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Hypersurfaces with free boundary in a convex conical domain in a space form

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Abstract. This study presents rigidity results for hypersurfaces with free boundary in a convex conical domain in a space form. Let C be a conical domain with a piecewise linear boundary in a space form. A compact hypersurface with free boundary $M \subset C$ is part of a geodesic sphere if either C is convex and M is an embedded and has constant higher order mean curvature, or H_l does not vanish and the ratio H_r/H_l, $l < r$, is constant on M.

1. Introduction. A geodesic sphere is the only totally umbilical non-planar hypersurface in space forms. Because every principal curvature is constant, it is clear that every higher order mean curvature (the rth elementary symmetric polynomial in the principal curvatures, see Section 3 for further details) and any ratio of two higher order mean curvatures is also constant. Ros [13] showed that a geodesic sphere is the only embedded closed hypersurface in Euclidean space with a constant higher order mean curvature. Montiel and Ros [9] extended Ros' result to space forms. Koh [5, 6, 7] provided a characterization of a geodesic sphere for a constant ratio of two higher order mean curvatures.

For any domain M of a geodesic sphere Σ, define C as the conical domain which is obtained by the union of geodesic rays connecting the origin of Σ to a point in M. Clearly, M is a *hypersurface with free boundary* in the cone C; that is, M intersects ∂C orthogonally along the boundary ∂M. This naturally raises the question what condition on a hypersurface with free boundary implies that it is part of a geodesic sphere.

In Euclidean space, Choe and Park [4] showed that a hypersurface with free boundary is part of a geodesic sphere when the hypersurface is embedded

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and has a constant higher order mean curvature. In Section 4, we show that the same result holds for other space forms.

In Section 5, following Koh's idea, we verify that an immersed hypersurface with free boundary of a constant ratio H_r/H_l, $l < r$, and nonvanishing H_l satisfies a similar rigidity result.

Both results are generalizations of the rigidity results of the author [10] regarding hypersurfaces with free boundary in a wedge in a space form.

2. Ros type inequality. Let $(\bar{M}^{n+1}, g)$ be an $(n+1)$-dimensional Riemannian manifold. Let $\Omega \subset \bar{M}^{n+1}$ be a compact domain with smooth boundary $\partial\Omega = M$. Let $\bar{\nabla}$, $\bar{\triangle}$, and $\bar{\nabla}^2$ denote the gradient, the Laplacian, and the Hessian on Ω, respectively, and let ∇, $\triangle$, N, σ, and H denote the gradient, the Laplacian, the unit outward normal vector field, the second fundamental form, and the normalized mean curvature on M, respectively. Let $d\Omega$ and dA be the canonical measures of Ω and M, respectively. In [11], Qiu and Xia generalized Reilly's formula as follows:

Reilly type formula. *Let $V : \bar{\Omega} \to \mathbb{R}$ be a given almost everywhere twice differentiable function. Given a smooth function f on Ω, denote $z = f|_M$ and $u = \bar{\nabla}_N f$. For any real number k, the following identity holds:*

$$\begin{aligned}(2.1)\quad \int_\Omega V\big((\bar{\triangle} f + k(n+1)f)^2 - |\bar{\nabla}^2 f + kfg|^2\big)\, d\Omega &= \int_M V\big(2u\triangle z + nHu^2 + \sigma(\nabla z, \nabla z) + 2nkuz\big)\, dA \\ &\quad + \int_M \bar{\nabla}_N V(|\nabla z|^2 - nkz^2)\, dA \\ &\quad + \int_\Omega (\bar{\nabla}^2 V - \bar{\triangle} V g - 2nkVg + V Ric)(\bar{\nabla} f, \bar{\nabla} f)\, d\Omega \\ &\quad + n\int_\Omega (k\bar{\triangle} V + (n+1)k^2 V) f^2\, d\Omega.\end{aligned}$$

We note that if $V = 1$ and $k = 0$ on Ω, then (2.1) becomes the classical Reilly formula. Using the Reilly type formula, Qiu and Xia [11] verified the following Ros type inequality:

Ros type inequality. *Let $\Omega \subset \mathbb{H}^{n+1}$ ($\mathbb{S}^{n+1}_+$ resp.) be a compact domain with smooth boundary M. For a fixed p, define $r(x) = \mathrm{dist}_{\mathbb{H}^{n+1}}(x, p)$ ($\mathrm{dist}_{\mathbb{S}^{n+1}}(x, p)$ resp.). Let $V(x) = \cosh r(x)$ ($\cos r(x)$ resp.). If the mean curvature H of M is positive, then*

$$\int_M \frac{V}{H}\, dA \ge (n+1)\int_\Omega V\, d\Omega.$$

Equality holds if and only if Ω is isometric to a geodesic ball.

Using a geometric flow, Brendle [3] provided another proof for the Ros type inequality. Batista, Cavalcante, and the present author [1] extended the Ros type inequality to weighted manifolds.

Choe and Park [4] extended the Ros type inequality to hypersurfaces with free boundary in a convex cone in Euclidean space. We now extend the Ros type inequality to hypersurfaces with nonempty free boundary in a convex cone in other space forms.

THEOREM 2.1. *Let $C \subset \mathbb{H}^{n+1}$ ($\mathbb{S}^{n+1}_+$ resp.) be a convex conical domain with piecewise smooth boundary and vertex p. Let $M \subset C$ be a compact embedded hypersurface in $\mathbb{H}^{n+1}$ ($\mathbb{S}^{n+1}_+$ resp.) with boundary ∂M such that M intersects ∂C orthogonally along ∂M. Let Ω be the domain enclosed by M and ∂C with piecewise smooth boundary. Let $V(x) = \cosh r(x)$ ($\cos r(x)$ resp.), where $r(x) = \mathrm{dist}_{\mathbb{H}^{n+1}}(x,p)$ ($\mathrm{dist}_{\mathbb{S}^{n+1}}(x,p)$ resp.). If the mean curvature H is positive on M, then*

$$\int_M \frac{V}{H}\, dA \geq (n+1) \int_\Omega V\, d\Omega. \tag{2.2}$$

Equality holds in (2.2) if and only if M is part of a geodesic sphere.

Proof. To apply the Reilly type formula, let $\Omega_\epsilon \subset \Omega$ be a domain with smooth boundary obtained from Ω by rounding smoothly the singular part of $\partial\Omega$ for a small distance $\epsilon > 0$.

In any case, $\bar\nabla^2 V = -kVg$. Then the Reilly type formula (2.1) becomes

$$\begin{aligned}\int_{\Omega_\epsilon} V\big((\bar\triangle f + k(n+1)f)^2 - |\bar\nabla^2 f + kfg|^2\big)\, d\Omega_\epsilon \\ = \int_{\partial\Omega_\epsilon} V\big(2u\triangle z + nHu^2 + \sigma(\nabla z, \nabla z) + 2nkuz\big)\, dA \\ + \int_{\partial\Omega_\epsilon} \bar\nabla_N V(|\nabla z|^2 - nkz^2)\, dA.\end{aligned}$$

Let $f : \Omega_\epsilon \to \mathbb{R}$ be the solution to the following mixed boundary value problem (for the existence of f, see [8]):

$$\begin{cases} \bar\triangle f + k(n+1)f = 1 & \text{in } \Omega_\epsilon, \\ f = 0 & \text{on } \partial\Omega_\epsilon \setminus \partial C, \\ u = \dfrac{\partial f}{\partial N} = 0 & \text{on } \partial\Omega_\epsilon \cap \partial C. \end{cases} \tag{2.3}$$

From the Cauchy–Schwarz inequality,

$$\begin{aligned}\frac{n}{n+1} \int_{\Omega_\epsilon} V(\bar\triangle f + k(n+1)f)^2\, d\Omega_\epsilon \\ \geq \int_{\Omega_\epsilon} V\big((\bar\triangle f + k(n+1)f)^2 - |\bar\nabla^2 f + kfg|^2\big)\, d\Omega_\epsilon.\end{aligned} \tag{2.4}$$

Because $f|_{\partial\Omega_\epsilon\setminus\partial C} = 0$ (that is, $z = 0$ on $\partial\Omega_\epsilon \setminus \partial C$),

$$\int\limits_{\partial\Omega_\epsilon\setminus\partial C} V(2u\triangle z + nHu^2 + \sigma(\nabla z, \nabla z) + 2nkuz)\, dA + \int\limits_{\partial\Omega_\epsilon\setminus\partial C} \bar{\nabla}_N V(|\nabla z|^2 - nkz^2)\, dA = \int\limits_{\partial\Omega_\epsilon\setminus\partial C} nVHu^2\, dA. \tag{2.5}$$

Because C is convex, we have $\sigma(\nabla z, \nabla z) \geq 0$. Since $V(x) = \cos r(x)$ ($\cosh r(x)$, resp.), we have

$$\bar{\nabla}_N V = -\sin r(x) g(\bar{\nabla} r(x), N) = 0 \quad (\sinh r(x) g(\bar{\nabla} r(x), N) = 0, \text{ resp.})$$

on ∂C. Because $u = 0$ on $\partial\Omega_\epsilon \cap \partial C$, we have

$$\int\limits_{\partial\Omega_\epsilon\cap\partial C} V(2u\triangle z + nHu^2 + \sigma(\nabla z, \nabla z) + 2nkuz)\, dA + \int\limits_{\partial\Omega_\epsilon\cap\partial C} \bar{\nabla}_N V(|\nabla z|^2 - nkz^2)\, dA \geq 0. \tag{2.6}$$

From (2.3)–(2.6), we have

$$\frac{1}{n+1} \int\limits_{\Omega_\epsilon} V\, d\Omega_\epsilon \geq \int\limits_{\partial\Omega_\epsilon\setminus\partial C} VHu^2\, dA. \tag{2.7}$$

Because $\bar{\triangle} V = -(n+1)kV$ and $\bar{\nabla}_N V = 0$ on $\partial\Omega_\epsilon \cap \partial C$, by Green's formula we have

$$\int\limits_{\Omega_\epsilon} V\, d\Omega_\epsilon = \int\limits_{\Omega_\epsilon} (V\bar{\triangle} f - f\bar{\triangle} V)\, d\Omega_\epsilon = \int\limits_{\partial\Omega_\epsilon} \left(V\frac{\partial f}{\partial N} - f\frac{\partial V}{\partial N} \right) dA = \int\limits_{\partial\Omega_\epsilon\setminus\partial C} Vu\, dA. \tag{2.8}$$

On the other hand,

$$\begin{aligned} \Big(\int\limits_{\Omega_\epsilon} V\, d\Omega_\epsilon \Big)^2 &= \Big(\int\limits_{\partial\Omega_\epsilon\setminus\partial C} Vu\, dA \Big)^2 \\ &\leq \int\limits_{\partial\Omega_\epsilon\setminus\partial C} VHu^2\, dA \int\limits_{\partial\Omega_\epsilon\setminus\partial C} \frac{V}{H}\, dA \\ &\leq \frac{1}{n+1} \int\limits_{\Omega_\epsilon} V\, d\Omega_\epsilon \int\limits_{\partial\Omega_\epsilon\setminus\partial C} \frac{V}{H}\, dA, \end{aligned} \tag{2.9}$$

where the first inequality comes from the Hölder inequality, and the second from (2.7).

Therefore, on letting $\epsilon \to 0$, we obtain (2.2).

Combining (2.4) and (2.7)–(2.9) we find that equality in (2.2) implies that $|\bar{\nabla}^2 f + kfg|^2 = \frac{1}{n+1}(\bar{\triangle} f + k(n+1)f)^2$. Because $\bar{\triangle} f + k(n+1)f = 1$, we get

$$\bar{\nabla}^2\Big(f+\frac{1}{n+1}\Big) = -k\Big(f+\frac{1}{n+1}\Big)g \quad \text{in } \Omega.$$

With $f+\frac{1}{n+1} = \frac{1}{n+1}$ on M, the conclusion follows from the Obata type result in [12, Theorem B] showing that M is part of a geodesic sphere. ■

3. Higher order mean curvatures and Minkowski type formulae. Let $\psi : M^n \to \bar{M}^{n+1}$ be an isometric immersion of an orientable n-dimensional manifold. For simplicity, we identify $\psi(M)$ with M. Since M is orientable, we consider a unit normal vector field N and write σ for the second fundamental form of the immersion and κ_i, $i = 1, \dots, n$, for the principal curvatures of M. For any $r = 1, \dots, n$, the *mean curvature of order* r, H_r, is defined by

$$P_n(t) := (1+\kappa_1 t)\cdots(1+\kappa_n t) = 1 + \binom{n}{1}H_1 t + \cdots + \binom{n}{n}H_n t^n \tag{3.1}$$

for a real variable t. Note that H_1 is the normalized mean curvature of M, H_2 is the scalar curvature of M up to a constant, and H_n is the Gauss–Kronecker curvature of M. For convenience, we define $H_0 = 1$.

For higher order mean curvatures, the following inequalities hold:

LEMMA 3.1. *If there is a point of M at which all the principal curvatures are positive, and if $H_r > 0$, $r = 1, \dots, n$, on M, then*

(i) $H_l > 0$ *for* $l < r$.
(ii) $H_r/H_l \le H_{r-1}/H_{l-1}$ *for any* $l < r$.
(iii) $H_s^{(s-1)/s} \le H_{s-1}$ *and* $H_s^{1/s} \le H_1 = H$, *where equality holds only at umbilical points if* $s > 1$.

Proof. For (i) and (iii) see, for example, [9, Lemma 1], and for (ii) see [2, Section 12] or [6, Lemma B]. ■

For closed hypersurfaces or submanifolds of constant higher order mean curvature, many geometric and rigidity results can be deduced from integral identities such as Minkowski formula (see, for example, [5, 9, 13]). Montiel and Ros [9] generalized the Minkowski formula in space forms and provided another characterization of a geodesic sphere.

Choe and Park [4] extended the Minkowski formula for compact hypersurfaces with boundary in Euclidean space. Here, we extend the Minkowski formula to compact hypersurfaces with boundary in other space forms. We include the proof of the Minkowski formula in space forms, for the reader's convenience, and then generalize it to hypersurfaces with nonempty boundary. For simplicity, $\bar{M}^{n+1}(k)$ denotes the $(n+1)$-dimensional simply connected space form of constant sectional curvature k; that is, $\mathbb{H}^{n+1} = \bar{M}^{n+1}(-1)$,

$\mathbb{S}^{n+1} = \bar{M}^{n+1}(1)$, and $\mathbb{R}^{n+1} = \bar{M}^{n+1}(0)$. If $k = 1$, we consider the usual embedding into $\mathbb{R}^{n+2}$. If $k = -1$, we use the hyperboloid model, more precisely, let $\mathbb{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz–Minkowski space with Lorentzian metric $\langle x, y\rangle = x_1y_1 + \cdots + x_{n+1}y_{n+1} - x_{n+2}y_{n+2}$, and let

$$\bar{M}^{n+1}(-1) = \{x \in \mathbb{L}^{n+2} \mid |x|^2 = -1,\, x_{n+2} \geq 1\}.$$

Assume that M^n is a closed hypersurface in $\bar{M}^{n+1}(k)$. Let $\psi : M \to \bar{M}^{n+1}(k)$ be an immersion. Here if $k = -1$, we regard it as $\psi : M \to \mathbb{L}^{n+2}$, and if $k = 1$ we regard it as $\psi : M \to \mathbb{R}^{n+2}$.

Because of the similarity between $\bar{M}^{n+1}(-1)$ and $\bar{M}^{n+1}(1)$, we focus on $k = -1$. From a direct computation, for any $v \in \mathbb{L}^{n+2}$ we have

$$\triangle\langle\psi, v\rangle = n(\langle\psi, v\rangle - H\langle N, v\rangle), \tag{3.2}$$

so by integration on M and application of the Stokes theorem,

$$\int_M (\langle\psi, v\rangle - H\langle N, v\rangle)\, dA = 0. \tag{3.3}$$

For a real number t close enough to 0, the parallel hypersurface is given by

$$\psi_t = \exp_\psi(tN) = \cosh t\psi + \sinh tN$$

and this is also an immersion.

If dA and $\kappa_1, \ldots, \kappa_n$ denote the volume form and the principal curvatures with respect to the principal directions of $e_1, \ldots, e_n$ of $\psi(M)$, respectively, then the volume form of $\psi_t(M) = M_t$ is given by

$$\begin{aligned} dA_t &= (\cosh t + \kappa_1 \sinh t)\cdots(\cosh t + \kappa_n \sinh t)dA \\ &= \cosh^n t \cdot P_n(\tanh t)dA, \end{aligned}$$

where P_n is as in (3.1). Since the unit normal vector field of ψ_t is $N_t = \sinh t\psi + \cosh tN$, $e_1, \ldots, e_n$ of $\psi(M)$ also represent the principal directions for ψ_t, and the corresponding principal curvatures are

$$\kappa_i(t) = \frac{\sinh t + \kappa_i \cosh t}{\cosh t + \kappa_i \sinh t}, \qquad i = 1, \ldots, n.$$

From a direct computation, the mean curvature $H(t)$ of M_t is

$$H(t) = \sum_{i=1}^{n} \kappa_i(t) = \frac{n \cosh t \sinh t \cdot P_n(\tanh t) + P_n'(\tanh t)}{n \cosh^2 t \cdot P_n(\tanh t)}. \tag{3.4}$$

Integrating (3.3) on M_t gives

$$\int_M \left[\big(nP_n(\tanh t) - \tanh t \cdot P_n'(\tanh t)\big)\langle\psi, v\rangle - P_n'(\tanh t)\langle N, v\rangle\right] dA = 0. \tag{3.5}$$

From (3.1) and (3.4) and on comparing the coefficients of $\tanh t$, we obtain the following Minkowski type identity:

$$\int_M (H_{r-1}\langle\psi, v\rangle - H_r\langle N, v\rangle)\, dA = 0, \qquad r = 1, \dots, n. \tag{3.6}$$

Similarly, the following result can be obtained for $k = 1$:

MINKOWSKI TYPE IDENTITY ([9]). *Let $\psi : M \to \bar{M}^{n+1}(k)$ be a closed orientable immersed hypersurface. Then, for any $r = 1, \dots, n$,*

(a) *if $k = -1$, then $\int_M (H_{r-1}\langle\psi, v\rangle - H_r\langle N, v\rangle)\, dA = 0$ for any $v \in \mathbb{L}^{n+2}$;*
(b) *if $k = 1$, then $\int_M (H_{r-1}\langle\psi, v\rangle + H_r\langle N, v\rangle)\, dA = 0$ for any $v \in \mathbb{R}^{n+2}$;*
(c) *if $k = 0$, then $\int_M (H_{r-1} - H_r\langle\psi, N\rangle)\, dA = 0$.*

We extend the Minkowski type identity to immersed free boundary hypersurfaces which are the nonempty boundary lying on a cone with piecewise linear boundary. A cone $C \subset \bar{M}^{n+1}(k)$ with vertex p has *piecewise linear boundary* if C is a conical domain in $\bar{M}^{n+1}(k)$, and each connected component of ∂C is contained in an n-dimensional totally geodesic submanifold passing through p.

PROPOSITION 3.2. *Let $C \subset \bar{M}^{n+1}(k)$ be a conical domain with piecewise linear boundary and with vertex p. Let M be a compact immersed hypersurface in $\bar{M}^{n+1}(k)$ with $\partial M \subset \partial C$ such that near ∂M, M lies inside of C and is perpendicular to ∂C. Then, for any $r = 1, \dots, n$,*

(a) *if $k = -1$, then $\int_M (H_{r-1}\langle\psi, p\rangle - H_r\langle N, p\rangle)\, dA = 0$;*
(b) *if $k = 1$, then $\int_M (H_{r-1}\langle\psi, p\rangle + H_r\langle N, p\rangle)\, dA = 0$;*
(c) *if $k = 0$, then $\int_M (H_{r-1} - H_r\langle\psi, N\rangle)\, dA = 0$.*

Proof. Because of the similarity between the two cases ($k = \pm 1$), we consider only the $k = -1$ case.

By an isometry in $\bar{M}^{n+1}(-1)$, we assume $p = (0, \dots, 0, 1) \in \mathbb{L}^{n+2}$. For any t sufficiently close to zero, the parallel hypersurface $\psi_t(M) = M_t$ becomes an immersed hypersurface. Since C is a cone with piecewise linear boundary, and for any $q \in \partial M$ the unit normal vector $N(q)$ is the same as the unit normal vector at $\psi_t(q) \in \partial M_t$, ∂M_t lies on ∂C and M_t intersects ∂C orthogonally along ∂M_t.

Integrating (3.2) on M_t and applying the Stokes theorem gives

$$\int_{M_t} (\langle\psi_t, p\rangle - H(t)\langle N_t, p\rangle)\, dA_t - \frac{1}{n}\int_{\partial M_t} \langle\nu_t, p\rangle = 0, \tag{3.7}$$

where ν_t is the outward unit conormal vector field to ∂M_t.

We note that for each smooth piece of ∂C, the vector p is perpendicular to that piece. Since M_t intersects ∂C orthogonally along ∂M_t, the conormal vector field ν_t is parallel to each smooth piece of ∂C; that is, $\langle\nu_t, p\rangle \equiv 0$

on ∂M_t. Thus, (3.7) coincides with (3.3). The argument in the closed case is the same. ∎

REMARK 3.3. For the $r = 1$ case in Proposition 3.2, the Minkowski type formula holds for a conical domain with piecewise smooth boundary by integrating (3.2) on M and applying of the Stokes theorem with $\langle \nu, p\rangle \equiv 0$ on ∂M.

4. Constant H_r embedded hypersurfaces with free boundary in a cone

THEOREM 4.1. *Let $C \subset \mathbb{H}^{n+1}$ (or $\mathbb{S}^{n+1}_+$) be a convex conical domain with piecewise linear boundary and vertex p. Let $M \subset C$ be a compact embedded constant rth order mean curvature hypersurface such that M intersects ∂C orthogonally along ∂M. Then M is part of a geodesic sphere.*

Proof. Let Ω denote the compact domain enclosed by M and ∂C. Let $\bar{M}^{n+1}(k)$ denote $\mathbb{H}^{n+1} = \bar{M}^{n+1}(-1)$ and $\mathbb{S}^{n+1}_+ = \bar{M}^{n+1}(1)$.

From a direct computation, we have $\bar{\triangle}\langle \psi, v\rangle = -k(n+1)\langle \psi, v\rangle$ for any $v \in \mathbb{R}^{n+2}$. Integrating on Ω and using the Stokes theorem, we have

$$-k(n+1)\int_{\Omega} \langle \psi, v\rangle \, d\Omega = \int_M \langle N, v\rangle \, dA + \int_{\partial\Omega\cap\partial C} \langle \nu, v\rangle \, dA,$$

where N and ν are the outward unit normal vector fields of M and $\partial\Omega \cap \partial C$, respectively.

With v taken to be p, the vertex of C, $\langle \nu, v\rangle \equiv 0$ on $\partial\Omega \cap \partial C$, that is,

$$-k(n+1)\int_{\Omega} \langle \psi, p\rangle \, d\Omega = \int_M \langle N, p\rangle \, dA. \tag{4.1}$$

Let $r(x) = \operatorname{dist}(x, p)$ be the distance function from p to x in $\bar{M}^{n+1}(k)$, defined to be the angle between the vectors $0p$ and $0x$. If $k = -1$ ($k = 1$, resp.), then $\langle \psi, p\rangle = -\cosh r(\psi)$ ($\cos r(\psi)$, resp.); that is, $k\langle x, p\rangle = V(x)$ in $\bar{M}^{n+1}(k)$.

From Proposition 3.2, we have

$$\int_M (H_{r-1}V(\psi) + kH_r\langle N, p\rangle)\, dA = 0.$$

Because H_r is constant and by (4.1), we have $(n+1)H_r\int_{\Omega} V\, d\Omega = \int_M H_{r-1}V\, dA$. At a point of M at which $r(x)$ attains its maximum value, all the principal curvatures are positive. From Lemma 3.1,

$$(n+1)H_r\int_{\Omega} V\, d\Omega = \int_M H_{r-1}V\, dA \geq \int_M H_r^{(r-1)/r}V\, dA,$$

so

$$(n+1)\int_{\Omega} V\, d\Omega \geq \int_M H_r^{-1/r}V\, dA \geq \int_M \frac{V}{H}\, dA. \tag{4.2}$$

Comparing (2.2) and (4.2), we find that M is part of a geodesic sphere by Theorem 2.1. ■

REMARK 4.2. Because the Minkowski type formula holds for a conical domain with piecewise smooth boundary for $r = 1$ in Proposition 3.2, Theorem 4.1 holds for a conical domain with piecewise smooth boundary for constant mean curvature hypersurfaces with free boundary.

5. Constant H_r/H_l immersed hypersurfaces with free boundary in a cone. Using the Minkowski formula and Newton's inequalities for normalized symmetric functions (Lemma 3.1), Koh [5, 7] provided characterizations of geodesic spheres in space forms. In Proposition 3.2, the Minkowski formula is extended to hypersurfaces with free boundary in space forms; hence Koh's results naturally extend to hypersurfaces with free boundary.

THEOREM 5.1. *Let $C \subset \bar{M}^{n+1}(k)$ (for $k = 1$, we assume $\bar{M}^{n+1}(1) = \mathbb{S}^{n+1}_{+}$) be a conical convex domain with piecewise linear boundary and vertex p. Let M be a compact immersed hypersurface in $\bar{M}^{n+1}(k)$ with $\partial M \subset \partial C$ such that near ∂M, M lies inside of C and is perpendicular to ∂C. If, for $r, l = 1, \ldots, n$ and $r > l$, the ratio H_r/H_l is constant and H_l does not vanish on M, then M is part of a geodesic sphere.*

Proof. By an isometry in $\bar{M}^{n+1}(-1)$, we can assume that the vertex p of C is $(0, \ldots, 0, 1) \in \mathbb{L}^{n+2}$. Let q be the point in M farthest from p. At q, all the principal curvatures are positive; clearly, both H_r and H_l are positive at q. Because $\alpha = H_r/H_l$ is constant and H_l does not vanish on M, H_r and H_l are positive on M and $\alpha > 0$. From Lemma 3.1(i), $H_s > 0$ if $s < r$.

By Lemma 3.1(ii),

$$0 < \alpha = \frac{H_r}{H_l} \le \frac{H_{r-1}}{H_{l-1}}. \tag{5.1}$$

Because $H_r = \alpha H_l$ and from Proposition 3.2,

$$\int_M H_{r-1}\langle \psi, p\rangle - \alpha H_l \langle N, p\rangle \, dA = 0. \tag{5.2}$$

Because $\alpha > 0$ is constant and from Proposition 3.2,

$$\int_M \alpha(H_{l-1}\langle \psi, p\rangle - H_l\langle N, p\rangle)\, dA = 0. \tag{5.3}$$

Combining (5.2) and (5.3) yields

$$\int_M (H_{r-1} - \alpha H_{l-1})\langle \psi, p\rangle \, dA = 0.$$

Because $\langle \psi, p\rangle \le -1$ on M and from (5.1),

$$\frac{H_r}{H_l} = \frac{H_{r-1}}{H_{l-1}} = \alpha \quad \text{on } M.$$

Proceeding inductively, and defining $p = r - l$, we obtain

$$\frac{H_{p+1}}{H_1} = \frac{H_p}{H_0} = H_p \quad \text{on } M, \tag{5.4}$$

that is, $H_{p+1}/H_p = H_1$.

On the other hand, from Lemma 3.1(ii),

$$H_{p+1}/H_p \leq H_p/H_{p-1} \leq \cdots \leq H_1. \tag{5.5}$$

Combination of (5.4) and (5.5) gives

$$H_{p+1}/H_p = H_p/H_{p-1} = \cdots = H_1,$$

and therefore

$$H_r = H_1^r, \qquad r = 1, \dots, p+1.$$

From Lemma 3.1(iii), M is part of a geodesic sphere.

For $k = 1$, we assume that the vertex of C is $p = (0, 0, \dots, 1) \in \mathbb{R}^{n+2}$. Because $\psi : M \to \mathbb{S}^{n+1}_+$, we have $\langle \psi, p \rangle > 0$. Now the conclusion follows by the same argument as for $k = -1$.

For $k = 0$, the Minkowski type identity is slightly different, but the rest of the argument is quite similar to that for $k = -1$.

By the same argument, we have

$$0 < \alpha = \frac{H_r}{H_l} \leq \frac{H_{r-1}}{H_{l-1}}, \tag{5.6}$$

$$H_r = \alpha H_l. \tag{5.7}$$

From Proposition 3.2(c),

$$\int_M (H_{r-1} - \alpha H_l \langle \psi, N \rangle)\, dA = 0. \tag{5.8}$$

Because $\alpha > 0$ is constant and from Proposition 3.2(c),

$$\int_M \alpha (H_{l-1} - H_l \langle \psi, N \rangle)\, dA = 0. \tag{5.9}$$

Combining (5.8) and (5.9) yields

$$\int_M (H_{r-1} - \alpha H_{l-1})\, dA = 0.$$

From (5.6),

$$\frac{H_r}{H_l} = \frac{H_{r-1}}{H_{l-1}} = \alpha \quad \text{on } M.$$

Proceeding inductively, and defining $p = r - l$, we obtain

$$\frac{H_{p+1}}{H_1} = \frac{H_p}{H_0} = H_p \quad \text{on } M, \tag{5.10}$$

that is, $H_{p+1}/H_p = H_1$. The remaining part is the same as for $k = -1$. ∎

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