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A fragment of Asperó–Mota's finitely proper forcing axiom and entangled sets of reals

by

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Abstract. We introduce a fragment $\mathsf{PFA}^{\text{s-fin}}(\omega_1)$ of Asperó–Mota's finitely proper forcing axiom $\mathsf{PFA}^{\text{fin}}(\omega_1)$. $\mathsf{PFA}^{\text{s-fin}}(\omega_1)$ implies some consequences of $\mathsf{PFA}^{\text{fin}}(\omega_1)$, for example $\mathsf{MA}_{\aleph_1}$ and the assertion that any two Aronszajn trees are club-isomorphic. For each integer $k \geq 2$, it is consistent that $\mathsf{PFA}^{\text{s-fin}}(\omega_1)$ holds, there exists a k-entangled set of reals, and $2^{\aleph_0} = \aleph_2$. This extends Abraham–Shelah's theorem that Martin's Axiom does not imply that any two $\aleph_1$-dense sets of reals are isomorphic.

1. introduction. It is a classical theorem due to Cantor that any two countable dense linear orders are isomorphic. But this may not be true for uncountable orders. For example, there is a set of $2^{\aleph_0}$-many pairwise non-order-isomorphic dense subsets of the real line of size $2^{\aleph_0}$ [8]. Baumgartner introduced $\aleph_1$-dense subsets of the real line, which are sets A of the reals such that any nonempty relatively open subset of A is of size $\aleph_1$, and proved that it is consistent that any two $\aleph_1$-dense subsets of the real line are isomorphic [5]. The assertion that any two $\aleph_1$-dense subsets of the real line are isomorphic is sometimes called *Baumgartner's Axiom.* Todorčević pointed out that the Proper Forcing Axiom PFA implies Baumgartner's Axiom [12, 8.3. Corollary], [14, §15.2].

Abraham–Shelah introduced the notion of *k-entangled sets of reals*, for an integer $k \geq 2$ [3]. A k-entangled set of reals is a counterexample to Baumgartner's Axiom, and also to two open coloring axioms due to Abraham–Rubin–Shelah [1] and Todorčević [12, §8], [14, §7.2] respectively. Abraham–Shelah proved that, for each integer $k \geq 2$, it is consistent that Martin's Axiom holds and a k-entangled set of reals exists, hence Baumgartner's Axiom fails [3].

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Asperó–Mota introduced a fragment of PFA, called the *Finitely Proper Forcing Axiom* $\mathsf{PFA}^{\mathrm{fin}}(\aleph_1)$ [2]. $\mathsf{PFA}^{\mathrm{fin}}(\aleph_1)$ implies Martin's Axiom and some consequences of PFA that cannot be deduced from Martin's Axiom. They showed that $\mathsf{PFA}^{\mathrm{fin}}(\aleph_1)$ is consistent with the continuum being larger than $\aleph_2$, by introducing an iteration technique of forcing notions along with Todorčević's side condition method. Since PFA implies that the size of the continuum is $\aleph_2$, $\mathsf{PFA}^{\mathrm{fin}}(\aleph_1)$ is a weak fragment of PFA.

In this paper, we introduce a fragment $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ of $\mathsf{PFA}^{\mathrm{fin}}(\aleph_1)$. Some applications of $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ come from those of $\mathsf{PFA}^{\mathrm{fin}}(\aleph_1)$. For example, $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ implies that $\mathsf{MA}_{\aleph_1}$ holds, that $\mho$ fails, that there are no weak club guessing ladder systems, and that any two Aronszajn trees are club-isomorphic. It is proved that, for each integer $k \geq 2$, it is consistent that $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ holds, there exists a k-entangled set of reals, and $2^{\aleph_0} = \aleph_2$. This extends Abraham–Shelah's theorem that Martin's Axiom does not imply Baumgartner's Axiom. To show this, we modify Asperó–Mota's iteration technique, from side conditions of models, to side conditions of relational structures. Our argument in this paper indicates both wide possibilities and limitations of application of Asperó–Mota iterations.

In §2, we explain k-entangled sets of reals and the related forcing notions. These forcing notions implicitly play a role in successor stages of our forcing iterations in §5. In §3, we introduce s-finite properness and its forcing axiom $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$. In §4, we introduce symmetric systems of relational structures that are key tools of our Asperó–Mota iterations. In §5, we define our forcing iterations, and in §6, we prove our main theorem.

2. k-entangled sets of reals. For each cardinal κ, H_κ denotes the set of all sets of hereditary cardinality less than κ. For ordinals α, β, we denote intervals of ordinals as follows:

$$[\alpha, \beta) := \{\gamma \in \beta : \alpha \leq \gamma\}.$$

Throughout the paper, a *basic open subset* of the finite-dimensional Euclidean space $\mathbb{R}^n$ means a product of rational open intervals of the real line $\mathbb{R}$. For any set X and integer k, $[X]^k$ denotes the set of all subsets of X of size k.

DEFINITION 2.1. Let $E = \{e_\xi : \xi \in \omega_1\}$ be a set of reals such that $e_\xi \neq e_\eta$ for $\xi \neq \eta$.

(1) For $\sigma \in [\omega_1]^k$ and $i \in k$, $\langle e_\xi : \xi \in \sigma\rangle_i$ denotes the ith element of the set $\{e_\xi : \xi \in \sigma\}$ with respect to the usual order $<_{\mathbb{R}}$ of the real line, and $\langle e_\xi : \xi \in \sigma\rangle$ denotes the sequence of length k whose ith coordinate is $\langle e_\xi : \xi \in \sigma\rangle_i$.

(2) For $d \in {}^k\{0,1\}$ and $\sigma, \tau \in [\omega_1]^k$, we define $\sigma <_d \tau$ iff

$$\forall i \in k \ \big(d(i) = 0 \iff \langle e_\xi : \xi \in \sigma\rangle_i <_{\mathbb{R}} \langle e_\xi : \xi \in \tau\rangle_i\big).$$

Define $\sigma \perp_d \tau$ iff neither $\sigma <_d \tau$ nor $\tau <_d \sigma$, and define $\sigma \not\perp_d \tau$ iff $\sigma \perp_d \tau$ does not hold.

(3) For an integer $k \geq 2$ and $d \in {}^k\{0,1\}$, a subset H of $[\omega_1]^k$ is called $\perp_d$*-homogeneous* if $\sigma \perp_d \tau$ for every $\{\sigma, \tau\} \in [H]^2$.

(4) (Abraham–Shelah [3, 20. Definition]) For an integer $k \geq 2$, E is called k*-entangled* if, for any $d \in {}^k\{0,1\}$, there are no uncountable pairwise disjoint $\perp_d$-homogeneous subsets of $[\omega_1]^k$; and E is *entangled* if it is k-entangled for every integer $k \geq 2$.

In other words, E is k-entangled iff, for any $d \in {}^k\{0,1\}$ and any uncountable pairwise disjoint subset I of $[\omega_1]^k$, there exists $\{\sigma, \tau\} \in [I]^2$ such that $\sigma \not\perp_d \tau$.

It follows from the definition that a 2-entangled set of reals is a counterexample to Baumgartner's Axiom. For integers k and k' with $2 \leq k \leq k'$, every k'-entangled set of reals is k-entangled. Note that Cohen forcing preserves any k-entangled set of reals. It is consistent that there exists an entangled set of reals. For example, a set of $\aleph_1$-many Cohen reals is entangled [3, 21. Remark (b)]. $\mathsf{MA}_{\aleph_1}$ implies that there are no entangled sets of reals [3, 21. Remark (c)], [11, Theorem 6]. Abraham–Shelah proved that, for each integer $k \geq 2$, it is consistent that $\mathsf{MA}_{\aleph_1}$ holds, there exists a k-entangled set of reals, and $2^{\aleph_0} = \aleph_2$ [3].

A k-entangled set of reals has the following useful and stronger property. The following proposition and its proof are due to Stevo Todorčević ([1]).

PROPOSITION 2.2 (Todorčević). *For an integer $k \geq 2$ and a set $E = \{e_\xi : \xi \in \omega_1\}$ of reals, the following are equivalent.*

(1) *E is k-entangled.*

(2) *For any $d \in {}^k\{0,1\}$ and any uncountable subset I of $[\omega_1]^k$ with the property that the set $\{\min(\tau) : \tau \in I\}$ is uncountable, there exists $\delta \in \omega_1$ such that, for any $\sigma \in I$ with $\delta \leq \min(\sigma)$, there exists $\tau \in I \cap [\delta]^k$ such that $\tau <_d \sigma$.*

Proof. Assertion (2) implies (1) by the definition of k-entangledness. Suppose now that E is k-entangled; we will show that E satisfies (2).

Let $d \in {}^k\{0,1\}$, and I an uncountable subset of $[\omega_1]^k$ with $\{\min(\tau) : \tau \in I\}$ uncountable. Let M be a countable elementary submodel of $H_{(2^{\aleph_0})^+}$ which

([1]) In the first draft, we introduced this stronger version (2) in Proposition 2.2, showed that a set of $\aleph_1$-many Cohen reals has property (2), and dealt with (2) instead of k-entangledness in the whole paper. Stevo Todorčević pointed out that our stronger version of k-entangledness is equivalent to the original one.

contains the set $\{\langle e_\xi : \xi \in \omega_1 \rangle, I\}$, and define $\delta := \omega_1 \cap M$. We note that $I \cap [\delta]^k = I \cap M$. We will show that, for any $\sigma \in I$ with $\delta \leq \min(\sigma)$, there exists $\tau \in I \cap [\delta]^k$ such that $\tau <_d \sigma$.

First, define

$$J := \{\mu \in I : \tau <_d \mu \text{ for some } \tau \in I\}.$$

We claim that the set $\{\min(\mu) : \mu \in I \setminus J\}$ is countable. Indeed, assume it is not. Then we can find an uncountable pairwise disjoint subset I' of $I \setminus J$. Since E is k-entangled, there exists $\{\sigma, \tau\} \in [I']^2$ such that either $\sigma <_d \tau$ or $\tau <_d \sigma$. However, then either τ or σ belongs to J, a contradiction.

Let $\sigma \in I$ with $\delta \leq \min(\sigma)$. Since the set J above belongs to M, so does $I \setminus J$. Hence $\{\min(\mu) : \mu \in I \setminus J\}$ is a subset of M, so $\sigma \in J$. Therefore, there exists $\tau \in I$ such that $\tau <_d \sigma$. Take a set $\{B_i : i \in 2k\}$ of pairwise disjoint basic open subsets of $\mathbb{R}$ (rational open intervals) such that $\langle e_\xi : \xi \in \tau \rangle \in \prod_{i \in k} B_i$ and $\langle e_\xi : \xi \in \sigma \rangle \in \prod_{i \in k} B_{k+i}$. Since both I and $\{B_i : i \in k\}$ belong to M, by elementarity of M we may assume that τ also belongs to M, which finishes the proof. ∎

NOTATION 2.3. *Throughout, we assume that k is an integer ≥ 2, and $E = \{e_\xi : \xi \in \omega_1\}$ is a k-entangled set of reals such that $e_\xi \neq e_\eta$ for any $\xi \neq \eta$. In the rest of the paper, we always refer to* (2) *of Proposition* 2.2 *as k-entangledness.*

Let P be a forcing notion (of size $\aleph_1$) which preserves ω_1 such that

$$\not\Vdash_P \text{“} E \text{ is } k\text{-entangled”}.$$

Then there are $p \in P$, $d \in {}^k\{0,1\}$ and a P-name $\dot{I}$ such that

$p \Vdash_P$ “$\dot{I} \subseteq [\omega_1]^k$ is uncountable with $\{\min(\tau) : \tau \in \dot{I}\}$ uncountable, and for any $\delta \in \omega_1$ there exists $\sigma \in \dot{I}$ such that $\delta \leq \min(\sigma)$ and for any $\tau \in \dot{I} \cap [\delta]^k$, $\tau \not<_d \sigma$”.

By shrinking $\dot{I}$, we may assume that

$$p \Vdash_{\mathbb{P}} \text{“ for every } \{\sigma, \tau\} \in [\dot{I}]^2 \text{ with } \max(\tau) < \min(\sigma), \ \tau \not<_d \sigma \text{”}.$$

Let κ be an (arbitrary) large enough regular cardinal such that the set $\{P, \dot{I}\}$ belongs to H_κ. Then we define the set $S(P, p, d, \dot{I}) = S(P)$ such that

$$S(P) := \left\{ \Sigma \in \left[[\omega_1]^k \right]^{k+2} : \text{for some } q \in \mathbb{P} \text{ with } q \leq_P p, \ q \Vdash_P \text{“ } \Sigma \subseteq \dot{I} \text{”} \right\}.$$

Define the forcing notion $\mathcal{A}(P, p, d, \dot{I}) = \mathcal{A}(P)$ that consists of the triples $\langle \mathcal{N}_0, \mathcal{N}_1, W \rangle$ such that

- $\mathcal{N}_0$ is a finite $\in$-chain of countable elementary submodels of H_κ which contain the set $\{E, P, p, d, \dot{I}\}$,
- $\mathcal{N}_1 \subseteq \mathcal{N}_0$,
- W is a finite subset of $S(P) \times \omega$; for each $x \in W$, write $x = \langle \Sigma^x, n^x \rangle$,

- for each $x \in W$, Σ^x is *separated* by $\mathcal{N}_0$, that is, for each $\{\sigma_0, \sigma_1\} \in [\Sigma^x]^2$, there are $N \in \mathcal{N}_0$ and $i \in \{0,1\}$ such that $\sigma_i \in N$ and $\sigma_{1-i} \cap N = \emptyset$; for each $x \in W$, we enumerate Σ^x as $\{\sigma^x_i : i \in k+2\}$ so that, for each $i \in k+1$, there is $N \in \mathcal{N}_0$ such that $\sigma^x_i \in N$ and $\sigma^x_{i+1} \notin N$ (that is, $\sigma^x_{i+1} \cap N = \emptyset$),
- the sequence $\langle \Sigma^x : x \in W \rangle$ is also separated by $\mathcal{N}_0$, that is, for each $\{x_0, x_1\} \in [W]^2$, there are $N \in \mathcal{N}_0$ and $i \in \{0,1\}$ such that $\Sigma^{x_i} \in N$ and $\Sigma^{x_{1-i}} \cap N = \emptyset$,
- for each $x \in W$, Σ^x is *not* separated by any member of $\mathcal{N}_1$, that is, for any $N \in \mathcal{N}_1$, either $\max(\bigcup \Sigma^x) < \omega_1 \cap N$ or $\omega_1 \cap N \leq \min(\bigcup \Sigma^x)$, and

★ for any $\{x, y\} \in [W]^2$, if $\forall \mu \in \Sigma^x\ \forall \nu \in \Sigma^y\ (\mu \not<_d \nu)$ or $\forall \mu \in \Sigma^x\ \forall \nu \in \Sigma^y\ (\nu \not<_d \mu)$, then $n^x \neq n^y$,

and for each $\langle \mathcal{N}_0, \mathcal{N}_1, W \rangle, \langle \mathcal{N}'_0, \mathcal{N}'_1, W' \rangle \in \mathcal{A}(P)$, define $\langle \mathcal{N}_0, \mathcal{N}_1, , W \rangle \leq_{\mathcal{A}(P)} \langle \mathcal{N}'_0, \mathcal{N}'_1, W' \rangle$ iff $\mathcal{N}_0 \supseteq \mathcal{N}'_0$, $\mathcal{N}_1 \supseteq \mathcal{N}'_1$, and $W \supseteq W'$. We notice that, for a condition $\langle \mathcal{N}_0, \mathcal{N}_1, W \rangle$ of $\mathcal{A}(P)$, W forms a function from a finite subset of $S(P)$ into ω.

For each $\alpha \in \omega_1$ and $p' \leq_P p$, the set

$$\Big\{ \langle \mathcal{N}_0, \mathcal{N}_1, W \rangle \in \mathcal{A}(P) : \text{for some } x \in W \text{ and } q \in P,\ q \leq_P p', \\ q \Vdash_{\mathbb{P}} \text{“} \Sigma^x \subseteq \dot{I} \text{”, and } \alpha \leq \min\Big(\bigcup \Sigma^x\Big) \Big\}$$

is dense in $\mathcal{A}(P)$. For $\langle \mathcal{N}_0, \mathcal{N}_1, W \rangle \in \mathcal{A}(P)$ and $x, y \in W$, if there exists $q \in \mathbb{P}$ such that

$$q \Vdash_P \text{“} \Sigma^x \cup \Sigma^y \subseteq \dot{I} \text{”},$$

then, by the property of $\dot{I}$ and the fact that either $\max(\bigcup \Sigma^x) < \min(\bigcup \Sigma^y)$ or $\max(\bigcup \Sigma^y) < \min(\bigcup \Sigma^x)$, it follows from ★ above that $n^x \neq n^y$. Thus,

$\Vdash_{\mathcal{A}(P)}$“ $\Vdash_P$ “if

$$\dot{Y} := \Big\{ \Sigma^x : x \in \bigcup_{\langle \mathcal{N}_0, \mathcal{N}_1, W \rangle \in \dot{G}_{\mathcal{A}(P)}} W, \text{ and for some } q \in \dot{G}_P,\ q \Vdash_P \text{“} \Sigma^x \subseteq \dot{I} \text{”} \Big\},$$

then the set $\{\min(\bigcup \Sigma) : \Sigma \in \dot{Y}\}$ is cofinal in ω_1^V, and the map $\dot{Y} \ni \Sigma \mapsto n \in \omega$ is injective, where n is unique such that $\langle \Sigma, n \rangle \in \bigcup_{\langle \mathcal{N}_0, \mathcal{N}_1, W \rangle \in \dot{G}_{\mathcal{A}(P)}} W$, and for some $q \in \dot{G}_P$, $q \Vdash_P$ “$\Sigma \subseteq \dot{I}$””””.

Therefore, if $\mathcal{A}(P)$ does not collapse $\aleph_1$, then

$$\Vdash_{\mathcal{A}(P)} \text{“} \aleph_1 \text{ is still uncountable, and } \Vdash_P \text{“} \aleph_1 \text{ is countable”}, \\ \text{hence } P \text{ collapses } \aleph_1 \text{”}.$$

In fact, it will be proved that $\mathcal{A}(P)$ is proper and forces that E is still k-entangled (Lemmas 2.6 and 2.7). Lemma 2.5 is the key to the proof of results about $\mathcal{A}(P)$. The next lemma is the first step towards Lemma 2.5.

Here, for each $\tau \in [\omega_1]^k$, we denote the (ordinal-)order increasing enumeration of the set τ by $\{\pi(\tau, i) : i \in k\}$.

LEMMA 2.4. *Suppose that*

- *κ is a regular cardinal greater than $\aleph_1$,*
- *M is a countable elementary submodel of H_κ,*
- *I is an uncountable subset of $[\omega_1]^k$ with $I \in M$,*
- *$\sigma \in I$ with $\omega_1 \cap M \le \min(\sigma)$ (then the set $\{\min(\tau) : \tau \in I\}$ is uncountable),*
- *$\{B_i : i \in k\}$ is a set of pairwise disjoint basic open subsets of $\mathbb{R}$ such that $\langle e_\xi : \xi \in \sigma\rangle \in \prod_{i\in k} B_i$ (then $\sup(B_i) < \inf(B_{i+1})$ by our notation of the sequence $\langle e_\xi : \xi \in \sigma\rangle$),*
- *$\gamma \in \omega_1 \cap M$, and*
- *$d \in {}^k\{0,1\}$.*

Then there are a set $\{C_i^0, C_i^1 : i \in k\}$ of basic open subsets of $\mathbb{R}$ and $\tau \in I \cap M$ such that

- *for each $i \in k$, both C_i^0 and C_i^1 are subsets of B_i,*
- *for each $i \in k$, $C_i^0 \cap C_i^1 = \emptyset$,*
- *$\langle e_\xi : \xi \in \sigma\rangle \in \prod_{i\in k} C_i^0$ and $\langle e_\xi : \xi \in \tau\rangle \in \prod_{i\in k} C_i^1$,*
- *$\gamma \le \min(\tau)$,*
- *for any $i, i' \in k$, $e_{\pi(\tau,i)} \in C_{i'}^1 \leftrightarrow e_{\pi(\sigma,i)} \in C_{i'}^0$, and*
- *$\tau <_d \sigma$.*

Proof. Define

$$I' := \{\mu \in I : \gamma \le \min(\mu) \ \&\ \forall i, i' \in k\ (e_{\pi(\mu,i)} \in B_{i'} \leftrightarrow e_{\pi(\sigma,i)} \in B_{i'})\}.$$

Though σ is not a member of M, the set

$$\{\langle i, i'\rangle \in k^2 : e_{\pi(\sigma,i)} \in B_{i'}\}$$

is a member of M. So I' belongs to M by the elementarity of M. Since $\sigma \in I'$ and $\sigma \cap M = \emptyset$ (that is, $\omega_1 \cap M \le \min(\sigma)$), the set $\{\min(\tau) : \tau \in I'\}$ is uncountable. As E is k-entangled in M, there exists $\delta \in \omega_1 \cap M$ which witnesses the k-entangledness for I', that is, for any $\nu \in I$ with $\delta \le \min(\nu)$, there exists $\mu \in I \cap [\delta]^k$ such that $\mu <_d \nu$. Therefore, we can find $\tau \in I' \cap [\delta]^k$ such that $\tau <_d \sigma$. For each $i \in k$, we take pairwise disjoint basic open subsets C_i^0 and C_i^1 of B_i (rational open intervals) such that for the unique $i' \in k$ with $e_{\pi(\sigma,i')} \in B_i$, we have $e_{\pi(\sigma,i')} \in C_i^0$ and $e_{\pi(\tau,i')} \in C_i^1$. This finishes the proof. ∎

LEMMA 2.5. *Suppose that*

- *κ is a regular cardinal greater than $\aleph_1$,*
- *$\{M_j : j \in h\}$ is an $\in$-chain of countable elementary submodels of H_κ,*
- *X is an uncountable subset of $\left[[\omega_1]^k\right]^h$ with $X \in M_0$,*
- *$\{\sigma_j : j \in h\} \in X$, $\sigma_j \cap M_j = \emptyset$ for each $j \in h$, and $\sigma_j \in M_{j+1}$ for each $j \in h-1$ (then $\{\min(\bigcup z) : z \in X\}$ is uncountable),*

- *$\{B_{j,i} : j \in h, i \in k\}$ is a set of pairwise disjoint basic open subsets of $\mathbb{R}$ such that, for each $j \in h$, $\langle e_\xi : \xi \in \sigma_j\rangle \in \prod_{i\in k} B_{j,i}$,*
- *$\gamma \in \omega_1 \cap M_0$, and*
- *$d_j \in {}^k\{0,1\}$ for each $j \in h$.*

Then there are a set $\{C^0_{j,i}, C^1_{j,i} : j \in h,\ i \in k\}$ of basic open sets of $\mathbb{R}$ and $\{\tau_j : j \in h\} \in X \cap M_0$ such that

- *for each $j \in h$ and $i \in k$, both $C^0_{j,i}$ and $C^1_{j,i}$ are subsets of $B_{j,i}$,*
- *for each $j \in h$ and $i \in k$, $C^0_{j,i} \cap C^1_{j,i} = \emptyset$,*
- *for each $j \in h$, $\langle e_\xi : \xi \in \sigma_j\rangle \in \prod_{i\in k} C^0_{j,i}$,*
- *for each $j \in h$, $\langle e_\xi : \xi \in \tau_j\rangle \in \prod_{i\in k} C^1_{j,i}$,*
- *$\gamma \le \min(\tau_0)$ and, for each $j \in h-1$, $\max(\tau_j) < \min(\tau_{j+1})$,*
- *for any $j \in h$ and $i, i' \in k$, $e_{\pi(\tau_j,i)} \in C^1_{j,i'} \leftrightarrow e_{\pi(\sigma_j,i)} \in C^0_{j,i'}$, and*
- *for each $j \in h$, $\tau_j <_{d_j} \sigma_j$.*

Proof. This is proved by induction on h. Define

$$X_0 := \Big\{\{\mu_j : j \in h\} \in X : \text{for each } j \in h-1,\ \max(\mu_j) < \min(\mu_{j+1}), \\ \text{and, for any } j \in h,\ \langle e_\xi : \xi \in \sigma_j\rangle \in \prod_{i\in k} B_{j,i}\Big\},$$

and

$$X_1 := \Big\{\{\mu_j : j \in h-1\} \in \big[[\omega_1]^k\big]^{h-1} : \{\sigma_0\} \cup \{\mu_j : j \in h-1\} \in X_0 \\ \text{and } \max(\sigma_0) < \min\Big(\bigcup_{i\in h-1} \mu_i\Big)\Big\}.$$

Then $X_0 \in M_0$ and $\{\sigma_{1+j} : j \in h-1\} \in X_1 \in M_1$. Thus by the inductive hypothesis, we obtain a set $\{C^0_{1+j,i}, C^1_{1+j,i} : j \in h-1,\ i \in k\}$ of basic open subsets of $\mathbb{R}$ and $\{\tau'_{1+j} : j \in h-1\} \in X_1 \cap M_1$ as in the conclusion of the lemma. Define

$$I := \Big\{\mu \in [\omega_1]^k : \text{there exists } \{\mu_{1+j} : j \in h-1\} \text{ such that} \\ \bullet\ \{\mu\} \cup \{\mu_{1+j} : j \in h-1\} \in X_0, \\ \bullet\ \text{for each } j \in h-1,\ \langle e_\xi : \xi \in \mu_{1+j}\rangle \in \textstyle\prod_{i\in k} C^1_{1+j,i}, \\ \bullet\ \max(\mu) < \min(\mu_1) \text{ and, for each } j \in h-2, \\ \max(\mu_{1+j}) < \min(\mu_{1+j+1})\Big\}.$$

Then I belongs to M_0 and the set $\{\tau'_{1+j} : j \in h-1\}$ witnesses that σ_0 belongs to I. Therefore, by applying Lemma 2.4, we obtain a set $\{C^0_{0,i}, C^1_{0,i} : i \in k\}$ of basic open subsets of $\mathbb{R}$ and $\tau_0 \in I \cap M_0$ as in the conclusion of Lemma 2.4. Take $\{\tau_{1+j} : j \in h-1\} \in M_0$ that witnesses $\tau_0 \in I$; this finishes the proof. ∎

By use of Lemma 2.5, it will be proved that $\mathcal{A}(P)$ is proper and preserves the k-entangledness of E. In the following proofs, we do not need to take care of what is the definition of the generic condition.

LEMMA 2.6. *$\mathcal{A}(P)$ is proper.*

Proof. Suppose that N is a countable elementary submodel of H_θ for some large enough regular cardinal θ such that N contains $\{E, P, \dot{I}, p, d, H_\kappa\}$, and that $\langle \mathcal{N}_0, \mathcal{N}_1, W\rangle \in \mathcal{A}(P) \cap N$. Then the triple $\langle \mathcal{N}_0 \cup \{N \cap H_\kappa\}, \mathcal{N}_1 \cup \{N \cap H_\kappa\}, W\rangle$ is an extension of $\langle \mathcal{N}_0, \mathcal{N}_1, W\rangle$ in $\mathcal{A}(P)$.

We now show that $\langle \mathcal{N}_0 \cup \{N \cap H_\kappa\}, \mathcal{N}_1 \cup \{N \cap H_\kappa\}, W\rangle$ is $(N, \mathcal{A}(P))$-generic. To do so, suppose that $\mathcal{D} \in N$ is a dense open subset of $\mathcal{A}(P)$ and $\langle \mathcal{N}_0', \mathcal{N}_1', W'\rangle \in \mathcal{D}$ is an extension of $\langle \mathcal{N}_0 \cup \{N \cap H_\kappa\}, \mathcal{N}_1 \cup \{N \cap H_\kappa\}, W\rangle$ in $\mathcal{A}(P)$. We will find $\langle \mathcal{N}_0'', \mathcal{N}_1'', W''\rangle \in \mathcal{D} \cap N$ which is compatible with $\langle \mathcal{N}_0', \mathcal{N}_1', W'\rangle$ in $\mathcal{A}(P)$.

Since $N \cap H_\kappa \in \mathcal{N}_1'$, N does not separate any member of W'. Hence,

$$\{\Sigma^x : x \in W'\} \setminus N = \{\Sigma^x : x \in W' \setminus N\}.$$

Note that $\bigcup\{\Sigma^x : x \in W' \setminus N\}$ is separated by $\mathcal{N}_0'$. Let $\{x_\zeta : \zeta \in l\}$ be the increasing enumeration of $W' \setminus N$ with respect to $\mathcal{N}_0'$, that is, if $\zeta < \zeta' < l$, then

$$\max\Big(\bigcup \Sigma^{x_\zeta}\Big) < \min\Big(\bigcup \Sigma^{x_{\zeta'}}\Big).$$

Take a set $\{B^\zeta_{j,i} : \zeta \in l,\ j \in k+2,\ i \in k\}$ of pairwise disjoint basic open sets of $\mathbb{R}$ such that for each $\zeta \in l$ and $j \in k+2$, $\langle e_\xi : \xi \in \sigma_j^{x_\zeta}\rangle \in \prod_{i\in k} B^\zeta_{j,i} =: B^\zeta_j$. Then, for any $d \in {}^k\{0,1\}$, $\zeta, \zeta' \in l$, $j, j' \in k+2$, and $\tau, \tau' \in [\omega_1]^k$ with $\langle e_\xi : \xi \in \tau\rangle \in B^\zeta_j$ and $\langle e_\xi : \xi \in \tau'\rangle \in B^{\zeta'}_{j'}$, we have

$$\tau <_d \tau' \iff \sigma_j^{x_\zeta} <_d \sigma_{j'}^{x_{\zeta'}}.$$

Define

$X := \big\{\{\Sigma^y : y \in W'' \setminus (W' \cap N)\} : -\ \langle \mathcal{N}_0'', \mathcal{N}_1'', W''\rangle \in \mathcal{D}$,

- $\mathcal{N}_0''$ end-extends $\mathcal{N}_0' \cap N$ and $\mathcal{N}_1''$ end-extends $\mathcal{N}_1' \cap N$,
- W'' end-extends $W' \cap N$ and $W'' \setminus (W' \cap N)$ is of size l; let $\langle y_\zeta : \zeta \in l\rangle$ be the increasing enumeration of $W'' \setminus (W' \cap N)$ with respect to $\mathcal{N}_0''$,
- for any $\zeta \in l$, $n^{y_\zeta} = n^{x_\zeta}$, and
- for any $\zeta \in l$ and $j \in k+2$, $\langle e_\xi : \xi \in \sigma_j^{y_\zeta}\rangle \in B^\zeta_j\big\}$.

We note that $X \in N$ and $\{\Sigma^{x_\zeta} : \zeta \in l\} \in X$. Moreover, $\bigcup\{\Sigma^{x_\zeta} : \zeta \in l\}$ is separated by $\mathcal{N}_0'$. So by applying Lemma 2.5 to the sets $\{\bigcup Y : Y \in X\}$, $\bigcup\{\Sigma^{x_\zeta} : \zeta \in l\}$, $\{B^\zeta_{j,i} : \zeta \in l,\ j \in k+2,\ i \in k\}$ and functions d and $1-d$ suitably, there exists $\{\Sigma^\zeta : \zeta \in l\} \in X \cap N$ such that, for each $\zeta \in l$, letting τ_0^ζ be the first member of Σ^ζ and τ_1^ζ the second member, we have $\tau_0^\zeta <_d \sigma_0^{x_\zeta}$ and $\tau_1^\zeta <_{1-d} \sigma_1^{x_\zeta}$ (that is, $\sigma_1^{x_\zeta} <_d \tau_1^\zeta$). Then we can take $\langle \mathcal{N}_0'', \mathcal{N}_1'', W''\rangle \in \mathcal{D} \cap N$ which witnesses that $\{\Sigma^\zeta : \zeta \in l\} \in X$, that is,

- $\mathcal{N}_0''$ end-extends $\mathcal{N}_0' \cap N$ and $\mathcal{N}_1''$ end-extends $\mathcal{N}_1' \cap N$,
- W'' end-extends $W' \cap N$ and $W'' \setminus (W' \cap N)$ is of size l; let $\langle y_\zeta : \zeta \in l\rangle$ be the increasing enumeration of $W'' \setminus (W' \cap N)$,
- for any $\zeta \in l$, $n^{y_\zeta} = n^{x_\zeta}$ and $\Sigma^{y_\zeta} = \Sigma^\zeta$, and
- for any $\zeta \in l$ and $j \in k+2$, $\langle e_\xi : \xi \in \sigma_j^{y_\zeta}\rangle \in B_j^\zeta$.

We claim that $W' \cup W''$ satisfies ★ in the definition of $\mathcal{A}(P)$. For each $\zeta \in l$, by the choice of Σ^ζ, Σ^{y_ζ} and Σ^{x_ζ} do not satisfy the assumption of ★. Suppose that $\{\zeta, \zeta'\} \in [l]^2$ and $n^{y_\zeta} = n^{x_{\zeta'}}$. Then since $n^{x_\zeta} = n^{y_\zeta} = n^{x_{\zeta'}}$, the sets Σ^{x_ζ} and $\Sigma^{x_{\zeta'}}$ do not satisfy the assumption of ★. Hence, by the role of intervals $B_{j,j}^\zeta$, the sets Σ^{y_ζ} and $\Sigma^{x_{\zeta'}}$ do not satisfy that assumption either.

Therefore $\langle \mathcal{N}_0' \cup \mathcal{N}_0'', \mathcal{N}_1' \cup \mathcal{N}_1'', W' \cup W''\rangle$ is a condition of $\mathcal{A}(P)$, and hence is a common extension of $\langle \mathcal{N}_0', \mathcal{N}_1', W'\rangle$ and $\langle \mathcal{N}_0'', \mathcal{N}_1'', W''\rangle$. ■

LEMMA 2.7. *$\mathcal{A}(P)$ preserves the k-entangledness of E.*

Proof. Suppose that $d' \in {}^k\{0,1\}$, $\dot J$ is an $\mathcal{A}(P)$-name for an uncountable subset of $[\omega_1]^k$, $\langle \mathcal{N}_0, \mathcal{N}_1, W\rangle \in \mathcal{A}(P)$, and N is a countable elementary submodel of H_θ for some large enough regular cardinal θ such that N contains $\{E, P, \dot I, p, \dot J, \langle \mathcal{N}_0, \mathcal{N}_1, W\rangle, H_\kappa\}$. As seen above, $\langle \mathcal{N}_0 \cup \{N \cap H_\kappa\}, \mathcal{N}_1 \cup \{N \cap H_\kappa\}, W\rangle$ is an $(N, \mathcal{A}(P))$-generic condition. We will show that

$$\langle \mathcal{N}_0 \cup \{N \cap H_\kappa\}, \mathcal{N}_1 \cup \{N \cap H_\kappa\}, W\rangle \Vdash_{\mathcal{A}(P)} \text{“ for every } \nu \in \dot J \text{ with}$$
$$\omega_1 \cap N \le \min(\nu), \text{ there exists } \tau \in \dot J \cap N[\dot G_{\mathcal{A}(P)}] \text{ such that } \tau <_{d'} \nu\text{”.}$$

Take $\langle \mathcal{N}_0', \mathcal{N}_1', W'\rangle \in \mathcal{A}(P)$ and $\nu \in [\omega_1]^k$ such that $\langle \mathcal{N}_0', \mathcal{N}_1', W'\rangle \le_{\mathcal{A}(P)} \langle \mathcal{N}_0 \cup \{N \cap H_\kappa\}, \mathcal{N}_1 \cup \{N \cap H_\kappa\}, W\rangle$, $\nu \cap N = \emptyset$ and

$$\langle \mathcal{N}_0', \mathcal{N}_1', W'\rangle \Vdash_{\mathcal{A}(P)} \text{“ } \nu \in \dot J\text{”.}$$

Let $\{x_\zeta : \zeta \in l\}$ be an increasing enumeration of $W' \setminus N$ with respect to $\mathcal{N}_0'$. Take a set $\{B_{j,i}^\zeta : \zeta \in l,\ j \in k+2,\ i \in k\}$ of pairwise disjoint basic open subsets of $\mathbb{R}$ such that, for each $\zeta \in l$ and $j \in k+2$, $\langle e_\xi : \xi \in \sigma_j^{x_\zeta}\rangle \in \prod_{i\in k} B_{j,i}^\zeta =: B_j^\zeta$. We recall that each Σ^{x_ζ} is separated by $\mathcal{N}_0'$ into $k+2$ pieces. Since ν is of size k, there exists $\{j_0, j_1\} \in [k+2]^2$ and a partition $\nu = \bigcup_{u\in h} \nu_u$ such that the set $\{\sigma_{j_0}^{x_\zeta}, \sigma_{j_1}^{x_\zeta} : \zeta \in l\} \cup \{\nu_u : u \in h\}$ is separated by $\mathcal{N}_0'$. For each $u \in h$, take $\bar\nu_u \in [\omega_1]^k$ such that

- $\bar\nu_u$ end-extends ν_u, and
- the set $\{\sigma_{j_0}^{x_\zeta}, \sigma_{j_1}^{x_\zeta} : \zeta \in l\} \cup \{\bar\nu_u : u \in h\}$ is also separated by $\mathcal{N}_0'$.

For each $u \in h$, let $\nu_u = \{\xi_v^u : v \in |\nu_u|\}$ be the increasing enumeration. For each $u \in h$, take $d_u' \in {}^k\{0,1\}$ such that, for each $v \in |\nu_u|$, letting $i, i' \in k$ be the unique indices such that

$$e_{\xi^u_v} = \langle e_\xi : \xi \in \nu \rangle_i = \langle e_\xi : \xi \in \bar{\nu}_u \rangle_{i'},$$

we have

$$d'_u(i') := d'(i).$$

Take a set $\{B_i^{\zeta,0}, B_i^{\zeta,1}, B_i^{l,u} : \zeta \in l,\ u \in h,\ i \in k\}$ of pairwise disjoint basic open subsets of $\mathbb{R}$ such that, for each $\zeta \in l$, $\langle e_\xi : \xi \in \sigma_{j_0}^{x_\zeta} \rangle \in \prod_{i \in k} B_i^{\zeta,0} =: B^{\zeta,0}$ and $\langle e_\xi : \xi \in \sigma_{j_1}^{x_\zeta} \rangle \in \prod_{i \in k} B_i^{\zeta,1} =: B^{\zeta,1}$, and, for each $u \in h$, $\langle e_\xi : \xi \in \bar{\nu}_u \rangle$ is in $\prod_{i \in k} B_i^{l,u} =: B^{l,u}$. Define

$X := \big\{ \{\sigma_{j_0}^{y_\zeta}, \sigma_{j_1}^{y_\zeta} : \zeta \in l\} \cup \{\mu_u : u \in h\} \in \big[[\omega_1]^k\big]^{2l+h}$: there exists $\langle \mathcal{N}_0'', \mathcal{N}_1'', W'' \rangle \in \mathcal{A}(P)$ such that

- $\mathcal{N}_0''$ end-extends $\mathcal{N}_0' \cap N$ and $\mathcal{N}_1''$ end-extends $\mathcal{N}_1' \cap N$,
- W'' end-extends $W' \cap N$ and $W'' \setminus (W' \cap N)$ is of size l; let $\langle y_\zeta : \zeta \in l \rangle$ be the increasing enumeration of $W'' \setminus (W' \cap N)$ with respect to $\mathcal{N}_0''$,
- for any $\zeta \in l$, $n^{y_\zeta} = n^{x_\zeta}$,
- $\{\mu_u : u \in h\} \in \big[[\omega_1]^k\big]^h$; for each $u \in h$, let $\mu_u = \{\eta_i^u : i \in k\}$ be the increasing enumeration,
- $\langle \mathcal{N}_0'', \mathcal{N}_1'', W'' \rangle \Vdash_{\mathcal{A}(P)}$ " $\bigcup_{u \in h} \{\eta_i^u : i \in |\nu_u|\} \in \dot{J}$",
- for any $\zeta \in l$ and $j \in k+2$, $\langle e_\xi : \xi \in \sigma_j^{y_\zeta} \rangle \in B_j^\zeta$,
- for each $\zeta \in l$, $\langle e_\xi : \xi \in \sigma_{j_0}^{y_\zeta} \rangle \in B^{\zeta,0}$ and $\langle e_\xi : \xi \in \sigma_{j_1}^{y_\zeta} \rangle \in B^{\zeta,1}$, and, for each $u \in h$, $\langle e_\xi : \xi \in \mu_u \rangle \in B^{l,u}$,
- the mapping

$$\begin{aligned} \sigma_{j_0}^{x_\zeta} &\mapsto \sigma_{j_0}^{y_\zeta}, && \text{for each } \zeta \in l, \\ \sigma_{j_1}^{x_\zeta} &\mapsto \sigma_{j_1}^{y_\zeta}, && \text{for each } \zeta \in l, \\ \bar{\nu}_u &\mapsto \mu_u, && \text{for each } u \in h, \end{aligned}$$

  preserves the order of the separation of $\{\sigma_{j_0}^{x_\zeta}, \sigma_{j_1}^{x_\zeta} : \zeta \in l\} \cup \{\bar{\nu}_u : u \in h\}$ by $\mathcal{N}_0'$ and of $\{\sigma_{j_0}^{y_\zeta}, \sigma_{j_1}^{y_\zeta} : \zeta \in l\} \cup \{\mu_u : u \in h\}$ by $\mathcal{N}_0''\big\}$.

We notice that

$$\{\sigma_{j_0}^{x_\zeta}, \sigma_{j_1}^{x_\zeta} : \zeta \in l\} \cup \{\bar{\nu}_u : u \in h\} \in X \in N,$$
$$(\{\sigma_{j_0}^{x_\zeta}, \sigma_{j_1}^{x_\zeta} : \zeta \in l\} \cup \{\bar{\nu}_u : u \in h\}) \cap N = \emptyset.$$

Therefore, by applying Lemma 2.5 to the sets X, $\{\sigma_{j_0}^{x_\zeta}, \sigma_{j_1}^{x_\zeta} : \zeta \in l\} \cup \{\bar{\nu}_u : u \in h\}$, $\{B_i^{\zeta,0}, B_i^{\zeta,1}, B_i^{l,u} : \zeta \in l,\ u \in h,\ i \in k\}$ and the functions d, $1-d$ and $\{d'_u : u \in h\}$ suitably, we obtain $\{\sigma_0^\zeta, \sigma_1^\zeta : \zeta \in l\} \cup \{\mu_u : u \in h\} \in X \cap N$ such that for each $\zeta \in l$, $\sigma_0^\zeta <_d \sigma_{j_0}^{x_\zeta}$ and $\sigma_1^\zeta <_{1-d} \sigma_{j_1}^{x_\zeta}$ (i.e., $\sigma_{j_1}^{x_\zeta} <_d \sigma_1^\zeta$), and for each

$u \in h$, $\mu_u <_{d'_u} \bar{\nu}_u$. Take $\langle \mathcal{N}''_0, \mathcal{N}''_1, W'' \rangle \in \mathcal{A}(P) \cap N$ which witnesses that $\{\sigma_0^\zeta, \sigma_1^\zeta : \zeta \in l\} \cup \{\mu_u : u \in h\} \in X$, so, for each $\zeta \in l$, $\sigma_{j_0}^{y_\zeta} = \sigma_0^\zeta$ and $\sigma_{j_1}^{y_\zeta} = \sigma_1^\zeta$. Similar to the previous proof, the triple $\langle \mathcal{N}'_0 \cup \mathcal{N}''_0, \mathcal{N}'_1 \cup \mathcal{N}''_1, W' \cup W'' \rangle$ satisfies assertion ★ in the definition of $\mathcal{A}(P)$. Therefore $\langle \mathcal{N}'_0 \cup \mathcal{N}''_0, \mathcal{N}'_1 \cup \mathcal{N}''_1, W' \cup W'' \rangle$ is a condition of $\mathcal{A}(P)$, and hence is a common extension of $\langle \mathcal{N}'_0, \mathcal{N}'_1, W' \rangle$ and $\langle \mathcal{N}''_0, \mathcal{N}''_1, W'' \rangle$. For each $u \in h$, let $\mu_u = \{\eta_i^u : i \in k\}$ be the increasing enumeration, and set

$$\mu := \bigcup_{u \in h} \{\eta_i^u : i \in |\nu_u|\}.$$

By the choice of the functions d'_u, $u \in h$, the set $\{\sigma_0^\zeta, \sigma_1^\zeta : \zeta \in l\} \cup \{\mu_u : u \in h\}$ and the condition $\langle \mathcal{N}''_0, \mathcal{N}''_1, W'' \rangle$, it follows that $\mu <_{d'} \nu$ and

$$\langle \mathcal{N}''_0, \mathcal{N}''_1, W'' \rangle \Vdash_{\mathcal{A}(P)} \text{“}\, \mu \in \dot{J}\,\text{”},$$

which finishes the proof. ■

3. The s-finitely proper forcing axiom. Asperó and Mota introduced the notion of finite properness. A forcing notion $\mathbb{Q}$ is called *finitely proper* if, for every large enough regular cardinal λ, every finite set $\{N_i : i \in n\}$ of countable elementary submodels of H_λ containing $\mathbb{Q}$ as a member, and every $p \in \mathbb{Q} \cap \bigcap_{i \in n} N_i$, there exists an extension of p in $\mathbb{Q}$ that is $(N_i, \mathbb{Q})$-generic for all $i \in n$ [2, Definitions 1.1]. $\mathsf{PFA}^{\mathrm{fin}}(\aleph_1)$ is the forcing axiom for all finitely proper forcing notions of size $\aleph_1$ and $\aleph_1$-many dense sets.

Definition 3.1.

(1) For a forcing notion $\mathbb{Q}$, a countable elementary submodel M of H_κ containing $\mathbb{Q}$ as a member, and a condition p of $\mathbb{Q}$, p is called a *solid* $(M, \mathbb{Q})$-generic condition if, for any countable elementary submodel N of H_κ containing $\mathbb{Q}$ as a member with $\omega_1 \cap N = \omega_1 \cap M$, p is $(N, \mathbb{Q})$-generic.

(2) A forcing notion $\mathbb{Q}$ is *s-finitely proper* if for any regular cardinal λ with $\lambda \geq \aleph_2$ and transitive model V_0 of $\mathsf{ZFC} - \mathsf{P}$ (the axioms of ZFC minus the Power Set Axiom) with $\{\mathbb{Q}, \lambda\} \in V_0$,

$V_0 \models$ “ for every finite set $\{N_i : i \in n\}$ of countable elementary submodels of H_λ containing $\mathbb{Q}$ as a member, and every $p \in \mathbb{Q} \cap \bigcap_{i \in n} N_i$, there exists an extension of p in $\mathbb{Q}$ that is solid $(N_i, \mathbb{Q})$-generic for all $i \in n$”.

(3) $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ is the forcing axiom for all s-finitely proper forcing notions of size $\aleph_1$ and $\aleph_1$-many dense sets.

An s-finite proper forcing notion is a proper forcing notion. So $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ is a fragment of PFA. A typical example of an s-finitely proper forcing notion is Baumgartner's adding a club subset of ω_1 by finite approximations

[6, §3]. A ccc forcing notion is also s-finitely proper. As another example, for given two Aronszajn trees S and T, the forcing notion that adds a club-isomorphism from S into T by finite approximations (defined in [4, 5.2. Definition] or [9, 5.10. Theorem]) is s-finitely proper. So $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ implies that $\mathsf{MA}_{\aleph_1}$ holds and any two Aronszajn trees are club-isomorphic, therefore every Countryman line contains an isomorphic copy of Todorčević's Countryman line $C(\rho_0)$ or its reverse (see [4, Introduction] or [13, §2.1]). In [2, §5], $\mathsf{PFA}^{\text{fin}}(\aleph_1)$ is applied to the axiom $\mho$ and weak club guessing sequences. $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ can also be applied to both situations, that is, $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ implies the failure of $\mho$ and the assertion that there are no weak club guessing ladder systems (for the proof, see also [15]). By the definition, $\mathsf{PFA}^{\text{fin}}(\aleph_1)$ implies $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$, but the authors do not know whether $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ may be a properly weak fragment of $\mathsf{PFA}^{\text{fin}}(\aleph_1)$, that is, whether it is consistent that $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ holds and $\mathsf{PFA}^{\text{fin}}(\aleph_1)$ fails.

4. Symmetric systems of relational structures. Recall that $k \geq 2$ is an integer and $E = \{e_\xi : \xi \in \omega_1\}$ is a k-entangled set of reals. Throughout the rest of the paper, suppose that

- κ is an uncountable regular cardinal such that $\kappa \geq \aleph_2$ and $2^{<\kappa} = \kappa$,
- Φ is a surjection from κ to H_κ such that for every $x \in H_\kappa$, $\Phi^{-1}[\{x\}]$ is unbounded in κ,
- $\mathfrak{M}_0$ is the set of countable elementary submodels of H_κ which contain E as a member, and contain Φ as a predicate.

For each $M \in \mathfrak{M}_0$, let $\overline{M}$ denote the transitive collapse of M, and let Ψ_M denote the transitive collapsing map from M onto $\overline{M}$. We always consider the members of $\mathfrak{M}_0$ as substructures of the structure $\langle H_\kappa, \in, \omega_1, E, \Phi\rangle$. (This situation is the same as in [10, §4].) So when M and M' in $\mathfrak{M}_0$ have the same transitive collapse in this sense, the composition $\Psi_{M'}^{-1} \circ \Psi_M$ is an isomorphism from $\langle M, \in, \omega_1 \cap M, E \cap M, \Phi\restriction M\rangle$ onto $\langle M', \in, \omega_1 \cap M', E \cap M', \Phi\restriction M'\rangle$. For each $M \in \mathfrak{M}_0$, since M is countable and ω_1 is of uncountable cofinality, $\omega_1 \cap M$ is a countable ordinal and $\omega_1 \cap M < \omega_1$. And if M and M' in $\mathfrak{M}_0$ are isomorphic in the above sense, then $\omega_1 \cap M = \omega_1 \cap M'$.

The following is the key tool of Todorčević's side condition method to build a proper forcing notion with models as side conditions to preserve all cardinalities (e.g. [10, §4]).

Definition 4.1. A finite subset S of $\mathfrak{M}_0$ is called a *symmetric system* if

(ho) for any $M, M' \in S$, if $\omega_1 \cap M = \omega_1 \cap M'$, then $\overline{M} = \overline{M'}$,
(up) for any $M, M' \in S$, if $\omega_1 \cap M' < \omega_1 \cap M$, then there exists $M'' \in S$ such that $\overline{M''} = \overline{M}$ and $M' \in M''$,

(down) ([2]) for any $M_0, M_1 \in S$ and $M' \in S \cap M_0$, if $\overline{M_0} = \overline{M_1}$, then $(\Psi_{M_1}^{-1} \circ \Psi_{M_0})(M')$ belongs to S, and

(id) ([3]) for any $M, M' \in S$, if $\omega_1 \cap M = \omega_1 \cap M'$, then the function

$$(\Psi_{M'}^{-1} \circ \Psi_M) \restriction (M \cap M')$$

is the identity.

We will deal with symmetric systems of relational structures. To introduce them, we define the following notions, and mention some necessary propositions.

Definition 4.2. Let $(\mathbb{P}, \leq_{\mathbb{P}})$ be a forcing notion with *the κ-chain condition* such that $\mathbb{P} \subseteq H_\kappa$. *The relational structure expanded by* $\mathbb{P}$ is the relational structure

$$\langle H_\kappa, \in, \mathbb{P}, \leq_{\mathbb{P}}, R_=^{\mathbb{P}}, R_\in^{\mathbb{P}}, H_\kappa^{\mathbb{P}}, E, \Phi \rangle,$$

where with $V^{\mathbb{P}}$ denoting the class of all $\mathbb{P}$-names,

$$\begin{aligned} R_=^{\mathbb{P}} &:= \{(p, \tau, \pi) \in (\mathbb{P} \times V^{\mathbb{P}} \times V^{\mathbb{P}}) \cap H_\kappa : p \Vdash_{\mathbb{P}} \text{“}\, \tau = \pi \,\text{”}\}, \\ R_\in^{\mathbb{P}} &:= \{(p, \tau, \pi) \in (\mathbb{P} \times V^{\mathbb{P}} \times V^{\mathbb{P}}) \cap H_\kappa : p \Vdash_{\mathbb{P}} \text{“}\, \tau \in \pi \,\text{”}\}, \\ H_\kappa^{\mathbb{P}} &:= V^{\mathbb{P}} \cap H_\kappa. \end{aligned}$$

For each $M \in \mathfrak{M}_0$, M is also considered as the substructure

$$\langle M, \in \cap M^2, \mathbb{P} \cap M, \leq_{\mathbb{P}} \cap M^2, R_=^{\mathbb{P}} \cap M^3, R_\in^{\mathbb{P}} \cap M^3, H_\kappa^{\mathbb{P}} \cap M, E \cap M, \Phi \cap M \rangle$$

of the relational structure expanded by $\mathbb{P}$, and we write $M \prec \mathbb{P}$ when the structure M is an elementary substructure of the relational structure expanded by $\mathbb{P}$.

The forcing notions we will define in §5 are *not* members of H_κ, but subsets of H_κ. The following can be proved in a similar way to [7, III §2]. In the following proposition, we need to take care of the definition of the $(M, \mathbb{P})$-generic condition because we consider the situation where $\mathbb{P}$ is a subset of M, but may *not* be a member of M. From now on, we say that $p \in \mathbb{P}$ is $(M, \mathbb{P})$-*generic* if, for any predense subset $\mathcal{D} \in M$ of $\mathbb{P}$, $\mathcal{D} \cap M$ is predense below p in $\mathbb{P}$. For the proof of the following proposition, see e.g. [7, III 2.11 Theorem].

([2]) This condition does not appear e.g. in [10, §4] (and appears in [2]). In this paper, it seems necessary for showing e.g. the properness of our forcing notions.

([3]) This property comes from the Asperó–Mota iteration [2]. It was only used in [2] to show that the Asperó–Mota iteration of length a limit ordinal of uncountable cofinality is proper. But in this paper, this property will be used frequently.

Proposition 4.3. *Suppose that* $\mathbb{P}$ *is a forcing notion with the* κ*-chain condition such that* $\mathbb{P} \subseteq H_\kappa$.

(1) *If* θ *is a large enough regular cardinal for* $\mathbb{P}$ *and* M^* *is a countable elementary submodel of* H_θ *which contains the set* $\{H_\kappa, \in, \mathbb{P}, E, \Phi\}$ *as a member, then* $M^* \cap H_\kappa \in \mathfrak{M}_0$ *and* $M^* \cap H_\kappa \prec \mathbb{P}$.

(2) *For any* $M \in \mathfrak{M}_0$ *with* $M \prec \mathbb{P}$,
$\Vdash_{\mathbb{P}}$ *"the structure*

$$\langle M[\dot{G}], \in \cap M[\dot{G}]^2, H_\kappa^V \cap M[\dot{G}], \mathbb{P} \cap M[\dot{G}], \leq_{\mathbb{P}} \cap M[\dot{G}]^2, \dot{G} \cap M[\dot{G}], R_=^{\mathbb{P}} \cap M[\dot{G}]^3, R_\in^{\mathbb{P}} \cap M[\dot{G}]^3, H_\kappa^{\mathbb{P}} \cap M[\dot{G}], E \cap M[\dot{G}], \Phi \cap M[\dot{G}]\rangle$$

is an elementary substructure of

$$\langle H_\kappa^{V[\dot{G}]}, \in, H_\kappa^V, \mathbb{P}, \leq_{\mathbb{P}}, \dot{G}, R_=^{\mathbb{P}}, R_\in^{\mathbb{P}}, H_\kappa^{\mathbb{P}}, E, \Phi\rangle \text{''}.$$

(3) *If* $M \in \mathfrak{M}_0$ *with* $M \prec \mathbb{P}$, *and* $p \in \mathbb{P}$, *then the following are equivalent:*
   - p *is* $(M, \mathbb{P})$*-generic,*
   - $p \Vdash_{\mathbb{P}}$ *"*$M[\dot{G}] \cap V = M$*", where* V *denotes the ground model, and*
   - $p \Vdash_{\mathbb{P}}$ *"*$M[\dot{G}] \cap \kappa = M \cap \kappa$*".*

Notation 4.4. For $\alpha \in \kappa + 1$ and a sequence $\langle X_\xi^i : i \in n,\ \xi \in \alpha\rangle$ of subsets of H_κ, we denote

$$\langle\!\langle X_\xi^i : i \in n,\ \xi \in \alpha\rangle\!\rangle := \{\langle i, \xi, x\rangle : i \in n,\ \xi \in \alpha,\ x \in X_\xi^i\}.$$

Then the tuple $\langle\!\langle X_\xi^i : i \in n,\ \xi \in \alpha\rangle\!\rangle$ is also a subset of H_κ.

Definition 4.5. Let $\alpha \in \kappa+1$, and $\langle \mathbb{P}_\xi : \xi \leq \alpha\rangle$ be a sequence of forcing notions such that $\mathbb{P}_\xi \subseteq H_\kappa$ and $\mathbb{P}_\xi$ has the κ-chain condition for each $\xi \leq \alpha$. The *relational structure expanded by* $\langle \mathbb{P}_\xi : \xi \leq \alpha\rangle$ is the structure

$$\big\langle H_\kappa, \in, \mathbb{P}_\alpha, \leq_{\mathbb{P}_\alpha}, \langle\!\langle R_=^{\mathbb{P}_\xi}, R_\in^{\mathbb{P}_\xi}, H_\kappa^{\mathbb{P}_\xi} : \xi \in \alpha\rangle\!\rangle\big\rangle.$$

For each $M \in \mathfrak{M}_0$, M is also considered as the substructure

$$\big\langle M, \in \cap M^2, \mathbb{P}_\alpha \cap M, \leq_{\mathbb{P}_\alpha} \cap M^2, \langle\!\langle R_=^{\mathbb{P}_\xi} \cap M^3, R_\in^{\mathbb{P}_\xi} \cap M^3, H_\kappa^{\mathbb{P}_\xi} \cap M : \xi \in \alpha \cap M\rangle\!\rangle\big\rangle$$

of the relational structure expanded by $\langle \mathbb{P}_\xi : \xi \leq \alpha\rangle$, and we write $M \prec \langle \mathbb{P}_\xi : \xi \leq \alpha\rangle$ when the structure M is an elementary substructure of the relational structure expanded by $\langle \mathbb{P}_\xi : \xi \leq \alpha\rangle$.

Note that if $\alpha < \kappa$ and $M \prec \langle \mathbb{P}_\xi : \xi \leq \alpha\rangle$, then α belongs to M. The following is a variation of Proposition 4.3 for iterated forcings.

Proposition 4.6. *Suppose that* $\alpha \in \kappa+1$, *and* $\langle \mathbb{P}_\xi : \xi \leq \alpha\rangle$ *is a sequence of forcing notions such that* $\mathbb{P}_\xi \subseteq H_\kappa$ *and* $\mathbb{P}_\xi$ *has the* κ*-chain condition for each* $\xi \leq \alpha$.

(1) *If θ is a large enough regular cardinal for the iteration $\langle \mathbb{P}_\xi : \xi \le \alpha \rangle$ and M^* is a countable elementary submodel of H_θ which contains the set $\{H_\kappa, \in, \langle \mathbb{P}_\xi : \xi \le \alpha \rangle, E, \Phi\}$ as a member, then $M^* \cap H_\kappa \in \mathfrak{M}_0$ and $M^* \cap H_\kappa \prec \langle \mathbb{P}_\xi : \xi \le \alpha \rangle$.*
(2) *If $\alpha < \kappa$, then for any $M \in \mathfrak{M}_0$ with $M \prec \langle \mathbb{P}_\xi : \xi \le \alpha \rangle$, α belongs to M.*
(3) *For any $M \in \mathfrak{M}_0$ with $M \prec \langle \mathbb{P}_\xi : \xi \le \alpha \rangle$ and $\beta \in \alpha$, if $\beta \in M$, then $M \prec \langle \mathbb{P}_\xi : \xi \le \beta \rangle$.*

The following is necessary for our symmetric systems of relational structures. This is the reason why we introduce the relational structures equipped with forcing notions that are subsets of H_κ.

Proposition 4.7. *Suppose that $M, N_0, N_1 \in \mathfrak{M}_0$ are elementary substructures of the relational structure expanded by $\langle \mathbb{P}_\xi : \xi \le \alpha \rangle$, N_0 and N_1 are isomorphic as such substructures (then $\Psi = \Psi_{N_1}^{-1} \circ \Psi_{N_0}$ is an isomorphism from N_0 onto N_1 as substructures of the relational structure expanded by $\langle \mathbb{P}_\xi : \xi \le \alpha \rangle$), $\beta \le \alpha$ is such that $\Psi(\beta) = \beta$, and $M \in \mathfrak{M}_0 \cap N_0$. Then*

- *if $M \prec \langle \mathbb{P}_\xi : \xi \le \alpha \rangle$, then the structure*
$$\big\langle M, \in \cap M^2, \mathbb{P}_\beta \cap M, \le_{\mathbb{P}_\beta} \cap M^2, \langle\!\langle R_=^{\mathbb{P}_\xi} \cap M^3, R_\in^{\mathbb{P}_\xi} \cap M^3, H_\kappa^{\mathbb{P}_\xi} \cap M : \xi \in \beta \cap M \rangle\!\rangle \big\rangle$$
belongs to N_0, and
- *the structure*
$$\big\langle \Psi(M), \in \cap \Psi(M)^2, \mathbb{P}_\beta \cap \Psi(M), \le_{\mathbb{P}_\beta} \cap \Psi(M)^2, \langle\!\langle R_=^{\mathbb{P}_\xi} \cap \Psi(M)^3, R_\in^{\mathbb{P}_\xi} \cap \Psi(M)^3, H_\kappa^{\mathbb{P}_\xi} \cap \Psi(M) : \xi \in \beta \cap \Psi(M) \rangle\!\rangle \big\rangle$$
belongs to $N_1 \cap \mathfrak{M}_0$ and is an elementary substructure of the relational structure expanded by $\langle \mathbb{P}_\xi : \xi \le \beta \rangle$.

The following is a key point of the proof of the properness of our forcing notions.

Proposition 4.8. *Suppose that $\alpha \in \omega_2$, $N_0, N_1 \in \mathfrak{M}_0$, and N_0 and N_1 are isomorphic as substructures of the relational structure expanded by $\langle \mathbb{P}_\xi : \xi \le \alpha \rangle$. Then $N_0 \cap \alpha = N_1 \cap \alpha$.*

Proof. We only handle the case where α is uncountable. By our assumption, $\alpha \in N_0 \cap N_1$ and $N_0 \cap \omega_1 = N_1 \cap \omega_1$. Then there exists a bijection $f : \omega_1 \to \alpha$ which is in both N_0 and N_1. Then

$$N_0 \cap \alpha = f[N_0 \cap \omega_1] = f[N_1 \cap \omega_1] = N_1 \cap \alpha. \ \blacksquare$$

5. Definition of the forcing iterations. We notice that, for each $M \in \mathfrak{M}_0$ and $\alpha \in \kappa + 1$, any initial segment of $\alpha \cap M$ is of the form $\beta \cap M$ for some (not necessarily unique) $\beta \in \alpha + 1$. For each $\alpha \in \kappa + 1$, we will define the forcing notion $\mathbb{P}_\alpha$ as a subset of

$$U_\alpha := [\mathfrak{M}_0]^{<\aleph_0} \times \Big\{ \bigcup_{\langle M,\beta\rangle \in Z} \{M\} \times (\beta \cap M) : Z \in [\mathfrak{M}_0 \times (\alpha+1)]^{<\aleph_0} \Big\} \times \bigcup_{D \in [\alpha]^{<\aleph_0}} (H_\kappa)^D.$$

Since $\mathfrak{M}_0$ is a subset of H_κ, for each $\alpha \in \kappa+1$ the forcing notion $\mathbb{P}_\alpha$ is a subset of H_κ.

To define $\mathbb{P}_\alpha$, we introduce the following notation. For each $\alpha \in \kappa+1$ and $p = (\mathcal{N}_p, R_p, A_p) \in U_\alpha$,

- $\mathrm{dom}(R_p) := \{M : \text{there is } \gamma \in \alpha \text{ such that } \langle M,\gamma\rangle \in R_p\}$,
- $\mathrm{ran}(R_p) := \{\gamma : \text{there is } M \in \mathfrak{M}_0 \text{ such that } \langle M,\gamma\rangle \in R_p\}$,
- for each $I \subseteq \alpha$,
$$R_p^{-1}[I] := \{M : \text{there is } \gamma \in I \text{ such that } \langle M,\gamma\rangle \in R_p\},$$
- for each $M \in \mathrm{dom}(R_p)$,
$$R_p(M) := \{\gamma \in \mathrm{ran}(R_p) : \langle M,\gamma\rangle \in R_p\},$$
- for each $\beta \in \alpha$, $p{\restriction}\beta = (\mathcal{N}_{p\restriction\beta}, R_{p\restriction\beta}, A_{p\restriction\beta})$ is defined as the member of U_β such that $\mathcal{N}_{p\restriction\beta} := \mathcal{N}_p$, $R_{p\restriction\beta} := R_p \cap (\mathfrak{M}_0 \times \beta)$, and $A_{p\restriction\beta} := A_p{\restriction}\beta$, the restriction of the function A_p to the set β.

We will define the forcing notion $\mathbb{P}_\alpha$, which will satisfy the κ-chain condition, by recursion on $\alpha \in \kappa+1$. Once we have defined the sequence $\langle \mathbb{P}_\xi : \xi \le \alpha\rangle$ of forcing notions, we will define

$$\mathfrak{M}^P_\alpha := \{M \in \mathfrak{M}_0 : M \prec \langle \mathbb{P}_\xi : \xi \le \alpha\rangle\}.$$

As seen below, for each $\alpha \in \kappa$, $\mathbb{P}_\alpha$ will be defined from the set

$$\{\omega_1, E, H_\kappa, \varPhi, \langle\!\langle \mathfrak{M}^P_\xi : \xi \in \alpha\rangle\!\rangle\}.$$

Definition 5.1. The forcing notion $\mathbb{P}_\alpha$ is defined by recursion on $\alpha \in \kappa+1$. However, each $\mathbb{P}_\alpha$ is defined uniformly. We assume recursively that, for each $\eta \in \alpha$,

- $\mathbb{P}_\eta \subseteq U_\eta$,
- $\mathbb{P}_\eta$ has the κ-cc (which is proved in Proposition 6.1),
- $\mathbb{P}_\eta$ preserves ω_1 (which is proved in Lemma 6.3), and
- $\mathbb{P}_\eta$ preserves E to be k-entangled (which is proved in Lemma 6.8).

Then $\mathbb{P}_\alpha$ consists of the members $p = (\mathcal{N}_p, R_p, A_p)$ of U_α which satisfy the following conditions:

(ob)
- $\mathcal{N}_p$ is finite and forms a symmetric system,
- $\mathrm{dom}(R_p) \subseteq \mathcal{N}_p$, and for each $M \in \mathrm{dom}(R_p)$, $R_p(M)$ is an initial segment of $\alpha \cap M$, and
- for each $\xi \in \mathrm{dom}(A_p)$, $A_p(\xi)$ is either a $\mathbb{P}_\xi$-name that belongs to H_κ or a finite subset of $\left[[\omega_1]^k\right]^{k+2} \times \omega$.

(el) For each $\eta \in \alpha$, $R_p^{-1}[\{\eta\}] \subseteq \mathfrak{M}_\eta^P$.

(ho) For each $\eta \in \alpha$ and $M_0, M_1 \in R_p^{-1}[\{\eta\}]$, if $\omega_1 \cap M_0 = \omega_1 \cap M_1$, then the structure $\langle M_0, \in, E, \Phi, \langle\!\langle \mathbb{P}_\xi : \xi \leq \eta \rangle\!\rangle\rangle$ is isomorphic to the structure $\langle M_1, \in, E, \Phi, \langle\!\langle \mathbb{P}_\xi : \xi \leq \eta \rangle\!\rangle\rangle$.

(up) For each $\eta \in \alpha$ and $M, N_0 \in R_p^{-1}[\{\eta\}]$, if $\omega_1 \cap M < \omega_1 \cap N_0$, then there exists $N_1 \in R_p^{-1}[\{\eta\}]$ such that $M \in N_1$ and $\omega_1 \cap N_1 = \omega_1 \cap N_0$.

(down) For each $\eta \in \alpha$ and $M, N_0, N_1 \in R_p^{-1}[\{\eta\}]$, if $M \in N_0$ and $\omega_1 \cap N_0 = \omega_1 \cap N_1$, then $(\Psi_{N_1}^{-1} \circ \Psi_{N_0})(M) \in R_p^{-1}[\{\eta\}]$.

(g) If $\xi \in \mathrm{dom}(A_p)$ and $p \restriction \xi$ belongs to $\mathbb{P}_\xi$, then either

(a) $\Phi(\xi) = \{\dot{\mathbb{Q}}_\xi\}$ where $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_\xi$-name and

$p \restriction \xi \Vdash_{\mathbb{P}_\xi}$ " $\dot{\mathbb{Q}}_\xi$ is an s-finitely proper forcing notion whose underlying set is ω_1, and preserves the k-entangledness of E", or

(b) $\Phi(\xi) = \{\dot{P}^\xi, \dot{I}^\xi, d^\xi\}$ where $\dot{P}^\xi$ and $\dot{I}^\xi$ are $\mathbb{P}_\xi$-names, $d^\xi \in {}^k\{0,1\}$, and

$p \restriction \xi \Vdash_{\mathbb{P}_\xi}$ " $\dot{P}^\xi$ is a forcing notion of size $\aleph_1$ which preserves ω_1,
$\dot{I}^\xi$ is an uncountable subset of $[\omega_1]^k$, and
$\Vdash_{\dot{P}^\xi}$" for every $\{\sigma, \tau\} \in [\dot{I}^\xi]^2$ with $\max(\tau) < \min(\sigma)$,
$\tau \not<_{d^\xi} \sigma$"".

Moreover:

(g-a) If $\xi \in \mathrm{dom}(A_p)$, $p \restriction \xi$ belongs to $\mathbb{P}_\xi$ and $\Phi(\xi) = \{\dot{\mathbb{Q}}_\xi\}$, then $A_p(\xi)$ is a $\mathbb{P}_\xi$-name for a condition of $\dot{\mathbb{Q}}_\xi$ such that for every $M \in R_p^{-1}[\{\xi\}]$,

$$p \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{" } A_p(\xi) \text{ is solid } (M[\dot{G}_{\mathbb{P}_\xi}], \dot{\mathbb{Q}}_\xi)\text{-generic".}$$

(g-b) If $\xi \in \mathrm{dom}(A_p)$, $p \restriction \xi$ belongs to $\mathbb{P}_\xi$ and $\Phi(\xi) = \{\dot{P}^\xi, \dot{I}^\xi, d^\xi\}$, then

(g-b-ob) $A_p(\xi)$ is a finite subset of $\left[[\omega_1]^k\right]^{k+2} \times \omega$ such that, for each $x = \langle \Sigma^x, n^x \rangle \in A_p(\xi)$,

$$p \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{" for some } a \in \dot{P}^\xi,\ a \Vdash_{\dot{P}^\xi} \text{" } \Sigma^x \subseteq \dot{I}^\xi \text{""},$$

and, for any $\{x, y\} \in [A_p(\xi)]^2$, either $\max(\bigcup \Sigma^x) < \min(\bigcup \Sigma^y)$ or $\max(\bigcup \Sigma^y) < \min(\bigcup \Sigma^x)$,

(g-b-sep) for any $M \in R_p^{-1}[\{\xi\}]$, the set $(\bigcup_{x \in A_p(\xi)} \Sigma^x) \setminus M$ is separated by some finite $\in$-chain with members in $\{N \in \mathfrak{M}_\xi^P : \{N\} \times (N \cap \xi) \subseteq R_p\}$, starting from M,

(g-b-nonsep) for any $M \in R_p^{-1}[\{\xi\}]$ and $x \in A_p(\xi)$, either $\max(\bigcup \Sigma^x) < \omega_1 \cap M$ or $\omega_1 \cap M \leq \min(\bigcup \Sigma^x)$, and

★ for any $\{x, y\} \in [A_p(\xi)]^2$, if $\forall \mu \in \Sigma^x\ \forall \nu \in \Sigma^y\ (\mu \not<_{d^\xi} \nu)$ or $\forall \mu \in \Sigma^x\ \forall \nu \in \Sigma^y\ (\nu \not<_{d^\xi} \mu)$, then $n^x \neq n^y$.

By definition, it can be proved that, for each $p \in \mathbb{P}_\alpha$ and $\eta \in \alpha$, $p \restriction \eta$ is a condition of $\mathbb{P}_\eta$. The order of $\mathbb{P}_\alpha$ is defined as follows: For each $p, q \in \mathbb{P}_\alpha$, $q \leq_{\mathbb{P}_\alpha} p$ iff

- $\mathcal{N}_q \supseteq \mathcal{N}_p$, $R_q \supseteq R_p$, $\mathrm{dom}(A_q) \supseteq \mathrm{dom}(A_p)$,
- for each $\xi \in \mathrm{dom}(A_p)$, if $q \restriction \xi \leq_{\mathbb{P}_\xi} p \restriction \xi$, then
  - whenever ξ is as in case (a), $q \restriction \xi \Vdash_{\mathbb{P}_\xi}$ " $A_q(\xi) \leq_{\dot{\mathbb{Q}}_\xi} A_p(\xi)$",
  - whenever ξ is as in case (b), $A_q(\xi) \supseteq A_p(\xi)$.

By definition, it can be proved that, for any $p, q \in \mathbb{P}_\alpha$ with $q \leq_{\mathbb{P}_\alpha} p$, and each $\eta \in \alpha$, we have $q \restriction \eta \leq_{\mathbb{P}_\eta} p \restriction \eta$.

LEMMA 5.2. *Suppose that $\alpha, \beta \in \kappa + 1$ with $\beta < \alpha$. Then $\mathbb{P}_\beta$ can be completely embedded into $\mathbb{P}_\alpha$.*

Proof. By definition, any condition of $\mathbb{P}_\beta$ is also a condition of $\mathbb{P}_\alpha$. Suppose that $q \in \mathbb{P}_\beta$, $p \in \mathbb{P}_\alpha$ and $q \leq_{\mathbb{P}_\beta} p \restriction \beta$. Then define $r := \langle \mathcal{N}_q \cup \mathcal{N}_p, R_q \cup R_p, A_q \cup (A_p \restriction [\beta, \alpha))\rangle$. By induction on $\eta \in \alpha$, we can check that $r \restriction \eta$ is a condition of $\mathbb{P}_\eta$. So r is a condition of $\mathbb{P}_\alpha$ and an extension of p in $\mathbb{P}_\alpha$. Such an r is a canonical common extension of q and p in $\mathbb{P}_\alpha$. Hence the identity map from $\mathbb{P}_\beta$ into $\mathbb{P}_\alpha$ is a complete embedding. ∎

REMARK 5.3. It follows from the previous lemma that the final forcing notion $\mathbb{P}_\kappa$ can be considered as the direct limit of the forcing notions $\mathbb{P}_\alpha$, defined as the union of the $\mathbb{P}_\alpha$.

6. Proofs

PROPOSITION 6.1. *For every $\alpha \in \kappa + 1$, $\mathbb{P}_\alpha$ has the $(2^{\aleph_0})^+$-chain condition. In fact, every subset of $\mathbb{P}_\alpha$ of size $(2^{\aleph_0})^+$ has a pairwise compatible subset of size $(2^{\aleph_0})^+$. In particular, if the Continuum Hypothesis* CH *holds, then $\mathbb{P}_\alpha$ has the $\aleph_2$-chain condition.*

Proof. Suppose that $\alpha \in \kappa$ and $\{p_\zeta : \zeta \in (2^{\aleph_0})^+\}$ is a set of $\big(2^{\aleph_0}\big)^+$-many conditions in $\mathbb{P}_\alpha$. By extending each condition p_ζ if necessary, we may assume that

- for every $\xi \in \mathrm{dom}(A_{p_\zeta})$, if $A_{p_\zeta}(\xi)$ is a $\mathbb{P}_\xi$-name for a condition of $\dot{\mathbb{Q}}_\xi$, then $A_{p_\zeta}(\xi)$ is an ordinal in ω_1.

This is possible because $\mathrm{dom}(A_{p_\zeta})$ is finite. By shrinking the set if necessary, we may assume that

- the set $\{\mathcal{N}_{p_\zeta} : \zeta \in (2^{\aleph_0})^+\}$ forms a Δ-system,
- the set $\{\mathrm{dom}(A_{p_\zeta}) : \zeta \in (2^{\aleph_0})^+\}$ forms a Δ-system with root D,

(•) the set $\{(\bigcup \mathcal{N}_{p_\zeta}) \cap \kappa : \zeta \in (2^{\aleph_0})^+\}$ forms a Δ-system with root K (which is a countable subset of κ),

- (•) for each $\gamma \in K$, the set $\{\overline{M} : M \in {R_{p_\zeta}}^{-1}\big[[\gamma,\kappa)\big] \cap \mathfrak{M}^P_\gamma\}$ does not depend on $\zeta \in (2^{\aleph_0})^+$,
- (•) for each $\zeta \in (2^{\aleph_0})^+$, $(\mathrm{dom}(A_{p_\zeta}) \setminus D) \cap K = \emptyset$,
- (•) for any $\zeta, \zeta' \in (2^{\aleph_0})^+$, $M \in \mathcal{N}_{p_\zeta}$ and $M' \in \mathcal{N}_{p_{\zeta'}}$, if $\overline{M} = \overline{M'}$, then $M \cap \kappa$ and $M' \cap \kappa$ are order isomorphic and the corresponding isomorphism fixes $\kappa \cap M \cap M'$ (which is a subset of K) (4), and
- for each $\gamma \in D$, the coordinate $A_{p_\zeta}(\gamma)$ (which is either a countable ordinal or a finite set of pairs in $H_{\aleph_1}$) does not depend on $\zeta \in (2^{\aleph_0})^+$.

Then we show that for any distinct ζ and ζ', p_ζ and $p_{\zeta'}$ are compatible in $\mathbb{P}_\alpha$. To see this, let $q \in U_\alpha$ be such that

- $\mathcal{N}_q := \mathcal{N}_{p_\zeta} \cup \mathcal{N}_{p_{\zeta'}}$, $R_q := R_{p_\zeta} \cup R_{p_{\zeta'}}$, and
- A_q is the function with domain $\mathrm{dom}(A_{p_\zeta}) \cup \mathrm{dom}(A_{p_{\zeta'}})$ such that for each $\gamma \in \mathrm{dom}(A_{p_\zeta}) \cup \mathrm{dom}(A_{p_{\zeta'}})$,

$$A_q(\gamma) := A_{p_\zeta}(\gamma) \cup A_{p_{\zeta'}}(\gamma)$$

(which is equal to $A_{p_\zeta}(\gamma)$ or $A_{p_{\zeta'}}(\gamma)$).

Such a q is a canonical amalgamation of p_ζ and $p_{\zeta'}$. Then by the above items (•), $\mathcal{N}_q$ and R_q satisfy (ob), (el), (ho), (up) and (down) of Definition 5.1. Recall that for each $M \in \mathfrak{M}_0$ and $\alpha \in \kappa$, if $\alpha \notin M$, then $M \notin \mathfrak{M}^P_\alpha$. So for any $\{\zeta, \zeta'\} \in \big[(2^{\aleph_0})^+\big]^2$ and $\alpha \in \mathrm{dom}(A_{p_\zeta}) \setminus D$, we have $\mathrm{dom}(R_{p_{\zeta'}}) \cap \mathfrak{M}^P_\alpha = \emptyset$. By induction on η and this observation, it can be proved that $q{\restriction}\eta$ is a common extension of $p_\zeta{\restriction}\eta$ and $p_{\zeta'}{\restriction}\eta$ in $\mathbb{P}_\eta$. Thus q is a condition of $\mathbb{P}_\alpha$, and is a common extension of p_ζ and $p_{\zeta'}$. ■

LEMMA 6.2. *For every $\alpha \in \kappa + 1$, $N \in \mathfrak{M}^P_\alpha$ and $p \in \mathbb{P}_\alpha \cap N$, there exists $q \in \mathbb{P}_\alpha$ such that $q \leq_{\mathbb{P}_\alpha} p$ and $\{N\} \times (\alpha \cap N) \subseteq R_q$.*

Proof. Suppose that $\alpha \in \kappa + 1$, $N \in \mathfrak{M}^P_\alpha$ and $p \in \mathbb{P}_\alpha \cap N$. We use induction on α.

If $\alpha = 0$, then the condition $q := \langle \mathcal{N}_p \cup \{N\}, \emptyset, \emptyset \rangle$ is what we want.

Suppose that α is a successor ordinal, say $\alpha = \beta + 1$. By the inductive hypothesis, there exists $q' \in \mathbb{P}_\beta$ such that $q' \leq_{\mathbb{P}_\beta} p{\restriction}\beta$, $\{N\} \times (\beta \cap N) \subseteq R_{q'}$. If either $\beta \notin \mathrm{dom}(A_p)$ or $\beta \in \mathrm{dom}(A_p)$ in case (b), the condition $q := \langle \mathcal{N}_{q'} \cup \{N\}, R_{q'} \cup R_p \cup \{\langle N, \beta \rangle\}, A_{q'} \rangle$ is what we want. Otherwise, that is, if $\beta \in \mathrm{dom}(A_p)$ is as in case (a), then we can choose a $\mathbb{P}_\beta$-name $A_q(\xi)$ in H_κ such that

$$q' \Vdash_{\mathbb{P}_\xi} \text{“} A_q(\xi) \leq_{\dot{\mathbb{Q}}_\xi} A_p(\xi) \text{ and } A_q(\xi) \text{ is solid } (N[\dot{G}_{\mathbb{P}_\xi}], \dot{\mathbb{Q}}_\xi)\text{-generic”}.$$

(4) In [2], Asperó and Mota point out that the corresponding isomorphism between M and M' fixes $\kappa \cap M \cap M'$ iff for any two consecutive ordinals ξ_0 and ξ_1, the order types of the sets $\{\mu \in \kappa \cap M : \xi_0 < \mu < \xi_1\}$ and $\{\mu \in \kappa \cap M' : \xi_0 < \mu < \xi_1\}$ are the same (these order types are countable ordinals).

Then

$$q := \langle \mathcal{N}_{q'} \cup \{N\}, R_{q'} \cup \{\langle N, \beta\rangle\}, A_{q'} \cup \{\langle \beta, A_q(\beta)\rangle\}\rangle$$

is as desired.

Suppose that α is a limit ordinal. Let $\beta := \max(\mathrm{dom}(A_p))$. By the inductive hypothesis, there exists $q' \in \mathbb{P}_{\beta+1}$ such that $q' \leq_{\mathbb{P}_{\beta+1}} p\restriction(\beta+1)$ and $\{N\} \times ((\beta+1) \cap N) \subseteq R_{q'}$. Then a canonical common extension of q' and p in $\mathbb{P}_\alpha$ is the desired condition. ∎

LEMMA 6.3. *Suppose* CH. *Then for every $\alpha \in \omega_2$, $p \in \mathbb{P}_\alpha$, and $N \in \mathfrak{M}^P_\alpha$ such that $\{N\} \times (\alpha \cap N) \subseteq R_p$, p is $(N, \mathbb{P}_\alpha)$-generic. Therefore, if* CH *holds and $\kappa = \omega_2$, then $\mathbb{P}_\kappa$ is proper.*

Proof. This is proved by induction on $\alpha \in \omega_2$.

Basic stage. This proof is very standard for the side condition method (see e.g. [10, Lemma 4]). Suppose that $N \in \mathfrak{M}_0$, $p \in \mathbb{P}_0$ satisfies $N \in \mathcal{N}_p$ (then $R_p = A_p = \emptyset$), $\mathcal{D} \in N$ is a predense subset of $\mathbb{P}_0$, and $q \leq_{\mathbb{P}_0} p$. We notice that q is of the form $(\mathcal{N}_q, \emptyset, \emptyset)$. It suffices to find $b' \in \mathcal{D} \cap N$ which is compatible with q in $\mathbb{P}_0$.

By extending q if necessary, we may assume that there exists $b \in \mathcal{D}$ such that $q \leq_{\mathbb{P}_0} b$. Take $\gamma \in \omega_1 \cap N$ such that

$$\{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap \gamma.$$

Define

$$\begin{aligned}\mathcal{E} := \bigl\{r \in \mathbb{P}_0 : {} & \text{there exists } b' \in \mathcal{D} \text{ such that } r \leq_{\mathbb{P}_0} b',\\ & \{\omega_1 \cap K : K \in \mathcal{N}_r\} \cap \gamma = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N =: \vec{\mathfrak{t}},\\ & \text{and } \mathcal{N}_r \supseteq \mathcal{N}_q \cap N\bigr\}.\end{aligned}$$

Since the set $\vec{\mathfrak{t}}$ belongs to N and the structure N has the parameter $\mathbb{P}_0$ as a predicate, $\mathcal{E}$ is definable in N. Moreover, we note that $q \in \mathcal{E}$. So by the elementarity of N, there exist $r \in \mathcal{E} \cap N$ and $b' \in \mathcal{D} \cap N$ such that $r \leq_{\mathbb{P}_0} b'$. Then $\mathcal{N}_r \subseteq N$. Define $q' \in U_0$ such that

$$\begin{aligned}\mathcal{N}_{q'} := \mathcal{N}_q \cup \mathcal{N}_r &\\ \cup \bigl\{(\Psi_M^{-1} \circ \Psi_N)(M') : {} & M \in \mathcal{N}_q \text{ with } \omega_1 \cap M = \omega_1 \cap N\\ & \text{and } M' \in \mathcal{N}_r \text{ with } \omega_1 \cap M' \notin \vec{\mathfrak{t}}\bigr\}.\end{aligned}$$

We note that q' is a condition of $\mathbb{P}_0$ and a common extension of q and r in $\mathbb{P}_0$.

Successor stage. Suppose that $\alpha \in \omega_2$, $N \in \mathfrak{M}^P_{\alpha+1}$, $p \in \mathbb{P}_{\alpha+1}$ satisfies $\{N\} \times ((\alpha+1) \cap N) \subseteq R_p$, $\mathcal{D} \in N$ is a predense subset of $\mathbb{P}_{\alpha+1}$, and $q \leq_{\mathbb{P}_{\alpha+1}} p$. By extending q if necessary, we may assume that there exists $b \in \mathcal{D}$ such that $q \leq_{\mathbb{P}_{\alpha+1}} b$. It suffices to find $b' \in \mathcal{D} \cap N$ which is compatible with q in $\mathbb{P}_{\alpha+1}$. We have the following three cases.

Subcase A: $\alpha \notin \mathrm{dom}(A_p)$. Define

$$\mathcal{D}' := \{c{\restriction}\alpha : c \in \mathcal{D} \text{ and } \alpha \notin \mathrm{dom}(A_c)\}.$$

Note that $q{\restriction}\alpha \in \mathcal{D}' \in N$. Since $q{\restriction}\alpha$ is $(N, \mathbb{P}_\alpha)$-generic, there exists r $(= b')$ $\in \mathcal{D} \cap N$ such that $r{\restriction}\alpha$ is compatible with $q{\restriction}\alpha$ in $\mathbb{P}_\alpha$ and $\alpha \notin \mathrm{dom}(A_r)$. Let $q' \in \mathbb{P}_\alpha$ be a common extension of $r{\restriction}\alpha$ and $q{\restriction}\alpha$. Define $q'' \in U_{\alpha+1}$ such that

$$\begin{aligned} R_{q''} &:= R_{q'} \cup R_q \cup R_r \\ &\quad \cup \big\{\langle (\Psi_M^{-1} \circ \Psi_N)(M'), \alpha\rangle : M \in {R_q}^{-1}[\{\alpha\}] \\ &\qquad\qquad \text{with } \omega_1 \cap M = \omega_1 \cap N, \text{ and } M' \in {R_r}^{-1}[\{\alpha\}]\big\}, \\ \mathcal{N}_{q''} &:= \mathcal{N}_{q'} \cup \mathcal{N}_q \cup \mathcal{N}_r \cup \mathrm{dom}(R_{q''}), \\ A_{q''} &:= A_{q'}. \end{aligned}$$

Since $R_{q''}{\restriction}\alpha = R_{q'}$, we have $q''{\restriction}\alpha = q'$. If $M' \in {R_r}^{-1}[\{\alpha\}]$, then $M' \in \mathfrak{M}^P_\alpha$ and $\{M'\} \times (\alpha \cap M') \subseteq R_r$. Since $r \in N$, we have $\mathcal{N}_r \cup R_r \subseteq N$. For each $M \in {R_q}^{-1}[\{\alpha\}]$ with $\omega_1 \cap M = \omega_1 \cap N$ and $M' \in {R_r}^{-1}[\{\alpha\}]$, it follows from Proposition 4.7 that $(\Psi_M^{-1} \circ \Psi_N)(M')$ belongs to $\mathfrak{M}^P_\alpha$. Thus, $\mathcal{N}_{q''}$ and $R_{q''}$ satisfy (el), (ho), (up) and (down) of Definition 5.1. Therefore, q'' is a condition of $\mathbb{P}_{\alpha+1}$, and is a common extension of q and r.

Subcase B: *$A_q(\alpha)$ is a $\mathbb{P}_\alpha$-name* (that is, α is as in case (a)). Define the $\mathbb{P}_\alpha$-name $\dot{\mathcal{D}}_{(\alpha)}$ such that

$$\Vdash_{\mathbb{P}_\alpha} \text{“} \dot{\mathcal{D}}_{(\alpha)} := \{A_c(\alpha) : c \in \mathcal{D} \text{ and } c{\restriction}\alpha \in \dot{G}_{\mathbb{P}_\alpha}\}\text{”}.$$

Then

$$\begin{aligned} q{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“} & \dot{\mathcal{D}}_{(\alpha)} \in N[\dot{G}_{\mathbb{P}_\alpha}], \dot{\mathcal{D}}_{(\alpha)} \subseteq \dot{\mathbb{Q}}_\alpha, \text{ and} \\ & A_q(\alpha) \text{ is solid } (N[\dot{G}_{\mathbb{P}_\alpha}], \dot{\mathbb{Q}}_\xi)\text{-generic”}. \end{aligned}$$

Since $q{\restriction}\alpha$ is $(N, \mathbb{P}_\alpha)$-generic, there are $q' \in \mathbb{P}_\alpha$, r $(= b') \in \mathcal{D} \cap N$ and a $\mathbb{P}_\alpha$-name $\dot{\nu}$ in H_κ such that

- $q' \leq_{\mathbb{P}_\alpha} q{\restriction}\alpha$,
- $A_r(\alpha)$ is a $\mathbb{P}_\alpha$-name in $H_\kappa \cap N$,
- $q' \Vdash_{\mathbb{P}_\alpha}$ “$r{\restriction}\alpha \in \dot{G}_{\mathbb{P}_\alpha}$ and $\dot{\nu}$ is a common extension of $A_r(\alpha)$ and $A_q(\alpha)$ in $\dot{\mathbb{Q}}_\alpha$”.

Then q' is also $(N, \mathbb{P}_\alpha)$-generic. By extending q' if necessary, we may assume that $q' \leq_{\mathbb{P}_\alpha} r{\restriction}\alpha$. Define $q'' \in U_{\alpha+1}$ such that

$$\begin{aligned} R_{q''} &:= R_{q'} \cup R_q \cup R_r \\ &\quad \cup \big\{\langle (\Psi_M^{-1} \circ \Psi_N)(M'), \alpha\rangle : M \in R_q^{-1}[\{\alpha\}] \\ &\qquad\qquad \text{with } \omega_1 \cap M = \omega_1 \cap N, \text{ and } M' \in R_r^{-1}[\{\alpha\}]\big\}, \\ \mathcal{N}_{q''} &:= \mathcal{N}_{q'} \cup \mathcal{N}_q \cup \mathcal{N}_r \cup \mathrm{dom}(R_{q''}), \\ A_{q''} &:= A_{q'} \cup \{\langle \alpha, \dot{\nu}\rangle\}. \end{aligned}$$

As in the previous case, $\mathcal{N}_{q''}$ and $R_{q''}$ satisfy (el), (ho), (up) and (down). For each $M \in R_q^{-1}[\{\alpha\}]$ with $\omega_1 \cap M = \omega_1 \cap N$ and $M' \in R_r^{-1}[\{\alpha\}]$, since

$$\begin{aligned} q' \Vdash_{\mathbb{P}_\alpha} \text{“}\omega_1 \cap \big((\Psi_M^{-1} \circ \Psi_N)(M')\big)[\dot{G}_{\mathbb{P}_\alpha}] &= \omega_1 \cap (\Psi_M^{-1} \circ \Psi_N)(M') \\ &= \omega_1 \cap M' = \omega_1 \cap M'[\dot{G}_{\mathbb{P}_\alpha}], A_q(\alpha) \text{ is solid } (M'[\dot{G}_{\mathbb{P}_\alpha}], \dot{\mathbb{Q}}_\alpha)\text{-generic,} \\ &\text{and } \dot{\nu} \leq_{\dot{\mathbb{Q}}_\alpha} A_q(\alpha)\text{”,} \end{aligned}$$

it follows that

$$q' \Vdash_{\mathbb{P}_\alpha} \text{“}\,\dot{\nu} \text{ is solid } ((\Psi_M^{-1} \circ \Psi_N)(M')[\dot{G}_{\mathbb{P}_\alpha}], \dot{\mathbb{Q}}_\alpha)\text{-generic”.}$$

Therefore, q'' is a condition of $\mathbb{P}_{\alpha+1}$, and hence a common extension of q and r.

SUBCASE C: *$A_q(\alpha)$ is a finite set of pairs* (that is, α is as in case (b)). Let $\{x^q_\zeta : \zeta \in l\}$ be the increasing enumeration of the set $A_q(\alpha) \setminus N$ with respect to $\mathcal{N}_q \cap \mathfrak{M}^P_\alpha$. Let $\{B^\zeta_{j,i} : \zeta \in l,\ j \in k+2,\ i \in k\}$ be a family of pairwise disjoint basic open subsets of $\mathbb{R}$ such that, for any $\zeta \in l$ and $j \in k+2$, $\langle e_\xi : \xi \in \sigma_j^{x^q_\zeta} \rangle \in \prod_{i\in k} B^\zeta_{j,i} =: B^\zeta_j$. Define a $\mathbb{P}_\alpha$-name $\dot{X}$ such that

$$\begin{aligned} \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{X} := \big\{\{\Sigma^x : x \in A_s(\alpha) \setminus (A_q(\alpha) \cap N)\} : s \in \mathbb{P}_{\alpha+1} \text{ and} \\ &- A_s(\alpha) \text{ is a finite set of pairs and } s{\restriction}\alpha \in \dot{G}_{\mathbb{P}_\alpha}, \\ &- \text{there is } b' \in \mathcal{D} \text{ such that } s \leq_{\mathbb{P}_{\alpha+1}} b', \\ &- |A_s(\alpha)| = |A_q(\alpha)| \text{ and } A_s(\alpha) \text{ end-extends } A_q(\alpha) \cap N; \\ &\quad \text{if } \{y^s_\zeta : \zeta \in l\} \text{ is the increasing enumeration of} \\ &\quad A_s(\alpha) \setminus (A_q(\alpha) \cap N), \text{ then} \\ &- \text{for each } \zeta \in l,\ n^{y^s_\zeta} = n^{x^q_\zeta}, \\ &- \text{for any } \zeta \in l \text{ and } j \in k+2,\ \langle e_\xi : \xi \in \sigma_j^{y^s_\zeta} \rangle \in B^\zeta_j \big\}\text{”.} \end{aligned}$$

We note that $\dot{X}$ is definable in N and

$$q{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“}\,\{\Sigma^x : x \in A_q(\alpha) \setminus N\} \in \dot{X} \setminus N[\dot{G}_{\mathbb{P}_\alpha}]\text{”.}$$

By Definition 5.1(g-b-sep) for α, the set $(\bigcup_{x\in A_q(\xi)} \Sigma^x) \setminus N$ is separated by some finite $\in$-chain $\overline{\mathcal{N}}$ with members in $\{N' \in \mathfrak{M}^P_\alpha : \{N'\} \times (\alpha \cap N') \subseteq R_q\}$, starting from N. By the inductive hypothesis, $q{\restriction}\alpha$ is $(N', \mathbb{P}_\alpha)$-generic for every $N' \in \overline{\mathcal{N}}$. Hence, for every $N' \in \overline{\mathcal{N}}$,

$$q{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“}\,\omega_1 \cap N'[\dot{G}_{\mathbb{P}_\xi}] = \omega_1 \cap N'\text{”.}$$

By (g-b-sep) for α,

$$\begin{aligned} q{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \text{“}\,&\text{the set } (\textstyle\bigcup_{x\in A_q(\xi)} \Sigma^x) \setminus N \text{ is separated by the finite} \\ &\in\text{-chain } \{N'[\dot{G}_{\mathbb{P}_\xi}] : N' \in \overline{\mathcal{N}}\} \text{ starting from } N[\dot{G}_{\mathbb{P}_\xi}]\text{”.} \end{aligned}$$

By Proposition 4.3,

$$q\restriction\alpha \Vdash_{\mathbb{P}_\alpha} \text{“ for each } N' \in \mathcal{N}_p,\ N'[\dot{G}_{\mathbb{P}_\xi}] \text{ is an elementary submodel of } H_\kappa^{V[\dot{G}_{\mathbb{P}_\alpha}]}\text{”}.$$

So, as in the proof of Lemma 2.6, we can apply Lemma 2.5 in the forcing extension by $q\restriction\alpha$ to obtain $q_0' \in \mathbb{P}_\alpha$ and a basic open set Π of a suitable finite-dimensional Euclidean space such that $q_0' \leq_{\mathbb{P}_\alpha} q\restriction\alpha$ and

$$q_0' \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{X} \cap \Pi \neq \emptyset \text{ and, for any } \{\Sigma^j : j < l\} \in \dot{X} \cap \Pi \text{ and } j < l, \text{ there are } \{\tau, \tau'\} \in [\Sigma^j]^2 \text{ and } \{\sigma, \sigma'\} \in [\Sigma^{x^q_j}]^2 \text{ such that } \tau <_{d^\alpha} \sigma \text{ and } \sigma' <_{d^\alpha} \tau'\text{”}.$$

Since

$$\{N\} \times (\alpha \cap N) \subseteq R_q\restriction\alpha \subseteq R_{q_0'},$$

by the induction hypothesis, q_0' is $(N, \mathbb{P}_\alpha)$-generic. So there are $q' \in \mathbb{P}_\alpha$, $r \in \mathbb{P}_{\alpha+1} \cap N$ and $b' \in \mathcal{D} \cap N$ such that $q' \leq_{\mathbb{P}_\alpha} q_0'$, $r \leq_{\mathbb{P}_{\alpha+1}} b'$, $A_r(\alpha)$ is a finite subset of pairs, $|A_r(\alpha)| = |A_q(\alpha)|$, $A_r(\alpha)$ end-extends $A_q(\alpha) \cap N$, for each $\zeta \in l$, $n^{y^r_\zeta} = n^{x^q_\zeta}$, and

$$q' \Vdash_{\mathbb{P}_\alpha} \text{“ } r\restriction\alpha \in \dot{G}_{\mathbb{P}_\alpha}, \text{ and } \{\Sigma^x : x \in A_r(\alpha) \setminus (A_q(\alpha) \cap N)\} \in \dot{X} \cap \Pi\text{”}.$$

Define $q'' \in U_{\alpha+1}$ such that

$$\begin{aligned}
R_{q''} :=\ & R_{q'} \cup R_r \\
& \cup \big\{\langle(\Psi_M^{-1} \circ \Psi_N)(M'), \alpha\rangle : M \in {R_q}^{-1}[\{\alpha\}] \\
& \qquad \text{with } \omega_1 \cap M = \omega_1 \cap N, \text{ and } M' \in R_r^{-1}[\{\alpha\}]\big\}, \\
\mathcal{N}_{q''} :=\ & \mathcal{N}_{q'} \cup \mathcal{N}_r \cup \mathrm{dom}(R_{q''}), \\
A_{q''} :=\ & A_{q'} \cup \{\langle\alpha, A_q(\alpha) \cup A_r(\alpha)\rangle\}.
\end{aligned}$$

By Proposition 4.7 and the fact that q and r satisfy (g-b-sep) for α, so does q''. By a similar observation to the previous case, q'' satisfies other conditions for the definition of $\mathbb{P}_{\alpha+1}$, that is, q'' is a condition of $\mathbb{P}_{\alpha+1}$. Hence q'' is a common extension of q, r and b' in $\mathbb{P}_{\alpha+1}$.

Limit stage. Suppose that $\alpha \in \omega_2 \cap \mathsf{Lim}$, $N \in \mathfrak{M}^P_\alpha$, $p \in \mathbb{P}_\alpha$ satisfies $\{N\} \times (\alpha \cap N) \subseteq R_p$, $\mathcal{D} \in N$ is a predense subset of $\mathbb{P}_\alpha$, and $q \leq_{\mathbb{P}_\alpha} p$ with $q \in \mathcal{D}$. By extending q if necessary, we may assume that there exists $b \in \mathcal{D}$ such that $q \leq_{\mathbb{P}_\alpha} b$. It suffices to find $b' \in \mathcal{D} \cap N$ compatible with q in $\mathbb{P}_\alpha$. We have two cases: α is of uncountable or countable cofinality. In the latter case, $\alpha \cap N$ is cofinal in α and hence we can take $\beta \in \alpha \cap N$ such that $\max(\mathrm{dom}(A_q)) < \beta$. But this may not be possible in the former case: it may happen that $\mathrm{dom}(A_q)$ is not bounded by $\sup(\alpha \cap N)$. So the argument for the former case is longer.

Subcase D: *α is of uncountable cofinality.* Since $\mathcal{N}_q$ forms a symmetric system, for each $M' \in \mathcal{N}_q$ with $\omega_1 \cap M' < \omega_1 \cap N$ there exists $M \in \mathcal{N}_q$ such that $\omega_1 \cap M = \omega_1 \cap N$ and $M' \in M$; then, by (id) in Definition 4.1,

$$\begin{aligned}\sup(M' \cap N \cap \alpha) &= \sup\big((\Psi_N^{-1} \circ \Psi_M)(M') \cap M \cap \alpha\big)\\ &\le \sup\big((\Psi_N^{-1} \circ \Psi_M)(M') \cap \alpha\big).\end{aligned}$$

Since N thinks that the set $(\Psi_N^{-1} \circ \Psi_M)(M')$ is countable and α is of uncountable cofinality,

$$\sup\big((\Psi_N^{-1} \circ \Psi_M)(M') \cap \alpha\big) \in N \cap \alpha.$$

So there exists $\beta \in \alpha \cap N$ such that

- $\max(\mathrm{dom}(A_q) \cap \sup(\alpha \cap N)) < \beta$,
- $\max(\{\sup(R_q(M)) : M \in \mathrm{dom}(R_q)\} \cap N) < \beta$, and
- for every $M' \in \mathcal{N}_q$ with $\omega_1 \cap M' < \omega_1 \cap N$,
$$\sup(M' \cap N \cap \alpha) < \beta.$$

By the second and third requirements on β, we observe that

⟨1⟩ for every $\eta \in [\beta, \alpha) \cap N$ and $K \in \mathcal{N}_q$ with $\omega_1 \cap K < \omega_1 \cap N$, $K \notin \mathfrak{M}_\eta^P$ (because if K was in $\mathfrak{M}_\eta^P$, η would be in K).

Take $\gamma \in \omega_1 \cap N$ such that

- $\{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap \gamma$, and
- for every $\xi \in \mathrm{dom}(A_q)$ as in case (b),
$$\{\Sigma^x : x \in A_q(\xi)\} \cap N = \{\Sigma^x : x \in A_q(\xi)\} \cap \big[[\gamma]^k\big]^{k+2}.$$

Define

$$\begin{aligned}\mathcal{E} := \big\{r\restriction\beta : r \in \mathbb{P}_\alpha \text{ and}\\ &- \text{there exists } b' \in \mathcal{D} \text{ such that } r \le_{\mathbb{P}_\alpha} b', \text{ and}\\ &- \{\omega_1 \cap K : K \in \mathcal{N}_r\} \cap \gamma = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N\big\}.\end{aligned}$$

We notice that $q\restriction\beta \in \mathcal{E}$ and $\mathcal{E}$ is definable in N. By Proposition 4.6, since $\beta \in N \in \mathfrak{M}_\alpha^P$, we have $N \in \mathfrak{M}_\beta^P$. Moreover,

$$\{N\} \times (\beta \cap N) \subseteq R_p \subseteq R_{q\restriction\beta}.$$

So, by the inductive hypothesis, $q\restriction\beta$ is $(N, \mathbb{P}_\beta)$-generic. Hence there exists p_1 in $\mathcal{E} \cap N$ which is compatible with the condition $q\restriction\beta$ in $\mathbb{P}_\beta$. Let $r \in \mathbb{P}_\alpha \cap N$ and $b' \in \mathcal{D} \cap N$ witness that $p_1 \in \mathcal{E}$. Let $p_2 \in \mathbb{P}_\beta$ be a common extension of $q\restriction\beta$ and p_1 $(= r\restriction\beta)$. We note that $\mathrm{dom}(A_{p_2}) \subseteq \beta$ and

$$\mathrm{dom}(A_q) \cap \mathrm{dom}(A_r) \cap [\beta, \alpha) = \emptyset,$$

more precisely,

$$\beta < \min(\mathrm{dom}(A_r) \setminus \beta) \le \max(\mathrm{dom}(A_r)) < \sup(\alpha \cap N) < \min(\mathrm{dom}(A_q) \setminus \beta),$$

because $\mathrm{dom}(A_r) \subseteq N$ and $\max(\mathrm{dom}(A_q) \cap \sup(\alpha \cap N)) < \beta$. Since $r \in N$,

⟨2⟩ $\mathcal{N}_r \subseteq N$,
⟨3⟩ $\mathrm{dom}(A_r) \subseteq \sup(\alpha \cap N) \cap N$,
⟨4⟩ for each $\eta \in [\beta, \alpha) \cap N$,

- if $\eta \in \mathrm{dom}(A_r)$, then $A_r(\eta) \subseteq N$, and
- $\{K \in R_q^{-1}[[\eta, \alpha)] \cap \mathfrak{M}^P_\eta : \omega_1 \cap K < \omega_1 \cap N\} = \emptyset$ (which follows from ⟨1⟩),

⟨5⟩ for each $\eta \in [\beta, \alpha) \setminus N$, $\mathcal{N}_r \cap \mathfrak{M}^P_\eta = \emptyset$.

Define the function $q' \in U_\alpha$ such that

$$\begin{aligned} R_{q'} &:= R_{p_2} \cup R_r \cup R_q \\ &\quad \cup \bigcup \big\{\{(\Psi_M^{-1} \circ \Psi_N)(K)\} \times ((\eta+1) \cap (\Psi_M^{-1} \circ \Psi_N)(K)) : \\ &\qquad\qquad \eta \geq \beta,\ \langle K, \eta\rangle \in R_r,\ \langle M, \eta\rangle \in R_q \text{ with } \omega_1 \cap M = \omega_1 \cap N\big\}, \\ \mathcal{N}_{q'} &:= \mathcal{N}_{p_2} \cup \mathcal{N}_r \cup \mathcal{N}_q \cup \mathrm{dom}(R_{q'}), \\ A_{q'} &:= \emptyset. \end{aligned}$$

By Lemma 4.7, $\mathcal{N}_{q'}$ satisfies (el) and (ho) of Definition 5.1. By (id), if M and M' in $R_{q'}^{-1}[\{\eta\}]$ are isomorphic and $K \in R_{q'}^{-1}[\{\eta\}] \cap M$ contains η as a member, then $(\Psi_{M'}^{-1} \circ \Psi_M)(K)$ also contains η as a member. So, by Propositions 4.7 and 4.8, $R_{q'}$ satisfies (down). Moreover, by applying (up) for R_q, we can show that $R_{q'}$ satisfies (up).

By Proposition 4.8 and the assumption that $\alpha < \omega_2$,

⟨6⟩ for every $\gamma \in [\sup(\alpha \cap N), \alpha)$, $M' \in \mathcal{N}_r$, and $\langle M, \gamma\rangle \in R_q$ with $\omega_1 \cap M = \omega_1 \cap N$, we have $\gamma \notin (\Psi_M^{-1} \circ \Psi_N)(M')$ (because $\gamma \notin M'$), hence, for every $\gamma \in [\sup(\alpha \cap N), \alpha)$, ${R_{q'}}^{-1}[\{\gamma\}] = {R_q}^{-1}[\{\gamma\}]$.

Therefore, by Lemma 4.7 and (id), $\mathcal{N}_{q'}$ satisfies (el), (ho), (up) and (down).

Define $q_\beta := p_2$. By induction on $\xi \in \mathrm{dom}(A_r) \setminus \beta$, we build $q_{\xi+1} \in \mathbb{P}_{\xi+1}$ such that

- $\mathcal{N}_{q_{\xi+1}} := \mathcal{N}_{q_\beta} \cup \mathcal{N}_{q'}$,
- for $\xi \in \mathrm{dom}(A_r) \setminus \beta$, letting $\xi' \in \mathrm{dom}(A_r) \setminus \beta$ be the predecessor of ξ in $\mathrm{dom}(A_r) \setminus \beta$ (or $\xi' = \beta$ when ξ is the least element of $\mathrm{dom}(A_r) \setminus \beta$),
$$R_{q_{\xi+1}} := R_{q_{\xi'+1}} \cup \big(R_{q'} \cap (\mathfrak{M}_0 \times (\xi+1))\big),$$
- if ξ is as in case (a), we extend $A_r(\xi)$ to $A_{q_{\xi+1}}(\xi)$ such that, for every $M' \in {R_{q'}}^{-1}[\{\xi\}]$ with $\omega_1 \cap M' \geq \omega_1 \cap N$,
$$q_{\xi+1}{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \text{“} A_{q_{\xi+1}}(\xi) \text{ is solid } (M'[\dot{G}_{\mathbb{P}_\xi}], \dot{\mathbb{Q}}_\xi)\text{-generic”}$$
(this can be done by ⟨4⟩), and if ξ is as in case (b), $A_{q_{\xi+1}}(\xi) := A_r(\xi)$.

Let $\xi_{\mathrm{m}} = \max(\mathrm{dom}(A_r))$ and suppose that $q_{\xi_{\mathrm{m}}}$ has been constructed. Define $q_\alpha \in U_\alpha$ such that

$$\mathcal{N}_{q_\alpha} := \mathcal{N}_{q_{\xi_{\mathrm{m}}}}, \quad R_{q_\alpha} := R_{q_{\xi_{\mathrm{m}}}} \cup R_{q'}, \quad A_{q_\alpha} := A_{q_{\xi_{\mathrm{m}}}} \cup (A_q{\restriction}[\beta, \alpha)).$$

It suffices to show that q_α is a condition of $\mathbb{P}_\alpha$, because then q_α is a common extension of q, r and b' in $\mathbb{P}_\alpha$. To do so, by induction on $\xi \in \mathrm{dom}(A_q)\setminus\beta$, it will be proved that $q_\alpha{\restriction}(\xi+1)$ is a condition of $\mathbb{P}_{\xi+1}$. We only need to check (g) of Definition 5.1 for $\xi \in \mathrm{dom}(A_q)\setminus\beta$. Suppose that $\xi \in \mathrm{dom}(A_q)\setminus\beta$ and $A_q(\xi)$ is a $\mathbb{P}_\xi$-name. Then by the inductive hypothesis up to ξ, $q_\alpha{\restriction}\xi \leq_{\mathbb{P}_\xi} q{\restriction}\xi$. Therefore, from $\langle 6\rangle$ and the fact that $q{\restriction}(\xi+1)$ satisfies (g-a), it follows that $q_\alpha{\restriction}(\xi+1)$ satisfies (g-a) automatically. Suppose that $\xi \in \mathrm{dom}(A_q)\setminus\beta$ and $A_q(\xi)$ is a finite set of pairs. Then $q_\alpha{\restriction}(\xi+1)$ satisfies (g-b), because of the role of γ.

Subcase E: *α is of countable cofinality.* Take $\beta \in \alpha\cap N$ and $\gamma \in \omega_1\cap N$ such that

- $\max(\mathrm{dom}(A_q)) < \beta$,
- for each $M \in \mathrm{dom}(R_q)$, either $R_q(M) \subseteq \beta$ or $R_q(M)$ is cofinal in α,
- $\{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap \gamma$, and
- for every $\xi \in \mathrm{dom}(A_q)$ as in case (b),

$$\{\Sigma^x : x \in A_q(\xi)\} \cap N = \{\Sigma^x : x \in A_q(\xi)\} \cap \big[[\gamma]^k\big]^{k+2},$$

and define

$$\begin{aligned}\mathcal{E} := \big\{r{\restriction}\beta : r \in \mathbb{P}_\alpha \text{ and } & \\ &- \text{there exists } b' \in \mathcal{D} \text{ such that } r \leq_{\mathbb{P}_\alpha} b',\\ &- \mathrm{dom}(A_r) \subseteq \beta,\\ &- \text{for each } M \in \mathrm{dom}(R_r), \text{ either } R_r(M) \subseteq \beta \text{ or}\\ &\quad R_r(M) \text{ is cofinal in } \alpha, \text{ and}\\ &- \{\omega_1\cap K : K \in \mathcal{N}_r\} \cap \gamma = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N\big\}.\end{aligned}$$

We note that $q \in \mathcal{E}$ and $\mathcal{E}$ is definable in N. By the inductive hypothesis and the fact that $\beta \in N \in \mathfrak{M}^P_\alpha$, $q{\restriction}\beta$ is $(N, \mathbb{P}_\beta)$-generic. So there exists $p_1 \in \mathcal{E}\cap N$ which is compatible with the condition $q{\restriction}\beta$ in $\mathbb{P}_\beta$. Let $r \in \mathbb{P}_\alpha \cap N$ and $b' \in \mathcal{D}\cap N$ witness that $p_1 \in \mathcal{E}$, and let $p_2 \in \mathbb{P}_\beta$ be a common extension of $q{\restriction}\beta$ and p_1 $(= r{\restriction}\beta)$.

Define a function q' in the set U_α such that

- $\mathcal{N}_{q'} := \mathcal{N}_{p_2} \cup \mathcal{N}_r \cup \mathcal{N}_q$ and $R_{q'} := R_{p_2} \cup R_r \cup R_q$,
- $\mathrm{dom}(A_{q'}) := \mathrm{dom}(A_{p_2})$ $(\subseteq \beta)$ and $A_{q'} := A_{p_2}$.

By a similar argument to the previous case, $q'{\restriction}\beta$ is the same as p_2, so $q'{\restriction}\beta$ is a condition of $\mathbb{P}_\beta$. Thus q' is a condition of $\mathbb{P}_\alpha$, and is a common extension of q, r and b' in $\mathbb{P}_\alpha$. ∎

Remark 6.4. The *solid* genericity has been used in the above proof. In Subcase B of the proof, we built q''. Note that $R^{-1}_{q''}[\{\alpha\}]$ may contain a model

M'' such that $M'' \notin N$ and $\omega_1 \cap M'' < \omega_1 \cap N$. For such an M'', there exist $M' \in R_q^{-1}[\{\alpha\}] \cap N$ and $M \in R_q^{-1}[\{\alpha\}]$ such that $\omega_1 \cap M = \omega_1 \cap N$ and $M'' = (\Psi_M{}^{-1} \circ \Psi_N)(M')$. In general, even if

$$q' \Vdash_{\mathbb{P}_\alpha} \text{“}A_q(\alpha) \text{ is } (M[\dot{G}_{\mathbb{P}_\alpha}], \dot{\mathbb{Q}}_\alpha)\text{-generic and } (M'[\dot{G}_{\mathbb{P}_\alpha}], \dot{\mathbb{Q}}_\alpha)\text{-generic”},$$

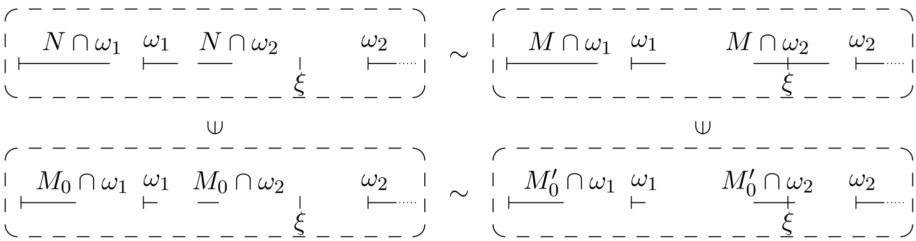
there is no guarantee that

$$q' \Vdash_{\mathbb{P}_\alpha} \text{“}A_q(\alpha) \text{ is } (M''[\dot{G}_{\mathbb{P}_\alpha}], \dot{\mathbb{Q}}_\alpha)\text{-generic”}.$$

This is the reason why solid genericity has been introduced.

Remark 6.5. The authors are afraid that the above lemma fails for $\alpha \geq \omega_2$. Suppose that α is a limit ordinal greater than ω_2 of uncountable cofinality. In this case, when we follow the argument as in Subcase D above, q' may fail assertion $\langle 6 \rangle$.

Suppose that the length of the iteration is $\omega_2 + 1$, $q \in \mathbb{P}_{\omega_2+1}$, $\{M, N\} \subseteq R_q^{-1}[\{\omega_2\}]$, $r \in \mathbb{P}_{\omega_2+1} \cap N$ is a nice copy of q inside N as in the proof of Lemma 6.3, $M_0 \in R_r^{-1}[\{\omega_2\}]$, and $M_0' := (\Psi_M{}^{-1} \circ \Psi_N)(M_0) \in \mathcal{N}_{q_2} \setminus (\mathcal{N}_q \cup \mathcal{N}_r)$, as in the following figure.



We want to find a common extension $q' \in \mathbb{P}_{\omega_2+1}$ of q and r.

Let $q_2 \in \mathbb{P}_{\omega_2}$ be a common extension of $q{\restriction}\omega_2$ and $r{\restriction}\omega_2$. We assume that $R_{q_2}(M_0')$ is a proper initial segment of the set $M_0' \cap \omega_2$. Then we may assume that no extension of q_2 in $\mathbb{P}_{\omega_2}$ is $(\mathbb{P}_{\omega_2}, M_0')$-generic. This is because there may be $\xi \in \mathrm{dom}(A_{q_2}) \cap M_0'$ $(\subseteq \omega_2)$ such that $\sup(R_{q_2}(M_0')) < \xi$ and

$$q_2{\restriction}\xi \Vdash_{\mathbb{P}_\xi} \text{“ } A_{q_2}(\xi) \text{ cannot be extended to solid } (M_0'[\dot{G}_{\mathbb{P}_\xi}], \dot{P}_\xi)\text{-generic”},$$

that is, no extension of $q_2{\restriction}(\xi+1)$ in $\mathbb{P}_{\xi+1}$ is $(\mathbb{P}_{\xi+1}, M_0')$-generic. Now we have $\{M, N\} \subseteq R_q{}^{-1}[\{\omega_2\}]$ and $M_0 \in R_r{}^{-1}[\{\omega_2\}]$. So q' has to satisfy $\langle M_0', \omega_2 \rangle \in R_{q'}$ by (down) of Definition 5.1. It follows that

$$R_{q'}(M_0') = M_0' \cap (\omega_2 + 1).$$

Hence $q'{\restriction}\omega_2$, which is an extension of q_2 in $\mathbb{P}_{\omega_2}$, must be $(\mathbb{P}_{\omega_2}, M_0')$-generic. This would be a contradiction.

LEMMA 6.6. *Suppose that $\alpha \in \omega_2$, $p \in \mathbb{P}_\alpha$, and $\gamma \in \mathrm{dom}(A_p)$ as in case* (b). *Then there exists an extension q of p in $\mathbb{P}_\alpha$ such that*

$$q \Vdash_{\mathbb{P}_\alpha} \text{“} \dot{P}^\gamma \text{ collapses } \omega_1 \text{”.}$$

Proof. Take $p'_{-1} \in \mathbb{P}_{\gamma+1}$ such that $p'_{-1} \le_{\mathbb{P}_{\gamma+1}} p{\restriction}(\gamma+1)$, $A_{p'_{-1}}(\gamma) = A_p(\gamma)$, and take a $\mathbb{P}_\gamma$-name $\dot{\varepsilon}_{-1}$ in H_κ such that

$$p'_{-1}{\restriction}\gamma \Vdash_{\mathbb{P}_\gamma} \text{“} \dot{\varepsilon}_{-1} \in \dot{P}^\gamma \text{”.}$$

By induction on $j \in k+2$, choose $N_j \in \mathfrak{M}^P_\gamma$, $p_j, p'_j \in \mathbb{P}_{\gamma+1}$, $\sigma_j \in [\omega_1]^k$ and a $\mathbb{P}_\gamma$-name $\dot{\varepsilon}_i$ in H_κ such that

- $\{p'_{j-1}\} \cup \{\sigma_{j'} : j' \in j\} \in N_j$,
- $\mathcal{N}_{p_j} := \mathcal{N}_{p'_{j-1}} \cup \{N_j\}$,
- $R_{p_j} := R_{p'_{j-1}} \cup (\{N_0\} \times ((\gamma+1) \cap N_0)) \cup (\{N_j\} \times (\gamma \cap N_j))$,
- $A_{p_j}{\restriction}\gamma = A_{p'_{j-1}}{\restriction}\gamma$, and $A_{p_j}(\gamma) := A_{p'_{j-1}}(\gamma)$ (then p_j is certainly a condition of $\mathbb{P}_{\gamma+1}$ and $p_j \le_{\mathbb{P}_{\gamma+1}} p'_{j-1}$), and
- $p'_j \le_{\mathbb{P}_{\gamma+1}} p_j$, $\omega_1 \cap N_j \le \min(\sigma_j)$, and

$$p'_j{\restriction}\gamma \Vdash_{\mathbb{P}_\gamma} \text{“} \dot{\varepsilon}_j \le_{\dot{P}^\gamma} \dot{\varepsilon}_{j-1} \text{ and } \dot{\varepsilon}_j \Vdash_{\dot{P}^\gamma} \text{“} \sigma_j \in \dot{I}^\gamma \text{””.}$$

Take $n \in \omega \setminus \{n^x : x \in A_p(\gamma)\}$, and define $q' \in U_{\gamma+1}$ such that $R_{q'} := R_{p'_k}$, $A_{q'}{\restriction}\gamma := A_{p'_k}{\restriction}\gamma$, and

$$A_{q'}(\gamma) := A_p(\gamma) \cup \{\langle \{\sigma_j : j \in k+2\}, n\rangle\}.$$

It follows from our construction that q' is a condition of $\mathbb{P}_{\gamma+1}$. Moreover, by Lemma 6.3, q' is $(N_0, \mathbb{P}_{\gamma+1})$-generic and

$$\begin{aligned} q' \Vdash_{\mathbb{P}_{\gamma+1}} \text{“}\{\sigma_j : j \in k+2\} \in \{\Sigma^x : r \in \dot{G}_{\mathbb{P}_{\gamma+1}}, \gamma \in \mathrm{dom}(A_r), x \in A_r(\gamma)\} \\ \in N_0[\dot{G}_{\mathbb{P}_{\gamma+1}}], \text{ and } \{\sigma_j : j \in k+2\} \cap N_0[\dot{G}_{\mathbb{P}_{\gamma+1}}] = \emptyset\text{”.} \end{aligned}$$

Therefore

$$\begin{aligned} q' \Vdash_{\mathbb{P}_{\gamma+1}} \text{“ the set } &\{\min(\Sigma^x) : r \in \dot{G}_{\mathbb{P}_{\gamma+1}}, \gamma \in \mathrm{dom}(A_r), x \in A_r(\gamma)\} \\ &\text{is uncountable”,} \end{aligned}$$

and so

$$q' \Vdash_{\mathbb{P}_{\gamma+1}} \text{“} \nVdash_{\dot{P}^\gamma} \text{“} \omega_1^V \text{ is uncountable””.}$$

A common extension of q' and p in $\mathbb{P}_\alpha$ is the desired condition. ∎

LEMMA 6.7. *Suppose* CH *and $\kappa = \omega_2$ (then $2^{\aleph_1} = \aleph_2$ by our initial assumption). Then $\mathbb{P}_\kappa$ forces* $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ *and $2^{\aleph_0} = \aleph_2$.*

Proof. $\mathbb{P}_\kappa$ forces $2^{\aleph_0} \ge \kappa = \aleph_2$. By Proposition 6.1, the number of isomorphic types of $\mathbb{P}_\kappa$-names for the reals is $\aleph_2^{\aleph_1}$, which is equal to $\aleph_2$. Thus $\mathbb{P}_\kappa$ forces $2^{\aleph_0} \le \aleph_2$. To show that $\mathbb{P}_\kappa$ forces $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$, we follow [2, proof of Theorem 1.3].

Suppose that $p \in \mathbb{P}_\kappa$ and $\{\dot{\mathbb{Q}}, \dot{D}_\zeta : \zeta \in \omega_1\}$ is a set of $\mathbb{P}_\kappa$-names such that

$$p \Vdash_{\mathbb{P}_\kappa} \text{“}\dot{\mathbb{Q}} \text{ is an s-finitely proper forcing notion and each } \dot{D}_\zeta \text{ is a dense subset of } \dot{\mathbb{Q}}\text{”}.$$

Then there exists a large enough ordinal $\alpha \in \kappa$ such that $p \in \mathbb{P}_\alpha$, $\dot{\mathbb{Q}}$ and $\dot{D}_\zeta$, $\zeta \in \omega$, are all $\mathbb{P}_\alpha$-names, and $\Phi(\alpha) = \{\dot{\mathbb{Q}}\}$. Then $\dot{\mathbb{Q}}$ is forced with $\mathbb{P}_\alpha$ to be s-finitely proper. (Notice that our s-finite properness is defined in downward absolute way.) So we notice that

$$p \Vdash_{\mathbb{P}_{\alpha+1}} \text{“for any } q \in \dot{G}_{\mathbb{P}_{\alpha+1}}, \text{ if } \alpha \in \mathrm{dom}(A_q), \text{ then } A_q(\alpha) \in \dot{\mathbb{Q}}_\alpha = \dot{\mathbb{Q}}\text{”}.$$

$\dot{G}^+_\alpha$ denotes a $\mathbb{P}_{\alpha+1}$-name such that

$$p \Vdash_{\mathbb{P}_{\alpha+1}} \text{“}\dot{G}^+_\alpha := \{A_q(\alpha) : q \in \dot{G}_{\mathbb{P}_{\alpha+1}}\}\text{”}.$$

Then, by [2, Lemma 4.9],

$$p \Vdash_{\mathbb{P}_{\alpha+1}} \text{“}\dot{G}^+_\alpha \text{ generates a } V[\dot{G}_{\mathbb{P}_\alpha}]\text{-generic filter of } \dot{\mathbb{Q}}_\alpha\ (= \dot{\mathbb{Q}})\text{”},$$

which finishes the proof. ∎

LEMMA 6.8. *Suppose* CH. *Then, for every $\alpha \in \omega_2$, $\mathbb{P}_\alpha$ preserves the k-entangledness of E. Therefore, if* CH *holds and $\kappa = \omega_2$, then $\mathbb{P}_\kappa$ also preserves the k-entangledness of E.*

Proof. This is also proved by induction on α. As in the proof of Lemma 6.3, we consider several cases.

Throughout the proof, suppose that $d' \in {}^k\{0,1\}$, $\dot{J}$ is a $\mathbb{P}_\alpha$-name for an uncountable subset of $[\omega_1]^k$, and $p \in \mathbb{P}_\alpha$. Take an $N \in \mathfrak{M}^P_\alpha$ which contains $\{E, \dot{J}, p\}$, and let q be an extension of p in $\mathbb{P}_\alpha$ such that $\{N\} \times (\alpha \cap N) \subseteq R_q$. By Lemma 6.3, q is $(N, \mathbb{P}_\alpha)$-generic. We show that

$$q \Vdash_{\mathbb{P}_\alpha} \text{“for every } \nu \in \dot{J} \text{ with } \omega_1 \cap N \le \min(\nu), \text{ there exists } \tau \in \dot{J} \cap N[\dot{G}_{\mathbb{P}_\alpha}] \text{ such that } \tau <_{d'} \nu\text{”}.$$

By extending q if necessary, suppose that $\nu \in [\omega_1]^k$, $\nu \cap N = \emptyset$, and

$$q \Vdash_{\mathbb{P}_\alpha} \text{“}\nu \in \dot{J}\text{”}.$$

Basic stage. Suppose that $\alpha = 0$. Let $\{B_i : i \in k\}$ be a set of pairwise disjoint basic open intervals of $\mathbb{R}$ such that $\langle e_\xi : \xi \in \nu\rangle \in \prod_{i\in k} B_i$. Take $\gamma \in \omega_1 \cap N$ such that

$$\{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap \gamma.$$

Define

$$X := \Big\{\mu \in [\omega_1]^k : \text{there exists } r \in \mathbb{P}_0 \text{ such that } \{\omega_1 \cap K : K \in \mathcal{N}_r\} \cap \gamma = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N, \ \mathcal{N}_r \supseteq \mathcal{N}_q \cap N, \text{ and } r \Vdash_{\mathbb{P}_0} \text{“}\mu \in \dot{J} \text{ and } \langle e_\xi : \xi \in \mu\rangle \in \prod_{i\in k} B_i\text{”}\Big\}.$$

We notice that X belongs to N and q witnesses that $\nu \in X$. Take $\delta \in \omega_1 \cap N$ which witnesses the k-entangledness for X. Then $\delta \leq \min(\nu)$. So there exists $\tau \in X \cap [\delta]^k$ such that $\tau <_{d'} \nu$. Let $r \in \mathbb{P}_0 \cap N$ witness that $\tau \in X$. Then, as seen in the proof of Lemma 6.3, q and r are compatible in $\mathbb{P}_0$. So a common extension of q and r in $\mathbb{P}_0$ forces that both τ and ν are in $\dot{J}$, which finishes the proof in this case.

Successor stage. Suppose that α is a successor ordinal, and, by renaming the ordinal α, suppose that $\dot{J}$ is a $\mathbb{P}_{\alpha+1}$-name, $p \in \mathbb{P}_{\alpha+1}$ and $N \in \mathfrak{M}^P_{\alpha+1}$.

Subcase A: $\alpha \notin \operatorname{dom}(A_p)$. This case can be handled in a similar way to the basic stage.

Subcase B: *$A_q(\alpha)$ is a $\mathbb{P}_\alpha$-name.* Then, by the definition,

$$q{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“} A_q(\alpha) \Vdash_{\dot{\mathbb{Q}}_\alpha} \text{“} E \text{ is } k\text{-entangled”}\text{”}.$$

We note that the condition $\langle q{\restriction}\alpha, A_q(\alpha)\rangle$ of the forcing iteration $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ is $(N, \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)$-generic. Define a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$-name $\dot{J}'$ such that, for every $r \in \mathbb{P}_\alpha$, a $\mathbb{P}_\alpha$-name $\dot{\varepsilon}$ for a condition of $\dot{\mathbb{Q}}_\alpha$, and $\mu \in [\omega_1]^k$,

$$r \Vdash_{\mathbb{P}_\alpha} \text{“} \dot{\varepsilon} \Vdash_{\dot{\mathbb{Q}}_\alpha} \text{“} \mu \in \dot{J}' \text{”}\text{”}$$

if and only if there exists $r' \in \mathbb{P}_{\alpha+1}$ such that $r' \Vdash_{\mathbb{P}_{\alpha+1}} \text{“} \mu \in \dot{J} \text{”}$, $r \leq_{\mathbb{P}_\alpha} r'{\restriction}\alpha$, $A_{r'}(\alpha)$ is a $\mathbb{P}_\alpha$-name, and $r \Vdash_{\mathbb{P}_\alpha} \text{“} \dot{\varepsilon} \leq_{\dot{\mathbb{Q}}_\alpha} A_{r'}(\alpha) \text{”}$. Then $\dot{J}' \in N$ and

$$q{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“} A_q(\alpha) \Vdash_{\dot{\mathbb{Q}}_\alpha} \text{“} \nu \in \dot{J}' \text{”}\text{”}.$$

Since

$$q{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“} A_q(\alpha) \Vdash_{\dot{\mathbb{Q}}_\alpha} \text{“} E \text{ is } k\text{-entangled”}\text{”},$$

there exists a $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$-name $\dot{\delta}$ in N such that

$q{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha}$“ $A_q(\alpha) \Vdash_{\dot{\mathbb{Q}}_\alpha}$“ $\dot{\delta} \in \omega_1$, which witnesses the k-entangledness for $\dot{J}'$””.

Then

$$q{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“} A_q(\alpha) \Vdash_{\dot{\mathbb{Q}}_\alpha} \text{“} \dot{\delta} < \omega_1 \cap N \leq \min(\nu) \text{”}\text{”}.$$

So there are $r \in \mathbb{P}_\alpha$, a $\mathbb{P}_\alpha$-name $\dot{\varepsilon}$ and $\tau \in [\omega_1]^k \cap N$ such that $r \leq_{\mathbb{P}_\alpha} q{\restriction}\alpha$,

$$r \Vdash_{\mathbb{P}_\alpha} \text{“} \dot{\varepsilon} \leq_{\dot{\mathbb{Q}}_\alpha} A_q(\alpha) \text{ and } \dot{\varepsilon} \Vdash_{\dot{\mathbb{Q}}_\alpha} \text{“} \tau \in \dot{J}' \cap [\dot{\delta}]^k \text{”}\text{”},$$

and $\tau <_{d'} \nu$. Since the condition $\langle r, \dot{\varepsilon}\rangle$ of $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ is $(N, \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha)$-generic, there exists $r' \in \mathbb{P}_{\alpha+1} \cap N$ such that $r' \Vdash_{\mathbb{P}_{\alpha+1}} \text{“} \tau \in \dot{J} \text{”}$, $r \leq_{\mathbb{P}_\alpha} r'{\restriction}\alpha$, $A_{r'}(\alpha) \in \omega_1$ and $r \Vdash_{\mathbb{P}_\alpha} \text{“} \dot{\varepsilon} \leq_{\dot{\mathbb{Q}}_\alpha} A_{r'}(\alpha) \text{”}$. Then, as seen in the proof of Lemma 6.3, q, r and r' are compatible in $\mathbb{P}_{\alpha+1}$, and a common extension of q and r in $\mathbb{P}_0$ forces that both τ and ν are in $\dot{J}$, which finishes the proof in this case.

Subcase C: *$A_q(\alpha)$ is a finite subset of pairs.* The proof in this case is similar to that of Lemma 2.7. Let $\{x^q_\zeta : \zeta \in l\}$ be the enumeration of $A_q(\alpha) \setminus N$ with respect to the set $\{N' \in \mathfrak{M}^P_\alpha : \{N'\} \times (\alpha \cap N') \subseteq R_q\}$ as in Subcase C of the proof of Lemma 6.3. We argue as in the proof of Lemma 2.7. Let $\{B^\zeta_{j,i} : \zeta \in l,\ j \in k+2,\ i \in k\}$ be a set of pairwise disjoint basic open subsets of $\mathbb{R}$ such that, for any $\zeta \in l$ and $j \in k+2$,

$$\langle e_\xi : \xi \in \sigma_j^{x^q_\zeta} \rangle \in \prod_{i\in k} B^\zeta_{j,i} =: B^\zeta_j.$$

As in Lemma 2.5 and Definition 5.1 (up), take $\{j_0, j_1\} \in [k+2]^2$ and a partition $\nu = \bigcup_{u\in h} \nu_u$ such that the set $\{\sigma_{j_0}^{x^q_\zeta}, \sigma_{j_1}^{x^q_\zeta} : \zeta \in l\} \cup \{\nu_u : u \in h\}$ is separated by some finite $\in$-chain with members in $\{N' \in \mathfrak{M}^P_\alpha : \{N'\} \times (\alpha \cap N') \subseteq R_q\}$. For each $u \in h$, take $\bar\nu_u \in [\omega_1]^k$ such that

- $\bar\nu_u$ end-extends ν_u, and
- the set $\{\sigma_{j_0}^{x^q_\zeta}, \sigma_{j_1}^{x^q_\zeta} : \zeta \in l\} \cup \{\bar\nu_u : u \in h\}$ is also separated by some finite $\in$-chain with members in $\{N' \in \mathfrak{M}^P_\alpha : \{N'\} \times (\alpha \cap N') \subseteq R_q\}$.

For each $u \in h$, let $\nu_u = \{\xi^u_v : v \in |\nu_u|\}$ be the increasing enumeration. For each $u \in h$, take $d'_u \in {}^k\{0,1\}$ such that, for each $v \in |\nu_u|$, letting $i, i' \in k$ be the unique indices such that

$$e_{\xi^u_v} = \langle e_\xi : \xi \in \nu \rangle_i = \langle e_\xi : \xi \in \bar\nu_u \rangle_{i'},$$

we have

$$d'_u(i') := d'(i).$$

Take a set $\{B^{\zeta,0}_i, B^{\zeta,1}_i, B^{l,u}_i : \zeta \in l, u \in h, i \in k\}$ of pairwise disjoint basic open sets of $\mathbb{R}$ such that, for each $\zeta \in l$, $\langle e_\xi : \xi \in \sigma_{j_0}^{x_\zeta} \rangle \in \prod_{i\in k} B^{\zeta,0}_i =: B^{\zeta,0}$ and $\langle e_\xi : \xi \in \sigma_{j_1}^{x_\zeta} \rangle \in \prod_{i\in k} B^{\zeta,1}_i =: B^{\zeta,1}$, and, for each $u \in h$, $\langle e_\xi : \xi \in \bar\nu_u \rangle \in \prod_{i\in k} B^{l,u}_i =: B^{l,u}$. Define a $\mathbb{P}_\alpha$-name $\dot X$ such that

$q{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha}$ "$\dot X = \big\{\{\sigma_{j_0}^{y^s_\zeta}, \sigma_{j_1}^{y^s_\zeta} : \zeta \in l\} \cup \{\mu_u : u \in h\} \in \big[[\omega_1]^k\big]^{2l+h}$:

there is $s \in \mathbb{P}_{\alpha+1}$ such that

- $A_s(\alpha)$ is a finite set of pairs, and $s{\restriction}\alpha \in \dot G_{\mathbb{P}_\alpha}$,
- $A_s(\alpha)$ end-extends $A_q(\alpha) \cap N$ and $A_s(\alpha) \setminus (A_q(\alpha) \cap N)$ is of size l; let $\langle y^s_\zeta : \zeta \in l \rangle$ be the increasing enumeration of $A_s(\alpha) \setminus (A_q(\alpha) \cap N)$ with respect to the set $\{N' \in \mathfrak{M}^P_\alpha : \{N'\} \times (\alpha \cap N') \subseteq R_s\}$,
- for any $\zeta \in l$, $n^{y^s_\zeta} = n^{x^q_\zeta}$,
- $\{\mu_u : u \in h\} \in \big[[\omega_1]^k\big]^h$; for each $u \in h$, let $\mu_u = \{\eta^u_i : i \in k\}$ be the increasing enumeration,
- $s \Vdash_{\mathbb{P}_{\alpha+1}}$ "$\bigcup_{u\in h}\{\eta^u_i : i \in |\nu_u|\} \in \dot J$",
- for any $\zeta \in l$ and $j \in k+2$, $\langle e_\xi : \xi \in \sigma_j^{y^s_\zeta} \rangle \in B^\zeta_j$,
- for each $\zeta \in l$, $\langle e_\xi : \xi \in \sigma_{j_0}^{y^s_\zeta} \rangle \in B^{\zeta,0}$ and $\langle e_\xi : \xi \in \sigma_{j_1}^{y^s_\zeta} \rangle \in B^{\zeta,1}$, and, for each $u \in h$, $\langle e_\xi : \xi \in \mu_u \rangle \in B^{l,u}$,

– the mapping

$$\sigma_{j_0}^{x_\zeta^q} \mapsto \sigma_{j_0}^{y_\zeta^s}, \quad \text{for each } \zeta \in l,$$
$$\sigma_{j_1}^{x_\zeta^q} \mapsto \sigma_{j_1}^{y_\zeta^s}, \quad \text{for each } \zeta \in l,$$
$$\bar{\nu}_u \mapsto \mu_u, \quad \text{for each } u \in h,$$

preserves the order of the separation of

$$\{\sigma_{j_0}^{x_\zeta^q}, \sigma_{j_1}^{x_\zeta^q} : \zeta \in l\} \cup \{\bar{\nu}_u : u \in h\} \quad \text{by} \quad \{N' \in \mathfrak{M}_\alpha^P : \{N'\} \times (\alpha \cap N') \subseteq R_q\}$$

and the separation of

$$\{\sigma_{j_0}^{y_\zeta^s}, \sigma_{j_1}^{y_\zeta^s} : \zeta \in l\} \cup \{\mu_u : u \in h\} \text{ by } \{N' \in \mathfrak{M}_\alpha^P : \{N'\} \times (\alpha \cap N') \subseteq R_s\}\}\text{''}.$$

We notice that $\dot{X}$ belongs to N and

$$q{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{``}\, \nu \in \dot{X}\,\text{''}.$$

As in the proof of Lemma 6.3, applying Lemma 2.5 to suitable objects in the extension by $q{\restriction}\alpha$ under $\mathbb{P}_\alpha$, we obtain an extension q' of $q{\restriction}\alpha$ in $\mathbb{P}_\alpha$, $\{\sigma_0^\zeta, \sigma_1^\zeta : \zeta \in l\} \cup \{\mu_u : u \in h\} \in N$ and $s \in \mathbb{P}_{\alpha+1} \cap N$ such that, for each $\zeta \in l$, $\sigma_0^\zeta <_{d^\alpha} \sigma_{j_0}^{x_\zeta^q}$ and $\sigma_{j_1}^{x_\zeta^q} <_{d^\alpha} \sigma_1^\zeta$, and for each $u \in h$, $\mu_u <_{d'_u} \bar{\nu}_u$, and

$$q' \Vdash_{\mathbb{P}_\alpha} \text{``}\, \{\sigma_0^\zeta, \sigma_1^\zeta : \zeta \in l\} \cup \{\mu_u : u \in h\} \in \dot{X} \text{ and } s \text{ witnesses it''}.$$

Then

$$\tau <_{d'} \bigcup_{u \in h} \{\eta_i^u : i \in |\nu_u|\},$$

and q', q and s are compatible in $\mathbb{P}_{\alpha+1}$, which finishes the proof in this case.

Limit stage. Suppose that α is a limit ordinal. As in the proof of Lemma 6.3, we consider two cases.

SUBCASE D: *α is of uncountable cofinality.* As in Lemma 6.3, we take $\beta \in \alpha \cap N$ and $\gamma \in \omega_1 \cap N$ such that

- $\max(\operatorname{dom}(A_q) \cap \sup(\alpha \cap N)) < \beta$,
- $\max(\{\sup(R_q(M)) : M \in \operatorname{dom}(R_q)\} \cap N) < \beta$,
- for every $M' \in \mathcal{N}_q$ with $\omega_1 \cap M' < \omega_1 \cap N$,
$$\sup(M' \cap N \cap \alpha) < \beta,$$
- $\{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap \gamma$, and
- for every $\xi \in \operatorname{dom}(A_q)$ as in case (b),
$$\{\Sigma^x : x \in A_q(\xi)\} \cap N = \{\Sigma^x : x \in A_q(\xi)\} \cap \left[[\gamma]^k\right]^{k+2}.$$

Define a $\mathbb{P}_\beta$-name $\dot{X}$ such that

$$q\restriction\beta \Vdash_{\mathbb{P}_\beta} \text{“}\, \dot{X} = \big\{\mu \in [\omega_1]^k : \text{there is } s \in \mathbb{P}_\alpha \text{ with } s\restriction\beta \in \dot{G}_{\mathbb{P}_\beta} \text{ such that } \{\omega_1 \cap K : K \in \mathcal{N}_r\} \cap \gamma = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N, \text{ and } s \Vdash_{\mathbb{P}_\alpha} \text{“}\, \mu \in \dot{J}\,\text{”}\big\}\,\text{”}.$$

We notice that $\dot{X}$ belongs to N and

$$q\restriction\beta \Vdash_{\mathbb{P}_\beta} \text{“}\, \nu \in \dot{X} \setminus N[\dot{G}_{\mathbb{P}_\beta}]\,\text{”}.$$

By the induction hypothesis,

$$q\restriction\beta \Vdash_{\mathbb{P}_\beta} \text{“}\, E \text{ is } k\text{-entangled}\,\text{”}.$$

So there is a $\mathbb{P}_\beta$-name $\dot{\delta}$ in N such that

$$q\restriction\beta \Vdash_{\mathbb{P}_\beta} \text{“}\, \dot{\delta} \in \omega_1 \text{ witnesses the } k\text{-entangledness for } \dot{X}\,\text{”}.$$

Then

$$q\restriction\beta \Vdash_{\mathbb{P}_\beta} \text{“}\, \dot{\delta} < \omega_1 \cap N \le \min(\nu)\,\text{”}.$$

Take an extension q' of $q\restriction\beta$ in $\mathbb{P}_\beta$, $\tau \in [\omega_1]^k \cap N$ and $s \in \mathbb{P}_\alpha \cap N$ such that $\tau <_{d'} \nu$ and

$$q' \Vdash_{\mathbb{P}_\beta} \text{“}\, \tau \in \dot{X} \cap [\dot{\delta}]^k \text{ and } s \text{ witnesses that } \tau \in \dot{X}\,\text{”}.$$

Then q', q and s are compatible in $\mathbb{P}_\alpha$, and a common extension of q', q and s forces that both μ and τ are in $\dot{J}$, which finishes the proof in this case.

Subcase E: *α is of countable cofinality.* Take $\beta \in \alpha \cap N$ and $\gamma \in \omega_1 \cap N$ such that

- $\max(\mathrm{dom}(A_q)) < \beta$,
- for each $M \in \mathrm{dom}(R_q)$, either $R_q(M) \subseteq \beta$ or $R_q(M)$ is cofinal in α,
- $\{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap N = \{\omega_1 \cap K : K \in \mathcal{N}_q\} \cap \gamma$, and
- for every $\xi \in \mathrm{dom}(A_q)$ as in case (b),

$$\{\Sigma^x : x \in A_q(\xi)\} \cap N = \{\Sigma^x : x \in A_q(\xi)\} \cap \big[[\gamma]^k\big]^{k+2}.$$

The rest of the proof is similar to the previous one. ∎

Consequently, the following has been proved.

Theorem 6.9. *Suppose that* CH *holds and $\kappa = \omega_2$ (then also $2^{\aleph_1} = \aleph_2$). Then $\mathbb{P}_\kappa$ forces that* $\mathsf{PFA}^{\text{s-fin}}(\aleph_1)$ *holds, E is a k-entangled set of reals, and $2^{\aleph_0} = \aleph_2$.*

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