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Stability of a suspension bridge with structural damping

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Abstract. We investigate the exponential stability of a rectangular plate equation assumed to be hinged on its vertical edges and free on its remaining horizontal edges. Our problem models the deformation of a suspension bridge under a structural damping.

1. Introduction. In the present paper, we consider a thin and narrow rectangular plate that represents the deck of either a footbridge or a suspension bridge. In the absence of forces, the plate lies flat horizontally and is represented by the planar domain $\Omega = (0,\pi) \times (-l,l)$ where $l \ll \pi$, with boundary Γ. The nonlocal evolution problem modeling the deformation of such a plate reads as follows:

(1.1)
$$\begin{cases} u_{tt} + \Delta^2 u - \phi(u)u_{xx} - \alpha u_{xxt} = h, & \text{in } \Omega \times (0,+\infty), \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & (y,t) \in (-l,l) \times (0,+\infty), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & (y,t) \in (-l,l) \times (0,+\infty), \\ u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) = 0, & (x,t) \in (0,\pi) \times (0,+\infty), \\ u_{yyy}(x,\pm l,t) + (2-\sigma)u_{xxy}(x,\pm l,t) = 0, & (x,t) \in (0,\pi) \times (0,+\infty), \\ u(x,y,0) = u_0(x,y), u_t(x,y,0) = u_1(x,y), & \text{in } \Omega, \end{cases}$$

where $\alpha > 0$ is a constant and the nonlinear term ϕ carries a nonlocal effect into the model and is defined by

$$\phi(u) = -P + S \int_\Omega u_x^2,$$

the constant σ is the Poisson ratio whose value for metals is around 0.3, while for concrete, it is between 0.1 and 0.2. For this reason we shall assume that $0 < \sigma < 1/2$.

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Here $S > 0$ depends on the elasticity of the deck material, the term $S \int_\Omega u_x^2$ measures the geometric nonlinearity of the plate under its stretching, and $P > 0$ is the prestressing constant. In fact, $P > 0$ if the plate is compressed and $P < 0$ if it is stretched. The function h represents the vertical load over the deck and may depend on time.

In [FGM16], Ferreira et al. analyzed in detail how a solution of (1.1) initially oscillating in an almost purely longitudinal fashion can suddenly start oscillating in a torsional fashion, even in the absence of external forces ($h = 0$). For this reason, we shall take $h = 0$ in the present work.

A large number of papers have been devoted to the study of suspension bridges. We start by citing the works of McKenna and Walter [MW87] and McKenna et al. [GLM89], where the existence of nonlinear oscillations is established. Bochicchio et al. [BGV13] and Ma and Zhong [MZ09] studied, respectively, the asymptotic dynamics and global attractors for coupled suspension bridge equations. Ferrero and Gazzola [FG15] gave a simplified model of suspension bridges as a rectangular plate where the two short edges are hinged and the two long edges are free. Al-Gwaiz et al. [ABG14] analyzed the bending and stretching energies of the model given in [FG15]. Berchio et al. [BFG16] discussed the structural instability of nonlinear plates modelling suspension bridges. See also the book by Gazzola [G15]. Concerning the stability of suspension bridges, we mention the work of Messaoudi and Mukiawa [MM17] where exponential decay was proved with a full (in the whole Ω) internal damping, of the form $\alpha u_t(x,y,t)$ with $\alpha > 0$, instead of $\alpha u_{xxt}(x,y,t)$, and with a different kind of nonlinearity. Cavalcanti et al. [CC$^+$18] showed exponential asymptotic stability with a localized damping, namely one of the form $\alpha(x,y)u_t(x,y,t)$, where $\alpha(x,y) \in L^\infty(\Omega)$ such that

$$\alpha(x,y) \geq \alpha_0 > 0 \quad \text{a.e. in } \omega,$$

on a small collar ω around the whole boundary.

We also mention Liu and Zhuang [LZ17], who extended the work of [W14] and studied the equation

$$u_{tt} + \Delta^2 u + au + |u_t|^{m-2}u_t = |u|^{p-2}u, \qquad m \geq 2,\ p > 2,$$

with the boundary conditions as in (1.1). More precisely, the authors gave necessary and sufficient conditions for global existence and energy decay results, and when $p > m$, a blow-up result was proved.

Let us also recall some recent results related to suspension bridges. In [MB$^+$16], Messaoudi et al. established a well-posedness result as well as the existence of a global attractor for the equation

$$u_{tt} + \delta u_t + \mu\Delta^2 u - \int_{-\infty}^{t} g(t-s)\Delta^2 u(s)\,ds + h(u) = f,$$

where $\delta, \mu > 0$, $f \in L^2(\Omega)$ is an external force, and the memory kernel g

and the nonlinear function h satisfy certain conditions. The same results were proved in [MMC16] when $g = 0$, but with a delay term of the form $\delta_2 u_t(x, y, t-\tau)$, where τ is the time delay. Finally, the well-posedness and a decay result were showed for a viscoelastic suspension bridge

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\, ds = 0$$

in [MM16].

Our paper deals with the exponential stability of solutions to a suspension bridge with a full structural damping. It is well known that this kind of damping has been well studied in the context of stabilization and attractors of wave and plate equations [CR82, T09, NP11, YLN16], and in the context of indirect stability of coupled plate equations in the recent work [HL19]. Our intention here is to extend those stability results to our problem. To the best of our knowledge, no one has discussed this issue with structural damping before.

The paper is organized as follows. The next section is devoted to the linear case, where the well-posedness and the exponential decay is proved. The same result is established for nonlinear model in the last section. We also discuss a nonhomogeneous case, which models a suspension bridge in the presence of a steady wind.

2. The linear model

2.1. Notation and preliminary results. We consider the following system

(2.1)
$$\begin{cases} u_{tt}(x,y,t) + \Delta^2 u(x,y,t) - \alpha u_{xxt}(x,y,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & (y,t) \in (-l,l) \times (0,+\infty), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & (y,t) \in (-l,l) \times (0,+\infty), \\ u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) = 0, & (x,t) \in (0,\pi) \times (0,+\infty), \\ u_{yyy}(x,\pm l,t) + (2-\sigma) u_{xxy}(x,\pm l,t) = 0, & (x,t) \in (0,\pi) \times (0,+\infty), \\ u(x,y,0) = u_0(x,y), u_t(x,y,0) = u_1(x,y), & \text{in } \Omega. \end{cases}$$

As in [FG15], we introduce the space

$$H^2_*(\Omega) = \{w \in H^2(\Omega) : w = 0 \text{ on } \{0,\pi\} \times (-l,l)\},$$

together with the inner product

$$(u,v)_{H^2_*} = \int_\Omega \Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}). \tag{2.2}$$

It is well known that $(H^2_*(\Omega),(\cdot,\cdot)_{H^2_*})$ is a Hilbert space, and the norm $\|\cdot\|^2_{H^2_*}$ is equivalent to the usual H^2 norm (see [FG15, Lemma 4.1]).

We recall the following results proved in [FG15] and [W14].

LEMMA 2.1. *Assume that $0<\sigma<1/2$ and $f\in L^2(\Omega)$. Then there exists a unique $u\in H^2_*(\Omega)$ such that*

$$(u,v)_{H^2_*(\Omega)}=\int_\Omega fv,\qquad \forall v\in H^2_*(\Omega). \tag{2.3}$$

Moreover, u is in $H^4(\Omega)$ and there exist constants $C=C(l,\sigma)>0$ and $C_e=C(\Omega,p)$ such that

$$\|u\|_{H^2_*(\Omega)}\le C\|f\|_{L^2(\Omega)}$$

and

$$\|u\|_p\le C_e\|u\|_{H^2_*(\Omega)}. \tag{2.4}$$

Now, we will exploit some ideas from [ABG14, Section 3], which will be useful for the remaining parts. Let us define

$$H^1_*(\Omega):=\{w\in H^1(\Omega): w=0 \text{ on } \{0,\pi\}\times(-l,l)\},$$
$$C^\infty_*(\Omega):=\{w\in C^\infty(\overline{\Omega}): \exists\varepsilon>0,\ w(x,y)=0 \text{ if } x\in[0,\varepsilon]\cup[\pi-\varepsilon,\pi]\},$$

which is a normed space when equipped with the norm

$$\|u\|_{H^1_*(\Omega)}:=\Big(\int_\Omega|\nabla u|^2\,dx\,dy\Big)^{1/2}. \tag{2.5}$$

One can easily verify that $H^1_*(\Omega)$ is the completion of $C^\infty_*(\Omega)$ with respect to the norm $\|\cdot\|_{H^1_*(\Omega)}$. In addition, it is not difficult to prove that the embedding $H^2_*(\Omega)\hookrightarrow H^1_*(\Omega)$ is compact and that the optimal embedding constant is given by

$$\Lambda_1:=\min_{w\in H^2_*(\Omega)}\frac{\|w\|^2_{H^2_*(\Omega)}}{\|w\|^2_{H^1_*(\Omega)}},$$

which gives the Poincaré-type inequality

$$\|w\|^2_{H^1_*(\Omega)}\le\Lambda_1^{-1}\|w\|^2_{H^2_*(\Omega)}\quad\text{for all } w\in H^2_*(\Omega). \tag{2.6}$$

We also need the following lemma:

LEMMA 2.2. *For any $u\in H^1_*(\Omega)$, we have*

$$\int_\Omega u^2(x,y)\,dx\,dy\le\frac{\pi^2}{2}\int_\Omega u_x^2(x,y)\,dx\,dy. \tag{2.7}$$

Proof. Since $u\in H^1_*(\Omega)$, we have

$$u^2(x,y)=\Big(\int_0^x u_x(s,y)\,ds\Big)^2.$$

By using Cauchy–Schwarz's inequality, we obtain

$$u^2(x,y) \le \Big(\int_0^x u_x^2(s,y)\,ds\Big)\Big(\int_0^x 1\,ds\Big).$$

Then

$$\int_0^\pi u^2(x,y)\,dx \le \int_0^\pi \Big(x\int_0^x u_x^2(s,y)\,ds\Big)\,dx \le \frac{\pi^2}{2}\int_0^\pi u_x^2(x,y)\,dx.$$

Consequently, by integrating over $(-l,l)$ with respect to y, (2.7) is established. ∎

2.2. Well-posedness and stability. Problem (2.1) can be written as

$$\begin{cases} U_t = AU, \\ U(0) = U_0, \end{cases}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad AU := \begin{pmatrix} v \\ -\Delta^2 u + \alpha v_{xx} \end{pmatrix}, \qquad U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

We define the Hilbert space $\mathcal{H} := H_*^2(\Omega)\times L^2(\Omega)$ endowed with the inner product

$$(U,V)_{\mathcal{H}} = (u,\tilde{u})_{H_*^2(\Omega)} + (v,\tilde{v})_{L^2(\Omega)},$$

where $U = (u,v)^T$, $V = (\tilde{u},\tilde{v})^T \in \mathcal{H}$. The domain of the operator A is defined by

$$D(A) := \{(u,v)\in\mathcal{H} : u \in H^4(\Omega) \text{ satisfies } (2.8), v\in H_*^2(\Omega)\},$$

where

$$\begin{cases} u_{xx}(0,y) = u_{xx}(\pi,y) = 0, \\ u_{yy}(x,\pm l) + \sigma u_{xx}(x,\pm l) = 0, \\ u_{yyy}(x,\pm l) + (2-\sigma)u_{xxy}(x,\pm l) = 0. \end{cases} \tag{2.8}$$

We define the energy of solutions of system (2.1) by

$$E(t) = \tfrac{1}{2}\|u_t(t)\|_{L^2}^2 + \tfrac{1}{2}\|u(t)\|_{H_*^2}^2, \qquad t \ge 0. \tag{2.9}$$

Multiplying the first equation of system (2.1) by u_t, integrating over Ω and using [MM17, Lemma 3.1], we get

$$\frac{dE(t)}{dt} = -\alpha\|u_{xt}\|_{L^2(\Omega)}^2 \le 0,$$

from which we deduce the following identity for the energy:

$$E(t_2) - E(t_1) = -\alpha\int_{t_1}^{t_2} \|u_{xt}\|_{L^2(\Omega)}^2\,dt, \qquad 0 \le t_1 \le t_2 < +\infty. \tag{2.10}$$

PROPOSITION 2.3. *The linear operator A generates a C_0 semigroup of contractions $(S(t))_{t\geq 0}$ on $\mathcal{H}$. Therefore problem* (2.1) *possesses a unique solution $U \in C([0,\infty);\mathcal{H})$. In addition, if $U_0 \in D(A)$, then problem* (2.1) *has a unique regular solution $U \in C([0+\infty);D(A)) \cap C^1([0,\infty);\mathcal{H})$.*

Proof. The proof goes exactly like that of [MM17, Lemma 3.3], so we omit it. ■

The main result of this section is stated as follows:

MAIN THEOREM 2.4. *Let $(u_0,u_1) \in D(A)$. Then there exist constants $K > 0$ and $\lambda > 0$ such that the energy functional* (2.9) *satisfies*

$$E(t) \leq Ke^{-\lambda t}, \quad \forall t \geq 0. \tag{2.11}$$

Proof. We will work with regular solutions; by standard density arguments, the decay result remains valid for weak solutions as well.

Multiplying (2.1) by u and integrating over $\Omega \times (s,T)$ for $0 < s < T$, we obtain

$$\int_s^T\int_\Omega (u_{tt}u + u\Delta^2 u - \alpha u u_{xxt}) = 0. \tag{2.12}$$

Recalling, by [MM17, Lemma 3.1], that

$$(\Delta^2 u, u)_{L^2(\Omega)} = \|u\|^2_{H^2_*(\Omega)},$$

we deduce from (2.12) that

$$\int_s^T\int_\Omega (u_t u)_t - \int_s^T\int_\Omega u_t^2 + \int_s^T \|u\|^2_{H^2_*(\Omega)} - \alpha\int_s^T\int_\Omega u u_{xxt} = 0. \tag{2.13}$$

This yields

$$\int_s^T E(t)\,dt + \int_s^T\int_\Omega (u_t u)_t - \frac{3}{2}\int_s^T\int_\Omega u_t^2 + \frac{1}{2}\int_s^T \|u\|^2_{H^2_*(\Omega)} - \alpha\int_s^T\int_\Omega u u_{xxt} = 0. \tag{2.14}$$

Then

$$\int_s^T E(t)\,dt \leq -\int_s^T\int_\Omega (u_t u)_t + \frac{3}{2}\int_s^T\int_\Omega u_t^2 + \alpha\int_s^T\int_\Omega u u_{xxt}. \tag{2.15}$$

Now, we estimate the terms on the right-hand side of (2.15). By using Lemma 2.1 and Young's inequality, the first term can be estimated as follows:

$$\begin{aligned} \Big|-\int_s^T\int_\Omega (u_t u)_t\Big| &\leq \Big|\int_\Omega u_t(s)u(s)\Big| + \Big|\int_\Omega u_t(T)u(T)\Big| \\ &\leq \frac{1}{2}\int_\Omega u_t^2(s) + \frac{1}{2}\int_\Omega u_t^2(T) + \frac{1}{2}\int_\Omega u^2(s) + \frac{1}{2}\int_\Omega u^2(T) \\ &\leq E(s) + E(T) + C\|u(s)\|^2_{H^2_*(\Omega)} + \|u(T)\|^2_{H^2_*(\Omega)} \leq CE(s), \end{aligned} \tag{2.16}$$

where C is a generic positive constant. For the second term, thanks to Lemma 2.2 we have

$$\frac{3}{2}\int_s^T\int_\Omega u_t^2 \le \frac{3}{2}\int_s^T\int_\Omega u_{tx}^2 = \frac{3}{2\alpha}\int_s^T(-E'(t))\,dt \le \frac{3}{2\alpha}E(s). \tag{2.17}$$

For the last term, by using Young's inequality and (2.6) we obtain, for every $\varepsilon > 0$,

$$\begin{aligned} \left|\alpha\int_s^T\int_\Omega uu_{xxt}\right| &= \left|-\alpha\int_s^T\int_\Omega u_x u_{xt}\right| \\ &\le C_\varepsilon\alpha\int_s^T\int_\Omega u_{xt}^2 + \alpha\frac{\varepsilon}{2}\int_s^T\int_\Omega u_x^2 \\ &\le C_\varepsilon\int_s^T(-E'(t))\,dt + \alpha\frac{\varepsilon}{2}\Lambda_1^{-1}\int_s^T\|u\|_{H_*^2(\Omega)}^2 \\ &\le C_\varepsilon E(s) + \alpha\varepsilon\Lambda_1^{-1}\int_s^T E(t)\,dt. \end{aligned} \tag{2.18}$$

Inserting (2.16)–(2.18) into (2.15), choosing ε such that $1-\alpha\varepsilon\Lambda_1^{-1} > 0$, we obtain the existence of a positive constant C_1 such that

$$\int_s^T E(t)\,dt \le C_1 E(s), \qquad \forall s > 0.$$

By letting $T \to +\infty$, and thanks to [K94, Theorem 8.1], we get the desired inequality (2.11). ∎

3. The nonlinear model

3.1. Well-posedness. Problem (1.1) can be written as

$$\begin{cases} U_t - AU = G, \\ U(0) = U_0, \end{cases}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad AU = \begin{pmatrix} v \\ -\Delta^2 u + \alpha v_{xx} \end{pmatrix}, \quad G(U) = \begin{pmatrix} 0 \\ \phi(u)u_{xx} \end{pmatrix}, \quad U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

where A and $D(A)$ are as in the previous section. In order to prove that problem (1.1) is well posed, it is sufficient to note that G is locally Lipschitz on $\mathcal{H}$, which is proved in [CC$^+$18, Lemma 3.1]. Thus, for a given $U_0 \in \mathcal{H}$, problem (1.1) has a unique local solution $U \in C([0,T);\mathcal{H})$.

3.2. Exponential stability. The energy associated to problem (1.1) is now defined by

$$E_u(t) = \underbrace{\tfrac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)}}_{\mathcal{K}_u(t)} + \underbrace{\tfrac{1}{2}\|u(t)\|^2_{H^2_*(\Omega)} - \tfrac{1}{2}P\|u_x(t)\|^2_{L^2(\Omega)} + \tfrac{1}{4}S\|u_x(t)\|^4_{L^2(\Omega)}}_{\mathcal{P}_u(t)}, \quad t \geq 0, \tag{3.1}$$

where $\mathcal{K}_u(t)$ and $\mathcal{P}_u(t)$ represent, respectively, the kinetic and the elastic potential energy of the model. Moreover, for $0 \leq t_1 \leq t_2 < +\infty$, one has the energy identity

$$E_u(t_2) - E_u(t_1) = -\alpha \int_{t_1}^{t_2}\int_{\Omega} u^2_{xt}(x,y,t)\,dx\,dy\,dt, \tag{3.2}$$

which shows that the energy is nonincreasing.

We observe that if $P < 0$, then $E_u(t) \geq 0$ for all $t \geq 0$. In elasticity, this situation corresponds to a plate that has been stretched rather than compressed, which does not occur in actual bridges. So, when $P > 0$, the most relevant case for bridges, the energy is no longer nonnegative, which plays an essential role in stabilization of distributed systems. To overcome this situation, we use the Poincaré-type inequality (2.6). So, for all $u \in H^2_*(\Omega)$ and since

$$\|u_x\|^2_{L^2(\Omega)} \leq \int_{\Omega} |\nabla u|^2\,dx\,dy \leq \Lambda_1^{-1}\,\|u\|^2_{H^2_*(\Omega)}, \tag{3.3}$$

we have

$$-\tfrac{1}{2}P\|u_x\|^2_{L^2(\Omega)} \geq -\tfrac{1}{2}P\Lambda_1^{-1}\|u\|^2_{H^2_*(\Omega)},$$

and therefore

$$\tfrac{1}{2}\|u\|^2_{H^2_*(\Omega)} - \tfrac{1}{2}P\|u_x\|^2_{L^2(\Omega)} \geq \tfrac{1}{2}\|u\|^2_{H^2_*(\Omega)}(1 - P\Lambda_1^{-1}).$$

Thus, if $0 \leq P \leq \Lambda_1$, then by the last inequality we deduce that $\frac{1}{2}\|u\|^2_{H^2_*(\Omega)} - \frac{1}{2}P\|u_x\|^2_{L^2(\Omega)} \geq 0$, and consequently $E_u(t) \geq 0$, which agrees with the assumption of [FGM16, Theorem 4]. In addition, we will have

$$\|u_t\|^2_{L^2(\Omega)} + \|u\|^2_{H^2_*(\Omega)} \leq CE(t) \leq CE(0).$$

Hence, the local solution obtained in Subsection 3.1 can be extended to $\mathbb{R}_+$. Hence, we have a global solution.

The main result of this section reads as follows:

Main Theorem 3.1. *Assume that* $0 \leq P \leq \gamma\Lambda_1$, *where* $0 < \gamma < 1$. *Then there exist constants* $K > 0$ *and* $\lambda > 0$ *such that the energy functional* (3.1) *satisfies*

$$E_u(t) \le K e^{-\lambda t}, \qquad \forall t \ge 0. \tag{3.4}$$

Proof. We will work with regular solutions; by standard density arguments, the decay remains valid for weak solutions as well.

Multiplying (1.1) by u and integrating over $\Omega \times (s,T)$ for $0 < s < T$, we obtain

$$\int_s^T \int_\Omega (u_{tt}u + u\Delta^2 u + Puu_{xx} - S\|u_x\|^2_{L^2(\Omega)} uu_{xx} - \alpha uu_{xxt}) = 0, \tag{3.5}$$

and therefore

$$\int_s^T \int_\Omega (u_t u)_t - \int_s^T \int_\Omega u_t^2 + \int_s^T \|u\|^2_{H^2_*(\Omega)} - P\int_s^T \|u_x\|^2_{L^2(\Omega)} + S\int_s^T \|u_x\|^4_{L^2(\Omega)} - \alpha \int_s^T \int_\Omega uu_{xxt} = 0. \tag{3.6}$$

This yields

$$\int_s^T E_u(t)\,dt + \int_s^T \int_\Omega (u_t u)_t - \frac{3}{2}\int_s^T \int_\Omega u_t^2 + \frac{1}{2}\int_s^T \|u\|^2_{H^2_*(\Omega)} - \frac{P}{2}\int_s^T \|u_x\|^2_{L^2(\Omega)} + \frac{3S}{4}\int_s^T \|u_x\|^4_{L^2(\Omega)} - \alpha \int_s^T \int_\Omega uu_{xxt} = 0. \tag{3.7}$$

Then, we obtain

$$\int_s^T E_u(t)\,dt \le -\int_s^T \int_\Omega (u_t u)_t + \frac{3}{2}\int_s^T \int_\Omega u_t^2 + \frac{P}{2}\int_s^T \|u_x\|^2_{L^2(\Omega)} + \alpha \int_s^T \int_\Omega uu_{xxt}. \tag{3.8}$$

Now, we estimate the terms on the right-hand side of (3.8). The first and the second terms can be, respectively, estimated as in (2.16) and (2.17), that is,

$$\Big| -\int_s^T \int_\Omega (u_t u)_t \Big| \le C E_u(s), \tag{3.9}$$

$$\frac{3}{2}\int_s^T \int_\Omega u_t^2 \le \frac{3}{2\alpha} E_u(s). \tag{3.10}$$

We can estimate the third term, thanks to (3.3), as

$$\frac{P}{2}\int_s^T \|u\|^2_{L^2(\Omega)} \le \frac{P}{2}\int_s^T \Lambda_1^{-1}\|u\|^2_{H^2_*(\Omega)} \le \gamma \int_s^T E_u(t)\,dt. \tag{3.11}$$

For the last term, as in (2.18) we obtain, for every $\varepsilon > 0$,

$$\left|\alpha \int_s^T \int_\Omega u u_{xxt}\right| \le C_\varepsilon E_u(s) + \alpha \Lambda_1^{-1} \varepsilon \int_s^T E_u(t)\, dt. \tag{3.12}$$

Inserting (3.9)–(3.12) into (3.8), choosing ε such that $1 - \gamma - \alpha\Lambda_1^{-1}\varepsilon > 0$, we obtain the existence of a positive constant C such that

$$\int_s^T E_u(t)\, dt \le C E_u(s), \qquad \forall s > 0.$$

By letting $T \to +\infty$, and invoking [K94, Theorem 8.1], we get the desired inequality (3.4). ∎

4. The nonhomogeneous linear problem. In this section, we discuss the nonhomogeneous equation

$$u_{tt}(x,y,t) + \Delta^2 u(x,y,t) - \alpha u_{xxt}(x,y,t) = h(x,y) \quad \text{in } \Omega \times (0,+\infty), \tag{4.1}$$

together with the initial and boundary conditions of (1.1). We assume that $h \in L^2(\Omega)$. This models a suspension bridge in the presence of a steady wind. Let us consider the following stationary problem

$$\begin{cases} \Delta^2 w(x,y) = h(x,y) & \text{in } \Omega, \\ w(0,y) = w(\pi,y) = w_{xx}(0,y) = w_{xx}(\pi,y) = 0 & \text{in } (-l,l), \\ w_{yy}(x,\pm l) + \sigma w_{xx}(x,\pm l) = 0 & \text{in } (0,\pi), \\ w_{yyy}(x,\pm l) + (2-\sigma) w_{xxy}(x,\pm l) = 0 & \text{in } (0,\pi). \end{cases} \tag{4.2}$$

We know, by a result of [FG15], that (4.2) has a unique solution

$$w \in H^2_*(\Omega) \cap H^4(\Omega). \tag{4.3}$$

Let $v(x,y,t) = u(x,y,t) - w(x,y)$, where u is the solution of (4.1) with the initial and boundary conditions of (1.1), and w is the solution of (4.2). Simple computation shows that v satisfies the equation and the boundary conditions of (2.1) and the initial conditions

$$v(x,y,0) = u_0(x,y) - w(x,y), \qquad v_t(x,y,t) = u_1(x,y). \tag{4.4}$$

By repeating the steps of the proof of Theorem 2.4, we obtain the following result.

Theorem 4.1. *Assume that $h \in L^2(\Omega)$ and $(u_0,u_1) \in D(A)$. Then there exist positive constants K and λ such that the solution of (4.1) with the initial and boundary conditions of (1.1) satisfies*

$$\|u_t(\cdot,\cdot,t)\|^2_{L^2(\Omega)} + \|u(\cdot,\cdot,t) - w\|^2_{H^2_*(\Omega)} \le K e^{-\lambda t}, \qquad \forall t \ge 0.$$

Remark 4.2. This last result shows that, in the presence of a steady wind, the bridge stabilizes to a steady position compatible with the wind.

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