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A set-valued extension of the Mazur–Ulam theorem

by

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Abstract. The Mazur–Ulam theorem states that every surjective isometry from a Banach space X to a Banach space Y is necessarily affine. Let $\mathfrak{K}(X)$ (resp. $\mathfrak{K}(Y)$) be the cone of all compact convex subsets of X (resp. Y) endowed with the Hausdorff metric. We extend the Mazur–Ulam theorem in the following manner: The restriction $T|_X$ of a surjective isometry $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ is an affine isometry from X onto Y; if, in addition, one of X and Y is either strictly convex, or Gâteaux smooth, then $T(C) = \bigcup\{T|_X(x) : x \in C\}$ for every $C \in \mathfrak{K}(X)$; and this is equivalent to "every surjective isometry $\mathfrak{K}(X) \to \mathfrak{K}(Y)$ is fully order preserving".

1. Introduction. The celebrated Mazur–Ulam theorem [21] states that every surjective isometry from a Banach space X to a Banach space Y is necessarily affine (see also [3, Thm. 14.1]). This theorem was the starting point of a series of interesting generalizations. See, for example, Figiel [11], Godefroy and Kalton [13] for nonsurjective isometries; Hyers and Ulam [16, 17], Gevietz [12], Gruber [14], Omladič and Šemrl [22], Vestfrid [29] for surjective ε-isometries; Qian [25], Šemrl and Väisälä [27], Cheng et al. [4], [6], [7], [9] for stability and weak stability of nonsurjective ε-isometries; Lindenstrauss and Szankowski [19] for surjective coarse isometries; and Cheng et al. [8] for nonsurjective coarse isometries.

In 1980, Gruber and Lettl [15] first showed a set-valued version of the Mazur–Ulam theorem for the Euclidean space $\mathbb{R}^n$: Let $\mathfrak{K}(\mathbb{R}^n)$ be the cone of all nonempty bounded closed convex subsets of $\mathbb{R}^n$ endowed with the Hausdorff metric d_H. Then for every self-isometry $T : \mathfrak{K}(\mathbb{R}^n) \to \mathfrak{K}(\mathbb{R}^n)$ there is an isometry $U : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(C) = \bigcup\{Ux : x \in C\}$ for every $C \in \mathfrak{K}(\mathbb{R}^n)$. 1986, Bandt [2] showed a generalized version of Gruber–Lettl's theorem: The same holds again if we substitute for $\mathfrak{K}(\mathbb{R}^n)$ all compact (not

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necessarily convex) subsets of a convex body in a finite-dimensional strictly convex normed space.

In 2018, Zhou, Zhang and Liu [31] first considered such a question in infinite-dimensional Banach spaces and showed the following result: Suppose that X and Y are Banach spaces such that w^*-exposed points of the dual unit balls B_{X^*} and B_{Y^*} are w^*-dense in the unit spheres S_{X^*}, S_{Y^*} of X^* and Y^* (they call such spaces "w^*-smooth"), in particular, both X and Y are Gâteaux smooth. Let $\mathfrak{K}(X)$ (resp. $\mathfrak{K}(Y)$) be the cone of all nonempty compact convex subsets of X (resp. Y). Suppose that $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ is a surjective isometry with $T(0) = 0$. Then (i) the restriction $T|_X$ of T is a surjective linear isometry from X to Y and (ii) $T(C) = \bigcup\{T|_X x : x \in C\}$ for every $C \in \mathfrak{K}(X)$. They further proposed the following questions [31, Problem 2.9]:

Problem 1.1. *Let X and Y be two (general) Banach spaces.*

(1) *If $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ is a surjective isometry with $T(0) = 0$, is the restriction $T|_X$ again a surjective linear isometry?*
(2) *What will happen if we drop the assumption that $T(0) = 0$?*

In this paper, motivated by Zhou, Zhang and Liu [31] but with a different approach, we show the answers to the two problems are affirmative:

Theorem 1.2. *Let X and Y be Banach spaces and $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ be a surjective isometry. Then the restriction $T|_X$ is an affine isometry from X onto Y.*

This, incorporating Zhou–Zhang–Liu's theorem [31, Theorem 2.8], implies that for every surjective isometry $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ we have $T(C) = \bigcup\{T|_X x : x \in C\}$ for every $C \in \mathfrak{K}(X)$ whenever either X or Y is w^*-smooth. We further show that this is still true if one of X and Y is strictly convex.

In this paper, we always assume that X is a Banach space, and B_X (resp. S_X) denotes the closed unit ball (resp. the unit sphere) of X; and $\mathfrak{K}(X)$ stands for the cone of all compact convex subsets of X endowed with the Hausdorff metric d_H, i.e.

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subset B + \varepsilon B_X,\ B \subset A + \varepsilon B_X\}, \qquad A, B \in \mathfrak{K}(X).$$

2. On order preserving isometries $\mathfrak{K}(X) \to \mathfrak{K}(Y)$. In this section, we consider representation of order preserving mappings from $\mathfrak{K}(X)$ to $\mathfrak{K}(Y)$. This was motivated by Artstein-Avidan and Milman [1]. The notion of "fully order preserving mapping" was introduced by Iusem, Reem and Svaiter [18].

Definition 2.1. Let S_1 and S_2 be partially ordered sets. A mapping $T : S_1 \to S_2$ is said to be *fully order preserving* if (1) T is surjective and (2) $Tx \geq Ty$ if and only if $x \geq y$.

EXAMPLE 2.2. Let X and Y be Banach spaces and $\varphi : X \to Y$ be an affine surjective isometry. We order $\mathfrak{K}(X)$ and $\mathfrak{K}(Y)$ by set inclusion, i.e. for $A, B \in \mathfrak{K}(X)$, $A \geq B$ if and only if $A \supset B$, and let $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ be defined for $A \in \mathfrak{K}(X)$ by $T(A) = \bigcup_{a\in A} \varphi a$. Then T is a fully order preserving isometry.

We define bounded-weak* continuous functions (or briefly b-w^*-*continuous functions*) to be functions whose restrictions to bounded sets are weak*-continuous. Let $K = (B_{X^*}, w^*)$, the closed unit ball of X^* endowed with the weak*-topology of X^*, let $C(K)$ be the Banach space of all b-w^*-continuous functions on K equipped with the sup-norm, and let $\mathfrak{V}(K)$ be the cone of $C(K)$ consisting of all b-w^*-continuous sublinear functions, i.e. every element of $\mathfrak{V}(K)$ is a b-w^*-continuous sublinear functional on X^*, but restricted to K. Note that the identity $\mathrm{id}_X : X \to \mathfrak{V}(K)$ is an isometric embedding. Then we can blur the distinction of $x \in X$ and $x \in \mathfrak{V}(K)$.

For a (continuous) convex function f defined on a Banach space X, we use $\partial f : X \to 2^{X^*}$ to denote the subdifferential mapping of f. If f is sublinear, then $x^* \in \partial f(x)$ if and only if $x^* \leq f$ and $\langle x^*, x\rangle = f(x)$. If, in addition, $X = Z^*$ for some Banach space Z and f is b-w^*-continuous, then $\partial f(x)$ is compact convex and contained in Z.

PROPOSITION 2.3. *Define* $J : \mathfrak{K}(X) \to \mathfrak{V}(K)$ *by*

$$J(C)(x^*) = \sup_{x\in C} \langle x^*, x\rangle \equiv \sigma_C(x^*), \qquad \forall C \in \mathfrak{K}(X), \forall x^* \in K. \tag{2.1}$$

Then J *is a fully order preserving isometry.*

Proof. Clearly, J is order preserving. It follows from [5, Theorem 2.1] that J is an isometry. It remains to show that J is surjective. Fix any $p \in \mathfrak{V}(K)$, and let $C = \partial p(0)$. Then $p = \sigma_C$, and the definition of $\partial p(0)$ and b-w^*-continuity of p on X^* entail that C is a compact convex subset of X. Thus, $J(C) = p$. ■

REMARK 2.4. Let $J : \mathfrak{K}(X) \to \mathfrak{V}(K)$ be defined as in (2.1) by $J(C) = \sigma_C$. If C is a singleton $\{x\}$, then we simply write $J\{x\} = x$, instead of $\sigma_{\{x\}}$.

The following result is a converse version of Example 2.2.

THEOREM 2.5. *Suppose that* $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ *is a fully order preserving isometry. Then* $\varphi \equiv T|_X : X \to Y$ *is a surjective affine isometry satisfying* $T(C) = \bigcup\{\varphi(x) : x \in C\}$.

Proof. Let $K_1 = (B_{X^*}, w^*)$ and $K_2 = (B_{Y^*}, w^*)$. For distinction, we denote the fully order preserving isometry $\mathfrak{K}(X) \to \mathfrak{V}(K_1)$ (resp. $\mathfrak{K}(Y) \to \mathfrak{V}(K_2)$) defined in Proposition 2.3 by J_1 (resp. J_2). For simplicity, we blur the distinction between a vector $x \in X$ and the singleton $\{x\}$. Since

$T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ is a fully order preserving isometry,

$$S \equiv J_2 \circ T \circ J_1^{-1} : \mathfrak{V}(K_1) \to \mathfrak{V}(K_2)$$

is again a fully order preserving isometry. Since $X \subset \mathfrak{V}(K_1)$ and $Y \subset \mathfrak{V}(K_2)$, and since every $x \in X$ (resp. $y \in Y$) is a minimal element of $\mathfrak{V}(K_1)$ (resp. $\mathfrak{V}(K_2)$) in the usual order of real-valued functions, S maps each singleton $x \in X$ into a singleton $y \in Y$. Thus, $S|_X : X \to Y$ is a surjective isometry. By the Mazur–Ulam theorem, $S|_X$ is affine. Hence, $T|_X = [J_2^{-1} \circ S \circ J_1]|_X : X \to Y$ is an affine isometry. Since both T and T^{-1} are fully order preserving, we obtain $T(C) = \bigcup \{T|_X(x) : x \in C\}$ for all $C \in \mathfrak{K}(X)$. ∎

3. Gâteaux differentiability of convex functions. In this section, we will discuss Gâteaux differentiability of convex functions defined on a closed convex set of a Banach space X with nonempty subset of its nonsupport points. To begin, we recall some notions and basic properties that will be used. For more information about Gâteaux differentiability of convex functions, we refer the reader to M. Fabian [10] and R. R. Phelps [23].

For a convex function f defined on a closed convex subset C of a Banach space X, its *subdifferential* $\partial f : C \to 2^{X^*}$ is defined by

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle,\ \forall y \in C\}. \tag{3.1}$$

In particular, if $\mathrm{int}(C) \neq \emptyset$ and $\partial f(x) = \{x^*\}$ is a singleton for some $x \in \mathrm{int}(C)$, then we say that f is *Gâteaux differentiable* at x, with *Gâteaux derivative* $df(x) = x^*$. Equivalently,

$$\lim_{t \to 0} \frac{f(x + tz) - f(x)}{t} = \langle x^*, z\rangle, \qquad \forall z \in X. \tag{3.2}$$

If f is a sublinear functional on X, then it is easy to check that $x^* \in \partial f(x)$ if and only if $x^* \leq f$ on X and $\langle x^*, x\rangle = f(x)$. If f is a continuous convex function on a convex open subset D, then ∂f is nonempty-valued and norm-to-w^* upper semicontinuous on D.

Definition 3.1. Let C be a closed convex set of a Banach space.

(i) A point $x \in C$ is said to be a *support point* of C if there exists a nonzero functional $x^* \in X^*$ such that $\langle x^*, x\rangle = \sup_{z \in C} \langle x^*, z\rangle$.
(ii) A point $x \in C$ is called a *nonsupport point* of C if it is not a support point of C. We use $N(C)$ to denote the set of all nonsupport points of C.

The following property is due to Phelps [24] (see also [26]).

Proposition 3.2. *Suppose that C is a closed convex subset of a Banach space X with $N(C) \neq \emptyset$. Then*

(i) *$N(C)$ is a dense G_δ subset of C;*
(ii) *for each $x_0 \in N(C)$, $V_{x_0} \equiv \bigcup_{\lambda \geq 0} \lambda(C - x_0)$ is a dense convex subset of X.*

REMARK 3.3. It follows from the separation theorem that $\mathrm{int}(C) \subset N(C)$, and if $\mathrm{int}(C) \neq \emptyset$, then $N(C) = \mathrm{int}(C)$. We should mention here that if C is separable, then $N(C) \neq \emptyset$ if X is replaced by the subspace $X_C \equiv \overline{V_0(C) - V_0(C)}$. In particular, for every closed reproducing cone C of a separable Banach space X, $N(C)$ is a dense G_δ-subset of C.

DEFINITION 3.4. Let C be a convex set of X with $N(C) \neq \emptyset$, and f be a convex function on C. We say that f is *relatively Gâteaux differentiable* at $x_0 \in N(C)$ (with respect to C) if there is $x^* \in X^*$ such that for all $z \in X$ for which there is $\delta > 0$ with $x_0 + tz \in C$ for all $0 < t < \delta$, we have

$$\lim_{t\to 0} \frac{f(x+tz) - f(x)}{t} = \langle x^*, z\rangle. \tag{3.3}$$

Note that V_{x_0} is a dense convex subset of X whenever $x_0 \in N(C)$. If $x_0 \in N(C)$ is a relatively Gâteaux differentiability point of a locally Lipschitz convex function f defined on C, then its relative Gâteaux derivative is unique. Thus, we can denote it again by $df(x_0)$.

DEFINITION 3.5. Let C^* be a convex set in the dual X^* of X.

(i) A functional $x^* \in C^*$ is said to be a *w^*-exposed point* of C^* if there is $x \in X$ such that

$$\langle x^*, x\rangle > \langle z^*, x\rangle, \qquad \forall z^* \,(\neq x^*) \in C^*. \tag{3.4}$$

(ii) A point $x \in X$ satisfying (3.4) is a *w^*-exposing functional* of C^* and exposing C^* at x^*.

(iii) We denote by $\exp(C^*)$ the set of all w^*-exposed points of C^*.

The following lemma is a consequence of [23, Prop. 6.9].

LEMMA 3.6. *Let X be a Banach space. Then the following statements are equivalent:*

(i) *x is a Gâteaux differentiability point of X with $x^* = d\|x\|$;*

(ii) *$x^* \in S_{X^*}$ is a w^*-exposed point of B_{X^*} and exposing B_{X^*} at x^* by x.*

Recall that a Banach space is a *Gâteaux differentiability* (resp. *weak Asplund*) *space* if every continuous convex function is densely Gâteaux differentiable in X (resp. differentiable at each point of a dense G_δ-subset of X). Mazur's theorem states that every separable Banach space is a weak Asplund space [20].

LEMMA 3.7. *Suppose that f is a continuous convex function defined on an open convex subset D of a weak Asplund space X, and that $C \subset D$ is a closed convex subset with $N(C) \neq \emptyset$. Then the set*

$$\{x \in C : f \text{ is Gâteaux differentiable at } x\}$$

contains a dense G_δ-subset C_0 of C.

Proof. Since f is a continuous convex function on D, it is locally Lipschitz on D. Since $C \subset D$ and $N(C) \neq \emptyset$, the restriction $f|_C$ is relatively Gâteaux differentiable at each point of a dense G_δ-subset C_0 of $N(C)$ [30]. This and Proposition 3.2 imply that for each $x_0 \in C_0$, there is $x^* \in X^*$ such that

$$\lim_{t\to 0^+} \frac{f|_C(x_0+tz) - f|_C(x_0)}{t} = \langle x^*, z\rangle, \quad \forall z \in V_{x_0} \equiv \bigcup_{\lambda>0} \lambda(C - x_0). \tag{3.5}$$

Note that $f = f|_C$ on C. Then

$$\lim_{t\to 0^+} \frac{f(x_0+tz) - f(x_0)}{t} = \langle x^*, z\rangle, \quad \forall z \in V_{x_0}. \tag{3.6}$$

Since V_{x_0} is a dense subset of X, we obtain $df(x_0) = x^*$. ∎

LEMMA 3.8 ([23, Theorem 6.2]). *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X is a Gâteaux differentiability space;*
(ii) *every w^*-compact convex set of X^* is the w^*-closed convex hull of its w^*-exposed points.*

PROPOSITION 3.9. *Suppose that X, Y are Banach spaces, and $U : X \to Y$ is a linear surjective isometry. Let $V = U^*$. Then the following statements are equivalent:*

(i) *$x \in X$ is a Gâteaux differentiability point of X with $d\|x\| = x^*$;*
(ii) *$Ux \in Y$ is a Gâteaux differentiability point of Y with $d\|Ux\| = V^{-1}x^*$.*

Proof. By Lemma 3.6, $x \in X$ is a Gâteaux differentiability point of X with $d\|x\| = x^*$ if and only x^* is a w^*-exposed point of B_{X^*} and exposed by x, i.e.

$$\langle x^*, x\rangle > \langle u^*, x\rangle, \quad \forall u^* \in B_{X^*} \setminus \{x^*\}. \tag{3.7}$$

Let $y = Ux$, $y^* = V^{-1}x^*$ and $v^* = V^{-1}u^*$. Then (3.7) is equivalent to the inequality

$$\langle y^*, y\rangle > \langle v^*, y\rangle, \quad \forall v^* \in B_{Y^*} \setminus \{y^*\}. \tag{3.8}$$

Clearly, (3.8) is equivalent to y^* being a w^*-exposed point of B_{Y^*} and exposed by y. We finish the proof by using Lemma 3.6 again. ∎

4. Gâteaux differentiability points in $\mathfrak{V}(K)$ and their derivatives. For a Banach space X, let $K = (B_{X^*}, w^*)$. We denote by $\mathfrak{V}(K)$ the cone of all b-w^*-continuous sublinear functionals on X^* but restricted to K, and set $E = \overline{\mathfrak{V}(K) - \mathfrak{V}(K)}$.

The following result directly follows from the definition of Gâteaux differentiability of convex functions.

LEMMA 4.1. *Suppose that X is a Banach space, and $\sigma \in \mathfrak{V}(K)$ is a Gâteaux differentiability point of E with $\varphi = d\|\sigma\|$. Then*

$$(4.1)\quad \langle \pm\varphi, \sigma_C - \sigma_D\rangle = \lim_{t\to\pm\infty}[\|t\sigma + (\sigma_C - \sigma_D)\|_E - \|t\sigma\|_E],\ \forall \sigma_C, \sigma_D \in \mathfrak{V}(K).$$

LEMMA 4.2. *Suppose that X is a separable Banach space. Then*

(i) $\exp(B_{E^*})$ *is a norming set of E, i.e. for each $u \in E$,*

$$\sup\{\langle u^*, u\rangle : u^* \in \exp(B_{E^*})\} = \|u\|_E;$$

(ii) *there is a dense G_δ-subset G of $\mathfrak{V}(K)$ such that at each point of G the norm $\|\cdot\|_E$ of E is Gâteaux differentiable.*

Proof. (i) Since X is separable, E is necessarily separable. By Mazur's theorem (see, for instance, [23, Theorem 1.20]), E is a Gâteaux differentiability space. By Lemma 3.8, B_{E^*} is the w^*-closed convex hull of $\exp(B_{E^*})$. Therefore, $\exp(B_{E^*})$ is a norming set of E.

(ii) Since E is separable, and since $\mathfrak{V}(K)$ is a reproducing cone of E, the conclusion follows from Lemma 3.7. ■

LEMMA 4.3. *Suppose that X is a nontrivial Banach space, and that G is the set of all Gâteaux differentiability points of X. Then*

(i) $d\|G\|_E \equiv \{d\|\sigma\|_E : \sigma \in G\} = \{\delta_{x^*} : x^* \in \exp(K)\}$;

(ii) *for each $0 \neq \sigma_C \in \mathfrak{V}(K)$, if $\sigma_C \notin X$, then*

$$(4.2)\quad \partial\|\sigma_C\|_E = w^*\text{-}\overline{\mathrm{co}}\,\{\delta_{x^*} : x^* \in S_{X^*} \text{ with } \langle x^*, x\rangle = \|\sigma_C\|_E \text{ and } x \in C\};$$

if $\sigma_C = x \in X$, then

$$(4.3)\quad \partial\|x\|_E = w^*\text{-}\overline{\mathrm{co}}\,\{\pm\delta_{\pm x^*} : x^* \in S_{X^*} \text{ with } \langle x^*, x\rangle = \|x\|\};$$

(iii) *in particular, if $x^* \in K$ is a w^*-exposed point of K and exposed by x, then*

$$\partial\|x\|_E = [-\delta_{-x^*}, \delta_{x^*}],\ \textit{the segment with endpoints } -\delta_{-x^*} \textit{ and } \delta_{x^*}.$$

Proof. (i) Given $\sigma_C \in \mathfrak{V}(K)$ with $\|\sigma_C\|_E = 1$, since σ_C is w^*-continuous on K, and since C is compact, there exist $x_0, y_0 \in C$ and $x_0^*, y_0^* \in K$ such that

$$\sigma_C(x_0^*) = \max \sigma_C = \langle x_0^*, x_0\rangle, \qquad \sigma_C(y_0^*) = \min \sigma_C = \langle y_0^*, y_0\rangle.$$

Therefore,

$$\|\sigma_C\|_E = \max\{\sigma_C(x_0^*), -\sigma_C(y_0^*)\} = \max\{\langle x_0^*, x_0\rangle, -\langle y_0^*, y_0\rangle\}.$$

If σ_C is a Gâteaux differentiability point of E with $d\|\sigma_C\|_E = \varphi$, then $\varphi = \delta_{x_0^*}$ when $\|\sigma_C\|_E = \sigma_C(x_0^*)$, while $\varphi = -\delta_{y_0^*}$ when $\|\sigma_C\|_E = -\sigma_C(y_0^*)$ $(= 1)$. Next, we show that the latter can never happen. Indeed, if it does, then $\sigma_C(y_0^*) = \sup_{x\in C}\langle y_0^*, x\rangle = -1$. Thus, $\langle y_0^*, x\rangle = -1$ for all $x \in C$. Consequently, $\langle \delta_{-y_0^*}, \sigma_C\rangle = \sup_{x\in C}\langle -y_0^*, x\rangle = 1$. Therefore, $-\delta_{y_0^*}, \delta_{-y_0^*} \in \partial\|\sigma_C\|_E$, and this contradicts σ_C being a Gâteaux differentiability point of E. Consequently, $d\|G\|_E \equiv \{d\|\sigma\|_E : \sigma \in G\} \subset \{\delta_{x^*} : x^* \in \exp(K)\}$.

Conversely, let $x^* \in \exp(K)$. By Lemma 3.6, there is $x \in S_X$ such that $d\|x\| = x^*$. Fix any $0 \le \varepsilon < 1$ and consider the segment $C = [(1-\varepsilon)x, x]$. Then x^* is the unique point of K such that $\|\sigma_C\|_E = \sigma_C(x^*) = \langle \delta_{x^*}, \sigma_C\rangle$. Thus, $d\|\sigma_C\|_E = \delta_{x^*}$.

(ii) For each $x^* \in S_{X^*}$ with $\langle x^*, x\rangle = \|\sigma_C\|_E$ and $x \in C$, we have $\langle \delta_{x^*}, \sigma_C\rangle = \sigma_C(x^*) = \|x\|$. Thus, $x^* \in \partial\|x\|$ and $\delta_{x^*} \in \partial\|\sigma_C\|$. Since $\partial\|\sigma_C\|$ is convex and w^*-compact in E^*, we obtain

$$w^*\text{-}\overline{\mathrm{co}}\,\{\delta_{x^*} : x^* \in S_{X^*} \text{ with } \langle x^*, x\rangle = \|\sigma_C\|_E \text{ and } x \in C\} \subset \partial\|\sigma_C\|_E. \tag{4.4}$$

Note that if σ_C is acting as an element of $C(K)$, then $\partial\|\sigma_C\|_{C(K)}$ is convex and w^*-compact in $S_{C(K)^*}$. It is also an extremal set of $B_{C(K)^*}$, that is, $B_{C(K)^*} \setminus \partial\|\sigma_C\|_{C(K)}$ is convex. Thus, every extreme point of $\partial\|\sigma_C\|_{C(K)}$ is an extreme point of $B_{C(K)^*}$. Therefore, $\mathrm{extr}(\partial\|\sigma_C\|_{C(K)}) \subset \mathrm{extr}(B_{C(K)^*}) = \{\pm\delta_{x^*} : x^* \in K\}$. Consequently,

$$\mathrm{extr}(\partial\|\sigma_C\|_{C(K)}) \subset \{\pm\delta_{x^*} : x^* \in S_{X^*} \text{ with } \langle \pm x^*, x\rangle = \|\sigma_C\|_E \text{ and } x \in C\}. \tag{4.5}$$

If $\sigma_C \notin X$, then $-\langle \delta_{-x^*}, \sigma_C\rangle < \|\sigma_C\|_E = \langle \delta_{x^*}, \sigma_C\rangle$, i.e. $-\delta_{x^*} \notin \partial\|\sigma_C\|_E$. Thus,

$$\mathrm{extr}(\partial\|\sigma_C\|_{C(K)}) \subset \{\delta_{x^*} : x^* \in S_{X^*} \text{ with } \langle \pm x^*, x\rangle = \|\sigma_C\|_E \text{ and } x \in C\}. \tag{4.6}$$

If $\sigma_C = x \in X$, then

$$\langle \delta_{x^*}, x\rangle = \|x\|_E = \langle -\delta_{-x^*}, x\rangle, \quad \text{i.e.} \quad \pm\,\delta_{\pm x^*} \in \partial\|x\|_E. \tag{4.7}$$

It follows from (4.5)–(4.7) and the Krein–Milman theorem that if $\sigma_C \notin X$ then

$$w^*\text{-}\overline{\mathrm{co}}\,\{\delta_{x^*} : x^* \in S_{X^*} \text{ with } \langle x^*, x\rangle = \|\sigma_C\|_E \text{ and } x \in C\} \supset \partial\|\sigma_C\|_E; \tag{4.8}$$

and if $\sigma_C = x \in X$,

$$\partial\|x\|_E = w^*\text{-}\overline{\mathrm{co}}\,\{\delta_{x^*}, -\delta_{-x^*} : x^* \in S_{X^*} \text{ with } \langle x^*, x\rangle = \|x\|\}. \tag{4.9}$$

We finish the proof of (ii) by combining (4.4), (4.7) and (4.8).

(iii) is just a consequence of (ii). ∎

Proposition 4.4. *Suppose that X is a Banach space. Then the following statements are equivalent:*

(i) $\exp(K)$ *is a norming set of* X*;*
(ii) $\{\delta_{x^*} : x^* \in \exp(K)\}$ *is a norming set of* $\mathfrak{V}(K)$.

Proof. (i)⇒(ii). Given $\sigma_C \in \mathfrak{V}(K)$, there exist $x_0^* \in S_{X^*}$ and $x_0 \in C$ such that $\|\sigma_C\|_E = \langle x_0^*, x_0\rangle = \|x_0\|$. Since $\exp(K)$ is a norming set of X, there exists a sequence $\{x_n^*\} \subset \exp(K)$ such that $\langle x_n^*, x_0\rangle \to \|x_0\|$. Thus,

$\|\sigma_C\|_E \geq \langle \delta_{x_n^*}, \sigma_C\rangle = \sigma_C(x_n^*) \geq \langle x_n^*, x_0\rangle \to \|x_0\|$. This entails that $\{\delta_{x^*} : x^* \in \exp(K)\}$ is a norming set of $\mathfrak{V}(K)$.

(ii)⇒(i). This is trivial. ■

LEMMA 4.5. *Let X be a Banach space, $E = \overline{\mathfrak{V}(K) - \mathfrak{V}(K)}$, $F = w^*\text{-}\overline{\text{span}}\{\delta_{x^*} : x^* \in \exp(K)\}$, $F^\perp = \{e \in E : \langle \varphi, e\rangle = 0, \forall \varphi \in F\}$, and let $Q : E \to E/F^\perp$ be the quotient mapping. If $\exp(K)$ is a norming set of X, then*

(i) $\|Q(\sigma_C)\|_{E/F^\perp} = \|\sigma_C\|_E$ *for all* $\sigma_C \in \mathfrak{V}(K)$;

(ii) *the restriction* $Q|_X : X \to E/F^\perp$ *is a linear isometric embedding;*

(iii) $\sigma_C \in \mathfrak{V}(K)$ *is a Gâteaux differentiability point of E with $d\|\sigma_C\|_E = \varphi$ if and only if $Q(\sigma_C)$ is a Gâteaux differentiability point of $E/F^\perp$ with $d\|Q(\sigma_C)\|_{E/F^\perp} = \varphi$.*

Proof. (i) By Lemma 4.3 and Proposition 4.4, $d\|G\|_E = \{\delta_{x^*} : x^* \in \exp(K)\}$, where G is the set of all Gâteaux differentiability points of $\mathfrak{V}(K)$, and $\{\delta_{x^*} : x^* \in \exp(K)\}$ is a norming set of $\mathfrak{V}(K)$. Consequently, for each $\sigma_C \in \mathfrak{V}(K)$, there is $\delta_{z^*} \in w^*\text{-}\overline{\{\delta_{x^*} : x^* \in \exp(K)\}}$ such that $\langle \delta_{z^*}, \sigma_C\rangle = \|\sigma_C\|_E$. This implies that $\|\sigma_C + h\|_E \geq \|\sigma_C\|_E$ for all $h \in F^\perp$, i.e. $\|Q(\sigma_C)\|_{E/F^\perp} = \|\sigma_C\|_E$.

(ii) is just a consequence of (i).

(iii) Assume that $\sigma_C \in \mathfrak{V}(K)$ is a Gâteaux differentiability point of E with $d\|\sigma_C\|_E = \varphi$. Then by Lemma 4.3, $\varphi = \delta_{x^*}$ for some $x^* \in \exp(K)$. Therefore, for all $h \in E$ and $t > 0$, we have

$$\frac{\|Q(\sigma_C + th)\|_{E/F^\perp} - \|Q(\sigma_C)\|_{E/F^\perp}}{t} \leq \frac{\|\sigma_C + th\|_E - \|\sigma_C\|_E}{t} \to \langle \delta_{x^*}, h\rangle \tag{4.10}$$

as $t \to 0^+$.

On the other hand, $d\| - \sigma_C\|_E = -\varphi = -\delta_{x^*}$. Thus

$$\frac{\|Q(\sigma_C - th)\|_{E/F^\perp} - \|Q(\sigma_C)\|_{E/F^\perp}}{-t} \geq \frac{\|\sigma_C - th\|_E - \|\sigma_C\|_E}{-t} \to \langle \delta_{x^*}, h\rangle \tag{4.11}$$

as $t \to 0^+$. The two inequalities entail

$$\lim_{t\to 0} \frac{\|Q(\sigma_C + th)\|_{E/F^\perp} - \|Q(\sigma_C)\|_{E/F^\perp}}{t} = \langle \delta_{x^*}, h\rangle, \quad \forall h \in E, \tag{4.12}$$

i.e. $d\|Q(\sigma_C)||_{E/F^\perp} = \delta_{x^*}$.

Conversely, assume that $Q(\sigma_C)$ is a Gâteaux differentiability point of $Q(\mathfrak{V}(K))$ for some $\sigma_C \in \mathfrak{V}(K)$. Then there is a unique $\varphi \in F$ with $\|\varphi\| = 1$ such that $\|Q(\sigma_C)\|_{E/F^\perp} = \langle \varphi, \sigma_C\rangle$. By (i), $\|Q(\sigma_C)\|_{E/F^\perp} = \|\sigma_C\|_E = \langle \delta_{z^*}, \sigma_C\rangle$ for some $\delta_{z^*} \in w^*\text{-}\overline{\{\delta_{x^*} : x^* \in \exp(K)\}}$. Thus, $\varphi = \delta_{z^*}$. Since C is compact, there is $z \in C$ such that $\|\sigma_C\|_E = \langle \delta_{z^*}, \sigma_C\rangle = \langle z^*, z\rangle = \|z\|$.

We claim that z^* is in $\exp(K)$ and is exposed by z. Indeed, otherwise there is $y^* \in \partial\|z\|$ with $y^* \neq z^*$ such that $\langle \delta_{y^*}, \sigma_C\rangle = \langle y^*, z\rangle = \|z\|$. Let $\{y_n^*\} \subset \mathrm{co}(\exp(K))$ with $y_n^* = \sum_j \lambda_{n,j} x_{n,j}^*$, $x_{n,j}^* \in \exp(K)$, $\lambda_{n,j} \geq 0$ for all n, j with $\sum_j \lambda_{n,j} = 1$ such that $y_n^* \to y^*$ in the w^*-topology of X^*. Then $\varphi_n \equiv \sum_j \lambda_{n,j}\delta_{x_{n,j}^*} \in S_F$ so that

$$\|\sigma_C\|_E \geq \langle \varphi_n, \sigma_C\rangle \geq \langle \delta_{y_n^*}, \sigma_C\rangle \to \sigma_C(y^*) = \|\sigma_C\|_E.$$

Note that every w^*-limit of evaluation functionals is again an evaluation functional. Let δ_{u^*} be an arbitrary w^*-cluster point of $\{\delta_{y_n^*}\}$. Then $\delta_{u^*} \neq \delta_{z^*}$ and $\langle \delta_{u^*}, \sigma_C\rangle = \|\sigma_C\|_E = \|Q\sigma_C\|_{E/F^\perp}$. This contradicts $Q(\sigma_C)$ being a Gâteaux differentiability point. ■

5. An extension of the Mazur–Ulam theorem. In this section, we will show that for every surjective isometry $f : \mathfrak{K}(X) \to \mathfrak{K}(Y)$, the restriction $f|_X$ is an affine isometry from X to Y, where X and Y are Banach spaces, and $\mathfrak{K}(X)$ and $\mathfrak{K}(Y)$ are the cones of all nonempty compact convex subsets of X and Y endowed with the Hausdorff metrics.

For Banach spaces X, Y, let $K_1 = (B_{X^*}, w^*)$, $K_2 = (B_{Y^*}, w^*)$ and let $\mathfrak{V}(K_j) \subset C(K_j)$ be the cones of all w^*-continuous sublinear functions corresponding to K_j, $j = 1, 2$. Let $E_j = \overline{\mathfrak{V}(K_j) - \mathfrak{V}(K_j)} \subset C(K_j)$, $j = 1, 2$.

LEMMA 5.1. *Let $T : \mathfrak{V}(K_1) \to \mathfrak{V}(K_2)$ be an isometry (not necessarily surjective). Then for every $\sigma \in \mathfrak{V}(K_1)$ with $\|\sigma\|_{E_1} = \langle \delta_{x^*}, x\rangle = \|x\|$ for some $x \in X\setminus\{0\}$ with $x^* \in \partial\|x\| \subset S_{X^*}$, there exists an extreme point $\psi \in \{\pm\delta_{y^*}\}$ of $K_2 = B_{E_2^*}$ for some $y^* \in S_{Y^*}$ such that*

$$\begin{aligned}\langle \psi, T(t\sigma) - T(-t\alpha x)\rangle &= \|T(t\sigma) - T(-t\alpha x)\|_{E_2}\\ &= \|T(t\sigma) - T(0)\|_{E_2} + \|T(-t\alpha x) - T(0)\|_{E_2}\\ &= t(1+\alpha)\|x\|, \quad \forall t, \alpha \in \mathbb{R}^+.\end{aligned}$$

Therefore,

$$\langle \psi, T(t\sigma) - T(0)\rangle = t\|x\|, \quad \forall t \in \mathbb{R}^+, \tag{5.1}$$

and

$$\langle \psi, T(0) - T(-tx)\rangle = t\|x\|, \quad \forall t \in \mathbb{R}^+. \tag{5.2}$$

Proof. For each $n \in \mathbb{N}$, let $\psi_n \in S_{E_2^*}$ be such that

$$\langle \psi_n, T(n\sigma) - T(-n\alpha x)\rangle = \|T(n\sigma) - T(-n\alpha x)\|_{E_2} = n(1+\alpha)\|x\|.$$

Since for each $n \in \mathbb{N}$, $T(n\sigma) - T(-n\alpha x) \in E_2^*$ is positively homogeneous and w^*-to-w^* continuous on $K_2 = (B_{Y^*}, w^*)$, there exists $y_n^* \in S_{Y^*}$ such that

$$\begin{aligned}\max\{\pm(T(n\sigma) - T(-n\alpha x))(y_n^*)\} &= |\langle \delta_{y_n^*}, T(n\sigma) - T(-n\alpha x)\rangle|\\ &= \|T(n\sigma) - T(-n\alpha x)\|_{E_2} = \|n\sigma + n\alpha x\|_{E_1} = n(1+\alpha)\|x\|.\end{aligned}$$

Therefore, we can claim $\psi_n \in \{\pm\delta_{y_n^*}\}$. We can further show that

$$\max\{\pm\langle\delta_{y_n^*}, T(t\sigma) - T(-t\alpha x))\rangle\} = t(1+\alpha)\|x\|, \qquad \forall t \in [0, n]. \tag{5.3}$$

Suppose, to the contrary, that there exist $n \in \mathbb{N}$ and $0 < t \le n$ such that

$$\langle\psi_n, T(t\sigma) - T(-t\alpha x))\rangle < t(1+\alpha)\|x\|.$$

Then

$$\begin{aligned} t(1+\alpha)\|x\| &> \langle\psi_n, T(t\sigma) - T(-t\alpha x))\rangle \\ &= \langle\psi_n, T(n\sigma) - T(-n\alpha x)\rangle + \langle\psi_n, T(t\sigma) - T(n\sigma)\rangle \\ &\quad - \langle\psi_n, T(-t\alpha x) - T(-n\alpha x)\rangle \\ &\ge n(1+\alpha)\|x\| - \|T(t\sigma) - T(n\sigma)\|_{E_2} \\ &\quad - \|T(-t\alpha x) - T(-n\alpha x)\|_{E_2} \\ &= n(1+\alpha)\|x\| - (n-t)\|x\| - (n-t)\alpha\|x\| = t(1+\alpha)\|x\|, \end{aligned}$$

a contradiction. It follows from (5.3) that the lemma holds for any w^*-accumulation point ψ of $\{\psi_n\} \subset \{\pm\delta_{y_n^*}\}$. Note that any w^*-cluster point ψ of $\{\delta_{y_n^*}\}$ is of the form δ_{y^*} for some $y^* \in K_2$. Then

$$\max\{\pm\langle\delta_{y^*}, T(t\sigma) - T(-t\alpha x)\rangle\} = t(1+\alpha)\|x\|, \qquad \forall t, \alpha \in \mathbb{R}^+.$$

Thus, $\psi \in \{\delta_{y^*}, -\delta_{y^*}\}$.

We can claim $y^* \in S_{Y^*}$ because $y^* \in K_2 \setminus \{0\}$ and

$$\begin{aligned} \|T(t\sigma) - T(-t\alpha x)\|_{E_2} &\ge \max\{\pm\langle\delta_{y^*/\|y^*\|}, T(t\sigma) - T(-t\alpha x)\rangle\} \\ &\ge \max\{\pm\langle\delta_{y^*}, T(t\sigma) - T(-t\alpha x)\rangle\} \\ &= \|T(t\sigma) - T(-t\alpha x)\|_{E_2}. \end{aligned}$$

Finally, we claim that ψ is an extreme point of $B_{E_2^*}$. Let

$$A = \{\psi \in S_{E_2^*} : (5.1) \text{ and } (5.2) \text{ hold}\}.$$

Then it is clear that A is a nonempty w^*-closed convex extremal subset of $B_{E_2^*}$. Therefore, it contains an extreme point of $B_{E_2^*}$. ∎

The following is a key result for the proof of the main results of this paper.

LEMMA 5.2 (Main lemma). *Let $T : \mathfrak{V}(K_1) \to \mathfrak{V}(K_2)$ be an isometry. Then*

(i) *for any Gâteaux differentiability point x of X with $x^* = d\|x\|$, there exists $\psi \in B_{E_2^*}$ with $\psi \in \{\pm\delta_{y^*}\} \subset S_{E_2^*}$ for some $y^* \in S_{Y^*}$, such that*

$$\langle\psi, T(\sigma_C) - T(\sigma_D)\rangle = \langle\delta_{x^*}, \sigma_C - \sigma_D\rangle, \qquad \forall \sigma_C, \sigma_D \in \mathfrak{V}(K_1); \tag{5.4}$$

(ii) *if, in addition, T is surjective, then there is a unique $y^* \in S_{Y^*}$ such that*

$$\langle\delta_{y^*}, T(\sigma_C) - T(\sigma_D)\rangle = \langle\delta_{x^*}, \sigma_C - \sigma_D\rangle, \qquad \forall \sigma_C, \sigma_D \in \mathfrak{V}(K_1), \tag{5.5}$$

$$\langle\delta_{-y^*}, T(\sigma_C) - T(\sigma_D)\rangle = \langle\delta_{-x^*}, \sigma_C - \sigma_D\rangle, \qquad \forall \sigma_C, \sigma_D \in \mathfrak{V}(K_1). \tag{5.6}$$

Proof. (i) Let σ be a Gâteaux differentiability point of $\mathfrak{V}(K)$ with $\|\sigma\|_{E_1} = \|x\|$ such that $d\|\sigma\|_{E_1} = \delta_{x^*}$. Then according to Lemma 5.1, there is $\psi \in \{\pm\delta_{y^*}\}$ for some $y^* \in S_{Y^*}$ such that

$$\langle \psi, T(t\sigma) - T(0)\rangle = t\|x\|, \quad \forall t \in \mathbb{R}^+,$$
$$\langle \psi, T(0) - T(-tx)\rangle = t\|x\|, \quad \forall t \in \mathbb{R}^+.$$

Lemma 4.1 now implies that for $\sigma_C \in \mathfrak{V}(K_1)$ (by letting $t \to +\infty$),

$$\begin{aligned} -\langle \psi, T(\sigma_C) - T(0)\rangle &= \langle -\psi, -T(t\sigma) + T(\sigma_C)\rangle - \langle \psi, T(t\sigma) - T(0)\rangle \\ &\le \|t(-\sigma) + \sigma_C\|_{E_1} - t\|\sigma\|_{E_1} \to -\langle \delta_{x^*}, \sigma_C\rangle \end{aligned}$$

and

$$\begin{aligned} \langle \psi, T(\sigma_C) - T(0)\rangle &= -\langle \psi, T(-tx) - T(\sigma_C)\rangle - \langle -\psi, T(-tx) - T(0)\rangle \\ &\le \|tx + \sigma_C\|_{E_1} - t\|x\| \to \max_{\varphi \in \partial\|x\|_{E_1}} \langle \varphi, \sigma_C\rangle. \end{aligned}$$

Therefore, for all $\sigma_C \in \mathfrak{V}(K_1)$,

$$\langle \delta_{x^*}, \sigma_C\rangle \le \langle \psi, T(\sigma_C) - T(0)\rangle \le \max_{\varphi \in \partial\|x\|_{E_1}} \langle \varphi, \sigma_C\rangle. \tag{5.7}$$

Since $\partial\|x\|_{E_1} = [-\delta_{-x^*}, \delta_{x^*}]$ and since $\delta_{x^*} \ge -\delta_{-x^*}$ on $\mathfrak{V}(K_1)$, we obtain

$$\max_{\varphi \in \partial\|x\|_{E_1}} \langle \varphi, \sigma_C\rangle = \langle \delta_{x^*}, \sigma_C\rangle, \quad \forall \sigma_C \in \mathfrak{V}(K_1). \tag{5.8}$$

Now (5.7) and (5.8) imply that

$$\langle \delta_{x^*}, \sigma_C\rangle = \langle \psi, T(\sigma_C) - T(0)\rangle, \quad \forall \sigma_C \in \mathfrak{V}(K_1). \tag{5.9}$$

Consequently, (5.4) holds.

(ii) Note that (5.9) is equivalent to $\psi \circ (T - T(0)) = \delta_{x^*}$. If $\psi_j \in E_2^*$, $j = 1, 2$, are such that

$$\psi_1 \circ (T - T(0)) = \delta_{x^*} = \psi_2 \circ (T - T(0)),$$

then

$$(\psi_1 - \psi_2) \circ (T - T(0)) = 0 \quad \text{on } \mathfrak{V}(K_1).$$

Consequently,

$$(\psi_1 - \psi_2) \circ (T(\sigma_C) - T(\sigma_D)) = 0, \quad \forall \sigma_C, \sigma_D \in \mathfrak{K}(X). \tag{5.10}$$

If T is surjective, then

$$\{T\sigma_C - T\sigma_D : C, D \in \mathfrak{K}(X)\} = \mathfrak{V}(K_2) - \mathfrak{V}(K_2)$$

is dense in E_2. This and (5.10) entail that $\psi_1 - \psi_2 = 0$ on $\mathfrak{V}(K_2) - \mathfrak{V}(K_2)$. Therefore, $\psi_1 - \psi_2 = 0$ on E_2. Consequently, there is a unique functional ψ ($\in \{\pm\delta_{y^*}\}$) satisfying (5.4).

Next, we will show that $\psi = \delta_{y^*}$. Indeed, otherwise $\psi = -\delta_{y^*}$. We consider $d\|-x\| = -x^*$. By (5.9) and the fact we have just proven, there exist a unique $y_1^* \in S_{E_2^*} \subset K_2$ and a unique $\psi_1 \in \{\pm\delta_{y_1^*}\}$ such that

$$(5.11) \qquad \langle \delta_{-x^*}, \sigma_C \rangle = \langle \psi_1, T(\sigma_C) - T(0) \rangle, \quad \forall \sigma_C \in \mathfrak{V}(K_1).$$

(5.9) and (5.11) entail

$$(5.12) \qquad \langle \delta_{x^*} + \delta_{-x^*}, \sigma_C \rangle = \langle \psi + \psi_1, T(\sigma_C) - T(0) \rangle, \quad \forall \sigma_C \in \mathfrak{V}(K_1).$$

Since $\delta_{x^*}+\delta_{-x^*}$ is nonnegative-valued on $\mathfrak{V}(K_1)$, $\psi+\psi_1$ is nonnegative-valued on

$$T(\mathfrak{V}(K_1)) - T(0) = \mathfrak{V}(K_2) - T(0) \supset \mathfrak{V}(K_2) \supset Y.$$

Consequently, $\psi_1 = -\psi$ on Y, which entails that either $\psi_1 = \delta_{y^*}$, or $\psi_1 = -\delta_{-y^*}$. If $\psi_1 = \delta_{y^*}$, then (5.12) implies $\delta_{x^*} + \delta_{-x^*} = 0$ on $\mathfrak{V}(K_1)$, and so X is the trivial space $\{0\}$. If $\psi_1 = -\delta_{-y^*}$, then (5.12) yields

$$\langle \delta_{x^*} + \delta_{-x^*}, \sigma_C \rangle = -\langle \delta_{y^*} + \delta_{-y^*}, T(\sigma_C) - T(0) \rangle, \quad \forall \sigma_C \in \mathfrak{V}(K_1).$$

Since $\delta_{x^*} + \delta_{-x^*} \geq 0$ and $-(\delta_{y^*} + \delta_{-y^*}) \leq 0$, we infer that $\delta_{x^*} + \delta_{-x^*} = 0$ on $\mathfrak{V}(K_1)$, which again implies $X = \{0\}$. Thus we have shown (5.5), i.e. $\psi = \delta_{y^*}$.

In order to show (5.6), note first that (5.12) implies $\psi_1 = -\psi$ on Y, which further entails that either $\psi_1 = -\delta_{y^*}$, or $\psi_1 = \delta_{-y^*}$. But $\psi_1 = -\delta_{y^*}$ is impossible, because in this case (5.12) would imply $\delta_{x^*}+\delta_{-x^*} = 0$ on $\mathfrak{V}(K_1)$. Therefore, $\psi_1 = \delta_{-y^*}$, that is, (5.6) holds. ∎

Recall that a bounded subset $A \subset X^*$ is a *norming* set of X provided $\sup_{x^* \in A}\langle x^*, x\rangle = \|x\|$ for all $x \in A$. Clearly, $A \subset X^*$ is a norming set of X if and only if $w^*\text{-}\overline{\text{co}}(A) = B_{X^*}$. By Lemma 3.8, if X is a Gâteaux differentiability space, then $\exp(K)$ is a norming set of X.

Lemma 5.3. *Let X and Y be Banach spaces and $T : \mathfrak{V}(K_1) \to \mathfrak{V}(K_2)$ be a surjective isometry. If $\exp(K_1)$ is a norming set of X, then $T(0) \in Y$. Consequently, $T(x) \in Y$ for all $x \in X$.*

Proof. By Proposition 4.4, $\{\delta_{x^*} : x^* \in \exp(K_1)\}$ is a norming set of $\mathfrak{V}(K_1)$. According to Lemma 5.2, for each $x^* \in \exp(K_1)$ there is a unique $y^* \in S_{E_2^*}$ such that

$$(5.13) \qquad \langle \delta_{y^*}, T(\sigma_C) - T(\sigma_D) \rangle = \langle \delta_{x^*}, \sigma_C - \sigma_D \rangle, \forall \sigma_C, \sigma_D \in \mathfrak{V}(K_1).$$

Let $\Delta = \Delta_{Y^*}$ be the set of all evaluation functionals δ_{y^*} for some $y^* \in S_{E_2^*}$ with respect to some $x^* \in \exp(K_1)$ so that the equality above holds. Then Δ_{Y^*} is a norming set of $T(\mathfrak{V}(K_1)) - T(0) = \mathfrak{V}(K_2)) - T(0) \supset \mathfrak{V}(K_2)$.

Suppose, to the contrary, that $T(0)$ is not in Y. Fix any $1 > \lambda > 0$, let $\sigma_D = 0$ and let $\sigma_C \in \mathfrak{V}(K_1)$ be such that $T(\sigma_C) = (1 - \lambda)T(0)$ in (5.13). Then by Lemma 5.2, for each $x^* \in \exp(K_1)$, there is a unique $y^* \in S_{E_2^*}$ such that

$$\langle \delta_{y^*}, -\lambda T(0)\rangle = \langle \delta_{y^*}, T(\sigma_C) - T(0)\rangle = \langle \delta_{x^*}, \sigma_C\rangle,$$
$$\langle \delta_{-y^*}, -\lambda T(0)\rangle = \langle \delta_{-y^*}, T(\sigma_C) - T(0)\rangle = \langle \delta_{-x^*}, \sigma_C\rangle.$$

Therefore,

$$0 \ge \langle \delta_{y^*}+\delta_{-y^*}, -\lambda T(0)\rangle = \langle \delta_{y^*}+\delta_{-y^*}, T(\sigma_C)-T(0)\rangle = \langle \delta_{x^*}+\delta_{-x^*}, \sigma_C\rangle \ge 0.$$

Consequently,

$$\langle \delta_{y^*} + \delta_{-y^*}, T(0)\rangle = 0, \quad \forall \delta_{\pm y^*} \in \varDelta.$$

Equivalently,

$$\langle \delta_{y^*}, T(0)\rangle = -\langle \delta_{-y^*}, T(0)\rangle, \quad \forall \delta_{\pm y^*} \in \varDelta.$$

Since $\varDelta$ is a norming set of E_2, $\{y^* : \delta_{y^*} \in \varDelta\}$ is a norming set of Y. Thus,

$$w^*\text{-}\overline{\mathrm{co}}\,\{y^* : \delta_{y^*} \in \varDelta\} = B_{Y^*}.$$

This fact, convexity and w^*-continuity of $\langle \delta_{\pm z^*}, T(0)\rangle$ with respect to $z^* \in K_2 = B_{Y^*}$ entail that

$$\langle \delta_{y^*}, T(0)\rangle = -\langle \delta_{-y^*}, T(0)\rangle, \quad \forall y^* \in K_2.$$

Since $\langle \delta_{y^*}, T(0)\rangle$ is convex and $-\langle \delta_{-y^*}, T(0)\rangle$ is concave, both are w^*-continuous linear functionals on E_2^* when restricted to K_2. This entails that $T(0)$ is a singleton, that is, $T(0) \in Y$.

To show that $T(x) \in Y$ for all $x \in X$, note that for each fixed $x \in X$, the mapping $T_x : \mathfrak{V}(K_1) \to \mathfrak{V}(K_2)$ defined by $T_x = T(x + \sigma_C)$ is again a surjective isometry. Therefore, $T(x) = T_x(0) \in Y$. ∎

COROLLARY 5.4. *Suppose that X and Y are Gâteaux differentiability spaces, and $T : \mathfrak{V}(K_1) \to \mathfrak{V}(K_2)$ is a surjective isometry. Then $T|_X$ is an affine isometry from X to Y.*

Proof. Since X and Y are Gâteaux differentiability spaces, $\exp(K_1)$ is a norming set of X and $\exp(K_2)$ is a norming set of Y. By Lemma 5.3, $T|_X$ ($T^{-1}|_Y$, resp.) is an isometric embedding from X (Y, resp.) to Y (X, resp.). Therefore, $T|_X : X \to Y$ is a surjective isometry. We finish the proof by applying the Mazur–Ulam theorem. ∎

COROLLARY 5.5. *Suppose that X and Y are Gâteaux differentiability spaces, and $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ is a surjective isometry. Then $T|_X$ is an affine isometry from X onto Y.*

Proof. Let $J_1 : \mathfrak{K}(X) \to \mathfrak{V}(K_1)$, $J_2 : \mathfrak{K}(Y) \to \mathfrak{V}(K_2)$ be defined by

$$J_1(C) = \sigma_C,\ C \in \mathfrak{K}(X), \quad \text{and} \quad J_2(D) = \sigma_D,\ D \in \mathfrak{K}(Y).$$

Then it follows from Proposition 2.3 that both J_1 and J_2 are fully order preserving affine isometries. Therefore, $J_2TJ_1^{-1} : \mathfrak{V}(K_1) \to \mathfrak{V}(K_2)$ is a surjective isometry. By Corollary 5.4, $J_2TJ_1^{-1}|_X$ is an affine surjective isometry from X onto Y. Consequently, $T|_X$ is an affine surjective isometry from X onto Y. ∎

LEMMA 5.6. *Let X and Y be Banach spaces, and $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ be a surjective isometry. Then for each separable subspace X_0, there exist separable subspaces $X_\infty \supset X_0$ of X and Y_∞ of Y such that $T|_{\mathfrak{K}(X_\infty)}$ is a surjective isometry from $\mathfrak{K}(X_\infty)$ to $\mathfrak{K}(Y_\infty)$.*

Proof. Let

$$Y_0 = \overline{\operatorname{span}}\Big\{\bigcup\{T(C) : C \in \mathfrak{K}(X_0)\}\Big\}.$$

Then $Y_0 \subset Y$ is again separable. Next, let

$$X_1 = \overline{\operatorname{span}}\Big\{\bigcup\{T^{-1}(C) : C \in \mathfrak{K}(Y_0)\}\Big\},$$
$$Y_1 = \overline{\operatorname{span}}\Big\{\bigcup\{T(C) : C \in \mathfrak{K}(X_1)\}\Big\}.$$

Inductively, for each $n \geq 1$, let

$$X_n = \overline{\operatorname{span}}\Big\{\bigcup\{T^{-1}(C) : C \in \mathfrak{K}(Y_{n-1})\}\Big\},$$
$$Y_n = \overline{\operatorname{span}}\Big\{\bigcup\{T(C) : C \in \mathfrak{K}(X_n)\}\Big\}.$$

Thus we obtain two sequences $\{X_n\}_{n=0}^\infty$ and $\{Y_n\}_{n=0}^\infty$ of separable subspaces with

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset X, \quad Y_0 \subset Y_1 \subset \cdots \subset Y_n \subset \cdots \subset Y,$$

satisfying

$$T\mathfrak{K}(X_{n-1}) \subset \mathfrak{K}(Y_{n-1}) \subset T\mathfrak{K}(X_n) \quad \text{for all } n \geq 1.$$

Let $X_\infty = \overline{\bigcup_{n=0}^\infty X_n}$ and $Y_\infty = \overline{\bigcup_{n=0}^\infty Y_n}$. Then it is easy to observe that $T(\mathfrak{K}(X_\infty)) = \mathfrak{K}(Y_\infty)$. ∎

The following theorem follows from Corollary 5.5 and Lemma 5.6.

THEOREM 5.7. *Suppose that X and Y are Banach spaces, and that $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ is a surjective isometry. Then $T|_X$ is an affine isometry from X onto Y.*

Proof. By Lemma 5.6, for each separable subspace $X_0 \subset X$, there exist a closed separable subspace $X_\infty \subset X$ and a separable closed subspace $Y_\infty \subset Y$ such that $T|_{X_\infty} : \mathfrak{K}(X_\infty) \to \mathfrak{K}(Y_\infty)$ is a surjective isometry. Since both X_∞ and Y_∞ are Gâteaux differentiability spaces, Corollary 5.5 shows that $T|_{X_\infty}$ is an affine surjective isometry from X_∞ onto Y_∞. We finish the proof by arbitrariness of X_0. ∎

6. Surjective isometries and order preserving mappings. In this section, we will show that every surjective isometry from $\mathfrak{V}(K_1)$ to $\mathfrak{V}(K_2)$

is order preserving assuming that X is either strictly convex or Gâteaux smooth.

Recall that a Banach space X is *strictly convex* if the conditions $x, y \in S_X$ and $\|x+y\| = 2$ imply $x = y$; and X is *Gâteaux smooth* if the norm of X is everywhere Gâteaux differentiable off the origin.

For a Banach space X, we denote again by $\mathfrak{K}(X)$ the family of all nonempty compact convex sets of X. Given $a \in X$, let

$$\mathfrak{K}_a(X) = \big\{C \in \mathfrak{K}(X) : \exists c = c_a \in X \text{ such that } \\ d_H(y, C) = \max\{\|(a+c-y)\|, \|(a-c)-y\|\},\ \forall y \in X\big\}.$$

LEMMA 6.1. *Suppose that X is a strictly convex Banach space and $a \in X$. Then for every $C \in \mathfrak{K}_a(X)$, we have $a \in C$.*

Proof. Assume $C \in \mathfrak{K}_a(X)$, and $c = c_a \in X$ is such that for all $y \in X$,

$$d_H(y, C) = d_H(a+c, y) \vee d_H(a-c, y) = \|a+c-y\| \vee \|a-c-y\|.$$

Let $y = a \pm c$. Then $d_H(a \pm c, C) = d_H(a \pm c, a \mp c) = 2\|c\|$, which implies

$$S(a \pm c, 2\|c\|) \cap C \neq \emptyset. \tag{6.1}$$

On the other hand, it follows from

$$d_H(a, C) = d_H(a+c, a) \vee d_H(a-c, a) = \|c\|$$

that

$$C \subset B(a, \|c\|). \tag{6.2}$$

Since X is strictly convex, (6.1) and (6.2) imply

$$B(a, \|c\|) \cap S(a \pm c, 2\|c\|) = \{a \pm c\}. \tag{6.3}$$

Thus, $a \pm c \in C$. Consequently, $a \in C$. ■

LEMMA 6.2. *Suppose X is a strictly convex Banach space, and $K \in \mathfrak{K}(X)$. Then the following statements are equivalent:*

(i) *$a \in K$;*
(ii) *$d_H(C, K) \le d_H(a, C) \vee d_H(a, K)$, for all $C \in \mathfrak{K}_a(X)$.*

Proof. (i)⇒(ii). Let $C \in \mathfrak{K}_a(X)$. It follows from $a \in C \cap K$ that

$$C \subset a + d_H(a, C)B_X \subset K + (d_H(a, C) \vee d_H(a, K))B_X, \tag{6.4}$$

$$K \subset a + d_H(a, K)B_X \subset C + (d_H(a, C) \vee d_H(a, K))B_X. \tag{6.5}$$

Thus, $d_H(C, K) \le d_H(a, C) \vee d_H(a, K)$.

(ii)⇒(i). Suppose $a \notin K$. Then there is $k \in K$ such that

$$\|a - k\| = \inf_{x \in K} \|a - x\| > 0.$$

Choose any $r > d_H(a, K)$, and let $C = \big[a - r\frac{k-a}{\|k-a\|}, a + r\frac{k-a}{\|k-a\|}\big]$. Then

$$C \in \mathfrak{K}_a(X) \quad \text{and} \quad d_H(a, C) = r. \tag{6.6}$$

On the other hand,

$$d_H(C,K) \ge \left\| a - r\frac{k-a}{\|k-a\|} - k \right\| = \|a-k\| + r > r. \tag{6.7}$$

Now (6.6), (6.7) and $r > d_H(a,K)$ together imply

$$d_H(C,K) > r = d_H(a,C) \vee d_H(a,K),$$

and this contradicts (ii). ■

THEOREM 6.3. *Suppose that X and Y are Banach spaces, and that $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ is a surjective isometry. If either X or Y is strictly convex, then $T|_X$ is an affine isometry from X onto Y such that $T(K) = \bigcup\{Tx : x \in K\}$ for all $K \in \mathfrak{K}(X)$.*

Proof. It follows from Theorem 5.7 that $T|_X$ is an affine isometry from X onto Y. Therefore, both X and Y are strictly convex if either of them is. It remains to show that

$$T(K) = \bigcup\{Tx : x \in K\} \quad \text{for all } K \in \mathfrak{K}(X). \tag{6.8}$$

Given $a \in X$ and $K \in \mathfrak{K}(X)$, we will show that $a \in K$ if and only if $Ta \in T(K)$. Since $T|_X : X \to Y$ is a surjective affine isometry,

$$\begin{aligned}\mathfrak{K}_{Ta}(Y) &= \{D \in \mathfrak{K}(Y) : \exists u \in Y\ \forall v \in Y,\\ &\qquad d_H(v,D) = d_H(Ta+u,v) \vee d_H(Ta-u,v)\}\\ &= \{T(C) : C \in \mathfrak{K}(X) : \exists x \in X\ \forall y \in X,\\ &\qquad d_H(y,C) = d_H(a+x,y) \vee d_H(a-x,y)\}\\ &= \{T(C) : C \in \mathfrak{K}_a(X)\}.\end{aligned}$$

This and Lemma 6.2 entail

$$\begin{aligned}a \in K &\iff \forall C \in \mathfrak{K}_a(X),\ d_H(C,K) \le d_H(a,C) \vee d_H(a,K)\\ &\iff \forall C \in \mathfrak{K}_a(X),\ d_H(T(C),T(K)) \le d_H(Ta,T(C)) \vee d_H(Ta,T(K))\\ &\iff \forall D \in \mathfrak{K}_{Ta}(Y),\ d_H(D,T(K)) \le d_H(Ta,D) \vee d_H(Ta,T(K))\\ &\iff Ta \in T(K).\end{aligned}$$

Thus, we have shown that $a \in K$ if and only if $Ta \in T(K)$. Consequently, $T(K) = \bigcup\{Ta : a \in K\}$. ■

THEOREM 6.4. *Let X and Y be Banach spaces, and $T : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ be a surjective isometry. If either X or Y is w^*-smooth (in particular, Gâteaux smooth), then $T|_X$ is an affine isometry from X onto Y such that $T(C) = \bigcup\{Tx : x \in C\}$ for all $C \in \mathfrak{K}(X)$.*

Proof. By Theorem 5.7, $T|_X$ is a surjective affine isometry from X onto Y. It follows that $f \equiv T - T(0) : \mathfrak{K}(X) \to \mathfrak{K}(Y)$ is again a surjective isometry with $f(0) = 0$. By [31, Theorem 2.8],

$$T(C) - T(0) = f(C) = \bigcup\{f(x) : x \in C\} = \bigcup\{T(x) - T(0) : x \in C\}$$

for all $C \in \mathfrak{K}(X)$. Therefore, $T(C) = \bigcup\{T(x) : x \in C\}$ for all $C \in \mathfrak{K}(X)$. ∎

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References

[1] S. Artstein-Avidan and V. Milman, *The concept of duality in convex analysis, and the characterization of the Legendre transform*, Ann. of Math. (2) 169 (2009), 661–674.

[2] C. Bandt, *On the metric structure of hyperspaces with Hausdorff metric*, Math. Nachr. 129 (1986), 175–183.

[3] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis I*, Amer. Math. Soc. Colloq. Publ. 48, Amer. Math. Soc., Providence, RI, 2000.

[4] L. Cheng, *On perturbed metric-preserving mappings of Banach spaces*, Sci. Sinica Math. 50 (2020), 1–28 (in Chinese).

[5] L. Cheng, Q. Cheng, Q. Shen, K. Tu and W. Zhang, *A new approach to measures of noncompactness of Banach spaces*, Studia Math. 240 (2018), 21–45.

[6] L. Cheng, Q. Cheng, K. Tu and J. Zhang, *A universal theorem for stability of ε-isometries on Banach spaces*, J. Funct. Anal. 269 (2015), 199–214.

[7] L. Cheng, Y. Dong and W. Zhang, *On stability of nonsurjective ε-isometries of Banach spaces*, J. Funct. Anal. 264 (2013), 713–734.

[8] L. Cheng, Q. Fang, S. Luo and L. Sun, *On non-surjective coarse isometries between Banach spaces*, Quaest. Math. 42 (2019), 347–362.

[9] L. Cheng and Y. Zhou, *On perturbed metric-preserved mappings and their stability characterizations*, J. Funct. Anal. 266 (2014), 4995–5015.

[10] M. Fabian, *Gâteaux Differentiability of Convex Functions and Topology: Weak Asplund Spaces*, Canad. Math. Soc. Ser. Monogr. Adv. Texts, Wiley, New York, 1997.

[11] T. Figiel, *On non linear isometric embeddings of normed linear spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 185–188.

[12] J. Gevirtz, *Stability of isometries on Banach spaces*, Proc. Amer. Math. Soc. 89 (1983), 633–636.

[13] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, Studia Math. 159 (2003), 121–141.

[14] P. M. Gruber, *Stability of isometries*, Trans. Amer. Math. Soc. 245 (1978), 263–277.

[15] P. M. Gruber and G. Lettl, *Isometries of the space of convex bodies in Euclidean space*, Bull. London Math. Soc. 12 (1980), 455–462.

[16] D. H. Hyers and S. M. Ulam, *On approximate isometries*, Bull. Amer. Math. Soc. 51 (1945), 288–292.

[17] D. H. Hyers and S. M. Ulam, *On approximate isometries on the space of continuous functions*, Ann. of Math. 48 (1947), 285–289.

[18] A. N. Iusem, D. Reem and B. F. Svaiter, *Order preserving and order reversing operators on the class of convex functions in Banach spaces*, J. Funct. Anal. 268 (2015), 73–92.

[19] J. Lindenstrauss and A. Szankowski, *Nonlinear perturbations of isometries*, in: Colloquium in Honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983), Astérisque 131 (1985), 357–371.
[20] S. Mazur, *Über konvexe Mengen in linearen normierten Räumen*, Studia Math. 4 (1933), 70–84.
[21] S. Mazur et S. Ulam, *Sur les transformations isométriques d'espaces vectoriels normés*, C. R. Acad. Sci. Paris 194 (1932), 946–948.
[22] M. Omladič and P. Šemrl, *On non linear perturbations of isometries*, Math. Ann. 303 (1995), 617–628.
[23] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Math. 1364, Springer, 1989.
[24] R. R. Phelps, *Some topological properties of support points of convex sets*, Israel J. Math. 13 (1972), 327–336.
[25] S. Qian, *ε-Isometric embeddings*, Proc. Amer. Math. Soc. 123 (1995), 1797–1803.
[26] J. Rainwater, *Yet more on the differentiability of convex functions*, Proc. Amer. Math. Soc. 103 (1988), 773–778.
[27] P. Šemrl and J. Väisälä, *Nonsurjective nearisometries of Banach spaces*, J. Funct. Anal. 198 (2003), 268–278.
[28] J. Väisälä, *A survey of nearisometries*, in: Papers on Analysis: A Volume Dedicated to Olli Martio on the Occasion of His 60th Birthday, Univ. of Jyväskylä, 2001, 305–315.
[29] I. A. Vestfrid, *Stability of almost surjective ε-isometries of Banach spaces*, J. Funct. Anal. 269 (2015), 2165–2170.
[30] C. Wu and L. Cheng, *A note on the differentiability of convex functions*, Proc. Amer. Math. Soc. 121 (1994), 1057–1062.
[31] Y. Zhou, Z. Zhang and C. Liu, *On representation of isometric embeddings between Hausdorff metric spaces of compact convex subsets*, Houston J. Math. 44 (2018), 917–925.

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