

COSET SPACES OF METRIZABLE GROUPS

BY

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Abstract. We characterize topological spaces which are coset spaces of (separable) metrizable groups and complete metrizable (Polish) groups. Moreover, it is shown that for a G -space X with a d -open action there is a topological group H of weight and cardinality less than or equal to the weight of X such that H admits a d -open action on X . This is further applied to show that if X is a separable metrizable coset space of some topological group, then X has a Polish extension which is a coset space of a Polish group.

1. Introduction. In the study of topological homogeneity it is natural to ask about topological groups (or their classes) which realize the homogeneity in question. This approach may give additional information about the phase space, its degree of homogeneity and type of the group action on it. The survey of A. V. Arhangel'skii and J. van Mill [8] can serve as a good introduction to the subject.

Throughout the paper the term *coset space* is used for a topological space X which is homeomorphic to the coset space G/H of some topological group G with respect to some subgroup $H \subset G$.

GENERAL QUESTION. Let X be a coset space and assume that X belongs to a class \mathcal{P} of topological spaces. Can X be a coset space of a topological group G from a given class \mathcal{P}' ? We can also ask about additional properties of the natural action $G \curvearrowright X = G/H$.

If a coset space X is locally compact, then X is a coset space of the group of homeomorphisms $G = \text{Hom}(X)$ in the g -topology [7, Theorems 3] which is a Raikov complete topological group [7, Theorem 6] and $w(G) \leq w(X)$. The inequality follows from the following facts: (1) $\text{Hom}(X)$ is a subgroup of $\text{Hom}(\alpha X)$, where αX is the Alexandroff compactification of X ; (2) the g -topology of $\text{Hom}(X)$ is induced by the compact-open topology on $\text{Hom}(\alpha X)$; (3) for the base of the compact-open topology on $\text{Hom}(\alpha X)$, whose weight

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is equal to $w(\alpha X) = w(X)$, one can take open sets as the sets from a big base of αX , and compact sets as their closures. Recall (see [2, p. 495]) that a *big base* of a space Z is a system \mathcal{B} of open subsets of Z such that for every closed subset $F \subset Z$ and an open neighborhood O of F , there exists $V \in \mathcal{B}$ such that $F \subset V \subset O$.

From Effros's theorem [12] and the above fact, G. Ungar [33] deduced that a homogeneous separable metrizable locally compact space is a coset space of a Polish group. F. Ancel [4, Question 3] asked whether every homogeneous Polish space is a coset space, preferably, of some Polish group. J. van Mill [29] gave an example of a homogeneous Polish space which need not be a coset space at all. However the following question still remains open.

QUESTION. Is a separable metrizable (respectively, Polish) coset space X a coset space of some separable metrizable (respectively, Polish) group?

This question has a positive answer in the case of strongly locally homogeneous spaces. Recall that a space X is called *strongly locally homogeneous* (abbreviated SLH) if it has an open base \mathbf{B} such that for every $B \in \mathbf{B}$ and any $x, y \in B$, there is a homeomorphism $f : X \rightarrow X$ which is supported on B (that is, f is the identity outside of B) and moves x to y [14]. Any homogeneous SLH space is a coset space [14]. J. van Mill made this result more precise by showing that a separable metrizable (respectively, Polish) SLH space is a coset space of a separable metrizable [28] (respectively, Polish [29]) group.

Recall that a topological space X is *metrically homogeneous* if there exists a metric ρ on X such that the natural action $\text{Iso}(X, \rho) \curvearrowright X$ of the group of isometries $\text{Iso}(X, \rho)$ is transitive. From [23] it follows that if a coset space G/H with respect to a compact subgroup $H \subset G$ is metrizable, then it is metrically homogeneous. On the other hand, the group $\text{Iso}(X)$ of a metric compactum X is a closed and equicontinuous subgroup of $\text{Hom}(X)$ in the compact-open topology. Hence, $\text{Iso}(X)$ in the compact-open topology (which coincides with the topology of pointwise convergence [11, Theorem 1, Ch. X, §2, 4]) is compact due to Ascoli's theorem. Therefore, a metrically homogeneous compactum is a coset space of a compact metrizable group. Thus, we get the following theorem of N. Okromeshko.

THEOREM 1.1 ([30]). *A metrizable compactum X is a coset space of a compact metrizable group iff it is metrically homogeneous.*

This result can be generalized as follows.

Recall that a metric ρ on a space X is called *proper* if every ball has compact closure. An action of a G -space (G, X, α) is called *proper* [10, Ch. IV] if the map $G \times X \rightarrow X \times X$, $(g, x) \mapsto (gx, x)$, is perfect. It is a well known fact that the natural action $G \curvearrowright G/H$ of a locally compact group on its

coset space is proper iff H is a compact subgroup of G . Then, from [1, Theorem 1.1] (necessity) and [15, Corollary 5.6] (sufficiency) we get the following generalization of Theorem 1.1.

THEOREM 1.2. *A locally compact separable metrizable space X is homeomorphic to a coset space G/H of a locally compact separable metrizable group G with respect to a compact subgroup $H < G$ iff X is metrically homogeneous with respect to some proper metric.*

REMARK 1.3. In [3, Proposition 2.7] it is shown, in fact, that the existence of a proper G -invariant metric on a coset space X of a locally compact separable metrizable group G is equivalent to the condition that X is metrically homogeneous and the stabilizers of the natural action $\text{Iso}(X) \curvearrowright X$ are compact.

A homeomorphism $g \in \text{Hom}(X)$ of a metric space (X, ρ) is a *Lipschitz homeomorphism* if there is $\lambda \geq 1$ such that

$$\lambda^{-1}\rho(x, y) \leq \rho(g(x), g(y)) \leq \lambda\rho(x, y) \quad \text{for all } x, y \in X.$$

Let $L(X, \rho)$ denote the group of all Lipschitz homeomorphisms.

A metrizable space X is called *Lipschitz homogeneous* if there exists a metric ρ on X such that the natural action of $L(X, \rho) \curvearrowright X$ is transitive. A. Hohti and H. Junnila [17] showed that every homogeneous locally compact separable metrizable space X is Lipschitz homogeneous. In the compact case, Hohti [16, Theorem 3.1] strengthened this result by showing that a homogeneous compact metrizable space X admits a metric ρ such that X is a coset space of $L(X, \rho)$ in the compact-open topology.

These results show that the approach of characterizing groups which realize homogeneity of metrizable spaces by using metrics and properties of homeomorphisms related to them is not new.

In Theorem 4.8 we characterize spaces which are coset spaces of (separable) metrizable groups. We use (totally bounded) metrics on coset spaces and the topology of uniform convergence on the groups of *uniform equivalences* with respect to these metrics.

In Theorem 4.10 and Corollary 4.12 we characterize complete metrizable spaces which are coset spaces of complete metrizable groups (in particular, of the same weight). Our approach is based on the possibility of extension of an action $G \times X \rightarrow X$ to an action $\rho G \times \beta_G X \rightarrow \beta_G X$, where ρG is the Raikov completion of the acting group G , and $\beta_G X$ is the maximal G -compactification. Emphasizing the invariant subset of points where the action is d -open, we use [19, Theorem 3] which implies openness of a d -open action of a Čech complete group (and hence of a complete metrizable group). Remark 4.15 shows how this approach works in the case of Polish SLH spaces.

In Theorem 5.6 (and Corollary 5.18) we show that for any d -open action $G \curvearrowright X$, there is a d -open action $H \curvearrowright X$ of a group H of cardinality and weight less than or equal to the weight of X . This may provide a possible approach to the questions we formulated. For example, Theorem 5.10 shows that if X is a coset space of a complete metrizable group then it is a coset space of a complete metrizable group G with $w(G) = w(X)$. This technique also allows us to construct equivariant compactifications preserving the weight and the d -openness of actions (Theorem 5.13). Moreover, we show in Corollary 5.21 that if X is a separable metrizable coset space, then X has a Polish extension \tilde{X} which is a coset space of a Polish group. Moreover, if we fix any countable set S of homeomorphisms of X then \tilde{X} can be constructed in such a way that each homeomorphism from S extends to \tilde{X} .

2. Preliminaries. All spaces, unless otherwise stipulated, are assumed to be Tychonoff, maps are continuous, notations and terminology are from [13]. A topological space (a set X with a topology τ) is denoted by (X, τ) if we want to emphasize which topology is considered. Uniformities are introduced by families of covers. For information about uniform spaces see [18], [31].

Nbd(s) is an abbreviation of open neighborhood(s), unless otherwise stated. $\text{cl}_\tau A$ or simply $\text{cl} A$, and $\text{int}_\tau A$ or simply $\text{int} A$ are the closure and the interior of a subset A of a space (X, τ) or X , respectively. For a family u of subsets of X and $Y \subset X$ we denote $\text{St}(Y, u) = \bigcup \{U \in u \mid Y \cap U \neq \emptyset\}$ the star of Y with respect to u . For covers u and v of X we denote $u \succ v$, $u * \succ v$ if u refines, respectively star-refines v . A metrizable space is called *Polish* if it is separable and has a complete metric. We denote by id various identity maps.

For information about topological groups, see [9] and [31]. We denote by $N_G(e)$ the family of nbds of the unity $e \in G$; ρG is the *Raikov completion* (or completion in the two-sided uniformity) of a topological group. $\text{Hom}(X)$ is the group of all homeomorphisms of X .

For a general information about group actions, see [34], [6], [24] and [20]. We denote by (G, X, α) an action α of a group G on a topological space X such that the map $\alpha^g : X \rightarrow X$, $\alpha^g(x) = \alpha(g, x)$, is continuous (is a homeomorphism) for every $g \in G$. Here the action does not depend on the topology of the acting group. If G is a topological group and the action $\alpha : G \times X \rightarrow X$ is continuous then (G, X, α) is called a *G-space* or a *topological transformation group*. The set $\{g \in G \mid \forall x \in X (gx = x)\}$ is called the *kernel of the action*. If the kernel of an action is $\{e\}$, the trivial subgroup, then the action is called *effective*.

By a *Polish G-space* (respectively, *separable metrizable G-space*) we mean a G -space (G, X, α) where X and G are Polish (respectively, separable metrizable) spaces.

If $X = G/H$ is a coset space of a topological group G , then (G, X, α) is a G -space endowed with the natural action $\alpha(g, xH) = (gx)H$, where $g, x \in G$.

A pair of maps $(\varphi : G \rightarrow H, f : X \rightarrow Y)$ of (G, X, α_G) to (H, Y, α_H) such that $\varphi : G \rightarrow H$ is a homomorphism and the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\varphi \times f} & H \times Y \\ \alpha_G \downarrow & & \downarrow \alpha_H \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, is called *equivariant*. Thus, the equivariance is defined not only for maps of G -spaces with the same acting group; it generalizes the traditional concept of equivariantness. We shall use the notation $(\varphi, f) : (G, X, \alpha_G) \rightarrow (H, Y, \alpha_H)$ in this case.

The commutativity of the above diagram may be written as the fulfillment of the condition

$$f(gx) = \varphi(g)f(x) \quad \text{for any } x \in X, g \in G.$$

The composition of equivariant pairs of maps is an equivariant pair. If f is an embedding then the pair (φ, f) is called an *equivariant embedding* of (G, X, α_G) into (H, Y, α_H) .

A G -space (G, X, α) is called *G -Tychonoff* if there is an equivariant embedding (id, f) of (G, X, α) into a G -space $(G, bX, \tilde{\alpha})$ where bX is a compactification of X . The *maximal G -compactification* is denoted $(G, \beta_G X, \alpha_\beta)$.

Let (G, X, α) be a G -space and let \mathbb{R}^X be the set of all real-valued functions on X . We can define an action of G on \mathbb{R}^X by $(gf)(x) = f(g^{-1}x)$, where $g \in G, f \in \mathbb{R}^X$ and $x \in X$.

A continuous function $f : X \rightarrow \mathbb{R}$ on a G -space (G, X, α) is called *G -uniform* if for any $\varepsilon > 0$ there is $O \in N_G(e)$ such that $|f(x) - f(gx)| < \varepsilon$ for any $x \in X, g \in O$. G -uniform maps were introduced by J. de Vries [35] under the name of α -uniform maps, where $\alpha : G \times X \rightarrow X$ is an action. The set $C_G^*(X)$ of all bounded G -uniform functions on X is an invariant subset of \mathbb{R}^X . Moreover, (G, X, α) is G -Tychonoff iff $C_G^*(X)$ separates points and closed subsets of X [6].

For an action $\alpha : G \times X \rightarrow X$ on a uniform space (X, \mathcal{U}) the following conditions are equivalent:

- (1) for any $u \in \mathcal{U}$, there exists $O \in N_G(e)$ such that the cover $\{Ox \mid x \in X\} = \{\alpha(O, x) \mid x \in X\}$ refines u ,
- (2) for any $u \in \mathcal{U}$, there are $O \in N_G(e)$ and $v \in \mathcal{U}$ such that $Ov = \{\alpha(O, V) \mid V \in v\}$ refines u .

An action satisfying either condition (1) or (2) is called *bounded* [34, Ch. III, §7.3]. The action is bounded with respect to some uniformity on X iff

(G, X, α) is G -Tychonoff. A uniformity \mathcal{U} on X is called an *equiuniformity* [25] if it is *saturated* (i.e., that all homeomorphisms from G are uniformly continuous) and the action is bounded with respect to \mathcal{U} . An action $\alpha : G \times X \rightarrow X$ on a uniform space (X, \mathcal{U}) equipped with an equiuniformity \mathcal{U} has a continuous extension $\tilde{\alpha}_\rho : \rho G \times \tilde{X} \rightarrow \tilde{X}$, where ρG is the Raikov completion of G and \tilde{X} is the completion of X with respect to \mathcal{U} [25]. Evidently, the restriction $\tilde{\alpha} : G \times \tilde{X} \rightarrow \tilde{X}$ of $\tilde{\alpha}_\rho$ is also continuous. Among equiuniformities there is a maximal one.

An action $\alpha : G \times X \rightarrow X$ is called:

- *open* if $x \in \text{int}(Ox)$ for any point $x \in X$ and $O \in N_G(e)$;
- *d-open* if $x \in \text{int}(\text{cl}(Ox))$ for any point $x \in X$ and $O \in N_G(e)$.

If (G, X, α) is a G -space with a d -open action, then X is a discrete sum of clopen subsets (*components of the action*). Each component of the action is the closure of the orbit of an arbitrary point of this component [22, Remark 2]. If (G, X, α) is a G -space with an open action and with one component of the action, then X is a coset space of G . Further information about (d -)open actions, their properties and natural uniform structures that are induced by (d -)open actions can be found in [22]–[21].

3. Compactification theorem. The following result is a part of [26, Theorem 2.13]. We shall present a proof for completeness. By $ib(G)$ will be denoted the index of narrowness of a topological group G [9, Chapter 5, §5.2].

LEMMA 3.1. *Let (G, X, α) be a G -space and T an invariant subset of $C_G^*(X)$ that separates points and closed subsets of X . Then the initial uniformity \mathcal{U}_T with respect to T is a totally bounded equiuniformity on X .*

Proof. The uniformity \mathcal{U}_T is compatible with the topology of X since the maps from T are continuous and separate points and closed subsets of X . Further, \mathcal{U}_T is totally bounded since it is the initial uniformity with respect to the maps to compacta, and it is saturated since T is an invariant subset of $C_G^*(X)$. Finally, since T consists of G -uniform functions, the action is bounded by \mathcal{U}_T . ■

Let (G, X, α) be a G -space. The set $C_G^*(X)$ of all bounded G -uniform functions $X \rightarrow \mathbb{R}$ endowed with pointwise defined algebraic operations and with the sup-norm $\|f\| = \sup_{x \in X} f(x)$ is a Banach space. It is well known that the action $\alpha^* : G \times C_G^*(X) \rightarrow C_G^*(X)$ defined by

$$\alpha^*(g, f)(x) = f(\alpha(g^{-1}, x)), \quad f \in C_G^*(X), g \in G, x \in X,$$

is linear, isometric and continuous (see, e.g., [6]).

LEMMA 3.2. *Let (G, X, α) be a G -space, $f \in C_G^*(X)$ and $T_f = Gf$. Then the initial pseudouniformity \mathcal{U}_{T_f} with respect to the maps from T_f has weight $\leq ib(G)$.*

Proof. For $f \in C_G^*(X)$ let $O_n \in N_G(e)$ be such that $|f(x) - f(gx)| < 1/n$ for $x \in X, g \in O_n, n \in \mathbb{N}$. For every $n \in \mathbb{N}$ there exists $G_n \subset G$ such that $|G_n| \leq ib(G)$ and $G_n O_n = G$. Put $G_\infty = \bigcup \{G_n \mid n \in \mathbb{N}\}$. Then the cardinality of $G_\infty f$ is $ib(G)$.

To finish the proof it suffices to show that the initial pseudouniformity with respect to the maps from $G_\infty f$ coincides with \mathcal{U}_{T_f} .

The subbase of \mathcal{U}_{T_f} consists of the covers of the form $(hf)^{-1}v$, where v is a uniform cover of $(hf)(X) \subset \mathbb{R}$ by intervals of diameter $\varepsilon > 0$, and $h \in G$. Let $O_n \in N_G(e), O_n^{-1} = O_n$, be such that

$$|f(x) - f(gx)| < 1/n < \varepsilon/3, \quad x \in X, g \in O_n.$$

Take $g_h \in G_\infty$ such that $h \in g_h O_n$ and let w be a uniform cover w of $(g_h f)(X) \subset \mathbb{R}$ consisting of intervals of diameter $\varepsilon/3$. Then $(g_h f)^{-1}w \succ (hf)^{-1}v$. In fact, for $x, y \in W \in (g_h f)^{-1}w$ we have

$$|(g_h f)(x) - (g_h f)(y)| < \varepsilon/3.$$

Then

$$\begin{aligned} & |(hf)(x) - (hf)(y)| \\ & \leq |(hf)(x) - (g_h f)(x)| + |(g_h f)(x) - (g_h f)(y)| + |(g_h f)(y) - (hf)(y)| \\ & = |f(h^{-1}x) - f(g_h^{-1}x)| + |(g_h f)(x) - (g_h f)(y)| + |f(g_h^{-1}y) - f(h^{-1}y)| \\ & = |f(t_h(g_h^{-1}x)) - f(g_h^{-1}x)| + |(g_h f)(x) - (g_h f)(y)| + |f(g_h^{-1}y) - f(t_h(g_h^{-1}x))| \\ & \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

where $t_h \in O_n$.

Thus, the initial pseudouniformity with respect to the maps from $G_\infty f$ is a base of \mathcal{U}_{T_f} . ■

THEOREM 3.3 ([26]). *For a G -Tychonoff space (G, X, α) , there is an equivariant embedding $(id, f) : (G, X, \alpha) \rightarrow (G, B, \tilde{\alpha})$, where B is compact and $w(B) \leq ib(G) \cdot w(X)$.*

Proof. The case of X finite is evident. Otherwise there exists a subset $T' \subset C_G^*(X)$ that separates points and closed subsets of X such that $|T'| \leq w(X)$. Therefore, the invariant subset $T = GT' = \{gf \mid g \in G, f \in T'\} \subset C_G^*(X)$ separates points and closed subsets of X .

By Lemma 3.1, \mathcal{U}_T is a totally bounded equiuniformity on X . Hence, the completion of X with respect to \mathcal{U}_T is a compactum B and $(G, B, \tilde{\alpha})$ is a G -compactification of (G, X, α) (see Section 2).

Since $w(\mathcal{U}_T) \leq |T'| \cdot \sup\{w(\mathcal{U}_{T_f}) \mid f \in T'\}$, by Lemma 3.2 we have $w(\mathcal{U}_T) \leq |T'| \cdot ib(G) \leq ib(G) \cdot w(X)$. Taking into account that the weight of a compactum coincides with the weight of its unique uniformity, we get $w(B) \leq ib(G) \cdot w(X)$. ■

Recall that a group G is ω -*narrow* if $ib(G) \leq \aleph_0$ [9, Chapter 5, §5.2]. The following corollary is an equivariant compactification result for actions of ω -narrow groups.

COROLLARY 3.4 ([5]). *Let G be an ω -narrow group. Then for a G -Tychonoff space (G, X, α) , there is an equivariant embedding $(id, f): (G, X, \alpha) \rightarrow (G, B, \tilde{\alpha})$ where B is compact and $w(B) \leq w(X)$.*

4. Characterizations of coset spaces of metrizable groups

THEOREM 4.1 ([14, Theorem 2.2]). *If X possesses a uniform structure \mathcal{U} under which every element of some homeomorphism group G is uniformly continuous, then G is a topological group relative to the uniform convergence topology induced by the uniformity.*

REMARK 4.2. (a) The sets $O_u = \{g \in G \mid \forall x \in X \ g(x) \in \text{St}(x, u)\}$, $u \in \mathcal{U}$, constitute a base of nbds (not open) of the unit element e .

(b) The uniformity of uniform convergence in Theorem 4.1 coincides with the right uniformity on G [31, Chapter 2, Exercise 2].

(c) We can add the following to Theorem 4.1. The uniformity \mathcal{U} is an equiuniformity with respect to the natural action $G \curvearrowright X$, and hence (G, X, α) is a G -Tychonoff space.

In fact, each element of G is a uniformly continuous map. Therefore, the uniformity \mathcal{U} is saturated. For arbitrary $u \in \mathcal{U}$ take $v \in \mathcal{U}$ such that $v * \succ u$, and a nbd O_v of e in G . Then $O_v v \succ u$. Hence, \mathcal{U} is bounded.

Recall that a bijective map f of uniform spaces is called a *uniform equivalence* if f and f^{-1} are uniformly continuous. We denote by $H(X, \mathcal{U})$ the group of self-uniform equivalences of a uniform space (X, \mathcal{U}) . An evident consequence of Remark 4.2(a) is the following corollary.

COROLLARY 4.3. *If $w(\mathcal{U}) \leq \aleph_0$ for a uniform space (X, \mathcal{U}) , then $H(X, \mathcal{U})$ endowed with the topology of uniform convergence is metrizable.*

It is well known that there is a unique uniformity \mathcal{U} on a compactum X and $\text{Hom}(X) = H(X, \mathcal{U})$.

PROPOSITION 4.4. *If X is a compactum then the topology of uniform convergence on any subgroup G of $\text{Hom}(X)$ coincides with the compact open topology.*

Proof. Let \mathcal{U} be the unique uniformity on X , $\tau_{\mathcal{U}}$ and τ_{CO} be the uniform convergence topology and the compact-open topology on G , respectively.

In order to prove the inclusion $\tau_{\mathcal{U}} \supset \tau_{CO}$ it is enough to show that any nbd of the unity $e \in G$ in the compact open topology contains a nbd of e in the topology of uniform convergence. A typical subbasic nbd of e in the topology τ_{CO} looks like $[K, O] = \{g \in G \mid g(K) \subset O\}$, where O is an open subset of X , $K \subset O$ and K is compact. There is $u \in \mathcal{U}$ such that $\text{St}(K, u) \subset O$. Then $O_u \subset [K, O]$.

For the proof of the converse inclusion, take an arbitrary nbd of e in the topology $\tau_{\mathcal{U}}$ of the form O_u , $u \in \mathcal{U}$. Let $w, v \in \mathcal{U}$ be such that $v *_{\succ} w *_{\succ} u$. Take a finite subcover $\{V_1, \dots, V_n\} \subset v$ of X and let $\text{St}(V_k, v) \subset W_k$, $W_k \in w$, $k = 1, \dots, n$. Then the set $W = \bigcap \{[\text{cl}(V_k), W_k] \mid k = 1, \dots, n\}$ is a nbd of e in the topology τ_{CO} and $W \subset O_u$. ■

COROLLARY 4.5. *If $w(\mathcal{U}) \leq \aleph_0$ for a totally bounded uniformity \mathcal{U} on X then $H(X, \mathcal{U})$ in the topology of uniform convergence is separable and metrizable.*

Proof. The completion bX of X with respect to \mathcal{U} is a metrizable compactum. Since each homeomorphism $g \in H(X, \mathcal{U})$ is uniformly continuous, it admits a unique extension $g' : bX \rightarrow bX$ which is uniformly continuous with respect to the unique uniformity on bX . Clearly, $(g \circ h)' = g' \circ h'$ and $1'_X = 1_{bX}$. In other words, the map $g \mapsto g'$ is a monomorphism of $H(X, \mathcal{U})$ into $\text{Hom}(bX)$. Thus, we can assume that $H(X, \mathcal{U})$ is a subgroup of $\text{Hom}(bX)$. Then by Proposition 4.4, the topology of the uniform convergence on $H(X, \mathcal{U})$ coincides with the compact-open topology. The latter topology is separable and metrizable. The rest follows from the coincidence of \mathcal{U} with the restriction on X of the unique uniformity of bX . ■

THEOREM 4.6. *Let (G, X, α) be a G -space with an open effective action and let \mathcal{U} be the maximal equiuniformity on X . Then*

- (1) *each element of G is a uniform equivalence (with respect to \mathcal{U}),*
- (2) *the topology of uniform convergence $\tau_{\mathcal{U}}$ on G is coarser than the original one,*
- (3) *$((G, \tau_{\mathcal{U}}), X, \alpha)$ is a G -space with an open action and \mathcal{U} is the maximal equiuniformity X .*

Proof. (1) follows immediately from the definition of an equiuniformity.

(2) Let $N_G(e)$ denote the base of nbds of e in the original topology. If the action is open then the covers $u = \{Ux \mid x \in X\}$, $U \in N_G(e)$, constitute a base for the maximal equiuniformity \mathcal{U} on X . Since for a neighborhood $V \in N_G(e)$ with $V = V^{-1}$, $V^3 \subset U$ we have $\{Vx \mid x \in X\} *_{\succ} \{Ux \mid x \in X\}$, the base of nbds (not open) of e in the topology of uniform convergence $\tau_{\mathcal{U}}$ consists of the sets $O_u = \{g \in G \mid \forall x \in X \ g(x) \in Ux\}$, $U \in N_G(e)$. Hence, it suffices to prove that for every set O_u , its interior $\text{int}_{\tau_{\mathcal{U}}} O_u$ is open in the original topology. In turn, it is sufficient to show that $U \subset \text{int}_{\tau_{\mathcal{U}}} O_u$.

Indeed, for arbitrary $g \in U$, there is $V \in N_G(e)$ such that $Vg \in U$. For any $h \in O_v$ we have $h(x) \in Vx$ for all $x \in X$. Thus, $hg(x) \in Vg(x) \subset Ux$ for all $x \in X$. Therefore, $hg \in O_u$ and $O_v g \subset O_u$, showing that $g \in \text{int}_{\tau_{\mathcal{U}}} O_u$. Hence, $U \subset \text{int}_{\tau_{\mathcal{U}}} O_u$, as required.

(3) From Remark 4.2(c) it follows that $((G, \tau_{\mathcal{U}}), X, \alpha)$ is a G -space, and the action is open since the topology $\tau_{\mathcal{U}}$ is coarser than the original one.

From the inclusions $U \subset \text{int}_{\tau_{\mathcal{U}}} O_u \subset O_u$ for $U \in N_G(e)$ and the equality $O_u x = Ux$, $x \in X$, it follows that $(\text{int}_{\tau_{\mathcal{U}}} O_u)x = Ux$, $x \in X$. Hence, the maximal equiuniformity on X for the action of G in the topology $\tau_{\mathcal{U}}$ is the same as the original one. ■

The following lemma immediately follows from the definition of the topology of uniform convergence.

LEMMA 4.7. *Let (G, X, α) be a G -space with equiuniformities $\mathcal{U} \supset \mathcal{V}$ on X . Then for the topologies $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{V}}$ of uniform convergence on G we have $\tau_{\mathcal{U}} \supset \tau_{\mathcal{V}}$.*

THEOREM 4.8. *The following conditions are equivalent for a (separable) metrizable space X :*

- (a) X is a coset space of a [separable] metrizable group G ,
- (b) there is a [totally bounded] metric ρ on X such that X is a coset space of the group of uniform equivalences with respect to ρ in the topology of uniform convergence.

Proof. (a) \Rightarrow (b) Let a [separable] metrizable space X be a coset space of a [separable] metrizable group G . Then (G, X, α) is a G -space with a natural action of G on X by left translations (open and transitive).

If G is separable and metrizable, by Theorem 3.3 there is an equivariant embedding $(\text{id}, f) : (G, X, \alpha) \rightarrow (G, B, \tilde{\alpha})$ where B is a metrizable compactum. There is a unique (up to a metric equivalence) totally bounded metric $\tilde{\rho}$ on B , and all elements of G are uniform equivalences with respect to the corresponding unique totally bounded uniformity $\mathcal{U}_{\tilde{\rho}}$ with a countable weight which is an equiuniformity. Let $\rho = \tilde{\rho}|_{X \times X}$. Then ρ is a totally bounded metric on X with respect to which all elements of G are uniform equivalences. Moreover, the corresponding totally bounded equiuniformity $\mathcal{U}_{\rho} = \mathcal{U}_{\tilde{\rho}}|_X$ is weaker than the maximal equiuniformity on X .

If G is metrizable then the weight of the maximal equiuniformity \mathcal{U} on X is countable. In fact, the covers $\{Ox \mid x \in X\}$, $O \in N_G(e)$, constitute a base of the maximal equiuniformity [22]. Since G is metrizable, we can take a countable local base $\mathcal{B}_G(e)$ at the unity $e \in G$. Then the covers $\{Ox \mid x \in X\}$, $O \in \mathcal{B}_G(e)$, constitute a countable base of the maximal equiuniformity on X . It is well known that every uniformity with a countable base is generated by a metric. Let ρ be a metric on X which generates this uniformity.

In both cases, by Theorem 4.6 and Lemma 4.7, the topology τ_ρ of uniform convergence with respect to ρ on G is weaker than the original topology of G . Hence, by Theorem 4.6, $((G, \tau_\rho), X, \alpha)$ is a G -space with an open action. Evidently, the natural action $\alpha' : H(X, \mathcal{U}) \times X \rightarrow X$ of the whole group of uniform equivalences is open since (G, τ_ρ) is a subgroup of $(H(X, \mathcal{U}), \tau_{\mathcal{U}})$ and $\alpha'|_{G \times X} = \alpha$.

(b) \Rightarrow (a) By Corollary 4.3 [Corollary 4.5], the group G of uniform equivalences with the topology of uniform convergence with respect to the [totally bounded] uniformity with countable weight (which corresponds to the [totally bounded] metric) is [separable] metrizable. Thus, G is [separable] metrizable whenever X is so. ■

LEMMA 4.9. *Let (G, X, α) be a G -space, and H be a dense subgroup of G . Then the set of points D_G where the action α is d -open coincides with the set of points D_H where the action $\alpha|_{H \times X}$ is d -open.*

Proof. Since $N_H(e) = H \cap N_G(e)$, for any element $O \in N_G(e)$ the set $O' = O \cap H \in N_H(e)$ is dense in O . Therefore, for any point $x \in X$ the set $O'x$ is dense in Ox and $\text{cl}(Ox) = \text{cl}(O'x)$. Hence, $x \in \text{int}(\text{cl}(Ox))$ iff $x \in \text{int}(\text{cl}(O'x))$. ■

THEOREM 4.10. *The following conditions are equivalent for a complete metrizable space X :*

- (a) X is a coset space of a complete metrizable group,
- (b) there exist a metrizable group G and a d -open action $G \times X \rightarrow X$ with only one component such that the set of points in $\beta_G X$ where the action $G \times \beta_G X \rightarrow \beta_G X$ is d -open coincides with X .

Proof. (a) \Rightarrow (b) Suppose a complete metrizable space X is a coset space of a complete metrizable group G . Then (G, X, α) is a G -Tychonoff space with the natural action of G on X by left translations which is open and transitive. Hence, the action α is d -open and has only one component.

We claim that in the G -space $(G, \beta_G X, \alpha_\beta)$, the set D of points where the action $G \times \beta_G X \rightarrow \beta_G X$ is d -open coincides with X . In fact, the inclusion $X \subset D$ is due to [19, Proposition 3]. By [19, Theorem 3] the restriction of action of G on D is open, and hence transitive. Thus, $D = X$.

(b) \Rightarrow (a) Let $\alpha_\beta : G \times \beta_G X \rightarrow \beta_G X$ and $\alpha_{\rho\beta} : \rho G \times \beta_G X \rightarrow \beta_G X$ be extensions of the action $\alpha : G \times X \rightarrow X$. The G -space $(\rho G, \beta_G X, \alpha_{\rho\beta})$ and the equivariant embedding $(i, \text{id}) : (G, \beta_G X, \alpha_\beta) \rightarrow (\rho G, \beta_G X, \alpha_{\rho\beta})$, where $i : G \rightarrow \rho G$ is the natural embedding, are well defined by [25]. The group ρG is complete metrizable being the Raikov completion of a metrizable group [31]. By Lemma 4.9, the set of points where the action $\alpha_{\rho\beta}$ is d -open coincides with the set of points where the action α_β is d -open. The latter set coincides with X by [19, Proposition 3].

Therefore, the G -space $(\rho G, X, \alpha_{\rho\beta}|_{\rho G \times X})$ with a d -open action with one component is well defined. Recall that by [19, Theorem 3], a d -open action of a Čech complete group is open. Since every complete metrizable group is Čech complete, we conclude that the action $\alpha_{\rho\beta}|_{\rho G \times X}$ is open. But an open action with one component is transitive. Hence, X is a coset space of ρG . ■

From the proof of Theorem 4.10 we get the following corollary.

COROLLARY 4.11. *Let G be a [separable] metrizable group, and let (G, X, α) be a G -space with a d -open action with only one component. Assume that the set of points in $\beta_G X$ where the action $G \times \beta_G X \rightarrow \beta_G X$ is d -open coincides with X . Then X is a complete metrizable [Polish] space.*

Since every metrizable group of infinite weight τ contains a dense subgroup of cardinality τ , Theorem 4.10 and [19, Lemma 3] imply the following corollary.

COROLLARY 4.12. *The following conditions are equivalent for a complete metrizable space X :*

- (a) X is a coset space of a complete metrizable group of infinite weight τ ,
- (b) there exist a metrizable group G with $|G| \leq \tau$ and a d -open action $G \times X \rightarrow X$ with only one component such that the set of points in $\beta_G X$ where the action $G \times \beta_G X \rightarrow \beta_G X$ is d -open coincides with X .

REMARK 4.13. According to [25], any action $G \times X \rightarrow X$ on a G -Tychonoff space can be extended to an action $\rho G \times \tilde{X} \rightarrow \tilde{X}$, where ρG is the Raikov completion of the acting group G and \tilde{X} is the completion of the phase space X with respect to any equiuniformity. Therefore, in Theorem 4.10 and Corollary 4.12 we can use the completion of a phase space X with respect to any equiuniformity (and not only the maximal totally bounded equiuniformity which corresponds just to $\beta_G X$).

QUESTION 4.14. *Can the group G , in Theorem 4.10, be taken as a subgroup of $H(X, \mathcal{U})$ with the topology of uniform convergence for some uniformity \mathcal{U} with a countable base?*

REMARK 4.15. An alternative proof that a Polish SLH space X is a coset space of a Polish group in [19, Theorem 5] is the following.

(a) By [19, Lemma 7, Corollary 2] there exist a metrizable compactification bX of X and a countable subgroup T of $\text{Hom}(bX)$ in the compact-open topology such that X is an invariant subset for the action of T on bX , and this action is d -open at every point of X .

(b) By [19, Lemmas 9, 10] we can strengthen the topology on T in such a way that it remains metrizable and the set of points where the action is d -open becomes equal to X .

(c) The required group is the Raikov completion of T in this topology.

5. Replacing the action but preserving d -openness

LEMMA 5.1. *Let (G, X, α) be a G -space with a d -open action, and Y be a dense subset of X . Then for every $O \in N_G(e)$,*

$$\lambda_O = \{\text{int}(\text{cl}(Oy)) \mid y \in Y\}$$

is a cover of X .

Proof. Fix $O \in N_G(e)$. Take $V \in N_G(e)$ such that $V^3 \subset O$ and $V^{-1} = V$. It is enough to show that $\gamma_V = \{\text{int}(\text{cl}(Vx)) \mid x \in X\}$ refines λ_O .

For the set $\text{int}(\text{cl}(Vx)) \in \gamma_V$, there exists a point $y \in Y$ such that $y \in \text{int}(\text{cl}(Vx))$ (since Y is dense in X). Lemma 3 of [22] establishes the inclusion $\text{St}(y, \gamma_V) \subset \text{int}(\text{cl}(Oy))$. Thus, $\text{int}(\text{cl}(Vx)) \subset \text{int}(\text{cl}(Oy))$, and hence $\gamma_V \succ \lambda_O$, as required. ■

PROPOSITION 5.2. *Let (G, X, α) be a G -space with a d -open action. Then there exist a family $\mathcal{O} \subset N_G(e)$ and a dense subset Y of X such that*

- (1) $|\mathcal{O}| \leq w(X)$, $|Y| \leq w(X)$,
- (2) $\gamma_{\mathcal{O}} = \{\text{int}(\text{cl}(Oy)) \mid y \in Y\}$, $O \in \mathcal{O}$, is a cover of X ,
- (3) $\{\text{int}(\text{cl}(Oy)) \mid y \in Y, O \in \mathcal{O}\}$ is a base of the topology of X ,
- (4) $\{\text{int}(\text{cl}(Oy)) \mid O \in \mathcal{O}\}$ is a local base at the point y for every $y \in Y$.

Proof. Since the action is continuous, the family $\{\text{int}(\text{cl}(Ox)) \mid x \in X, O \in N_G(e)\}$ is a base for X . It has a subfamily \mathcal{B} of cardinality $w(X)$ which is a base of X . Each element $B \in \mathcal{B}$ is of the form $\text{int}(\text{cl}(O_Bx_B))$, where $x_B \in X$, $O_B \in N_G(e)$.

Put $Y = \{x_B \mid B \in \mathcal{B}\}$. Evidently, Y is a dense subset of X and $|Y| \leq w(X)$. For every $y \in Y$, let $\mathcal{O}_y \subset N_G(e)$ be such that $|\mathcal{O}_y| \leq w(X)$ and $\{\text{int}(\text{cl}(Oy)) \mid O \in \mathcal{O}_y\}$ is a local base at the point y .

Put

$$\mathcal{O} = \{O_B \mid B \in \mathcal{B}\} \cup \bigcup \{\mathcal{O}_y \mid y \in Y\}.$$

Evidently, $|\mathcal{O}| \leq w(X)$. Conditions (3) and (4) follow from the definitions of Y and \mathcal{O} , and condition (2) is a consequence of Lemma 5.1. ■

REMARK 5.3. In Proposition 5.2 one can assume that the family \mathcal{O} is a filter basis of the same cardinality such that

- (a) all the sets $O \in \mathcal{O}$ are symmetric, i.e., $O = O^{-1}$,
- (b) for every $O \in \mathcal{O}$, there is $V \in \mathcal{O}$ such that $V^2 \subset O$.

Therefore, $H = \bigcap \{O \mid O \in \mathcal{O}\}$ is a subgroup of G , and since $Hy = y$ for any $y \in Y$, and Y is dense in X , it follows that H belongs to the kernel of the action. Moreover, $\psi(H, G) \leq w(X)$, where $\psi(H, G)$ is a pseudocharacter of H in G .

The following conditions are sufficient for an action to be d -open.

PROPOSITION 5.4. *Let (G, X, α) be a G -space and Y be a dense subset of X such that*

- (1) *the action is d -open at all points of Y ,*
- (2) *for any $O \in N_G(e)$ the family $\{\text{int}(\text{cl}(Oy)) \mid y \in Y\}$ is a cover of X ,*
- (3) *for any $y \in Y$ and $O, U, V \in N_G(e)$ with $U^{-1} = U$, $U^2 \subset O$, one has*

$$\text{cl}(Uy) \subset O \text{int}(\text{cl}(Vy)) = \bigcup \{g \text{int}(\text{cl}(Vy)) \mid g \in O\}.$$

Then the action on X is d -open.

Proof. For arbitrary $O \in N_G(e)$ with $O^{-1} = O$, take $U \in N_G(e)$ such that $U^{-1} = U$ and $U^4 \subset O$. By condition (2), for any $x \in X$ there is $y \in Y$ such that $x \in \text{int}(\text{cl}(Uy))$. For any $V \in N_G(e)$, from condition (3) we have $x \in \text{int}(\text{cl}(Uy)) \subset U^2 \text{int}(\text{cl}(Vy))$. Hence, there is $g \in U^2$ such that $gx \in \text{int}(\text{cl}(Vy))$ and $Ox \cap \text{int}(\text{cl}(Uy))$ is dense in $\text{int}(\text{cl}(Uy))$. Therefore, $x \in \text{int}(\text{cl}(Uy)) \subset \text{cl}(Ox)$, showing that the action is d -open at x . ■

LEMMA 5.5. *Let G be a group of cardinality τ , and $\mathcal{O} \subset N_G(e)$ be a family of cardinality τ . Then there exists a filter in $N_G(e)$ containing \mathcal{O} , with a basis \mathcal{B} of cardinality τ , satisfying the conditions (a) and (b) of Remark 5.3 and the following condition:*

- (c) *for any $O \in \mathcal{B}$ and $g \in G$ there is $V \in \mathcal{B}$ such that $gVg^{-1} \subset O$.*

Proof. Starting with $\mathcal{O}_1 = \mathcal{O}$ we can assume that the sets $O \in \mathcal{O}_1$ are symmetric (take the sets $O \cap O^{-1}$ instead of O). Then for any $O \in \mathcal{O}$ take $O_n = O_n^{-1}$ ($O_1 = O$), $O_{n+1}^2 \subset O_n$, $n \in \mathbb{N}$. And at last, let \mathcal{O}'_1 be the family of finite intersections of sets $\{O_n \mid O \in \mathcal{O}_1, n \in \mathbb{N}\}$.

The resulting family \mathcal{O}'_1 is a filter basis of cardinality τ , satisfying the conditions (a) and (b) of Remark 5.3. Put

$$\mathcal{O}_2 = \{gOg^{-1} \mid O \in \mathcal{O}'_1, g \in G\}.$$

Obviously $|\mathcal{O}_2| = \tau$.

Applying the same procedure to \mathcal{O}_2 we obtain the family \mathcal{O}_3 of cardinality τ , and so on. The family $\bigcup \{\mathcal{O}_n \mid n \in \mathbb{N}\}$ is the required filter basis $\mathcal{B} \subset N_G(e)$ of cardinality τ . ■

THEOREM 5.6. *Let (G, X, α) be a G -space with a d -open action. Then there exist a group H and a continuous d -open action $\gamma : H \times X \rightarrow X$ such that $|H| \leq w(X)$ and $w(H) \leq w(X)$.*

Proof. Let the family $\mathcal{O} \subset N_G(e)$ and the dense subset Y of X be taken as in Proposition 5.2. We may also assume that the family \mathcal{O} is a filter basis satisfying the conditions (a) and (b) of Remark 5.3. The following procedure will allow us to construct a group of cardinality $w(X)$ and a filter basis of elements of $N_G(e)$ of cardinality $w(X)$ which satisfy the conditions (a), (b) of Remark 5.3 and the condition (c) of Lemma 5.5.

Since α is a d -open action, for $y \in Y$ and $O \in \mathcal{O}$, we can choose a set $G(O, y) \subset O$ of cardinality $w(X)$ such that $G(O, y)y$ is dense in $\text{int}(\text{cl}(Oy))$. By [22, Lemma 4], for any pair $O, V \in \mathcal{O}$ and $x \in X$, we have

$$\text{cl}(Ox) \subset O^2 \text{int}(\text{cl}(Vx)).$$

There is a subset $H(O, V, x) \subset O^2$ such that $|H(O, V, x)| \leq w(X)$ and $\text{cl}(Ox) \subset H(O, V, x) \text{int}(\text{cl}(Vx))$.

Let H_1 be the subgroup of G generated by the subset

$$\bigcup \{H(O, V, y) \mid O, V \in \mathcal{O}, y \in Y\} \cup \bigcup \{G(O, y) \mid O \in \mathcal{O}, y \in Y\}.$$

Evidently, $|H_1| \leq w(X)$.

By Lemma 5.5, for the group H_1 and the family $\mathcal{O} \subset N_G(e)$, there exists a filter in $N_G(e)$ containing \mathcal{O} , with a basis \mathcal{B}_1 of cardinality $w(X)$, satisfying the conditions (a), (b) of Remark 5.3 and the condition (c) of Lemma 5.5.

Applying the same procedure to the family $\mathcal{B}_1 \subset N_G(e)$ we get a subgroup H_2 with $|H_2| \leq w(X)$, and a filter in $N_G(e)$ containing \mathcal{O} , with a basis \mathcal{B}_2 of cardinality $w(X)$, satisfying the conditions (a), (b) of Remark 5.3 and the condition (c) of Lemma 5.5, and so on. The subgroup $H' = \bigcup \{H_n \mid n \in \mathbb{N}\}$ is of cardinality $w(X)$ and the family $\mathcal{B} = \bigcup \{\mathcal{B}_n \mid n \in \mathbb{N}\}$ is the filter basis of cardinality τ satisfying the conditions (a), (b) of Remark 5.3 and the condition (c) of Lemma 5.5.

Let H be the quotient group H'/T , where $T = (\bigcap \{O \mid O \in \mathcal{B}\}) \cap H'$ is a normal subgroup of H' and let $q : H' \rightarrow H'/T$ be the quotient homomorphism. Following [31, Proposition 1.21], we shall consider the topology on H for which the local basis at the unity (not necessarily of open sets) is $\{q(O \cap H') \mid O \in \mathcal{B}\}$. Evidently $|H| \leq w(X)$ and $w(H) \leq w(X)$.

Since H' is a subgroup of G , and T lies in the kernel of the action α , the action $\gamma : H \times X \rightarrow X$ given by $\gamma(h, x) = \alpha(h', x)$, $h' \in q^{-1}(h)$, is well defined. It is continuous due to the choice of \mathcal{B} ($\mathcal{O} \subset \mathcal{B}$) and Proposition 5.2. In order to prove the d -openness of γ we shall check the fulfillment of conditions in Proposition 5.4.

The fulfillment of (1). Take $q(O \cap H')$ with $\text{int}(q(O \cap H')) \in N_H(e)$, $O \in \mathcal{B}$, and let $y \in Y$. Since the set $G(O, y)y$ is dense in $\text{int}(\text{cl}(Oy))$ and $G(O, y) \subset H'$, we have $y \in \text{int}(\text{cl}(Oy)) \subset \text{int}(\text{cl}(q(O \cap H')y))$, as required.

The fulfillment of (2) is due to Lemma 5.1 and the conditions $G(O, y) \subset O$ and $\text{cl}(G(O, y)y) \supset \text{int}(\text{cl}(Oy))$ for $y \in Y$, $O \in \mathcal{B}$.

The fulfillment of (3) follows from $\text{cl}(Ox) \subset H(O, V, x) \text{int}(\text{cl}(Vx))$ and $H(O, V, y) \subset H'$. ■

QUESTION 5.7. *Do the maximal equiuniformities on X with respect of the actions α and γ coincide?*

REMARK 5.8. If in the proof of Theorem 5.6 we suppose that the family \mathcal{O} contains as an element the whole group G , then the components of the actions α and γ coincide.

REMARK 5.9. If in Theorem 5.6, $\chi(G) \leq w(X)$, then in the first step of the proof we can take \mathcal{O} , a local base at e , with $|\mathcal{O}| \leq w(X)$. Then we obtain a subgroup $H = H_1$ of G with $w(H) \leq w(X)$, $|H| \leq w(X)$ which acts d -openly on X .

In particular, if the group G is metrizable and the phase space X is separable metrizable then we obtain a metrizable countable subgroup H of G which acts d -openly on X .

THEOREM 5.10. *Let X be a coset space of a complete metrizable group G . Then X is a coset space of a complete metrizable subgroup H of G with $w(H) \leq w(X)$.*

Proof. By Remark 5.9, there is a metrizable subgroup $H' \subset G$ with $|H'| \leq w(X)$ which acts d -openly on X . Its Raikov completion $H = \rho H'$ is the closure of H' in G and is a complete metrizable group. Thus, the action $H \times X \rightarrow X$ is well defined and d -open. Moreover, from [19, Theorem 3] it follows that it is open, and hence X is a coset space of the complete metrizable group H with $w(H) \leq w(X)$. ■

COROLLARY 5.11. *Suppose a Polish space X is a coset space of a complete metrizable group G . Then X is a coset space of a Polish subgroup H of G .*

QUESTION 5.12. *Let X be a coset space of a metrizable group G . Can it be a coset space of a metrizable group H with $w(H) \leq w(X)$?*

Since for a topological group G , $\text{ib}(G) \leq w(G)$, we get from Theorems 3.3 and 5.6 the following result.

THEOREM 5.13. *For a G -space (G, X, α) with a d -open action, there exist a group H and an equivariant compactification $(H, bX, \tilde{\gamma})$ of (H, X, γ) such that*

- (1) $w(bX) = w(X)$,
- (2) $|H| \leq w(X)$ and $w(H) \leq w(X)$,
- (3) γ is a d -open action,
- (4) components of actions α and γ coincide.

REMARK 5.14. The group H in Theorem 5.13 can be considered as a subgroup of $\text{Hom}(bX)$ with the compact-open topology.

In [7] it is proved that the compact-open topology is the least admissible topology on $\text{Hom}(bX)$. In [32] it is noted that the compact open topology is the least admissible topology on any subgroup H of $\text{Hom}(bX)$. In fact, let $[K, W] = \{h \in H \mid h(K) \subset W\}$ be an element of a subbase of the compact-open topology, where $K \subset bX$ is compact and $W \subset bX$ is open.

Fix $h \in [K, W]$. For any $x \in K$ there are a nbd Ox of x and a nbd $V_x h$ of h (in the original topology) such that $\gamma(V_x h, Ox) \subset W$. Take a finite set of nbds Ox_1, \dots, Ox_n such that $K \subset O = Ox_1 \cup \dots \cup Ox_n$, and denote $V := V_{x_1} \cap \dots \cap V_{x_n}$. Then $\gamma(Vh, O) \subset W$. This shows that the set $[K, W]$ is open in H (in the original topology).

Since the restriction $\gamma|_{H \times X} : H \times X \rightarrow X$ is d -open, it will evidently be d -open when H is equipped with a weaker topology.

REMARK 5.15. From the proof of Theorem 5.6 it is easy to see that any subgroup (not topological) of G of cardinality $w(X)$ can be a subgroup of the group H in Theorems 5.6 and 5.13.

QUESTION 5.16. *Can Theorem 5.13 be strengthened in such a way that the action $\gamma : H \times bX \rightarrow bX$ is d -open?*

If the answer to this question is positive, then the following question is also interesting.

QUESTION 5.17. *Can Theorem 5.13 be strengthened in such a way that the action $\gamma : H \times X \rightarrow X$ is d -open and its extension to the completion of X with respect to the maximal equiuniformity is d -open too?*

The following corollary gives an answer to Question 2.6 from [21].

COROLLARY 5.18. *Let (G, X, α) be a G -space with a d -open action, where X is a separable metrizable space. Then there exist a countable metrizable group H , a metrizable compactification bX of X and a continuous action $\gamma : H \times bX \rightarrow bX$ such that $\gamma|_{H \times X} : H \times X \rightarrow X$ is a d -open action. Moreover, if we fix an arbitrary countable set S of homeomorphisms from G , then H can be chosen in such a way that $S \subset H$.*

COROLLARY 5.19. *Let (G, X, α) be a G -space with a d -open action, where X is a separable metrizable space. Then there exist a Polish G -space (T, Y, δ) with an open action and a countable subgroup H' of G such that $(H', X, \alpha|_{H' \times X})$ is equivariantly embedded in (T, Y, δ) and the image of H' is dense in T .*

Proof. From the proof of Theorem 5.6 it follows that there exists a countable subgroup H' of G such that $(q, \text{id}) : (H', X, \alpha|_{H' \times X}) \rightarrow (H, X, \gamma)$ is an equivariant embedding in a separable metrizable topological transformation group (H, X, γ) with a d -open action. From Corollary 3.4 it follows that (H, X, γ) has an equivariant compactification $(H, bX, \tilde{\gamma})$, where bX is metrizable. Let $Y \subset bX$ be the set of points at which the action $\tilde{\gamma}$ is d -open. It is an invariant subset of bX and $X \subset Y$ by [19, Proposition 3].

By [25], $(H, bX, \tilde{\gamma}_\rho)$ is equivariantly embedded in $(\rho H, bX, \tilde{\gamma})$, and by Lemma 4.9, the action of ρH on bX is d -open at the same invariant set of

points Y . Put $T = \rho H$. Then T is Polish, and as proved in [4], the action of T on Y is open. This implies that Y is Polish. ■

REMARK 5.20. Alternatively, Corollaries 5.18 and 5.19 can be proved using M. Megrelishvili's Theorem 2.5 from [27]. Accordingly, one can equivariantly embed a separable metrizable G -space (G, X, α) in $(I^\omega, \text{Hom}(I^\omega), \pi)$ and then take the closures of G in $\text{Hom}(I^\omega)$ and of X in I^ω , respectively.

COROLLARY 5.21. *Every separable metrizable space X which is a coset space of some group has a Polish extension \tilde{X} which is a coset space of a Polish group. Moreover, if we fix any countable set S of homeomorphisms of X then \tilde{X} can be constructed in such a way that each homeomorphism from S extends to \tilde{X} .*

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