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Local k-connectedness of an inverse limit of polyhedra

by

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Abstract. We provide an easily verifiable condition for local k-connectedness of an inverse limit of polyhedra.

1. Introduction. The Nöbeling space characterization theorem [3, 1, 2, 10, 12] states that if a space is strongly universal in the class of n-dimensional separable complete metric spaces and is k-connected and locally k-connected for each $k < n$, then it is homeomorphic to the n-dimensional Nöbeling space N_n^{2n+1}.

In geometric group theory many spaces arise naturally as inverse limits of polyhedra (simplicial complexes endowed with the metric topology). In particular, boundaries at infinity of hyperbolic spaces can be expressed as such. Striking examples of applications of the Nöbeling space characterization theorem are proofs that the boundary at infinity of the curve complex of the $(n+5)$-punctured 2-dimensional sphere is homeomorphic to the n-dimensional Nöbeling space N_n^{2n+1} [8, 7].

In the present paper we prove a condition for local k-connectedness of an inverse limit of polyhedra that is easy to verify. It is designed to aid detection of local k-connectedness in many examples of spaces arising in geometric group theory. We prove the following theorem.

Definition 1.1. Let K and L be simplicial complexes. We say that a map $p\colon K \to L$ is *n-regular* if it is quasi-simplicial (i.e. it is simplicial into the first barycentric subdivision βL of L) and if for each simplex δ of βL the inverse image $p^{-1}(\delta)$ has vanishing homotopy groups in dimensions less than n (regardless of the choice of basepoint).

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THEOREM 1.2. *Let*

$$X = \varprojlim\big(K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots\big).$$

Assume that for each i the following conditions are satisfied:

(I) *K_i is a locally finite-dimensional simplicial complex endowed with the metric topology; and*
(II) *p_i is a quasi-simplicial map that is surjective and n-regular.*

Then X is locally k-connected for each $k < n$, and each short projection $\pi_i^m\colon K_m \to K_i$ and each long projection $\pi_i\colon X \to K_i$ is a weak n-homotopy equivalence (*it induces homomorphisms on homotopy groups in dimensions less than n, regardless of the choice of basepoint*).

Note that there is no assumption of local finiteness of complexes in Theorem 1.2.

It is known that (under suitable assumptions) if the bonding maps in the inverse limit are n-soft, then the inverse limit is k-connected for $k < n$ [5]. The condition for local k-connectedness given in the present paper may be regarded as a combinatorial analog of this statement for inverse limits of polyhedra. Note that n-regular bonding maps need not be n-soft.

2. Preliminaries.

In this section we set the basic definitions and reference known results that will be used in the later sections.

2.1. Absolute extensors in dimension n

DEFINITION 2.1. We say that a space X is *k-connected* if each map $\varphi\colon S^k \to X$ from a k-dimensional sphere into X is null-homotopic in X.

DEFINITION 2.2. We say that a space X is *locally k-connected* if for each point $x \in X$ and each open neighborhood $U \subset X$ of x there exists an open neighborhood V of x such that each map $\varphi\colon S^k \to V$ from a k-dimensional sphere into V is null-homotopic in U.

DEFINITION 2.3. We say that a metric space X is an *absolute neighborhood extensor in dimension n* if every map from a closed subset A of an n-dimensional metric space into X extends over an open neighborhood of A. The class of absolute neighborhood extensors in dimension n is denoted by $ANE(n)$ and its elements are called *$ANE(n)$-spaces.*

DEFINITION 2.4. We say that a metric space X is an *absolute extensor in dimension n* if every map from a closed subset of an n-dimensional metric space Y into X extends over the entire space Y. The class of absolute extensors in dimension n is denoted by $AE(n)$ and its elements are called *$AE(n)$-spaces.*

DEFINITION 2.5. Let $\mathcal{C}$ be a class of topological spaces. We let $AE(\mathcal{C})$ denote the class of *absolute extensors for all spaces from the class* $\mathcal{C}$. We write $AE(X)$ for $AE(\{X\})$.

Absolute extensors and absolute neighborhood extensors in dimension n were characterized by Dugundji in the following theorem.

THEOREM 2.6 ([6]). *Let X be a metric space. Then*

(1) $X \in ANE(n) \Leftrightarrow X$ *is locally k-connected for all $k < n$; and*
(2) $X \in AE(n) \Leftrightarrow X \in ANE(n)$ *and X is k-connected for all $k < n$.*

LEMMA 2.7 (cf. [15, Theorem 6.1.8]). *Assume that $A_1 \subset A_2 \subset \cdots$ is a sequence of subsets of a metric space such that each A_i is closed and for each i, $A_i \subset \operatorname{Int} A_{i+1}$. If A_i is $AE(n)$ for each i, then $A = \bigcup_i A_i$ is $AE(n)$.*

2.2. Polyhedra. For a simplicial complex, the underlying polyhedron has two topologies, the Whitehead (weak) topology and the metric topology. The metric topology is the topology of point-wise convergence of barycentric coordinates [9, p. 100]. The weak topology is metrizable if and only if the complex is locally finite, and it coincides with the metric topology in this case. Since we work in the metric category with complexes that are not locally finite, *we always assume the metric topology on simplicial complexes* [11, 9].

DEFINITION 2.8. Let K be a simplicial complex. We let $\tau(K)$ denote the *triangulation* of K (the set of simplices of K). We let $V(K)$ denote the *vertex set* of K. We let βK denote the *barycentric subdivision* of K (i.e., the same space but with a finer triangulation $\tau(\beta K)$).

DEFINITION 2.9. Let K be a simplicial complex. Let $\kappa > 0$. Let $\ell_1(V(K))$ denote the Banach space $\{x \in \mathbb{R}^{V(K)} : \sum_{v \in V(K)} |(x)_v| < \infty\}$, where $(x)_v$ denotes the vth coordinate, equipped with the standard $\|\cdot\|_1$ norm. For $v \in v(K)$ define $e_v : V(K) \to \mathbb{R}$ by the formula $e_v(w) = 0$ for $w \neq v$ and $e_v(v) = \kappa$. We embed each vertex v of K as $e_v \in \ell_1(V(K))$ and extend this embedding to K to be affine on each simplex of K. We consider K to be a subspace of $\ell_1(V(K))$. We call the induced metric on K the *metric of scale* κ on K. For $\kappa = 1$ it is the standard metric, as defined in [9, p. 100]. The topology induced by this metric is called the *metric topology.*

DEFINITION 2.10. A *polyhedron* is a simplicial complex endowed with the metric topology.

DEFINITION 2.11. Let K and L be polyhedra. We say that K is a *full subpolyhedron* of L if whenever the vertices $v_0, \ldots, v_n$ span a simplex in K and each v_i is a vertex in L, then the v_i span a simplex in L.

LEMMA 2.12. *A locally finite-dimensional polyhedron is a complete metric $ANE(\infty)$-space.*

Proof. It is complete by [9, Lemma 11.5]. It is an $ANE(\infty)$-space by [9, Theorem 11.3]. ∎

DEFINITION 2.13. Let K and L be simplicial complexes and let $p\colon K \to L$. We say that p is *quasi-simplicial* if it is a simplicial map into βL.

LEMMA 2.14 (cf. [4]). *Assume that K and L are polyhedra endowed with metrics of scale κ and λ respectively. If $p\colon K \to L$ is quasi-simplicial, then it is $\frac{\lambda}{2\kappa}$-Lipschitz.*

2.3. Weak n-homotopy

DEFINITION 2.15. We say that a map is a *weak n-homotopy equivalence* if it induces isomorphisms on homotopy groups of dimensions less than n, regardless of the choice of basepoint.

DEFINITION 2.16. Let $\mathcal{F}$ be a cover of a space X. We say that maps $f, g\colon Y \to X$ are *$\mathcal{F}$-close* if for each $y \in Y$ there exists $F \in \mathcal{F}$ such that $f(y), g(y) \in F$.

DEFINITION 2.17. Let $\mathcal{U}$ be a cover of a space X. We say that maps $f, g\colon Y \to X$ are *$\mathcal{U}$-homotopic* if there exists a homotopy $H\colon Y \times [0,1] \to X$ whose paths refine $\mathcal{U}$, i.e. for each $y \in Y$ there exists $U \in \mathcal{U}$ such that $H(\{y\} \times [0,1]) \subset U$.

2.4. Carrier Theorem

DEFINITION 2.18. Let $\mathcal{C}$ be a class of topological spaces. We say that a cover $\mathcal{F}$ of a topological space is a $\mathcal{C}$-cover, if for each $\mathcal{A} \subset \mathcal{F}$ the intersection $\bigcap \mathcal{A}$ is either empty or belongs to $\mathcal{C}$.

DEFINITION 2.19. We say that a cover is *locally finite-dimensional* if its nerve is locally finite-dimensional.

DEFINITION 2.20. A *carrier* is a function $C\colon \mathcal{F} \to \mathcal{G}$ from a cover $\mathcal{F}$ of a space X into a collection $\mathcal{G}$ of subsets of a topological space such that for each $\mathcal{A} \subset \mathcal{F}$ if $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$, then $\bigcap_{A \in \mathcal{A}} C(A) \neq \emptyset$. We say that a map f is *carried by C* if it is defined on a closed subset of X and $f(F) \subset C(F)$ for each $F \in \mathcal{F}$. Here we write $f(F)$ for $f(F \cap \operatorname{dom}(f))$.

CARRIER THEOREM ([13]). *Assume that $C\colon \mathcal{F} \to \mathcal{G}$ is a carrier such that $\mathcal{F}$ is a cover of a space X and $\mathcal{G}$ is an $AE(X)$-cover of another space. If $\mathcal{F}$ is closed, locally finite, and locally finite-dimensional, then each map carried by C extends to a map of the entire space X, also carried by C.*

Corollary 2.21 ([13]). *If $\mathcal{F}$ is a closed locally finite locally finite-dimensional $AE(n)$-cover of a space Y, then any two $\mathcal{F}$-close maps from a metric space of dimension less than n into Y are $\mathcal{F}$-homotopic.*

2.5. Covers. We regard covers as indexed collections of sets and use the usual notation $\mathcal{F} = \{F_i\}_{i\in I}$, where I denotes the indexing set.

Definition 2.22. Let K be a polyhedron. Let $L \subset K$ be a subcomplex of K. The *open star* $\operatorname{ost}_K L$ of L in K is the complement of the union of all simplices of K that do not intersect L:

$$\operatorname{ost}_K L = K \setminus \bigcup\{\delta \in \tau(K) \colon \delta \cap L = \emptyset\}.$$

The *barycentric star* $\operatorname{bst}_K L$ of L in K is the union of all simplices of βK that intersect L:

$$\operatorname{bst}_K L = \bigcup\{\delta \in \tau(\beta K) \colon \delta \cap L \neq \emptyset\}.$$

We let

$$\mathcal{O}_K = \{\operatorname{ost}_K\{v\} \colon v \in V(K)\}$$

denote the *cover of K by open stars of vertices.* We let

$$\mathcal{B}_K = \{\operatorname{bst}_K\{v\} \colon v \in V(K)\}$$

denote the *cover of K by barycentric stars of vertices.*

Lemma 2.23 ([14]). *Let K be a polyhedron. The cover $\mathcal{B}_K$ by barycentric stars of vertices is a closed locally finite $AE(\infty)$-cover of K. Moreover, if K is locally finite-dimensional, then $\mathcal{B}_K$ is locally finite-dimensional. The cover $\mathcal{O}_K$ by open stars of vertices is an open $AE(\infty)$-cover of K.*

Definition 2.24. Let $X = \varprojlim(K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots)$. Let $v(K_i)$ denote the set of vertices of K_i. We let

$$\mathcal{O}_{K_i} = \{O_v = \operatorname{ost}_{K_i} v\}_{v\in v(K_i)}$$

be the cover of K_i by open stars of vertices of K_i (see Definition 2.22) and

$$\mathcal{O}_i = \{\pi_i^{-1}(\operatorname{ost}_{K_i} v)\}_{v\in v(K_i)}$$

be the cover of X by sets of threads that pass through elements of $\mathcal{O}_{K_i}$.

Definition 2.25. Let $X = \varprojlim(K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots)$. Let $v(K_i)$ denote the set of vertices of K_i. We let

$$\mathcal{B}_{K_i} = \{\operatorname{bst}_{K_i} v\}_{v\in v(K_i)}$$

be the cover of K_i by barycentric stars of vertices of K_i (see Definition 2.22) and

$$\mathcal{B}_i = \{\pi_i^{-1}(\operatorname{bst}_{K_i} v)\}_{v\in v(K_i)}$$

be the cover of X by sets of threads that pass through elements of $\mathcal{B}_{K_i}$.

DEFINITION 2.26. We say that a cover $\mathcal{F} = \{F_i\}_{i\in I}$ is *isomorphic* to a cover $\mathcal{G} = \{G_i\}_{i\in I}$ if for each $J \subset I$ we have

$$\bigcap_{j\in J} F_j \neq \emptyset \iff \bigcap_{j\in J} G_j \neq \emptyset.$$

Note the same indexing set of $\mathcal{F}$ and $\mathcal{G}$.

DEFINITION 2.27. Let $p\colon Y \to Z$ be a map. Let $\mathcal{F} = \{F_i\}_{i\in I}$ be a cover of Z. A *pull-back of* $\mathcal{F}$ is a cover $p^{-1}(\mathcal{F})$ of Y defined by the formula

$$p^{-1}(\mathcal{F}) = \{p^{-1}(F_i)\}_{i\in I}.$$

The pull-back $p^{-1}(\mathcal{F})$ retains the indexing set I of $\mathcal{F}$.

LEMMA 2.28. *Let $f\colon X \to Y$ be a map, let $\mathcal{G}$ be a cover of Y and let $f^{-1}(\mathcal{G})$ be a pull-back of $\mathcal{G}$.*

(1) *If f is surjective, then the covers $f^{-1}(\mathcal{G})$ and $\mathcal{G}$ are isomorphic.*
(2) *If $\mathcal{G}$ is open (respectively closed, locally finite, locally finite-dimensional), then so is $f^{-1}(\mathcal{G})$.*

3. Nerve Theorem. The Nerve Theorem in its abstract form states that if two spaces admit isomorphic $AE(n)$-covers, then they are weak n-homotopy equivalent. For general spaces, some local finiteness and dimension restrictions are placed on the covers [13]. In our case we need such a theorem without these restrictions, which we are able to prove for polyhedra covered by subcomplexes.

The goal of this section is to prove the following theorem.

THEOREM 3.1. *If a quasi-simplicial map $p\colon K \to L$ of polyhedra is surjective and n-regular, then it is a weak n-homotopy equivalence and the pull-back $p^{-1}(\mathcal{B}_L)$ is an $AE(n)$-cover of K.*

The main tool used in the proof is Theorem 3.5, which when applied to a canonical map into the nerve of a cover [13] implies the usual Nerve Theorem.

3.1. Nerve Theorem for non-locally-finite covers

LEMMA 3.2. *If K is a full subpolyhedron of a polyhedron L, then*

(1) *K is a deformation retract of $\operatorname{bst}_L K$; and*
(2) *K is a deformation retract of $\operatorname{ost}_L K$.*

Proof. We let V_L denote the set of vertices of L and $V_K \subset V_L$ denote the set of vertices of K. We regard L as a subspace of $\ell_1(V_L)$ (see Definition 2.9). Let $p\colon L \to \ell_1(V_L)$ be the map that sends all coordinates that do not belong to K to 0, defined by

$$(p(x))_v = \begin{cases} 0, & v \in V_L \setminus V_K, \\ (x)_v, & v \in V_K. \end{cases}$$

We have

$$\operatorname{ost}_L K = \{x \in L\colon p(x) \neq 0\}.$$

Let $q\colon \operatorname{ost}_L K \to L$ be defined by the formula

$$q(x) = \frac{p(x)}{\|p(x)\|_1}.$$

We define a map $\Phi\colon \operatorname{ost}_L K \times [0,1] \to L$ by

$$\Phi(x,t) = t \cdot q(x) + (1-t) \cdot x.$$

Observe that $\operatorname{supp} \Phi(x,t) \subset \operatorname{supp}(p(x))$ and $V_K \cap \operatorname{supp} \Phi(x,t) \neq \emptyset$, hence $\Phi(x,t) \in \operatorname{ost}_L K$. Note that $\Phi(x,1) \in K$ because K is a full subcomplex. Hence Φ is a deformation retraction of $\operatorname{ost}_L K$ to K.

Observe that

$$\operatorname{bst}_L K = \big\{x \in L\colon \exists_{v \in V_K}\ (x)_v \geq \max\{(x)_w\colon w \in V_L\}\big\}.$$

Since Φ preserves the relation $\exists_{v \in V_K} (x)_v \geq \max\{(x)_w\colon w \in V_L\}$, Φ restricted to $\operatorname{bst}_L K \times [0,1]$ maps into $\operatorname{bst}_L K$, hence it is a deformation retraction of $\operatorname{bst}_L K$ to K. ∎

COROLLARY 3.3. *If K is a polyhedron and $\mathcal{K} = \{K_i\}_{i \in I}$ is an $AE(n)$-cover of K by subcomplexes, then the collection*

$$\operatorname{ost}_{\beta K} \mathcal{K} = \{\operatorname{ost}_{\beta K} K_i\}_{i \in I}$$

is an open $AE(n)$-cover of K. Moreover, $\operatorname{ost}_{\beta K} \mathcal{K}$ and $\mathcal{K}$ are isomorphic covers of K.

Proof. Observe that if $\{A_j\}_{j \in J}$ is a collection of subcomplexes of a polyhedron K, then

$$\bigcap_{j \in J} \operatorname{ost}_{\beta K} A_j = \operatorname{ost}_{\beta K} \bigcap_{j \in J} A_j.$$

An application of Lemma 3.2 finishes the proof. ∎

LEMMA 3.4. *If $\mathcal{U}$ is an open $AE(n)$-cover of a space Y, then any two $\mathcal{U}$-close maps from a metric space of dimension less than n into Y are $\mathcal{U}$-homotopic.*

Proof. Let $\mathcal{U}$ be an open $AE(n)$-cover of a metric space Y. Let X be a metric space of dimension less than n. Let $f, g\colon X \to Y$ be $\mathcal{U}$-close. We have to show that f and g are homotopic by a homotopy whose paths refine $\mathcal{U}$.

Let $\mathcal{V} = \{V_U\}_{U \in \mathcal{U}}$ be the collection of subsets of X defined by $V_U = f^{-1}(U) \cap g^{-1}(U)$. Since f and g are $\mathcal{U}$-close, $\mathcal{V}$ is an open cover of X.

Let $\mathcal{W}$ be a closed, locally finite cover of X with multiplicity at most n that refines $\mathcal{V}$. Let $\mathcal{F} = \{W \times [0,1] \colon W \in \mathcal{W}\}$ be a cover of $X \times [0,1]$. Note that $\mathcal{F}$ is a closed, locally finite, locally finite-dimensional cover of $X \times [0,1]$.

Let $H_0 \colon K \times \{0,1\} \to Y$ be the map defined by $H_0(x,0) = f(x)$ and $H_0(x,1) = g(x)$.

Let $C \colon \mathcal{V} \to \mathcal{U}$ be the map defined by $C(V_U) = U$. It is a carrier and both f and g are carried by C, directly from the definition of $\mathcal{V}$. Let $D \colon \mathcal{W} \to \mathcal{V}$ be a map such that for each $W \in \mathcal{W}$ we have $W \subset D(W)$. Such a D exists because $\mathcal{W}$ refines $\mathcal{V}$. Let $E \colon \mathcal{F} \to \mathcal{W}$ be a map defined by $D(W \times [0,1]) = W$. The composition $C \circ D \circ E$ is a carrier and H_0 is carried by it. Observe that $\mathcal{U}$ is an $AE(X \times [0,1])$-cover of Y (see Definition 2.5). By the Carrier Theorem, H_0 can be extended to a map $H \colon X \times [0,1] \to Y$ that is carried by $C \circ D \circ E$. Clearly, H is a homotopy between f and g. For each path $\{x\} \times [0,1] \subset X \times [0,1]$ there exists an element $W \times [0,1] \in \mathcal{F}$ such that $\{x\} \times [0,1] \subset W \times [0,1]$, as $\mathcal{W}$ is a cover of X. Since H is carried into $\mathcal{U}$, the whole path lies in an element of $\mathcal{U}$. Therefore H is a $\mathcal{U}$-homotopy. ∎

THEOREM 3.5. *Let $\mathcal{K} = \{K_i\}_{i \in I}$ be a cover of a polyhedron K by subcomplexes. Let $\mathcal{L} = \{L_i\}_{i \in I}$ be a cover of a polyhedron L by subcomplexes. Let $p \colon K \to L$ be a surjective simplicial map that maps elements of $\mathcal{K}$ into the corresponding elements of $\mathcal{L}$ (i.e. $p(K_i) \subset L_i$ for each $i \in I$). If $\mathcal{K}$ and $\mathcal{L}$ are isomorphic $AE(n)$-covers, then p is a weak n-homotopy equivalence.*

Proof. To begin, we will show that p induces monomorphisms on homotopy groups of dimensions less than n. Let $m < n$. Let $\varphi \colon S^m \to K$. Assume that $p \circ \varphi$ is null-homotopic in L. Let $\Phi \colon B^{m+1} \to L$ denote such a null-homotopy ($\Phi|_{S^m} = p \circ \varphi$). We have to show that φ is null-homotopic in K.

Let $\mathcal{F}$ be a finite closed cover of B^{m+1} with mesh small enough so that for each $F \in \mathcal{F}$ we can pick $i_F \in I$ such that

$$(*)\ \varphi(F \cap S^m) \subset \operatorname{ost}_{\beta K} K_{i_F} \quad \text{and} \quad (**)\ \Phi(F) \subset \operatorname{ost}_{\beta L} L_{i_F}.$$

By $(**)$, a map $C \colon \mathcal{F} \to \operatorname{ost}_{\beta L} \mathcal{L}$ defined by $C(F) = \operatorname{ost}_{\beta L} L_{i_F}$ is a carrier. Since $\mathcal{K}$ and $\mathcal{L}$ are isomorphic, the map $C' \colon \mathcal{F} \to \operatorname{ost}_{\beta K} \mathcal{K}$ defined by $C'(F) = \operatorname{ost}_{\beta K} K_{i_F}$ is a carrier as well. By $(*)$, φ is carried by C'. By Corollary 3.3, $\operatorname{ost}_{\beta K} \mathcal{K}$ is an $AE(n)$-cover. By the Carrier Theorem, φ extends to a map $\tilde{\varphi} \colon B^{m+1} \to K$ that is carried by C'. This is a null-homotopy of φ in K and we are done with the proof that p induces monomorphisms on homotopy groups of dimensions less than n.

Next we will show that p induces epimorphisms on homotopy groups of dimensions less than n. Let $m < n$. Let $\psi \colon S^m \to L$. Let $\mathcal{G}$ be a closed finite cover of S^m that refines $\psi^{-1}(\operatorname{ost}_{\beta L} \mathcal{L})$. For each $G \in \mathcal{G}$ pick $i_G \in I$ such that $\psi(G) \subset \operatorname{ost}_{\beta L} L_{i_G}$. Then the map $D \colon \mathcal{G} \to \operatorname{ost}_{\beta L} \mathcal{L}$ defined by $D(G) = \operatorname{ost}_{\beta L} L_{i_G}$ is a carrier and ψ is carried by D. Since $\mathcal{K}$ and $\mathcal{L}$ are isomorphic, the map $D'(G) = \operatorname{ost}_{\beta K} K_{i_G}$ is a carrier as well. By Corollary 3.3, $\operatorname{ost}_{\beta K} \mathcal{K}$ is an $AE(n)$-cover. By the Carrier Theorem, there exists a map $\tilde{\psi}\colon S^m \to K$ that is carried by D'. Observe that if $x \in G \in \mathcal{G}$, then $\tilde{\psi}(x) \in \operatorname{ost}_{\beta_K} K_{i_G}$ and $\psi(x) \in \operatorname{ost}_{\beta L} L_{i_G}$. Since $p(K_{i_G}) \subset L_{i_G}$, the maps ψ and $p \circ \tilde{\psi}$ are $\operatorname{ost}_{\beta L} \mathcal{L}$-close. By Corollary 3.3, $\operatorname{ost}_{\beta L} \mathcal{L}$ is an $AE(n)$-cover and by Lemma 3.4, ψ and $p \circ \tilde{\psi}$ are homotopic. Hence p induces epimorphisms on homotopy groups of dimensions less than n. This concludes the second part of the proof. ■

3.2. Proof of Theorem 3.1

LEMMA 3.6. *Let K and L be polyhedra. If $p\colon K \to L$ is a simplicial, surjective map such that for each $\delta \in \tau(L)$ the inverse image $p^{-1}(\delta)$ is an $AE(n)$, then p is a weak n-homotopy equivalence.*

Proof. Apply Theorem 3.5 with the cover $\mathcal{L} = \tau(L)$ of L and the cover $\mathcal{K} = p^{-1}(\mathcal{L})$ of K. ■

Proof of Theorem 3.1. The map p is simplicial onto βK and satisfies the conditions of Lemma 3.6, hence it is a weak n-homotopy equivalence.

Let $\mathcal{A} \subset \mathcal{B}_L$ satisfy $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$. By Lemma 2.23, $\bigcap_{A \in \mathcal{A}} A$ is an absolute extensor in dimension n. We have $B = \bigcap_{A \in \mathcal{A}} p^{-1}(A) = p^{-1}(\bigcap_{A \in \mathcal{A}} A)$. The map $p|_B\colon B \to \bigcap_{A \in \mathcal{A}} A$ satisfies the conditions of Lemma 3.6, hence it is a weak n-homotopy equivalence. Therefore B has vanishing homotopy groups in dimensions less than n and by Theorem 2.6, it is an absolute extensor in dimension n. Hence $p^{-1}(\mathcal{B}_L)$ is an $AE(n)$-cover. ■

4. A lifting condition

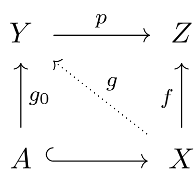
LEMMA 4.1. *Let $p\colon Y \to Z$ be a surjective map of metric spaces. Let $\mathcal{F}$ be a cover of Z. Assume that either $\mathcal{F}$ is an open cover or $\mathcal{F}$ is a closed, locally finite, locally finite-dimensional cover. Assume that $p^{-1}(\mathcal{F})$ is an $AE(n)$-cover.*

Let X be an at most n-dimensional metric space. Let $f\colon X \to Z$. Let A be a closed subset of X. Let $g_0\colon A \to Y$ be a map such that $p \circ g_0 = f|_A$ (a lift of f on A).

Then there exists a map $g\colon X \to Y$ that satisfies the following conditions:

(1) $g|_A = g_0$ *(g extends g_0); and*
(2) $p \circ g$ *is $\mathcal{F}$-close to f.*

Proof. We have the following commutative diagram:



Let $\mathcal{P} = p^{-1}(\mathcal{F})$ and $\mathcal{G} = f^{-1}(\mathcal{F})$. Let $C\colon \mathcal{G} \to \mathcal{P}$ be defined by the formula $C(f^{-1}(F)) = p^{-1}(F)$. Since p is surjective, C is a carrier. For each $F \in \mathcal{F}$ and each $x \in f^{-1}(F) \cap A$ we have $g_0(x) \in p^{-1}(f(x)) \in p^{-1}(F)$; hence g_0 is carried by C.

Let $\mathcal{H}$ be a closed locally finite locally finite-dimensional cover of X such that $\mathcal{H}$ refines $\mathcal{G}$. If $\mathcal{F}$ is an open cover, then such an $\mathcal{H}$ exists as a refinement of an open cover $\mathcal{G}$. If $\mathcal{F}$ is closed locally finite locally finite-dimensional, then we can take $\mathcal{H} = \mathcal{G}$, which has the required properties by Lemma 2.28. Let $D\colon \mathcal{H} \to \mathcal{G}$ be any map such that $H \subset D(H)$ for each $H \in \mathcal{H}$.

Observe that $C \circ D\colon \mathcal{H} \to \mathcal{P}$ is a carrier, g_0 is carried by $C \circ D$, $\mathcal{P}$ is an $AE(n)$-cover, X is at most n-dimensional metric space and $\mathcal{H}$ is closed, locally finite and locally finite-dimensional. By the Carrier Theorem, there exists a map $g\colon X \to Y$ that extends g_0 and that is carried by $C \circ D$. This implies that for each $x \in X$ there exists $F \in \mathcal{F}$ such that $x \in f^{-1}(F)$ and $g(x) \subset p^{-1}(F)$. Therefore $p(g(x)) \in F$ so $p \circ g$ is $\mathcal{F}$-close to f. ■

4.1. A lifting condition for inverse limits

Definition 4.2. Let $Z = \varprojlim(Z_1 \xleftarrow{p_1} Z_2 \xleftarrow{p_2} \cdots)$ and let $\mathcal{F}_i$ be a cover of Z_i. We define the following conditions:

(A) For each i, Z_i is a complete metric space.
(B) For each i, the bonding map $p_i\colon Z_{i+1} \to Z_i$ is surjective and 1-Lipschitz.
(C) For each i, $\mathcal{F}_i$ is either an open cover or a closed, locally finite, locally finite-dimensional cover.
(D) For each i, the pull-back $p_i^{-1}(\mathcal{F}_i)$ is an $AE(n)$-cover.
(E) $\sum_i \operatorname{mesh} \mathcal{F}_i < \infty$.

We denote short projections in the inverse limit by $\pi_i^k\colon Z_k \to Z_i$ $(k > i)$ and long projections in the inverse limit by $\pi_k\colon Z \to Z_k$.

Lemma 4.3. *Let $Z = \varprojlim(Z_1 \xleftarrow{p_1} Z_2 \xleftarrow{p_2} \cdots)$. Let $\mathcal{F}_i$ be a cover of Z_i. Assume that the conditions of Definition 4.2 are satisfied.*

Let X be an at most n-dimensional metric space. Let $f\colon X \to Z_1$. Let A be a closed subset of X. Let $g_0\colon A \to Z$ be a map such that $\pi_1 \circ g_0 = f|_A$ (a lift of f on A).

Then there exists a map $g\colon X \to Z$ such that $g|_A = g_0$ (g extends g_0).

Proof. Let $f_1 = f$. Applying Lemma 4.1 for each $i > 1$ we construct a map $f_i\colon X \to Z_i$ that satisfies the following conditions:

(i) $p_{i-1} \circ f_i$ is $\mathcal{F}_{i-1}$-close to f_{i-1}; and
(ii) $\pi_i \circ g_0 = f_i|_A$.

For each k and each $m > k$ we let

$$a_m^k = \pi_k^m \circ f_m : X \to Z_k.$$

By (i), we have $d_{\sup}(f_m, p_m \circ f_{m+1}) \le \operatorname{mesh} \mathcal{F}_m$. By (B) the short projection π^m_k is 1-Lipschitz, hence $d_{\sup}(a^k_m, a^k_{m+1}) = d_{\sup}(\pi^m_k \circ f_m, \pi^m_k \circ p_m \circ f_{m+1})$ $\le \operatorname{mesh} \mathcal{F}_m$. Therefore for $l > m$ we have

$$\begin{aligned} d_{\sup}(a^k_l, a^k_m) &\le d_{\sup}(a^k_l, a^k_{l-1}) + d_{\sup}(a^k_{l-1}, a^k_{l-2}) + \cdots + d_{\sup}(a^k_{m+1}, a^k_m) \\ &\le \operatorname{mesh} \mathcal{F}_{l-1} + \operatorname{mesh} \mathcal{F}_{l-2} + \cdots + \operatorname{mesh} \mathcal{F}_{m+1} + \operatorname{mesh} \mathcal{F}_m. \end{aligned}$$

By (E), the sequence a^k_m is uniformly convergent. Let

$$a^k = \lim_{m\to\infty} a^k_m.$$

By (A), Z_k is complete, hence $a^k \colon X \to Z_k$ is well defined.

It follows from the definition that $p_k \circ a^{k+1}_m = a^k_m$. Passing to the limit we have

$$p_k \circ a^{k+1} = \lim_{m\to\infty} p_k \circ a^{k+1}_m = \lim_{m\to\infty} a^k_m = a^k.$$

Therefore, for each x the sequence $(a^k(x))_k$ is a thread in Z and we can define a map $g \colon X \to Z$ by the formula

$$(g(x))_k = a^k(x).$$

Observe that by (ii) and by the definition of a^k_m we have

$$a^k_m|_A = \pi^m_k \circ f_m|_A = \pi^m_k \circ \pi_m \circ g_0|_A = \pi_k \circ g_0|_A.$$

Therefore $a^k|_A = \pi_k \circ g_0$ for each k, hence $g|_A = g_0|_A$. ∎

THEOREM 4.4. *Let $Z = \varprojlim(Z_1 \xleftarrow{p_1} Z_2 \xleftarrow{p_2} \cdots)$. Let $\mathcal{F}_i$ be a cover of Z_i. If the conditions of Definition* 4.2 *are satisfied and Z_1 is an $ANE(n)$, then Z is an $ANE(n)$.*

Proof. We will show that Z is an $ANE(n)$ directly from the definition. Let X be an at most n-dimensional metric space and let $A \subset X$ be a closed subset. Take $g_0 \colon A \to Z$. By assumption, Z_1 is an $ANE(n)$ so we can extend $\pi_1 \circ g_0 : A \to Z_1$ to a map $f_1 : U \to Z_1$, where U is an open neighborhood of A in X. By Lemma 4.3, f_1 can be lifted to a map $g \colon U \to Z$ such that $g|_A = g_0$. This is the extension of f we sought. ∎

5. Local k-connectedness of inverse limits of polyhedra

DEFINITION 5.1. Let

(L) $$X = \varprojlim\bigl(K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots\bigr).$$

Fix m and let $A \subset K_m$ be a subcomplex of K_m. A *restriction of* (L) *to* A is the inverse limit

(R) $$X' = \varprojlim\bigl(K'_m \xleftarrow{p'_m} K'_{m+1} \xleftarrow{p'_{m+1}} \cdots\bigr),$$

where $K'_m = A$ and for each $j \ge m$, $K'_{j+1} = p_j^{-1}(K'_j)$ and $p'_j = p_j|_{K'_{j+1}}$.

LEMMA 5.2. *Let* (R) *be the restriction of* (L) *to a subcomplex* $A \subset K_m$. *If, for each* i, (L) *satisfies the following conditions:*

(I) K_i *is a polyhedron; and*
(II) p_i *is a quasi-simplicial map that is surjective and* n*-regular,*

then so does (R). *The inverse limit* X' *is homeomorphic to* $\pi_m^{-1}(A)$, *where* $\pi_m \colon X \to X_m$ *denotes the long projection.*

Proof. We have

$$\pi_m^{-1}(A) = \varprojlim\bigl(K_1' \xleftarrow{p_1'} K_2' \xleftarrow{p_2'} \cdots\bigr),$$

where $K_j' = p_j(K_{j+1}')$ and $p_j' = p_j|_{K_{j+1}'}$ for $j < m$. The restriction (R) is the same sequence with the first $m-1$ elements removed. This changes the metric on the limit, but not the topology, hence X' is homeomorphic to $\pi_m^{-1}(A)$.

The other conditions are trivial to verify. ∎

THEOREM 5.3. *Let* $X = \varprojlim(K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots)$. *Assume that for each* i *the following conditions are satisfied:*

(I) K_i *is a locally finite-dimensional polyhedron; and*
(II) p_i *is a quasi-simplicial map that is surjective and* n*-regular.*

Then

(1) X *is an* $ANE(n)$;
(2) *each short projection* $\pi_i^k \colon K_k \to K_i$ *and each long projection* $\pi_i \colon X \to X_i$ *is a weak* n*-homotopy equivalence;*
(3) *for each* i, *the covers* $\mathcal{O}_i$ *and* $\mathcal{B}_i$ *are* $AE(n)$*-covers of* X.

Proof. Fix a metric of scale 2^{-i} on K_i (see Definition 2.9). Let $(*)$ denote the inverse limit $X = \varprojlim\bigl(K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots\bigr)$ along with a sequence $\mathcal{B}_{K_i}$ of covers, where $\mathcal{B}_{K_i}$ is a cover of K_i by barycentric stars of vertices.

We will verify that $(*)$ satisfies conditions (A)–(E) of Definition 4.2.

(A) By Lemma 2.12, every K_i is a complete metric space.

(B) By (II), p_i is surjective. By the choice of scale on K_i and Lemma 2.14, each bonding map is 1-Lipschitz.

(C) By Lemma 2.23, each $\mathcal{B}_{K_i}$ is a closed, locally finite, locally finite-dimensional cover.

(D) By assumption, p_i is quasi-simplicial and n-regular. Hence by Theorem 3.1, the pull-back $p_i^{-1}(\mathcal{B}_{K_i})$ is an $AE(n)$-cover.

(E) By the choice of the metric on K_i, we have $\sum_i \operatorname{mesh} \mathcal{B}_{K_i} \le \sum_i \operatorname{diam} K_i < \infty$.

To prove (1) it is enough to verify the conditions of Theorem 4.4 for $(*)$. We have just verified the conditions of Definition 4.2. By Lemma 2.12, K_1 is an $ANE(n)$. We are done.

By Theorem 3.1 each p_i is a weak n-homotopy equivalence and therefore all the short projections π_i^k are weak n-homotopy equivalences. To finish the proof of (2), we must show that

(mono) for fixed $i > 0$ and $m < n$, the long projection $\pi_i : X \to K_i$ induces a monomorphism on the homotopy group of dimension m, regardless of the choice of basepoint; and

(epi) for fixed $i > 0$ and $m < n$, the long projection $\pi_i : X \to K_i$ induces an epimorphism on the homotopy group of dimension m, regardless of the choice of basepoint.

We prove (mono). Fix $i > 0$ and $m < n$. Let $\varphi\colon S^m \to X$. Assume that $\pi_i \circ \varphi$ is null-homotopic in K_i. Let $\Phi\colon B^{m+1} \to K_i$ denote the null-homotopy (B^{m+1} denotes the $m+1$-dimensional unit ball in $\mathbb{R}^{m+1}$). We have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi_i} & K_i \\ \uparrow{\scriptstyle\varphi} & \nwarrow^{\tilde{\Phi}} & \uparrow{\scriptstyle\Phi} \\ S^m & \hookrightarrow & B^{m+1} \end{array}$$

The inverse limit

$$X = \varprojlim\bigl(K_i \xleftarrow{p_i} K_{i+1} \xleftarrow{p_{i+1}} \cdots\bigr)$$

along with the sequence of covers by barycentric stars of vertices (which we obtain by truncating $(*)$) satisfies the conditions of Definition 4.2. Since B^{m+1} is at most n-dimensional, by Lemma 4.3, there exists a lift $\tilde{\Phi}$ such that the diagram is commutative. This lift is a null-homotopy of φ in X, hence π_i induces a monomorphism on the homotopy group of dimension m.

We will show (epi) at the end of the proof.

Next we prove (3). To begin, we will show that for each i, $\mathcal{B}_i$ is an $AE(n)$-cover. Let $\mathcal{A}$ be a collection of elements of $\mathcal{B}_i$ such that the intersection $\bigcap \mathcal{A}$ is non-empty. By the definition of $\mathcal{B}_i$, we have $\mathcal{A} = \{\pi_i^{-1}(\mathrm{bst}_{K_i} v)\}_{v \in V}$ for some set of vertices V of K_i. Let $A = \bigcap_{v \in V} p_i^{-1}(\mathrm{bst}_{K_i} v)$. Since p_i is quasi-simplicial, A is a subcomplex of K_{i+1}. By Theorem 3.1, A has vanishing homotopy groups in dimensions less than n. Let

$$X' = \varprojlim\bigl(K'_{i+1} \xleftarrow{p'_{i+1}} K'_{i+2} \xleftarrow{p'_{i+2}} \cdots\bigr),$$

where $K'_{i+1} = A$ and for each $j \geq i+1$, $K'_{j+1} = p_j^{-1}(K'_j)$ and $p'_j = p_j|_{K'_{j+1}}$, be the restriction of $(*)$ to A. By Lemma 5.2 it satisfies assumptions (I) and (II) and is homeomorphic to $\bigcap \mathcal{A}$. Hence from what we have already proven, $X' = \bigcap \mathcal{A}$ is an $ANE(n)$ and the long projection $\pi_{i+1}\colon \bigcap \mathcal{A} \to A$ induces monomorphisms on homotopy groups of dimensions less than n.

Since A has vanishing homotopy groups in these dimensions, so does $\bigcap \mathcal{A}$. By Theorem 2.6, $\bigcap \mathcal{A}$ is an $AE(n)$ hence $\mathcal{B}_i$ is an $AE(n)$-cover.

It follows from Lemma 2.7 that $\mathcal{O}_i$ is an $AE(n)$-cover, as open stars are (infinite) unions of iterated barycentric stars.

Finally, we prove (epi). Fix $i > 0$ and $m < n$. Let $\varphi\colon S^m \to K_i$. Let $\mathcal{O}$ be a cover of K_i by open stars of vertices. We have just shown that $\pi_i^{-1}(\mathcal{O})$ is an $AE(n)$-cover. Hence by Lemma 4.1, there exists a map $\psi\colon S^m \to X$ such that $\pi_i \circ \psi$ and φ are $\mathcal{O}$-close. By Lemma 3.4, these maps are homotopic. This shows that π_i induces epimorphisms on homotopy groups of dimensions less than n. Hence π_i is a weak n-homotopy equivalence. ∎

Corollary 5.4. *Let*

$$X = \varprojlim \big(K_1 \xleftarrow{p_1} K_2 \xleftarrow{p_2} \cdots \big).$$

Assume that for each i, (I) K_i *is a locally finite-dimensional polyhedron, and* (II) p_i *is a quasi-simplicial map that is surjective and* n*-regular. Let* Y *be a metric space of dimension less than* n *and let* $f, g\colon Y \to X$*. Then if* f *is* $\mathcal{O}_i$*-close to* g*, then* f *is* $\mathcal{O}_i$*-homotopic to* g*.*

Proof. By Theorem 5.3(3), $\mathcal{O}_i$ is an $AE(n)$-cover of X. By Lemma 3.4, f and g are $\mathcal{O}_i$-homotopic. ∎

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