using (3.9) and Lemma 3.3 we get

$$A_1 \ll (q_1 q_2)^{1/2+\varepsilon} \sum_{1 \leq m \leq H_0} \frac{(q_1 q_2, m)^{1/2}}{m} = (q_1 q_2)^{1/2+\varepsilon} A_0,$$

where

$$A_0 = \sum_{1 \leq m \leq H_0} \frac{(q_1 q_2, m)^{1/2}}{m}.$$

We have

$$A_0 \ll \sum_{r | q_1 q_2} r^{1/2} \sum_{m \leq H_0 \atop m \equiv 0 (r)} \frac{1}{m} \ll (\log H_0) \sum_{r | q_1 q_2} r^{-1/2} \ll (q_1 q_2 H_0)^\varepsilon.$$

From (3.11) and (3.12) we deduce the first inequality in (3.10).

Using (3.9), (3.12) and Lemma 3.3 we obtain

$$A_2 \ll (q_1 q_2)^{1/2+\varepsilon} \sum_{1 \leq m, n \leq H_0} \frac{(q_1 q_2, m, n)^{1/2}}{mn}$$

$$\ll (q_1 q_2)^{1/2+\varepsilon} \sum_{1 \leq m, n \leq H_0} \frac{(q_1 q_2, m)^{1/2}(q_1 q_2, n)^{1/2}}{mn}$$

$$= (q_1 q_2)^{1/2+\varepsilon} A_0^2 \ll (q_1 q_2)^{1/2+\varepsilon} H_0^\varepsilon,$$

which proves the second inequality in (3.10). ■

The final lemma we need gives us important expansions.
**Lemma 3.6 ([10]).** For any $H_0 \geq 2$, we have

$$\psi(t) = -\sum_{1 \leq |m| \leq H_0} \frac{e(mt)}{2\pi im} + O(f(H_0, t)),$$

where $f(H_0, t)$ is a positive, infinitely differentiable and 1-periodic function of $t$. It can be expanded into the Fourier series

$$f(H_0, t) = \sum_{m=-\infty}^{+\infty} b_{H_0}(m)e(mt),$$

with coefficients $b_{H_0}(m)$ such that

$$b_{H_0}(m) \ll \frac{\log H_0}{H_0} \text{ for all } m$$

and

$$\sum_{|m| > H_0^{1+\varepsilon}} |b_{H_0}(m)| \ll H_0^{-A}.$$

Here $A > 0$ is arbitrarily large and the constant in the $\ll$ symbol depends on $A$ and $\varepsilon$.

4. **Proof of the theorem.** Using (2.2) and the well-known identity $\mu^2(n) = \sum_{d \mid n} \mu(d)$ we get

$$\Gamma(H) = \sum_{d_1, d_2 \leq z} \mu(d_1)\mu(d_2) \sum_{\substack{1 \leq x, y \leq H \\% \atop x^2+y^2+1 \equiv 0 (d_1^2) \\% \atop x^2+y^2+2 \equiv 0 (d_2^2)}} 1 = \Gamma_1(H) + \Gamma_2(H),$$

where

$$\Gamma_1(H) = \sum_{d_1d_2 \leq z} \mu(d_1)\mu(d_2) \Sigma(H, d_1^2, d_2^2),$$

$$\Gamma_2(H) = \sum_{d_1d_2 > z} \mu(d_1)\mu(d_2) \Sigma(H, d_1^2, d_2^2),$$

$$\Sigma(H, d_1^2, d_2^2) = \sum_{\substack{1 \leq x, y \leq H \\% \atop x^2+y^2+1 \equiv 0 (d_1^2) \\% \atop x^2+y^2+2 \equiv 0 (d_2^2)}} 1,$$

$$\sqrt{H} \leq z \leq H.$$

**Estimation of $\Gamma_1(H)$.** When estimating $\Gamma_1(H)$ we will suppose that $q_1 = d_1^2$, $q_2 = d_2^2$, where $d_1$ and $d_2$ are square-free, $(q_1, q_2) = 1$ and $d_1d_2 \leq z$. 
Denote

\[ \Omega(H, q_1, q_2, x) = \sum_{h \leq H \atop h \equiv x (q_1 q_2)} 1. \]

Apparently

\[ \Omega(H, q_1, q_2, x) = H q_1^{-1} q_2^{-1} + O(1). \]

Using (2.4), (4.4) and (4.6) we obtain

\[ \Sigma(H, q_1, q_2) = \sum_{x, y : (2.4)} \Omega(H, q_1, q_2, x) \Omega(H, q_1, q_2, y). \]

On the other hand, (4.6) gives us

\[ \Omega(H, q_1, q_2, y) = \left( \frac{H - y}{q_1 q_2} \right) - \left( \frac{-y}{q_1 q_2} \right) = \frac{H}{q_1 q_2} + \psi \left( \frac{-y}{q_1 q_2} \right) - \psi \left( \frac{H - y}{q_1 q_2} \right). \]

From (4.8) and (4.9) we find that

\[ \Sigma(H, q_1, q_2) = \sum_{x, y : (2.4)} \Omega(H, q_1, q_2, x) \left( \frac{H}{q_1 q_2} - \psi \left( \frac{H - y}{q_1 q_2} \right) \right) + \Sigma', \]

where

\[ \Sigma' = \sum_{x, y : (2.4)} \Omega(H, q_1, q_2, x) \psi \left( \frac{-y}{q_1 q_2} \right). \]

To estimate \( \Sigma' \), we decompose it as

\[ \Sigma' = \Sigma'' + \Sigma''', \]

where

\[ \Sigma'' = \sum_{1 \leq x \leq q_1 q_2} \Omega(H, q_1, q_2, x) \sum_{1 \leq y \leq q_1 q_2 \atop x^2 + 1 \equiv 0 (q_1) \atop x^2 + 2 \equiv 0 (q_2)} \psi \left( \frac{-y}{q_1 q_2} \right), \]

\[ \Sigma''' = \sum_{1 \leq x \leq q_1 q_2} \Omega(H, q_1, q_2, x) \sum_{1 \leq y \leq q_1 q_2 \atop x^2 + 1 \equiv 0 (q_1) \atop x^2 + 2 \equiv 0 (q_2)} \psi \left( \frac{-y}{q_1 q_2} \right). \]

First, we consider \( \Sigma''' \). We note that the sum over \( y \) in (4.13) does not contain terms with \( y = q_1 q_2 / 2 \) or \( y = q_1 q_2 \). Moreover for any \( y \) satisfying the congruences and such that \( 1 \leq y < q_1 q_2 / 2 \) the number \( q_1 q_2 - y \) satisfies the same congruences and we have \( \psi \left( \frac{-y}{q_1 q_2} \right) + \psi \left( \frac{-y}{q_1 q_2} \right) = 0 \). Consequently,

\[ \Sigma''' = 0. \]

Next we consider the sum \( \Sigma'' \). According to the above considerations the sum over \( y \) in (4.12) reduces to a sum with at most two terms (corresponding to
Consecutive square-free \( x^2 + y^2 + 1, \ x^2 + y^2 + 2 \)

\[ y = q_1 q_2 / 2 \text{ and } y = q_1 q_2. \] Therefore

\[
(4.15) \quad \Sigma'' \ll \sum_{1 \leq x \leq q_1 q_2} \Omega(H, q_1, q_2, x).
\]

Now taking into account (4.7), (4.15), the Chinese remainder theorem and the fact that the number of solutions of the congruence \( x^2 \equiv a (q_1 q_2) \) is \( O((q_1 q_2)\varepsilon) = O(H^\varepsilon) \), we get

\[
(4.16) \quad \Sigma'' \ll \sum_{1 \leq x \leq q_1 q_2} \Omega(H, q_1, q_2, x) \ll H^\varepsilon(Hq_1^{-1}q_2^{-1} + 1).
\]

Here \( a \) depends on \( q_1 \) and \( q_2 \).

From (4.11), (4.14) and (4.16) it follows that

\[
(4.17) \quad \Sigma' \ll H^\varepsilon(Hq_1^{-1}q_2^{-1} + 1).
\]

By (4.10) and (4.17) we obtain

\[
(4.18) \quad \Sigma(H, q_1, q_2) = \sum_{x, y: (2.4)} \Omega(H, q_1, q_2, x) \left( \frac{H}{q_1 q_2} - \psi\left(\frac{H - y}{q_1 q_2}\right) \right)
+ O(H^\varepsilon(Hq_1^{-1}q_2^{-1} + 1)).
\]

Proceeding in the same way with the sum \( \Omega(H, q_1, q_2, x) \) we find that

\[
(4.19) \quad \Sigma(H, q_1, q_2) = \sum_{x, y: (2.4)} \left( \frac{H}{q_1 q_2} - \psi\left(\frac{H - x}{q_1 q_2}\right) \right) \left( \frac{H}{q_1 q_2} - \psi\left(\frac{H - y}{q_1 q_2}\right) \right)
+ O(H^\varepsilon(Hq_1^{-1}q_2^{-1} + 1)).
\]

Bearing in mind (2.3), (2.5) and (4.19) we get

\[
(4.20) \quad \Sigma(H, q_1, q_2) = \frac{H^2 \lambda(q_1, q_2)}{q_1^2 q_2^2} - 2 \frac{H}{q_1 q_2} \Sigma_1(H, q_1, q_2)
+ \Sigma_2(H, q_1, q_2) + O(H^\varepsilon(Hq_1^{-1}q_2^{-1} + 1)),
\]

where

\[
(4.21) \quad \Sigma_1(H, q_1, q_2) = \sum_{x, y: (2.4)} \psi\left(\frac{H - x}{q_1 q_2}\right),
\]

\[
(4.22) \quad \Sigma_2(H, q_1, q_2) = \sum_{x, y: (2.4)} \psi\left(\frac{H - x}{q_1 q_2}\right) \psi\left(\frac{H - y}{q_1 q_2}\right).
\]

First we consider \( \Sigma_1(H, q_1, q_2) \). Using (2.3) and Lemma 3.6 with \( H_0 = H \) we obtain

\[
(4.23) \quad \Sigma_1(H, q_1, q_2) = \Sigma'(H, q_1, q_2) + O(\Sigma''(H, q_1, q_2)),
\]
where

\begin{equation}
\Sigma_1''(H, q_1, q_2) = \sum_{x, y \in \mathbb{Z}} \left( \frac{e\left( m\left( H - x \right) \right)}{2\pi im} \right)
\end{equation}

(4.24)

\begin{equation}
\Sigma_1''(H, q_1, q_2) = \sum_{x, y \in \mathbb{Z}} f \left( H, \frac{H - x}{q_1 q_2} \right).
\end{equation}

(4.25)

Formula (4.24) and Lemma 3.5 give us

(4.26)

\[ \Sigma_1'(H, q_1, q_2) \ll H^{\varepsilon} (q_1 q_2)^{1/2}. \]

In order to estimate \( \Sigma_1''(H, q_1, q_2) \) we use (2.3), (2.5), (4.25) and Lemmas 3.3, 3.5, 3.6 to get

(4.27)

\[ \Sigma_1''(H, q_1, q_2) = \sum_{x, y \in \mathbb{Z}} \left( b_H(0) + \sum_{1 \leq |m| \leq H^{1+\varepsilon}} b_H(m) e\left( m\left( H - x \right) \right) \right) + O(1) \]

\[ = b_H(0) \lambda(q_1, q_2) + \sum_{1 \leq |m| \leq H^{1+\varepsilon}} b_H(m) e\left( \frac{mH}{q_1 q_2} \right) \lambda(q_1, q_2, -m, 0) + O(1) \]

\[ \ll H^{\varepsilon-1} q_1 q_2 + 1 + H^{\varepsilon-1} \sum_{1 \leq |m| \leq H^{1+\varepsilon}} |\lambda(q_1, q_2, -m, 0)| \]

\[ \ll H^{\varepsilon-1} q_1 q_2 + 1 + H^\varepsilon \sum_{1 \leq |m| \leq H^{1+\varepsilon}} \frac{|\lambda(q_1, q_2, m, 0)|}{m} \ll H^{\varepsilon-1} q_1 q_2 + H^\varepsilon (q_1 q_2)^{1/2}. \]

From (4.23), (4.26) and (4.27) it follows that

(4.28)

\[ \Sigma_1(H, q_1, q_2) \ll H^{\varepsilon-1} q_1 q_2 + H^\varepsilon (q_1 q_2)^{1/2}. \]

Next we consider the sum \( \Sigma_2(H, q_1, q_2) \). Bearing in mind (4.22), (4.25), (4.27) and Lemmas 3.5 and 3.6 we find that

(4.29)

\[ \Sigma_2(H, q_1, q_2) = \sum_{x, y \in \mathbb{Z}} \sum_{1 \leq |m|, |n| \leq H} \left( \frac{e\left( \frac{(m+n)H}{q_1 q_2} \right)}{(2\pi i)^2 mn} \right) \]

\[ + O\left( H^\varepsilon \Sigma_1''(H, q_1, q_2) \right) \]

\[ = \sum_{1 \leq |m|, |n| \leq H} \left( \frac{e\left( \frac{(m+n)H}{q_1 q_2} \right)}{(2\pi i)^2 mn} \lambda(q_1, q_2, -m, -n) \right) + O\left( H^{\varepsilon-1} q_1 q_2 + H^\varepsilon (q_1 q_2)^{1/2} \right) \]
Consecutive square-free $x^2 + y^2 + 1$, $x^2 + y^2 + 2$

\[
\ll \sum_{1 \leq |m|, |n| \leq H} \left| \frac{\lambda(q_1, q_2, m, n)}{|mn|} \right| + H^{\varepsilon-1} q_1 q_2 + H^\varepsilon (q_1 q_2)^{1/2}
\]

\[
\ll H^{\varepsilon-1} q_1 q_2 + H^\varepsilon (q_1 q_2)^{1/2}.
\]

Taking into account (4.20), (4.28) and (4.29) we get

\[
\Sigma(H, q_1, q_2) = H^2 \frac{\lambda(q_1, q_2)}{q_1^2 q_2^2} + O(H^\varepsilon H q_1^{-1/2} q_2^{-1/2} + q_1^{1/2} q_2^{1/2} + H^{-1} q_1 q_2)).
\]

From (4.2), (4.5) and (4.30) we obtain

\[
\Gamma_1(H) = H^2 \sum_{d_1 d_2 \leq z \atop (d_1, d_2) = 1} \frac{\mu(d_1) \mu(d_2) \lambda(d_1^2, d_2^2)}{d_1^4 d_2^4} + O(H^\varepsilon z^2)
\]

\[
= \sigma H^2 - H^2 \sum_{d_1 d_2 > z \atop (d_1, d_2) = 1} \frac{\mu(d_1) \mu(d_2) \lambda(d_1^2, d_2^2)}{d_1^4 d_2^4} + O(H^\varepsilon z^2),
\]

where

\[
\sigma = \sum_{d_1, d_2 = 1 \atop (d_1, d_2) = 1} ^\infty \frac{\mu(d_1) \mu(d_2) \lambda(d_1^2, d_2^2)}{d_1^4 d_2^4}.
\]

Using (3.2) we find that

\[
\sum_{d_1 d_2 > z \atop (d_1, d_2) = 1} \frac{\mu(d_1) \mu(d_2) \lambda(d_1^2, d_2^2)}{d_1^4 d_2^4} \ll \sum_{d_1 d_2 > z \atop (d_1, d_2) = 1} \frac{(d_1 d_2)^{2+\varepsilon}}{(d_1 d_2)^4} \ll \sum_{n > z} \frac{\tau(n)}{n^{2-\varepsilon}} \ll z^{\varepsilon-1}.
\]

It remains to see that the product (2.7) and the sum (4.32) coincide. From (2.5), (3.6) and $(d_1, d_2) = 1$ it follows that

\[
\lambda(d_1^2, d_2^2) = \lambda(d_1^2, 1) \lambda(1, d_2^2).
\]

Bearing in mind (4.32) and (4.34) we get

\[
\sigma = \sum_{d_1 = 1} ^\infty \frac{\mu(d_1) \lambda(d_1^2, 1)}{d_1^4} \sum_{d_2 = 1} ^\infty \frac{\mu(d_2) \lambda(1, d_2^2)}{d_2^4} f_{d_1}(d_2),
\]

where

\[
f_{d_1}(d_2) = \begin{cases} 
1 & \text{if } (d_1, d_2) = 1, \\
0 & \text{if } (d_1, d_2) > 1.
\end{cases}
\]

Clearly the function

\[
\frac{\mu(d_2) \lambda(1, d_2^2)}{d_2^4} f_{d_1}(d_2)
\]
is multiplicative with respect to $d_2$ and the series
\[
\sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^4} f_{d_1}(d_2)
\]
is absolutely convergent.

Applying the Euler product we obtain
\[
(4.36) \quad \sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^2)}{d_2^4} f_{d_1}(d_2)
= \prod_{p|d_1} \left(1 - \frac{\lambda(1, p^2)}{p^4}\right) \prod_{p} \left(1 - \frac{\lambda(1, p^2)}{p^4}\right)^{-1}.
\]

From (4.35) and (4.36) it follows that
\[
(4.37) \quad \sigma = \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^4} \prod_{p} \left(1 - \frac{\lambda(1, p^2)}{p^4}\right) \prod_{p|d_1} \left(1 - \frac{\lambda(1, p^2)}{p^4}\right)^{-1}
= \prod_{p} \left(1 - \frac{\lambda(1, p^2)}{p^4}\right) \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^4} \prod_{p|d_1} \left(1 - \frac{\lambda(1, p^2)}{p^4}\right)^{-1}.
\]

Obviously the function
\[
\frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^4} \prod_{p|d_1} \left(1 - \frac{\lambda(1, p^2)}{p^4}\right)^{-1}
\]
is multiplicative with respect to $d_1$ and the series
\[
\sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2, 1)}{d_1^4} \prod_{p|d_1} \left(1 - \frac{\lambda(1, p^2)}{p^4}\right)^{-1}
\]
is absolutely convergent. Applying again the Euler product from (4.37) we obtain
\[
(4.38) \quad \sigma = \prod_{p} \left(1 - \frac{\lambda(1, p^2)}{p^4}\right) \prod_{p} \left(1 - \frac{\lambda(p^2, 1)}{p^4} \left(1 - \frac{\lambda(1, p^2)}{p^4}\right)^{-1}\right)
= \prod_{p} \left(1 - \frac{\lambda(p^2, 1) + \lambda(1, p^2)}{p^4}\right).
\]

Bearing in mind (4.5), (4.31), (4.33) and (4.38) we get
\[
(4.39) \quad \Gamma_1(H) = \sigma H^2 + \mathcal{O}(H^\varepsilon (z^2 + H z^{-1})),
\]
where $\sigma$ is given by the product (2.7).
**Estimation of \( \Gamma_2(H) \).** Using (4.3) we write

\[
\left| \Gamma_2(H) \right| \ll (\log H)^2 \sum_{D_1 \leq d_1 < 2D_1} \sum_{D_2 \leq d_2 < 2D_2} \sum_{k \leq (2H^2+1)d_1^{-2}} \sum_{1 \leq x, y \leq H} 1, \quad \text{where} \quad 1/2 \leq D_1, D_2 \leq \sqrt{2H^2+2}, \quad D_1D_2 \geq z/4.
\]

On the one hand, (4.40) gives us

\[
\left| \Gamma_2(H) \right| \ll H^\varepsilon \sum_{D_1 \leq d_1 < 2D_1} \sum_{k \leq (2H^2+1)d_1^{-2}} \sum_{D_2 \leq d_2 < 2D_2} \sum_{l \leq (2H^2+2)D_2^{-2}} 1 \ll H^\varepsilon D_1^{-1}.
\]

On the other hand, (4.40) implies

\[
\left| \Gamma_2(H) \right| \ll H^\varepsilon \sum_{D_2 \leq d_2 < 2D_2} \sum_{l \leq (2H^2+2)D_2^{-2}} \sum_{D_1 \leq d_1 < 2D_1} \sum_{k \leq (2H^2+1)d_1^{-2}} 1 \ll H^{2+\varepsilon} D_2^{-1}.
\]

By (4.41)–(4.43) it follows that

\[
\left| \Gamma_2(H) \right| \ll H^{2+\varepsilon} z^{-1/2}.
\]

**End of the proof.** Bearing in mind (4.1), (4.39) and (4.44) and choosing \( z = H^{4/5} \) we obtain the asymptotic formula (2.6).

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