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Monochromatic sumset without large cardinals

by

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Abstract. We show in this note that in the forcing extension by $\mathrm{Add}(\omega, \beth_\omega)$, the following Ramsey property holds: for any $r \in \omega$ and any $f : \mathbb{R} \to r$, there exists an infinite $X \subset \mathbb{R}$ such that $X + X$ is monochromatic under f. We also show the Ramsey statement above is true in ZFC when $r = 2$. This answers two questions of Komjáth et al. (2019).

1. Introduction

Definition 1.1. Let $(A, +)$ be an additive structure and κ, r be cardinals. Let $A \to^+ (\kappa)_r$ abbreviate the statement: for any $f : A \to r$, there exists $X \subset A$ with $|X| = \kappa$ such that $X + X := \{a + b : a, b \in X\}$ is monochromatic under f.

There have been recent developments on additive partition relations for real numbers. Hindman et al. [7] showed that if $2^\omega < \aleph_\omega$ then there exists some $r \in \omega$ such that $\mathbb{R} \not\to^+ (\aleph_0)_r$. On the other hand, Komjáth et al. [10] showed that relative to the existence of an ω_1-Erdős cardinal, it is consistent that $\mathbb{R} \to^+ (\aleph_0)_r$ for any $r \in \omega$. These results are optimal in a sense, because there are the following restrictions:

(1) Komjáth [9] and independently Soukup and Weiss [14] showed that $\mathbb{R} \not\to^+ (\aleph_1)_2$;
(2) Soukup and Vidnyánszky showed that there exists a finite coloring f on $\mathbb{R}$ such that no infinite $X \subset \mathbb{R}$ has $X + \cdots + X$ (k summands) monochromatic for $k \geq 3$.

It should be emphasized that the difficulty comes from the fact that repetitions are allowed. If we only want some infinite $X \subset \mathbb{R}$ such that

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$X \oplus X = \{a+b : a \neq b \in X\}$ is monochromatic, then the classical theorem of Ramsey implies this already. In fact, Hindman's finite-sum theorem is a much stronger Ramsey-type statement: for any finite coloring of $\mathbb{N}$, there exists some infinite $X \subset \mathbb{N}$ such that $\mathrm{FS}(X) := \{\sum_{0\leq i<k} a_i : \{a_0, a_1, \dots, a_{k-1}\} \in [X]^{<\omega}\}$ is monochromatic. However, if repeated sums are allowed, things turn towards the other direction: Hindman [6] showed $\mathbb{N} \nrightarrow^+ (\aleph_0)_3$ and Owings asked (and it is still open) whether $\mathbb{N} \nrightarrow^+ (\aleph_0)_2$ is true. Interestingly, Fernández-Bretón and Rinot [5] showed that the uncountable analogs of Hindman's theorem must necessarily fail in a strong way.

There have been other results in the similar vein showing that for every finite partition of some large uncountable structure, there must be some countable substructure that is "large" and "well-behaved". The exact meaning of "large" and "well-behaved" depends on specific situations. Sometimes it is not possible to get a homogeneous substructure satisfying some "largeness" requirement, but there are ways to find such a substructure as close to homogeneous as possible. For example, recent work of Raghavan and Todorcevic [11] shows the existence of certain large cardinals in the universe *directly implies* that for any finite partition of $[\mathbb{R}]^2$, there exists a homeomorphic copy of $\mathbb{Q}$, say $X \subset \mathbb{R}$, such that $[X]^2$ intersects at most two elements in the partition.

The following questions among others were asked by the authors of [10].

(1) Is the use of large cardinals necessary to establish the consistency $\mathbb{R} \rightarrow^+ (\aleph_0)_r$ for all $r \in \omega$?
(2) Is $\mathbb{R} \rightarrow^+ (\aleph_0)_2$ true in ZFC?

We answer the first question negatively and the second positively.

THEOREM 1.2.

(1) *In the forcing extension by* $\mathrm{Add}(\omega, \beth_\omega)$, $\mathbb{R} \rightarrow^+ (\aleph_0)_r$ *for any* $r \in \omega$.
(2) $\mathbb{R} \rightarrow^+ (\aleph_0)_2$.

REMARK 1.3. The continuum in the model of [10] is an $\aleph$-fixed point, which is very large. Over a ground model of GCH, Theorem 1.2 suggests that the most natural way to eliminate the obstacles from cardinal arithmetic works since by a result of Hindman et al. [7], if $\mathbb{R} \rightarrow^+ (\aleph_0)_r$ for all $r < \omega$, then $2^\omega \geq \aleph_{\omega+1}$.

NOTATION 1.4. We will identify $(\mathbb{R}, +)$, as a vector space over $\mathbb{Q}$, with $\bigoplus_{i<2^\omega} \mathbb{Q}$. The latter is the direct sum of 2^ω copies of $(\mathbb{Q}, +)$. More concretely, any $s \in \bigoplus_{i<2^\omega}(\mathbb{Q}, +)$ is a finitely supported function whose range is contained in $\mathbb{Q}$. The addition in the direct sum is defined coordinatewise. Similarly for some cardinal κ, $\bigoplus_{i<\kappa} \mathbb{N}$ is the direct sum of κ copies of $(\mathbb{N}, +)$. It is easy to see that if $\kappa \leq 2^\omega$, $\bigoplus_{i<\kappa} \mathbb{N}$ is an additive substructure of $\mathbb{R}$.

2. Proof of Theorem 1.2

DEFINITION 2.1 ([8], [7], [10]). For any $r \geq 2$, define a sequence of finite strings of natural numbers $\langle s_{r,l} : l \leq r\rangle$ such that for each $l \leq r$, $|s_l| = r + l$ and

$$s_l(k) = \begin{cases} 2 & \text{if } k < 2l, \\ 4 & \text{otherwise.} \end{cases}$$

In other words, each s_l is formed by $2l$ many 2's followed by $r - l$ many 4's.

REMARK 2.2. Since r will be fixed in the proof of Theorem 1.2, we will suppress the subscripts and write $\langle s_l : l \leq r\rangle$ for $\langle s_{r,l} : l \leq r\rangle$.

DEFINITION 2.3 (The star operation; see [8], [10]). Let K be either $\mathbb{N}$ or $\mathbb{Q}$. For $k \in \omega$, $s \in (K - \{0\})^k$ and a finite subset of ordinals $a = \{\xi_i : i < k\}_< \subset \lambda$, let $s * a$ denote the function from λ to K supported on a that sends ξ_i to $s(i)$.

We first prove Theorem 1.2(2), which is simpler but contains some ideas which will be used later in the proof of Theorem 1.2(1).

Proof of Theorem 1.2(2). First we use the following partition theorem, which goes back to Dushnik–Miller (see [3, Theorem 11.3]).

CLAIM 2.4. *For any $n, k \in \omega$ and any $f : [\omega_1]^n \to k$, there exists $A \subset \omega_1$ of order type $\omega + 1$ such that $f\restriction A$ is constant.*

Proof of the claim. Given n, k, f as in the claim, pick a countable elementary submodel $N \prec H(\omega_2)$ containing f. We may assume $n \geq 2$. Let $\delta := N \cap \omega_1$. Recursively, we can build an infinite set $A \subset \delta$ such that for any increasing $\{x_i : i < n\} \subset A$, $f(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_{n-2}, \delta)$. To see how this is done, suppose at some stage we have built a finite set A' satisfying the above requirement; we demonstrate how to augment the set by one more element. For each $\bar{y} \in [A']^{n-1}$, let $i_{\bar{y}} = f(\bar{y} \cup \{\delta\}) \in k$. Clearly $\langle i_{\bar{y}} : \bar{y} \in [A']^{n-1}\rangle \in N$. For each $\bar{y} \in [A']^{n-1}$, consider $B_{\bar{y}} = \{\alpha < \omega_1 : f(\bar{y} \cup \{\alpha\}) = i_{\bar{y}}\}$. Then $\bigcap_{\bar{y}\in[A']^{n-1}} B_{\bar{y}}$ is unbounded in δ since $\delta \in \bigcap_{\bar{y}\in[A']^{n-1}} B_{\bar{y}}$ and $\bigcap_{\bar{y}\in[A']^{n-1}} B_{\bar{y}} \in N$. Picking any element from $(\delta \cap \bigcap_{\bar{y}\in[A']^{n-1}} B_{\bar{y}}) - (\sup A' + 1)$ and adding it to A' will maintain the satisfaction of the requirement.

Finally, consider $g : [A]^{n-1} \to k$ defined by $g(\bar{a}) = f(\bar{a} \cup \{\delta\})$. Applying the Ramsey theorem, we get an infinite set $A^* \subset A$ such that g is constant on $[A^*]^{n-1}$ with color $j < k$. It is clear that $f\restriction[A^* \cup \{\delta\}]^n \equiv j$. ∎

Returning back to the proof of Theorem 1.2(2), we actually prove a stronger statement: $\bigoplus_{i<\omega_1} \mathbb{N} \to^+ (\aleph_0)_2$. Recall $\langle s_i : i \leq 2\rangle = \langle s_{2,i} : i \leq 2\rangle$ as in Definition 2.1 and Remark 2.2. Let $f : \bigoplus_{i<\omega_1} \mathbb{N} \to 2$ be given. Let $d_i(\bar{a}) = f(s_i * \bar{a})$ be defined for $i < 3$. In particular, the domainof d_i is

$[\omega_1]^{i+2}$ for $i < 3$. Apply Claim 2.4 to get $A = \{\alpha_j : j \leq \omega\} \in [\omega_1]^{\omega+1}$ such that $d_i \restriction [A]^{i+2} \equiv \rho_i < 2$ for all $i < 3$. By the pigeon-hole principle we have the following cases and we will define $X = \{x_i : i \in \omega\}$ for each case.

(1) $\rho_0 = \rho_1 = \rho$. Let $x_i = \frac{1}{2} s_0 * (\alpha_i, \alpha_\omega)$. Then $f(2x_i) = f(s_0 * (\alpha_i, \alpha_\omega)) = d_0(\alpha_i, \alpha_\omega) = \rho_0 = \rho$. For any $i < j \in \omega$, $f(x_i + x_j) = f(s_1 * (\alpha_i, \alpha_j, \alpha_\omega)) = d_1(\alpha_i, \alpha_j, \alpha_\omega) = \rho_1 = \rho$.
(2) $\rho_0 = \rho_2 = \rho$. Let $x_i = \frac{1}{2} s_0 * (\alpha_{2i}, \alpha_{2i+1})$. Then $f(2x_i) = f(s_0 * (\alpha_{2i}, \alpha_{2i+1})) = d_0(\alpha_{2i}, \alpha_{2i+1}) = \rho_0 = \rho$. For any $i < j \in \omega$, $f(x_i + x_j) = f(s_2 * (\alpha_{2i}, \alpha_{2i+1}, \alpha_{2j}, \alpha_{2j+1})) = d_2(\alpha_{2i}, \alpha_{2i+1}, \alpha_{2j}, \alpha_{2j+1}) = \rho_2 = \rho$.
(3) $\rho_2 = \rho_1 = \rho$. Let $x_i = \frac{1}{2} s_0 * (\alpha_0, \alpha_1, \alpha_{i+2})$. Then we have $f(2x_i) = f(s_0 * (\alpha_0, \alpha_1, \alpha_{i+2})) = d_0(\alpha_0, \alpha_1, \alpha_{i+2}) = \rho_0 = \rho$. For any $i < j \in \omega$, $f(x_i + x_j) = f(s_2 * (\alpha_0, \alpha_1, \alpha_{i+2}, \alpha_{j+2})) = d_2(\alpha_0, \alpha_1, \alpha_{i+2}, \alpha_{j+2}) = \rho_2 = \rho$. ■

Clearly the above proof does not generalize to the case when $r = 3$ since $2^\omega \nrightarrow (\omega+2)^3_2$. A more fundamental restriction is that by a result of Hindman et al. [7], there exists some $r \in \omega$ such that $\bigoplus_{i<\omega_1} \mathbb{N} \nrightarrow^+ (\aleph_0)_r$.

We dedicate the rest of the article to proving Theorem 1.2(1). Let $\lambda = \beth_\omega$ and $\mathbb{P} = \mathrm{Add}(\omega, \lambda)$. In fact, we show that in $V^{\mathbb{P}}$, $\bigoplus_{i<\lambda} \mathbb{N} \rightarrow^+ (\aleph_0)_r$ for any $r \in \omega$.

Definition 2.5. Suppose $W, W' \subset \lambda$ are such that $\mathrm{type}(W) = \mathrm{type}(W')$. Let $h_{W,W'} : W \to W'$ be the unique order isomorphism. For $A, A' \subset \lambda$ with $\mathrm{type}(A) = \mathrm{type}(A')$, $h_{A,A'}$ naturally induces a map from $\mathbb{P}\restriction A$ to $\mathbb{P}\restriction A'$ where any $p \in \mathbb{P}\restriction A$ is mapped to $p' \in \mathbb{P}\restriction A'$ such that $\mathrm{dom}(p') = h_{A,A'}(\mathrm{dom}(p))$ and $p'(j) = p(h^{-1}_{A,A'}(j))$. We will abuse the notation by using $h_{A,A'}$ to denote the induced map from $\mathbb{P}\restriction A$ to $\mathbb{P}\restriction A'$. This can be easily inferred from the context.

We will use the following combinatorial lemma due to Todorčević [15].

Lemma 2.6 (The higher dimensional Δ-system lemma). *Fix $r, d \in \omega$. Let $\langle \dot{d}_i : [\lambda]^i \to r \mid i \leq d\rangle$ be a sequence of $\mathbb{P}$-names for colorings. Then there exist $E \subset \lambda$ of order type ω_1 and $W : [E]^{\leq d} \to [\lambda]^{\leq \aleph_0}$ such that*

CL.1. *For all $u \in [E]^{\leq d}$, $u \subset W(u)$ and $\mathbb{P}\restriction W(u)$ contains a maximal antichain deciding the value of $\dot{d}_{|u|}(u)$.*
CL.2. *For any $u, v \in [E]^{\leq d}$ such that $|u| = |v|$, $\mathrm{type}(W(u)) = \mathrm{type}(W(v))$, $h_{W(u),W(v)}(u) = v$ and for any $p \in \mathbb{P}\restriction W(u)$ and any $n < r$, $p \Vdash \dot{d}_{|u|}(u) = n$ iff $h_{W(u),W(v)}(p) \Vdash \dot{d}_{|v|}(v) = n$.*
CL.3. *For any $u, v \in [E]^{\leq d}$, $W(u) \cap W(v) = W(u \cap v)$.*
CL.4. *For any $u_1 \subset u_2$ and $u'_1 \subset u'_2$ where $u_2, u'_2 \in [E]^{\leq d}$, if $(u_2, u_1, <) \simeq (u'_2, u'_1, <)$, then $h_{W(u_2),W(u'_2)}\restriction W(u_1) = h_{W(u_1),W(u'_1)}$.*

Remark 2.7. The concept of a *double Δ-system* appeared in [15, Sections 2 and 3]; it can be regarded as a blueprintfor the 2-dimensional (that is, when $d = 2$) Δ-system lemma as stated in Lemma 2.6. Different variations of Lemma 2.6 appeared in [13], [12, Lemma 4.1], [2, Claim 7.2.a] and [16, Appendix]. The fact that $\lambda \to (\aleph_1)^{2d}_{2^\omega}$ suffices to get the conclusion of Lemma 2.6. The interested readers are directed to the proofs in [2, Claim 7.2.a] (for **CL.1**–**3**) and in [16, Appendix] (for **CL.4**).

Remark 2.8. The referee pointed out that Lemma 2.6 can also be obtained using Theorem 1 of Baumgartner [1], which is a generalization of the Erdős-Rado theorem [4].

Let $G \subset \mathbb{P}$ be generic over V. In $V[G]$, suppose $f : \bigoplus_{i<\lambda} \mathbb{N} \to r$ is the given coloring. Define $d_i : [\lambda]^{r+i} \to r$ such that $d_i(\bar{a}) = f(s_i * \bar{a})$ for $i \leq r$. Let $\dot{d}_i$ for $i \leq r$ be the corresponding names.

Back in V, apply Lemma 2.6 to $d = 2r$ and $\langle \dot{d}_i : i \leq r\rangle$, and find the desired E and W (strictly speaking, we should apply Lemma 2.6 to the sequence $\langle \dot{d}'_{i+r} : i \leq r\rangle$ where $\dot{d}'_{i+r} = \dot{d}_i$ for $i \leq r$). Enumerate E increasingly as $\{e_i : i \in \omega_1\}$. Let $A_i = \{e_{\omega\cdot i+j} : 1 \leq j \leq \omega\}$ for each $i < r$. For each $i < r$, $j \leq \omega$, let $\alpha^i_j = e_{\omega\cdot i+(1+j)}$.

Definition 2.9. For any $l \leq r$ and any tuple $\bar{s} \in \prod_{i<l}[A_i]^2 \times \prod_{i\geq l, i<r} A_i$, we naturally identify $\bar{s}$ as an $(r+l)$-tuple. To be more concrete, we take two elements from each of the first l sets ordered naturally and one element from each of the remaining sets.

(1) $\bar{s}$ is *l-canonical* if it is of the form

$$(\alpha^0_{i_0}, \alpha^0_{i'_0}, \ldots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_{i'_{l-1}}, \alpha^l_{i_l}, \alpha^{l+1}_{i_{l+1}}, \ldots, \alpha^{r-1}_{i_{r-1}})$$

such that $i_k < i'_k \leq \omega$ and $\max\{i_m : m < r, i_m < \omega\} < i'_k$, for any $k < l$. If, in addition, we are given a sequence $\langle D_i \subset A_i : i < r\rangle$, then we say that $\bar{s}$ is *from* $\langle D_i : i < r\rangle$ if $\bar{s} \in \prod_{i<l}[D_i]^2 \times \prod_{i\geq l, i<r} D_i$.

(2) We call $\bar{i} = \langle i_k : k < r\rangle$ the *index* of an l-canonical tuple $\bar{s}$. We say that $\bar{s}$ is *index-strictly-increasing* if $i_k \leq i_{k'}$ whenever $k < k' < r$, and if $i_k \in \omega$, then $i_k < i_{k'}$.

(3) For any two ordinals α, α', let $\bar{s}_{\alpha\to\alpha'}$ denote the tuple obtained by replacing the occurrence of α in $\bar{s}$ by α'. Similarly for any two sequences of ordinals $\bar{\alpha}, \bar{\alpha}'$ of the same length, $\bar{s}_{\bar{\alpha}\to\bar{\alpha}'}$ denotes the tuple obtained by replacing the occurrence of α_i in $\bar{s}$ by α'_i for each $i < |\bar{\alpha}|$. This notation is used to make the statements of Claims 2.11 and 2.12 precise.

Notation 2.10. In what follows, we frequently confuse a tuple with the set that consists of elements of the tuple, namely $\bar{s} = \langle s_i : i < n\rangle$ is identified with $\{s_i : i < n\}$. It can be mostly inferred from the context, for example $W(\bar{s}) = W(\{s_i : i < n\})$ and $W(\bar{s} \cap \bar{t}) = W(\{s_i : i < n\} \cap \{t_j : j < m\})$ where $\bar{t} = \langle t_j : j < m\rangle$.

Claim 2.11. *In $V[G]$, for any $j < r$ and for any finite $B_i \subset A_i$ with $a^i_\omega \in B_i$ for $i < r$, there exists arbitrarily large $\alpha \in A_j \backslash \{\alpha^j_\omega\}$ such that $\alpha > B_j \backslash \{\alpha^j_\omega\}$ and the following is true: for any $l \leq r$ and any l-canonical tuple $\bar{s}$ from $\langle B_i : i < r\rangle$ containing α^j_ω, $d_l(\bar{s}') = d_l(\bar{s})$ where $\bar{s}' = \bar{s}_{\alpha^j_\omega \to \alpha}$.*

Proof. Fix $j < r$. Work in V. For any given $p \in \mathbb{P}$ and $\gamma \in A_j \backslash \{\alpha^j_\omega\}$, we want to find $p' \leq p$ and $\alpha > \max\{\gamma, \max B_j \backslash \{\alpha^j_\omega\}\}$ in $A_j \backslash \{\alpha^j_\omega\}$ such that p' forces the conclusion above is true for this α. This clearly suffices by the density argument.

Given $p \in \mathbb{P}$, extending it if necessary, we may assume that for each $l \leq r$ and each l-canonical tuple $\bar{s}$ from $\langle B_i : i < r\rangle$, $p \restriction W(\bar{s})$ decides the value of $\dot{d}_l(\bar{s})$. Find $\alpha \in A_j \backslash \{\alpha^j_\omega\}$ large enough such that

- $\alpha > \max\{\max B_j \backslash \{\alpha^j_\omega\}, \gamma\}$,
- $\mathrm{dom}(p) \cap (W(u \cup \{\alpha\}) - W(u)) = \emptyset$ for all $u \in [\bigcup_{i<r} B_i]^{\leq 2r-1}$.

This is possible since $\mathrm{dom}(p)$ is finite and for any fixed $u \in [\bigcup_{i<r} B_i]^{\leq 2r-1}$, $W(u \cup \{\alpha\}) \cap W(u \cup \{\alpha'\}) = W(u)$ for any $\alpha \neq \alpha' > \max u + 1$.

Define $p' = p \cup \bigcup_{l \leq r} \{h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s})) : \bar{s}$ is an l-canonical tuple from $\langle B_i : i < r\rangle, \alpha^j_\omega \in \bar{s}$, and $\bar{s}' = \bar{s}_{\alpha^j_\omega \to \alpha}\}$. We claim that p' is the desired condition. To verify this, it suffices to show the following:

(1) *p' is a condition.* We show p is compatible with $h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s}))$ and $h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s}))$ is compatible with $h_{W(\bar{t}),W(\bar{t}')}(p \restriction W(\bar{t}))$ for each $\bar{s}, \bar{t}$ as above.

- Fix $\bar{s}$. To see p is compatible with $p^* := h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s}))$, notice that $\mathrm{dom}(p) \cap \mathrm{dom}(p^*) \subset \mathrm{dom}(p) \cap W(\bar{s}') \subset W(\bar{s}' - \{\alpha\})$ by the choice of α. By **CL.4**, $h_{W(\bar{s}),W(\bar{s}')} \restriction W(\bar{s} - \{\alpha^j_\omega\})$ is the identity function on $W(\bar{s} - \{\alpha^j_\omega\})$ since $(\bar{s}, \bar{s} - \{\alpha^j_\omega\}, <) \simeq (\bar{s}', \bar{s}' - \{\alpha\}, <)$ and $\bar{s} - \{\alpha^j_\omega\} = \bar{s}' - \{\alpha\}$. Hence $p^* \restriction W(\bar{s}' - \{\alpha\}) = p \restriction W(\bar{s} - \{\alpha^j_\omega\}) = p \restriction W(\bar{s}' - \{\alpha\})$.
- Fix $\bar{s}, \bar{t}$ as above. Let $q_0 := h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s})), q_1 := h_{W(\bar{t}),W(\bar{t}')}(p \restriction W(\bar{t}))$. Then $\mathrm{dom}(q_0) \cap \mathrm{dom}(q_1) \subset W(\bar{s}') \cap W(\bar{t}') = W(\bar{s}' \cap \bar{t}') = W((\bar{s} \cap \bar{t})_{\alpha^j_\omega \to \alpha})$. Observe that $(\bar{s}, \bar{s} \cap \bar{t}, <) \simeq (\bar{s}', \bar{s}' \cap \bar{t}', <)$ and $(\bar{t}, \bar{s} \cap \bar{t}, <) \simeq (\bar{t}', \bar{s}' \cap \bar{t}', <)$. By **CL.4**, we have $h_{W(\bar{s}),W(\bar{s}')}(W(\bar{s} \cap \bar{t})) = W(\bar{s}' \cap \bar{t}')$ and $h_{W(\bar{t}),W(\bar{t}')}(W(\bar{s} \cap \bar{t})) = W(\bar{s}' \cap \bar{t}')$. Hence
$$\begin{aligned} q_0 \restriction W((\bar{s} \cap \bar{t})_{\alpha^j_\omega \to \alpha}) &= h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s} \cap \bar{t})) \\ &= h_{W(\bar{s} \cap \bar{t}),W(\bar{s}' \cap \bar{t}')}(p \restriction W(\bar{s} \cap \bar{t})), \\ q_1 \restriction W((\bar{s} \cap \bar{t})_{\alpha^j_\omega \to \alpha}) &= h_{W(\bar{t}),W(\bar{t}')}(p \restriction W(\bar{s} \cap \bar{t})) \\ &= h_{W(\bar{s} \cap \bar{t}),W(\bar{s}' \cap \bar{t}')}(p \restriction W(\bar{s} \cap \bar{t})). \end{aligned}$$
Since q_0 and q_1 agree on their common domain, it follows that they are compatible.

(2) *p' forces $\dot{d}_l(\bar{s}) = \dot{d}_l(\bar{s}')$ for any l-canonical tuple $\bar{s}$ from $\langle B_i : i < r\rangle$ containing α^j_ω where $\bar{s}' = \bar{s}_{\alpha^j_\omega \to \alpha}$ for any $l \le r$.* Fix l and $\bar{s}$. By the initial assumption about p, we know there exists $n < r$ such that $p \restriction W(\bar{s}) \Vdash \dot{d}_l(\bar{s}) = n$. By **CL.2**, $h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s})) \Vdash \dot{d}_l(\bar{s}') = n$. Hence $p' \Vdash \dot{d}_l(\bar{s}') = n = \dot{d}_l(\bar{s})$. ∎

Claim 2.12. *There exist $C^i \subset A_i$ containing α^i_ω for $i < r$ such that*

(1) *for each $i < r$,* $\mathrm{type}(C^i) = \omega + 1$,
(2) *for each $l \le r$ and each index-strictly-increasing l-canonical tuple*

$$\bar{s} = (\alpha^0_{i_0}, \alpha^0_{i'_0}, \ldots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_{i'_{l-1}}, \alpha^l_{i_l}, \alpha^{l+1}_{i_{l+1}}, \ldots, \alpha^{r-1}_{i_{r-1}})$$

from $\langle C^i : i < r\rangle$ we have $d_l(\bar{s}) = d_l(\bar{s}')$, where

$$\begin{aligned}\bar{s}' &= \bar{s}_{\langle \alpha^j_{i'_j} : j<l\rangle^\frown \langle \alpha^j_{i_j} : r>j\ge l\rangle \to \langle \alpha^j_\omega : j<r\rangle} \\ &= (\alpha^0_{i_0}, \alpha^0_\omega, \ldots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_\omega, \alpha^l_\omega, \alpha^{l+1}_\omega, \ldots, \alpha^{r-1}_\omega).\end{aligned}$$

In particular, the color $\bar{s}$ gets under d_l only depends on its index.

Proof. We will build these sets in ω steps. We will pick one point at a time from sets listed in the following order:

$$A_0, A_1, \ldots, A_{r-1}, A_0, A_1, \ldots, A_{r-1}, A_0, A_1, \ldots, A_{r-1}, \ldots.$$

In particular, we will find $J^i = \{j^i_k : k \in \omega\} \subset \omega$ such that $C^i = \{\alpha^i_{j^i_k} : k \in \omega\} \cup \{\alpha^i_\omega\}$ for each $i < r$. For fixed i, k, let C^i_k denote the set $\{\alpha^i_{j^i_{k'}} : k' \le k\} \cup \{\alpha^i_\omega\}$.

Let $C^i_{-1} = \{\alpha^i_\omega\}$ for all $i < r$. Recursively, suppose for some $i < r$ and $k \in \omega$ we have defined j^p_q for all $p < r$, $q \in \omega \cup \{-1\}$ and $\langle q, p\rangle <_{\mathrm{lex}} \langle k, i\rangle$ (i.e. either $q < k$ or $q = k$ and $p < i$). Apply Claim 2.11 to pick $j^i_k \in \omega$ such that

- $j^i_k > j^p_q$ for all $\langle q, p\rangle <_{\mathrm{lex}} \langle k, i\rangle$,
- for any $l \le r$ and any l-canonical tuple $\bar{s}$ containing α^i_ω from $\langle C^p_{k_p} : p < r\rangle$ where $k_p = k$ if $p < i$ and $k_p = k - 1$ if $p \ge i$, it is true that $d_l(\bar{s}) = d_l(\bar{s}_{\alpha^i_\omega \to \alpha^i_{j^i_k}})$.

We now verify that $\langle C^i : i < r\rangle$ satisfies (2). Fix $l \le r$ and some index-strictly-increasing l-canonical tuple $\bar{s}$ from $\langle C^i : i < r\rangle$, say

$$\bar{s} = (\alpha^0_{i_0}, \alpha^0_{i'_0}, \ldots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_{i'_{l-1}}, \alpha^l_{i_l}, \alpha^{l+1}_{i_{l+1}}, \ldots, \alpha^{r-1}_{i_{r-1}}).$$

By the hypothesis, we know $\max\{i_m : m < r, i_m < \omega\} < i'_k$ for any $k < l$. By the conclusion of Claim 2.11 and the index management in our recursive process, we know that

$$d_l(\bar{s}) = d_l(\bar{s}_{\langle\alpha^j_{i'_j}:j<l\rangle^\frown\langle\alpha^j_{i_j}:r>j\geq l\rangle\to\langle\alpha^j_\omega:j<r\rangle})$$
$$= d_l(\alpha^0_{i_0}, \alpha^0_\omega, \dots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_\omega, \alpha^l_\omega, \alpha^{l+1}_\omega, \dots, \alpha^{r-1}_\omega). \blacksquare$$

By Claim 2.12, we may assume without loss of generality that the sets $\langle A_i : i < r\rangle$ already satisfy the following: for each $l \leq r$, each index-strictly-increasing l-canonical tuple $\bar{s}$ from $\langle A_i : i < r\rangle$ satisfies (2) in the conclusion of Claim 2.12.

To finish the proof, we basically need similar arguments to those in [10, Claim 2.9, Step 5]. We supply a proof for completeness.

Claim 2.13. *There exist* $\langle B_i \subset A_i : i < r,\ \alpha^i_\omega \in B_i, \mathrm{type}(B_i) = \omega+1\rangle$ *and* $\langle \rho_l < r : l \leq r\rangle$ *such that for each* $l \leq r$*, for each index-strictly-increasing* l*-canonical tuple* $\bar{s}$ *from* $\langle B_i : i < r\rangle$*,* $d_l(\bar{s}) = \rho_l$*.*

Proof. Fix $l \leq r$ and $W \in [\omega]^{\aleph_0}$. Define $g : [W]^l \to r$ such that for each $\bar{i} = \{i_0 < i_1 < \cdots < i_{l-1}\}$,

$$g(\bar{i}) = d_l(\alpha^0_{i_0}, \alpha^0_\omega, \dots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_\omega, \alpha^l_\omega, \alpha^{l+1}_\omega, \dots, \alpha^{r-1}_\omega).$$

Let $I \in [W]^{\aleph_0}$ be a monochromatic subset with color ρ_l for g. For any index-strictly-increasing l-canonical tuple

$$\bar{s} = (\alpha^0_{j_0}, \alpha^0_{j'_0}, \dots, \alpha^{l-1}_{j_{l-1}}, \alpha^{l-1}_{j'_{l-1}}, \alpha^l_{j_l}, \alpha^{l+1}_{j_{l+1}}, \dots, \alpha^{r-1}_{j_{r-1}})$$

such that $j_k, j'_t \in I \cup \{\omega\}$ for any $k < r$ and $t < l$, by Claim 2.12 and the remark that follows we know that

$$d_l(\bar{s}) = d_l(\alpha^0_{j_0}, \alpha^0_\omega, \dots, \alpha^{l-1}_{j_{l-1}}, \alpha^{l-1}_\omega, \alpha^l_\omega, \alpha^{l+1}_\omega, \dots, \alpha^{r-1}_\omega)$$
$$= g(\{j_0 < \cdots < j_{l-1}\}) = \rho_l.$$

To get the conclusion of the claim, apply the above procedure repeatedly to get $\omega \supset I_0 \supset I_1 \supset \cdots \supset I_{r-1} \supset I_r$. It is clear that $B_i = \{\alpha^i_j : j \in I_r\} \cup \{\alpha^i_\omega\}$ for $i < r$ will be the desired sets. $\blacksquare$

By Claim 2.13, we may assume without loss of generality that the sets $\langle A_i : i < r\rangle$ already satisfy the following: there exist $\langle \rho_l : l \leq r\rangle$ such that for each $l \leq r$, for each index-strictly-increasing l-canonical tuple $\bar{s}$ from $\langle A_i : i < r\rangle$, $d_l(\bar{s}) = \rho_l$. By the pigeon-hole principle, there exist $l' < l$ such that $\rho_{l'} = \rho_l = \rho$.

Claim 2.14. *There exists an infinite* X *such that* $f{\restriction}(X + X) \equiv \rho$*.*

Proof. For $i < \omega$, let

$$\bar{a}_i = (\alpha^0_0, \alpha^0_\omega, \alpha^1_1, \alpha^1_\omega, \dots, \alpha^{l'-1}_{l'-1}, \alpha^{l'-1}_\omega,$$
$$\alpha^{l'}_{l'+i\cdot(l-l')}, \alpha^{l'+1}_{l'+1+i\cdot(l-l')}, \dots, \alpha^{l-1}_{l-1+i\cdot(l-l')}, \alpha^l_\omega, \dots, \alpha^{r-1}_\omega),$$

that is, we take

- $\{\alpha^k_k, \alpha^k_\omega\}$ from A_k for each $k < l'$,
- $\{\alpha^k_{k+i(l-l')}\}$ from A_k for each $k \geq l'$ and $k < l$,
- $\{\alpha^k_\omega\}$ from A_k for each $k \geq l$.

Define $x_i = \frac{1}{2} s_{l'} * \bar{a}_i$. For $i < j \in \omega$, consider

$$\bar{b}_{i,j} = (\alpha^0_0, \alpha^0_\omega, \ldots, \alpha^{l'-1}_{l'-1}, \alpha^{l'-1}_\omega, \\ \alpha^{l'}_{l'+i\cdot(l-l')}, \alpha^{l'}_{l'+j\cdot(l-l')}, \ldots \alpha^{l-1}_{l-1+i\cdot(l-l')}, \alpha^{l-1}_{l-1+j\cdot(l-l')}, \alpha^l_\omega, \ldots, \alpha^{r-1}_\omega),$$

that is, we take

- $\{\alpha^k_k, \alpha^k_\omega\}$ from A_k for each $k < l'$,
- $\{\alpha^k_{k+i(l-l')}, \alpha^k_{k+j(l-l')}\}$ from A_k for each $k \geq l'$ and $k < l$,
- $\{\alpha^k_\omega\}$ from A_k for each $k \geq l$.

It is not hard to notice that $x_i + x_j = s_l * \bar{b}_{i,j}$. To check that $\bar{b}_{i,j}$ is an l-canonical tuple, let $k < l$ be given. If $k < l'$, then it is immediate that $k < \omega$. If $l' \leq k < l$, then $k+i(l-l') < k+(i+1)(l-l') \leq k+j(l-l')$ since $l-l' > 0$. Also notice that $l'+j(l-l') \geq l'+(i+1)(l-l') = l+i(l-l') > l-1+i(l-l')$.

Therefore, for any $i < j \in \omega$, $\bar{a}_i$ ($\bar{b}_{i,j}$ respectively) is easily seen to be an index-strictly-increasing l'-canonical (l-canonical) tuple. Therefore, $f(2x_i) = f(s_{l'} * \bar{a}_i) = d_{l'}(\bar{a}_i) = \rho_{l'} = \rho$ and $f(x_i+x_j) = f(s_l * \bar{b}_{i,j}) = d_l(\bar{b}_{i,j}) = \rho_l = \rho$. We conclude that $X = \{x_i : i \in \omega\}$ is as desired. ■

Claim 2.14 finishes the proof of Theorem 1.2(1).

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