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EINSTEIN–WEYL STRUCTURES ON REAL HYPERSURFACES OF COMPLEX TWO-PLANE GRASSMANNIANS

BY

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Abstract. We study real Hopf hypersurfaces with Einstein–Weyl structures in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. First we prove that a real Hopf hypersurface with a closed Einstein–Weyl structure $W = (g, \theta)$ is of type (B) if $\nabla_\xi \theta = 0$, where ξ denotes the Reeb vector field of the hypersurface. Next, for a Hopf hypersurface with non-vanishing geodesic Reeb flow, we prove that there does not exist an Einstein–Weyl structure $W = (g, k\eta)$, where k is a non-zero constant and η is a one-form dual to ξ. Finally, it is proved that a real Hopf hypersurface with two closed Einstein–Weyl structures $W^{\pm} = (g, \pm\theta)$ is of type (A) or type (B).

1. Introduction. Let $G_2(\mathbb{C}^{m+2})$ be a complex two-plane Grassmannian, consisting of all complex two-dimensional linear subspaces of $\mathbb{C}^{m+2}$, which is the unique compact, irreducible, Kähler, quaternionic Kähler manifold that is not a hyper-Kähler manifold. In this article we always assume $m \geq 3$ because it is well known that $G_2(\mathbb{C}^3)$ is isometric to $\mathbb{C}P^2$ and $G_2(\mathbb{C}^4)$ is isometric to the real Grassmannian manifold $G_2^+(\mathbb{R}^6)$ of oriented 2-dimensional linear subspaces of $\mathbb{R}^6$. Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$. It is known that the Kähler structure J on $G_2(\mathbb{C}^{m+2})$ induces a structure vector field ξ called the *Reeb vector field* on M and given by $\xi := -JN$, where N is the local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$, and the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$ induces the almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ on M with $\xi_v := -J_v N$, $v = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ is the canonical basis of $\mathfrak{J}$. On the real hypersurface M there exist two natural distributions (that is, subbundles of the tangent bundle): $[\xi] = \mathrm{Span}\{\xi\}$ and $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$. Denote by $\mathfrak{D}$ the orthogonal complement of the distribution $\mathfrak{D}^\perp$. By using these distributions, Berndt and Suh [BS99] proved that the Reeb vector field ξ belongs either to $\mathfrak{D}$ or $\mathfrak{D}^\perp$ and gave the following classification:

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THEOREM 1.1 ([BS99]). *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If $\mathfrak{D}^\perp$ and $[\xi]$ are invariant under the shape operator, then either*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ if $\xi \in \mathfrak{D}^\perp$, or*
(B) *M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$ if $\xi \in \mathfrak{D}$, where $m = 2n$.*

If the Reeb vector field ξ is invariant under the shape operator, i.e. $A\xi = \alpha\xi$ for some function α, then M is said to be a *Hopf hypersurface*. Based on the classification of Theorem 1.1, Berndt and Suh later proved that real hypersurfaces of type (B) are the only hypersurfaces with the Reeb vector ξ belonging to the distribution $\mathfrak{D}$, namely they proved the following theorem.

THEOREM 1.2 ([LS10]). *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

By using Theorems 1.1 and 1.2, many geometers have given characterizations for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with conditions imposed on geometric quantities such as the shape, normal (or structure) Jacobi operator, Ricci tensor, etc. (see [PSW14, PJS07, LSW16, SH15, S10]). On the other hand, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be *Einstein* if the Ricci tensor Q is given by $g(QX, Y) = ag(X, Y)$ for a constant a and any vector fields X and Y on M. For this we observe the following conclusion.

THEOREM 1.3 (see [PSW10] or [S13, Corollary 0.1]). *There do not exist any Einstein Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.*

It is natural to ask whether there exists a condition weaker than having an Einstein metric that can be realized on the real Hopf hypersurfaces of $G_2(\mathbb{C}^{m+2})$. Our answer is affirmative. For example, Jeong–Suh [JS14] proved that if a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, admits a Ricci soliton with potential vector field W being Reeb vector field ξ, then $\lambda = 4(m+1)$. Here a *Ricci soliton* is a generalization of of an Einstein metric, namely, a Riemannian metric satisfying

$$\tfrac{1}{2}\mathcal{L}_W g + Ric - \lambda g = 0,$$

where λ is a constant and Ric is the Ricci tensor. The vector field W is called the *potential vector field*. Also, we note that Suh [S06] gave a classification of Hopf pseudo-Einstein real hypersurfaces.

As a generalization of Einstein metrics in terms of affine connections, N. Hitchin [HI82] introduced a Weyl structure satisfying a certain Ricci tensor condition, called an Einstein–Weyl structure. We refer also to [H82].

A Weyl structure $W = (D, [g])$ on a smooth manifold consists of a torsion free affine connection D preserving a conformal structure $[g]$. Namely, for some (or any) metric g within the conformal structure, there exists a unique 1-form θ such that $Dg = -2\theta \otimes g$. Since the previous equation is invariant under the transformations $g' = e^{2f}g$ and $\theta' = \theta - df$, where $f \in C^\infty(M)$, we always denote by $W = (g, \theta)$ the Weyl structure in question (see [G09]). A Weyl structure is called an *Einstein–Weyl structure* if the symmetrized Ricci tensor is proportional to a metric g representing $[g]$:

$$Ric^D(Y, X) + Ric^D(X, Y) = \Lambda g(Y, X), \qquad \Lambda \in C^\infty(M). \tag{1.1}$$

Further, if the unique 1-form θ is closed, then W is said to be a *closed Einstein–Weyl structure* (closedness of θ is clearly independent of the choice of g within $[g]$). The Einstein–Weyl condition plays a key role in physics, the pure Einstein theory being too strong as a system model for various physical questions. By contrast, Einstein–Weyl structures appear naturally as the background of the static Yang–Mills–Higgs theory. On the other hand, Matzeu [M00] proved that several classes of almost contact manifolds also naturally carry Einstein–Weyl structures. Einstein–Weyl structures have received a lot of attention in the context of almost contact metric manifolds (see [GM17, G09, M02, M11, N98]).

Since real hypersurfaces of $G_2(\mathbb{C}^{m+2})$ also have almost contact structures, motivated by the above background, in this paper we study the real Hopf hypersurfaces of $G_2(\mathbb{C}^{m+2})$ admitting Einstein–Weyl structures $W = (g, \theta)$ such that the induced metric g lies within the conformal structure $[g]$ and obtain the following results.

THEOREM 1.4. *Let M be a real Hopf hypersurface of $G_2(\mathbb{C}^{m+1})$ admitting a closed Einstein–Weyl structure $W = (g, \theta)$ with $\nabla_\xi \theta = 0$. Then M is a real hypersurface of type* (B).

THEOREM 1.5. *For a real Hopf hypersurface of $G_2(\mathbb{C}^{m+1})$ with non-vanishing geodesic Reeb flow, there does not exist an Einstein–Weyl structure $W = (g, k\eta)$, where k is a non-zero constant.*

THEOREM 1.6. *Let M be a Hopf real hypersurface of $G_2(\mathbb{C}^{m+1})$ admitting two closed Einstein–Weyl structures $W^\pm = (g, \pm\theta)$. Then M is of either type* (A) *or type* (B).

In order to prove our results, we need to recall some definitions and related conclusions on real hypersurfaces as well as Einstein–Weyl structures, which are presented in Sections 2 and 3, respectively. The proofs of our results will be given in Section 4.

2. Real hypersurfaces of $G_2(\mathbb{C}^{m+2})$. In this section we summarize some basic notations and formulas concerning the complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. For more details we refer to [BS99, BS02, S10, S06, S12]. Let $G_2(\mathbb{C}^{m+2})$ be the complex Grassmannian of all complex 2-dimensional linear subspaces of $\mathbb{C}^{m+2}$. In fact $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space $SU(m+2)/(S(U(2)\times U(m))$. Up to scaling there exists a unique $S(U(2)\times U(m))$-invariant Riemannian metric $\widetilde{g}$ on $G_2(\mathbb{C}^{m+2})$. The Grassmannian manifold $G_2(\mathbb{C}^{m+2})$ equipped with such a metric becomes a symmetric space of rank two, which is both Kähler and quaternionic Kähler. Denote by J and $\mathfrak{J}$ the Kähler structure and the quaternionic Kähler structure on $G_2(\mathbb{C}^{m+2})$, respectively. A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of almost Hermitian structures J_v such that $J_vJ_{v+1} = J_{v+2} = -J_{v+1}J_v$, where the index is taken modulo 3. As is well known, the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$ satisfy the following relations:

$$JJ_v = J_vJ, \quad \mathrm{trace}(JJ_v) = 0, \quad v = 1, 2, 3.$$

We denote by $\widetilde{\nabla}$ the Livi-Civita connection with respect to $\widetilde{g}$. There exist 1-forms q_1, q_2, q_3 such that

$$\widetilde{\nabla}_X J_v = q_{v+2}(X)J_{v+1} - q_{v+1}(X)J_{v+2}$$

for any vector field X on $G_2(\mathbb{C}^{m+2})$.

Let M be an immersed real hypersurface of $G_2(\mathbb{C}^{m+2})$ with induced metric g. There exists a locally defined unit normal vector field N on M and we write $\xi := -JN$ for the structure vector field of M. An induced one-form η defined by $\eta(\cdot) = \widetilde{g}(J\cdot, N)$ is dual to ξ. For any vector field X on M the tangent part of JX is denoted by $\phi X = JX - \eta(X)N$. Moreover, the following identities hold:

$$\phi^2 = -\mathrm{Id} + \eta\otimes\xi, \quad \eta\circ\phi = 0, \quad \phi\circ\xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{2.2}$$

where $X, Y \in \mathfrak{X}(M)$. These formulas show that (ϕ, η, ξ, g) is an almost contact metric structure on M. Similarly, every almost Hermitian structure J_v induces an almost contact structure $(\phi_v, \eta_v, \xi_v, g)$ on M, given by

$$\xi_v = -J_vN, \quad \eta_v(X) = g(\xi_v, X), \quad \phi_vX = J_vX - \eta_v(X)N,$$

for any vector field X. Thus, relations (2.1) and (2.2) hold for $(\phi_v, \eta_v, \xi_v, g)$.

Denote by ∇, A the induced Riemannian connection and the shape operator on M, respectively. Then the Gauss and Weingarten formulas read

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX, \tag{2.3}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on $G_2(\mathbb{C}^{m+2})$ corresponding to $\widetilde{g}$. Also, we have

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX. \tag{2.4}$$

Moreover, the following equations are proved in [BS99]:

$$\phi_{v+1}\xi_v = -\xi_{v+2}, \qquad \phi_v\xi_{v+1} = \xi_{v+2}, \tag{2.5}$$

$$\phi\xi_v = \phi_v\xi, \qquad \eta(\xi_v) = \eta_v(\xi), \tag{2.6}$$

$$\phi\phi_v X = \phi_v\phi X + \eta_v(X)\xi - \eta(X)\xi_v, \tag{2.7}$$

$$\nabla_X\xi_v = q_{v+2}(X)\xi_{v+1} - q_{v+1}(X)\xi_{v+2} + \phi_v AX, \tag{2.8}$$

$$\begin{aligned}\nabla_X(\phi_v\xi) = q_{v+2}(X)\phi_{v+1}\xi - q_{v+1}(X)\phi_{v+2}\xi \\ + \phi_v\phi AX - g(AX,\xi)\xi_v + \eta(\xi_v)AX.\end{aligned} \tag{2.9}$$

The curvature tensor R and the Codazzi equation of M are respectively given as follows:

$$\begin{aligned} &R(X,Y)Z \\ &= g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + 2g(X,\phi Y)\phi Z \\ &\quad + \sum_{v=1}^{3}\{g(\phi_v Y,Z)\phi_v X - g(\phi_v X,Z)\phi_v Y - 2g(\phi_v X,Y)\phi_v Z\} \\ &\quad + \sum_{v=1}^{3}\{g(\phi_v\phi Y,Z)\phi_v\phi X - g(\phi_v\phi X,Z)\phi_v\phi Y\} \\ &\quad - \sum_{v=1}^{3}\{\eta(Y)\eta_v(Z)\phi_v\phi X - \eta(X)\eta_v(Z)\phi_v\phi Y\} \\ &\quad - \sum_{v=1}^{3}\{\eta(X)g(\phi_v\phi Y,Z) - \eta(Y)g(\phi_v\phi X,Z)\}\xi_v \\ &\quad + g(AY,Z)AX - g(AX,Z)AY,\end{aligned} \tag{2.10}$$

$$\begin{aligned} &(\nabla_X A)Y - (\nabla_Y A)X \\ &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ &\quad + \sum_{v=1}^{3}\{\eta_v(X)\phi_v Y - \eta_v(Y)\phi_v X - 2g(\phi_v X,Y)\xi_v\} \\ &\quad + \sum_{v=1}^{3}\{\eta_v(\phi X)\phi_v\phi Y - \eta_v(\phi Y)\phi_v\phi X\} \\ &\quad + \sum_{v=1}^{3}\{\eta(X)\eta_v(\phi Y) - \eta(Y)\eta_v(\phi X)\}\xi_v\end{aligned} \tag{2.11}$$

for any vector fields X, Y, Z on M.

In view of (2.10), a straightforward computation gives, for the Ricci tensor,

$$QX = (4m+7)X - 3\eta(X)\xi - 3\sum_{v=1}^{3}\eta_v(X)\xi_v + \sum_{v=1}^{3}\{\eta_v(\xi)\phi_v\phi X - \eta_v(\phi X)\phi\xi_v - \eta(X)\eta_v(\xi)\xi_v\} + hAX - A^2X. \tag{2.12}$$

Moreover, for a Hopf hypersurface, that is, when $A\xi = \alpha\xi$, where α is a smooth function, the following equations are satisfied.

PROPOSITION 2.1 ([BS02]). *If M is a Hopf hypersurface, then for all vector fields X on M we have*

$$\begin{aligned} A\phi AX = {} & \tfrac{1}{2}\alpha(A\phi X + \phi AX) + \phi X \\ & - 2\sum_{v=1}^{3}\{\eta(\xi_v)\eta(\phi_v X)\xi + \eta(\xi_v)\eta(X)\phi_v\xi\} \\ & + \sum_{v=1}^{3}\{\eta_v(X)\phi\xi_v + \eta(\phi_v X)\xi_v + \eta(\xi_v)\phi_v X\}, \end{aligned} \tag{2.13}$$

$$X(\alpha) = \xi(\alpha)\eta(X) - 4\sum_{v=1}^{3}\eta_v(\xi)\eta_v(\phi X), \tag{2.14}$$

$$(A\phi)^2X = (\phi A)^2X. \tag{2.15}$$

For the real hypersurfaces of type (B) and (A), Berndt and Suh provided descriptions of their principal curvatures and the corresponding eigenspaces.

PROPOSITION 2.2 ([BS99]). *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}\,r), \qquad \beta = \sqrt{2}\cot(\sqrt{2}\,r), \qquad \lambda = -\sqrt{2}\tan(\sqrt{2}\,r), \qquad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \qquad m(\beta) = 2, \qquad m(\lambda) = 2m-2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N, \\ T_\lambda &= \{X : X \perp \mathbb{H}\xi, JX = J_1X\}, \\ T_\mu &= \{X : X \perp \mathbb{H}\xi, JX = -J_1X\}. \end{aligned}$$

PROPOSITION 2.3 ([BS99]). *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \delta = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\delta) = 4n-4 = m(\mu),$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathfrak{J}J\xi, \quad T_\gamma = \mathfrak{J}\xi, \quad T_\delta, \quad T_\mu,$$

where

$$T_\delta \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\delta = T_\delta, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\delta = T_\mu.$$

3. Einstein–Weyl structures. In this section we recall some concepts related to Einstein–Weyl structures. Suppose that (M, c) is a conformal manifold with a conformal class c. A *Weyl connection* D in (M, c) is a torsion-free linear connection which preserves the conformal class $c = [g]$. For any metric g in c there exists a 1-form θ, called thee *Lee form* with respect to g, such that $Dg = -2\theta \otimes g$. It is related to the Levi-Civita connection ∇ by the relation

$$D_X Y = \nabla_X Y + \theta(X)Y + \theta(Y)X - g(X,Y)B \tag{3.1}$$

for any vector fields X, Y, where B is dual to θ with respect to g. A Weyl structure on M is defined by the pair $W = (D, [g])$. Note that the relation $Dg = -2\theta \otimes g$ and equation (3.1) are invariant under the transformations $g' = e^{2f}g$, $\theta' = \theta - df$, where f is a smooth function on M. Therefore we always denote the Weyl structure by $W = (g, \theta)$, and it is said to be *closed*, respectively *exact*, if its Lee form is closed, respectively exact, with respect to any metric in c. A Weyl structure $W = (g, \theta)$ is called *Einstein–Weyl* if the trace-free component of the symmetric part of Ric^D is identically zero, namely, there exists a smooth function Λ such that relation (1.1) holds.

By (3.1), a straightforward computation implies the curvature tensor and the Ricci tensor of the Weyl connection D are expressed by

$$R^D(X,Y)Z = R(X,Y)Z + \Sigma_g(X,Y)Z - \Sigma_g(Y,X)Z, \tag{3.2}$$

$$\begin{aligned} Ric^D(Y,Z) = Ric(Y,Z) &- 2n(\nabla_Z\theta)(Y) + (\nabla_Y\theta)(Z) \\ &+ (2n-1)\theta(Z)\theta(Y) + (\delta\theta - (2n-1)|\theta|^2)g(Y,Z), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \Sigma_g(X,Y)Z = (\nabla_X\theta)(Y)Z &+ (\nabla_X\theta)(Z)Y - g(Y,Z)\nabla_X B \\ &- g(Y,Z)|\theta|^2 X - g(X,Z)\theta(Y)B + \theta(Y)\theta(Z)X \end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$ and $\delta\theta$ denotes the codifferential of θ with respect to g.

Moreover, the following characterization of closed Weyl connections was proved in [M11].

PROPOSITION 3.1 ([M11]). *Let* (M, ξ, η, ϕ, g) *be a* $(2n+1)$*-dimensional almost contact manifold. Then the Weyl structure* $W = (g, \theta)$ *is closed if and only if* $\eta(R^D(X,Y)\xi) = 0$ *for all* $X, Y \in \mathfrak{X}(M)$*, where the Weyl connection* D *is induced from the almost contact metric structure* (ξ, η, ϕ, g)*.*

It is well known that for an almost contact manifold (M, ξ, η, ϕ, g) its tangent bundle TM can be decomposed as $TM = \mathbb{R}\xi \oplus \mathcal{D}$, where $\mathcal{D} = \{X \in TM : \eta(X) = 0\}$. Applying Proposition 3.1, we immediately obtain the following result.

PROPOSITION 3.2. *Let* (M, ξ, η, ϕ, g) *be a* $(2n+1)$*-dimensional almost contact manifold. If the Weyl structure* $W = (g, \theta)$ *is closed, then either* $B \in \mathbb{R}\xi$ *or* $B \in \mathcal{D}$*.*

Proof. By Proposition 3.1, we deduce from (3.2) that for all vector fields X, Y,

$$\begin{aligned}(\nabla_X\theta)(\xi)\eta(Y) - \eta(Y)\eta(\nabla_X B) - \eta(X)\theta(Y)\eta(B)\\ - [(\nabla_Y\theta)(\xi)\eta(X) - \eta(X)\eta(\nabla_Y B) - \eta(Y)\theta(X)\eta(B)] = 0.\end{aligned}$$

Here we have used the relation $(\nabla_X\theta)Y = (\nabla_Y\theta)X$, which follows as $d\theta = 0$. Since

$$(\nabla_X\theta)(\xi) = \nabla_X(g(B,\xi)) - \theta(\nabla_X\xi) = g(\nabla_X B, \xi) = \eta(\nabla_X B),$$

the above relation is simplified as

$$[-\eta(X)\theta(Y) + \eta(Y)\theta(X)]\eta(B) = 0.$$

Thus by taking $Y = \xi$, we see that $\theta(X)\eta(B) = 0$ for all $X \in \mathcal{D}$, meaning that either $B \in \mathcal{D}$ or $B \in \mathbb{R}\xi$. ∎

In the following we assume that M is a real hypersurface of $G_2(\mathbb{C}^{m+2})$ admitting a Einstein–Weyl structure $W = (g, \theta)$ such that the induced metric g lies within the conformal structure $[g]$. Thus it follows from (3.3) and (1.1) that

$$Ric(X,Y) - \frac{4m-3}{2}\big((\nabla_X\theta)Y + (\nabla_Y\theta)X\big) + (4m-3)\theta(X)\theta(Y) = \sigma g(X,Y), \tag{3.4}$$

where $\sigma = \delta\theta - (4m-3)|\theta|^2 + \Lambda/2$. Furthermore, if M admits two Einstein–Weyl structures with $W^{\pm} = (g, \pm\theta)$, then the following two equations hold for arbitrary vector fields X, Y on M (Higa [H93]):

$$(\nabla_X\theta)Y + (\nabla_Y\theta)X + \frac{2}{4m-1}\delta\theta g(X,Y) = 0, \tag{3.5}$$

$$\begin{aligned} Ric(X,Y) - \frac{r}{4m-1}g(X,Y) \\ = \frac{4m-3}{4m-1}|\theta|^2 g(X,Y) - (4m-3)\theta(X)\theta(Y). \end{aligned} \tag{3.6}$$

Here r denotes the scalar curvature of M.

For a closed Einstein–Weyl structure, using (3.4) we further obtain

$$(4m-3)\nabla_X B = QX + (4m-3)\theta(X)B - \sigma X. \tag{3.7}$$

This easily yields

$$\begin{aligned} (4m-3)R(X,Y)B = (\nabla_X Q)Y - (\nabla_Y Q)X + \theta(Y)QX \\ - \theta(X)QY - \sigma[\theta(Y)X - \theta(X)Y] \\ - (X\sigma)Y + (Y\sigma)X. \end{aligned} \tag{3.8}$$

4. Proofs of theorems. First we consider the hypersurface M admitting a closed Einstein–Weyl structure. We need the following lemma.

LEMMA 4.1. *Let M be a real hypersurface of $G_2(\mathbb{C}^{m+1})$ admitting a closed Einstein–Weyl structure $W = (g,\theta)$ with $\nabla_\xi\theta = 0$. Then $B \in \mathcal{D}$. Further, if M is Hopf then ξ belongs to either $\mathfrak{D}$ or $\mathfrak{D}^\perp$.*

Proof. First, by Proposition 3.2, we know that either $B \in \mathbb{R}\xi$ or $B \in \mathcal{D}$. If $B \in \mathbb{R}\xi$ we can set $\theta = f\eta$ for $f \in C^\infty(M)$. Then $0 = \nabla_\xi B = \xi(f)\xi + f\phi A\xi$, that is, $f\phi A\xi = -\xi(f)\xi$. Since $d\theta = 0$, i.e. $g(\nabla_X B, Y) = g(\nabla_Y B, X)$ for all $X, Y \in \mathfrak{X}(M)$, from the second relation of (2.4) we have

$$X(f)\eta(Y) + fg(\phi AX, Y) = Y(f)\eta(X) + fg(\phi AY, X). \tag{4.1}$$

Clearly, taking $X = \xi$ gives $Y(f) = 0$, i.e. f is constant. Thus $\phi A\xi = 0$, that is, M is Hopf and we can write $A\xi = \alpha\xi$ where $\alpha = \eta(A\xi)$. Further, we derive from (4.1) that $\phi AX + A\phi X = 0$ for all $X \in \mathfrak{X}(M)$. Then $h = \alpha$ (see [JLS08, Lemma 3.1]). Since $\nabla_\xi\theta = 0$, from (3.7) and (2.12) we have

$$0 = (4m+4)\xi - 4\sum_{v=1}^{3}\eta_v(\xi)\xi_v + [(4m-3)f^2 - \sigma]\xi.$$

For any vector field X, taking an inner product of the previous equation with $\phi^2 X$ gives

$$\sum_{v=1}^{3}\eta_v^2(\xi)\eta(X) = \sum_{v=1}^{3}\eta_v(\xi)\eta_v(X). \tag{4.2}$$

It follows that $\eta(X)\sum_{v=1}^{3}\eta_v^2(\xi) = 0$ for any $X \in \mathfrak{D}$. That means that either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.

For $\xi \in \mathfrak{D}$, M is of type (B) by Theorem 1.1. But real hypersurfaces of type (B) cannot satisfy the anti-commuting formula $\phi A + A\phi = 0$ by Proposition 2.3. For $\xi \in \mathfrak{D}^\perp$, from the proof of [JLS08, Proposition 3.4], we know that it is impossible, hence in this case $B \in \mathcal{D}$.

If M is Hopf and $B \in \mathcal{D}$, i.e. $A\xi = \alpha\xi$, using (2.12) we derive from (3.7) with $X = \xi$ that

$$(4m + 4 + h\alpha - \alpha^2)\xi - 4\sum_{v=1}^{3} \eta_v(\xi)\xi_v = \sigma\xi.$$

From this, by taking an inner product with $\phi^2 X$ for any vector field X, we also deduce (4.2), which completes the proof. ∎

Next we consider a special Einstein–Weyl structure on a real hypersurface.

Lemma 4.2. *Let M be a real Hopf hypersurface of $G_2(\mathbb{C}^{m+1})$ admitting an Einstein–Weyl structure $W = (g, \theta)$ with $\theta = k\eta$, where k is a non-zero constant. Then ξ belongs to $\mathfrak{D}$ or $\mathfrak{D}^\perp$.*

Proof. From (2.12) and (3.4), we have

$$\begin{aligned}(4.3)\qquad & (4m + 7 - \sigma)X - 3\eta(X)\xi - 3\sum_{v=1}^{3} \eta_v(X)\xi_v \\ & + \sum_{v=1}^{3}\{\eta_v(\xi)\phi_v\phi X - \eta_v(\phi X)\phi\xi_v - \eta(X)\eta_v(\xi)\xi_v\} + hAX - A^2X \\ & - \frac{4m-3}{2}k(\phi AX - A\phi X) + (4m-3)k^2\eta(X)\xi = 0.\end{aligned}$$

Using (2.4), it follows from (4.3) with $X = \xi$ that

$$\big(4m + 4 - \sigma + h\alpha - \alpha^2 + (4m-3)k^2\big)\xi = 4\sum_{v=1}^{3} \eta_v(\xi)\xi_v.$$

Taking the inner product with $\phi^2 Y$ implies $\sum_{v=1}^{3} \eta_v(\xi)\eta_v(\phi^2 Y) = 0$. The rest of the proof is the same as for Lemma 4.1. ∎

Finally, if M admits two Einstein–Weyl structures $W^\pm = (g, \pm\theta)$, we immediately prove the following result.

Lemma 4.3. *Let M be a Hopf real hypersurface of $G_2(\mathbb{C}^{m+1})$ admitting two closed Einstein–Weyl structures $W^\pm = (g, \pm\theta)$. Then ξ belongs to $\mathfrak{D}$ or $\mathfrak{D}^\perp$.*

Proof. By (2.12) and (3.6), we get

$$
\begin{aligned}
(4.4)\qquad & -3\eta(X)\xi - 3\sum_{v=1}^{3}\eta_v(X)\xi_v + hAX - A^2X \\
& + \sum_{v=1}^{3}\{\eta_v(\xi)\phi_v\phi X - \eta_v(\phi X)\phi\xi_v - \eta(X)\eta_v(\xi)\xi_v\} \\
& = \left(\frac{r}{4m-1} + \frac{4m-3}{4m-1}|\theta|^2 - 4m - 7\right)X - (4m-3)\theta(X)B.
\end{aligned}
$$

By Proposition 3.2, $B \in \mathcal{D}$ or $B \in \mathbb{R}\xi$. For $B \in \mathbb{R}\xi$, by the proof of Lemma 4.1 the conclusion is clear, thus we only consider the case where $B \in \mathcal{D}$. Taking the inner product of (4.4) with ξ gives

$$
-4\sum_{v=1}^{3}\eta_v(X)\eta(\xi_v) = \left(\frac{r}{4m-1} + \frac{4m-3}{4m-1}|\theta|^2 - 4m - 4 - h\alpha + \alpha^2\right)\eta(X).
$$

Replacing X by ϕ^2X gives $\sum_{v=1}^{3}\eta_v(\phi^2X)\eta(\xi_v) = 0$, which implies $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$ as in the proof of Lemma 4.2. ∎

In the following we apply the above lemmas to prove our theorems.

Proof of Theorem 1.4. From (3.7) and (2.12), we get

$$
\begin{aligned}
(4.5)\qquad (4m-3)\nabla_X B = {} & (4m+7-\sigma)X - 3\eta(X)\xi - 3\sum_{v=1}^{3}\eta_v(X)\xi_v \\
& + \sum_{v=1}^{3}\{\eta_v(\xi)\phi_v\phi X - \eta_v(\phi X)\phi\xi_v - \eta(X)\eta_v(\xi)\xi_v\} \\
& + hAX - A^2X + (4m-3)\theta(X)B.
\end{aligned}
$$

By Lemma 4.1, we first set $\xi \in \mathfrak{D}$. Then M is a real hypersurface of type (B) by Theorem 1.2. Thus (4.5) becomes

$$
\begin{aligned}
(4m-3)\nabla_X B = {} & (4m+7-\sigma)X - 3\eta(X)\xi - 3\sum_{v=1}^{3}\eta_v(X)\xi_v - \sum_{v=1}^{3}\eta_v(\phi X)\phi\xi_v \\
& + hAX - A^2X + (4m-3)\theta(X)B.
\end{aligned}
$$

Since $\nabla_\xi\theta = 0$, the above relation with $X = \xi$ yields $\sigma = 4m + 4 + h\alpha - \alpha^2$.

In the following we assume $\xi \in \mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$. In this case there exists an Hermitian structure $J_1 \in \mathfrak{J}$ such that $J_1N = JN$, that is, $\xi = \xi_1$. From (2.5) we have

$$
(4.6)\qquad \phi\xi_2 = \phi_2\xi_1 = -\xi_3, \qquad \phi_1\xi_2 = \xi_3, \qquad \phi\xi_3 = \phi_3\xi_1 = \xi_2.
$$

By (4.6), the formula (4.5) becomes

$$(4m-3)\nabla_X B = (4m+7-\sigma)X - 7\eta(X)\xi - 2\eta_2(X)\xi_2 - 2\eta_3(X)\xi_3 + hAX - A^2X + (4m-3)\theta(X)B.$$

By $\nabla_\xi\theta = 0$, we get $\sigma = 4m + h\alpha - \alpha^2$. Further, since θ is closed and $B \in \mathcal{D}$, for all X we have $0 = g(\nabla_\xi B, X) = g(\nabla_X B, \xi) = -g(B, \phi AX)$, i.e. $A\phi B = 0$. Now differentiating this along ξ implies $(\nabla_\xi A)\phi B = 0$.

Using (2.10), we compute

$$\begin{aligned}(\nabla_\xi A)\phi B - (\nabla_{\phi B}A)\xi &= \phi^2 B + \phi_1\phi B - 2\eta_2(\phi B)\xi_3 + 2\eta_3(\phi B)\xi_2\\ &= -B + \phi_1\phi B - 2\eta_3(B)\xi_3 - 2\eta_2(B)\xi_2.\end{aligned}$$

Because $(\nabla_{\phi B}A)\xi = \phi B(\alpha)\xi + \alpha\phi A\phi B - A\phi A\phi B$, we obtain

$$-B + \phi_1\phi B - 2\eta_3(B)\xi_3 - 2\eta_2(B)\xi_2 = -\phi B(\alpha)\xi.$$

By taking the inner product with ξ_2 and ξ_3, we find respectively $\eta_2(B) = \eta_3(B) = 0$ and $\phi B + \phi_1 B = 0$. This shows that B belongs to $\mathfrak{D}_1$, where $\mathfrak{D}_1 = \{X \in \mathfrak{D} : \phi_1 X = -\phi X\}$.

For the curvature tensor (2.10) with $X = \xi$ and $Z = B$, we obtain

$$R(\xi,Y)B = g(Y,B)\xi + \sum_{v=1}^{3}\{g(\phi_v Y, B)\phi_v\xi - 2g(\phi_v\xi, Y)\phi_v B\} - \sum_{v=1}^{3} g(\phi_v\phi Y, B)\xi_v + \alpha g(AY,B)\xi. \tag{4.7}$$

On the other hand, putting $X = \xi$ in (3.8) gives

$$(4m-3)R(\xi,Y)B = (\nabla_\xi Q)Y - (\nabla_Y Q)\xi + \theta(Y)Q\xi - \sigma\theta(Y)\xi - (\xi\sigma)Y + (Y\sigma)\xi. \tag{4.8}$$

Next we compute the right side of (4.8). Since the Ricci tensor (2.12) is simplified as

$$QY = (4m+7)Y - 7\eta(Y)\xi - 2[\eta_2(Y)\xi_2 + \eta_3(Y)\xi_3] + \phi_1\phi Y + hAY - A^2Y,$$

we have

$$Q\xi = (4m + h\alpha - \alpha^2)\xi, \quad \text{i.e.} \quad Q\xi = \sigma\xi.$$

Hence (4.8) becomes

$$(4m-3)R(\xi,Y)B = (\nabla_\xi Q)Y - (\nabla_Y Q)\xi - (\xi\sigma)Y + (Y\sigma)\xi \tag{4.9}$$

and

$$\begin{aligned}(\nabla_Y Q)\xi &= (-7 + h\alpha - \alpha^2)\phi AY + Y(\sigma)\xi\\ &\quad - \{-2[\eta_3(AY)\xi_2 - \eta_2(AY)\xi_3] - \phi_1 AY + hA\phi AY - A^2\phi AY\}.\end{aligned}$$

On the other hand, making use of (2.7)–(2.9) and the Codazzi equation (2.11), we compute (see also [S13, Eq. (5.1)])

$$\begin{aligned}
(\nabla_\xi Q)Y &= (\xi h)AY + h(\nabla_\xi A)Y - (\nabla_\xi A^2)Y \\
&= (\xi h)AY + h[(Y\alpha)\xi + \alpha\phi AY - A\phi AY + \phi Y + \phi_1 Y - 2\eta_2(Y)\xi_3 + 2\eta_3(Y)\xi_2] \\
&\quad - (Y\alpha)A\xi - \alpha A\phi AY + A^2\phi AY - A\phi Y - A\phi_1 Y + 2\eta_2(Y)A\xi_3 - 2\eta_3(Y)A\xi_2 \\
&\quad - (AY\alpha)\xi - \alpha\phi A^2 Y + A\phi A^2 Y - \phi AY - \phi_1 AY + 2\eta_2(AY)\xi_3 - 2\eta_3(AY)\xi_2.
\end{aligned}$$

Thus

$$\begin{aligned}
(\nabla_\xi Q)Y - (\nabla_Y Q)\xi \\
= (\xi h)AY + h[(Y\alpha)\xi + \phi Y + \phi_1 Y - 2\eta_2(Y)\xi_3 + 2\eta_3(Y)\xi_2] \\
- (Y\alpha^2)\xi - \alpha A\phi AY - A\phi Y - A\phi_1 Y + 2\eta_2(Y)A\xi_3 - 2\eta_3(Y)A\xi_2 \\
- \alpha\phi A^2 Y + A\phi A^2 Y - 2\phi_1 AY + 4\eta_2(AY)\xi_3 - 4\eta_3(AY)\xi_2 \\
+ (6+\alpha^2)\phi AY - Y(\sigma)\xi.
\end{aligned}$$

Here we have used the relation $Y(\alpha) = \xi(\alpha)\eta(Y)$, which follows from (2.14).

Substituting the above formula into (4.9) and combining with (4.7), we obtain

$$\begin{aligned}
(4m-3)[g(Y,B)\xi + \alpha g(AY,B)\xi + \{g(\phi_3 Y,B) - g(\phi_2\phi Y,B)\}\xi_2 \\
- \{g(\phi_2 Y,B) + g(\phi_3\phi Y,B)\}\xi_3 + 2g(\xi_3,Y)\phi_2 B - 2g(\xi_2,Y)\phi_3 B] \\
= (\xi h)AY + h[(Y\alpha)\xi + \phi Y + \phi_1 Y - 2\eta_2(Y)\xi_3 + 2\eta_3(Y)\xi_2] \\
- (Y\alpha^2)\xi - \alpha A\phi AY - A\phi Y - A\phi_1 Y + 2\eta_2(Y)A\xi_3 - 2\eta_3(Y)A\xi_2 \\
- \alpha\phi A^2 Y + A\phi A^2 Y - 2\phi_1 AY + 4\eta_2(AY)\xi_3 - 4\eta_3(AY)\xi_2 \\
+ (6+\alpha^2)\phi AY - (\xi\sigma)Y.
\end{aligned} \tag{4.10}$$

Since $B \in \mathfrak{D}_1$, in terms of (2.13) and (2.15), we have

$$\tfrac{1}{2}\alpha\phi AB + \phi B + \phi_1 B = 0, \quad \text{i.e.} \quad \alpha AB = 0.$$

If $\alpha \neq 0$, we have $AB = 0$ and we infer from (4.10) with $Y = B$ that $(4m-3)g(B,B) = 0$, i.e. $B = 0$. Here we have used the fact that $B(\alpha) = 0$ follows from (2.14). If $\alpha = 0$, taking $Y = B$ in (4.10) and an inner product with ξ also yields $B = 0$. This is impossible.

This completes the proof of Theorem 1.4. ∎

Proof of Theorem 1.5. By Lemma 4.2, we know that either $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$ under the assumption of Theorem 1.5. Next we consider these two cases separately.

We first assume $\xi \in \mathfrak{D}$. Then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$ by Theorem 1.2, and equation (4.3) becomes

$$(4.11)\qquad (4m+7-\sigma)X-3\eta(X)\xi-3\sum_{v=1}^{3}\eta_v(X)\xi_v-\sum_{v=1}^{3}\eta_v(\phi X)\phi\xi_v$$
$$+hAX-A^2X-\frac{4m-3}{2}k\Big(\phi AX-A\phi X\Big)+(4m-3)k^2\eta(X)\xi=0,$$

where $\sigma=\delta\theta-(4m-3)|\theta|^2+\Lambda/2=k\ \mathrm{trace}(\phi A)-(4m-3)k^2+\Lambda/2$. According to Proposition 2.3, putting $X=\xi_1\in T_\beta$ in (4.11), we have

$$(4m+4-\sigma+h\beta-\beta^2)\xi_1-\frac{4m-3}{2}k\beta\phi\xi_1=0.$$

This means that $\beta=0$. Therefore M cannot be a real hypersurface of type (B).

In the following we set $\xi=\xi_1\in\mathfrak{D}^\perp$ as in the proof of Theorem 1.4. Equation (4.3) is simplified as

$$(4.12)\qquad (4m+7-\sigma)X-7\eta(X)\xi-2\eta_2(X)\xi_2-2\eta_3(X)\xi_3+\phi_1\phi X$$
$$+hAX-A^2X-\frac{4m-3}{2}k(\phi AX-A\phi X)+(4m-3)k^2\eta(X)\xi=0.$$

When $X=\xi$, we get

$$\sigma=4m+h\alpha-\alpha^2+(4m-3)k^2.$$

Using the Codazzi equation (2.11), we compute

$$\begin{aligned}R(X,Y)B&=\nabla_X\nabla_YB-\nabla_Y\nabla_XB-\nabla_{[X,Y]}B\\&=k\big((\nabla_X\phi)AY-(\nabla_Y\phi)AX\big)+k\phi\big((\nabla_XA)Y-(\nabla_YA)X\big)\\&=k\big(\eta(AY)AX-\eta(Y)A^2X\big)+k\phi\Big[\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\\&\quad+\sum_{v=1}^{3}\{\eta_v(X)\phi_vY-\eta_v(Y)\phi_vX-2g(\phi_vX,Y)\xi_v\}\\&\quad+\sum_{v=1}^{3}\{\eta_v(\phi X)\phi_v\phi Y-\eta_v(\phi Y)\phi_v\phi X\}\\&\quad+\sum_{v=1}^{3}\{\eta(X)\eta_v(\phi Y)-\eta(Y)\eta_v(\phi X)\}\xi_v\Big].\end{aligned}$$

Putting $X=\xi$ and using (4.6) gives

$$\begin{aligned}(4.13)\qquad R(\xi,Y)B&=k\big(\eta(AY)A\xi-\eta(Y)A^2\xi\big)\\&\quad+k\phi\Big[\phi Y+\sum_{v=1}^{3}\{\eta_v(\xi)\phi_vY-\eta_v(Y)\phi_v\xi-3g(\phi_v\xi,Y)\xi_v\}\Big]\\&=k\phi[\phi Y+\phi_1Y+2\eta_3(Y)\xi_2-2\eta_2(Y)\xi_3].\end{aligned}$$

On the other hand, the curvature tensor (2.10) with $X = \xi$ and $Z = B$ is expressed as

$$
\begin{aligned}
(4.14)\qquad R(\xi, Y)B &= k\eta(Y)\xi - kY - 3k\sum_{v=1}^{3} g(\phi_v Y, \xi)\phi_v\xi \\
&\quad + k\sum_{v=1}^{3}\eta_v(\xi)\phi_v\phi Y - k\sum_{v=1}^{3} g(\phi_v\phi Y, \xi)\xi_v + k\alpha^2\eta(Y)\xi - k\alpha AY \\
&= k\eta(Y)\xi - kY - 2k[\eta_2(Y)\xi_2 + \eta_3(Y)\xi_3] \\
&\quad + k\phi_1\phi Y + k\alpha^2\eta(Y)\xi - k\alpha AY.
\end{aligned}
$$

Comparing (4.13) and (4.14), we thus obtain

$$\phi\phi_1 Y = \phi_1\phi Y + \alpha^2\eta(Y)\xi - \alpha AY.$$

Letting $Y = \xi_2$ and $Y = \xi_3$, respectively, we have $\alpha A\xi_2 = \alpha A\xi_3 = 0$. If $\alpha \neq 0$ then $A\xi_2 = A\xi_3 = 0$. Now in terms of Proposition 2.2 we consider $X = \xi_2 \in T_\beta$ in (4.12). Then

$$(4m + 6 - \sigma)\xi_2 + hA\xi_2 - \frac{4m-3}{2}k(\phi A\xi_2 - A\phi\xi_2) = A^2\xi_2,$$

i.e. $(4m+6-\sigma)\xi_2 + \frac{4m-3}{2}k\beta\phi\xi_2 = 0$, which shows $\beta = 0$. This is impossible, which completes the proof. ∎

Proof of Theorem 1.6. By virtue of Lemma 4.3, we first assume $\xi \in \mathfrak{D}$. Then M is a real hypersurface of type (B) as before.

In the following we consider $\xi = \xi_1 \in \mathfrak{D}^\perp$. Formula (4.4) becomes

$$
\begin{aligned}
(4.15)\qquad &-7\eta(X)\xi + hAX - A^2X + \phi_1\phi X - 2\{\eta_2(X)\xi_2 + \eta_3(X)\xi_3\} \\
&= \left(\frac{r}{4m-1} + \frac{4m-3}{4m-1}|\theta|^2 - 4m - 7\right)X - (4m-3)\theta(X)B.
\end{aligned}
$$

Putting $X = \xi$ gives

$$(4.16)\qquad \left(\frac{r}{4m-1} + \frac{4m-3}{4m-1}|\theta|^2 - 4m - h\alpha + \alpha^2\right)\xi = (4m-3)\theta(\xi)B.$$

On the other hand, by (3.5) we have

$$(4.17)\qquad \nabla_X B = -\frac{\delta\theta}{4m-1}X,$$

thus

$$
\begin{aligned}
(4.18)\qquad R(X, Y)B &= \nabla_X\nabla_Y B - \nabla_Y\nabla_X B - \nabla_{[X,Y]}B \\
&= -\frac{X(\delta\theta)}{4m-1}Y + \frac{Y(\delta\theta)}{4m-1}X.
\end{aligned}
$$

Next we need to prove $A\mathfrak{D} \subset \mathfrak{D}$. According to Proposition 3.1, we will divide the argument into two cases.

Case I: $B \in \mathbb{R}\xi$. Set $B = f\xi$ for some $f \in C^\infty(M)$. By (2.10),

$$R(\xi, Y)\xi = \eta(Y)\xi - Y - 3\sum_{v=1}^{3} g(\phi_v\xi, Y)\phi_v\xi + \sum_{v=1}^{3}\eta_v(\xi)\phi_v\phi Y - \sum_{v=1}^{3} g(\phi_v\phi Y, \xi)\xi_v + \alpha^2\eta(Y)\xi - \alpha AY. \tag{4.19}$$

Combining (4.18) with (4.19) gives

$$-\xi(\delta\theta)Y + Y(\delta\theta)\xi = (4m-1)f\Big[\eta(Y)\xi - Y - 3\sum_{v=1}^{3} g(\phi_v\xi, Y)\phi_v\xi + \phi_1\phi Y - \sum_{v=1}^{3} g(\phi_v\phi Y, \xi)\xi_v + \alpha^2\eta(Y)\xi - \alpha AY\Big].$$

Taking the inner product with ξ_2 and ξ_3 respectively, we get $\eta_2(AY) = \eta_3(AY) = 0$ when $Y \in \mathfrak{D}$. This shows $A\mathfrak{D} \subset \mathfrak{D}$.

Case II: $B \in \mathcal{D}$. Then from (4.16) it yields

$$\frac{r}{4m-1} + \frac{4m-3}{4m-1}|\theta|^2 - 4m - h\alpha + \alpha^2 = 0.$$

Therefore (4.15) becomes

$$-7\eta(X)\xi + hAX - A^2X + \phi_1\phi X - 2\{\eta_2(X)\xi_2 + \eta_3(X)\xi_3\} = (h\alpha - \alpha^2 - 7)X - (4m-3)\theta(X)B. \tag{4.20}$$

On the other hand, we know $g(\nabla_X B, \xi) = -g(B, \nabla_X\xi) = -g(B, \phi AX)$, which implies by (4.17) that $\delta\theta = 0$. That means B is parallel by (4.17), thus $R(X,Y)B = 0$ for all vector fields X, Y on M. From the curvature tensor (2.10) with $X = \xi$ and $Z = B$, we obtain

$$0 = g(Y,B)\xi + \sum_{v=1}^{3}\{g(\phi_v Y, B)\phi_v\xi - g(\phi_v\xi, B)\phi_v Y - 2g(\phi_v\xi, Y)\phi_v B\} + \sum_{v=1}^{3}\eta_v(B)\phi_v\phi Y - \sum_{v=1}^{3} g(\phi_v\phi Y, B)\xi_v + g(AY, B)A\xi.$$

Let us put $Y = \xi_2$ in the above formula and use (4.6). Then we obtain $2\phi_3 B = \alpha g(A\xi_2, B)\xi$, which yields $\eta_2(B) = 0$ by taking the inner product with ξ_2. In the same way, taking $Y = \xi_3$ in the above formula gives $\eta_3(B) = 0$. That is, B belongs to $\mathfrak{D}$. Thus from (4.20) with $X = \xi_2$ and $X = \xi_3$ we can obtain respectively

$$hA\xi_2 - A^2\xi_2 = (h\alpha - \alpha^2 - 6)\xi_2, \quad hA\xi_3 - A^2\xi_3 = (h\alpha - \alpha^2 - 6)\xi_3.$$

As in the proof of [S06, Theorem 3], we also have $A\mathfrak{D} \subset \mathfrak{D}$.

Summing up these two cases shows that M is a real hypersurface of type (A) by Theorem 1.1. This completes the proof of Theorem 1.6. ∎

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