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LIE MAPS ON ALTERNATIVE RINGS PRESERVING IDEMPOTENTS

BY

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Abstract. Let $\mathfrak{R}$ and $\mathfrak{R}'$ be unital 2,3-torsion free alternative rings and $\varphi : \mathfrak{R} \to \mathfrak{R}'$ be a surjective Lie multiplicative map that preserves idempotents. Assume that $\mathfrak{R}$ has a nontrivial idempotent. Under certain assumptions on $\mathfrak{R}$, we prove that φ is of the form $\psi + \tau$, where ψ is either an isomorphism or the negative of an anti-isomorphism of $\mathfrak{R}$ onto $\mathfrak{R}'$ and τ is an additive mapping of $\mathfrak{R}$ into the centre of $\mathfrak{R}'$ which maps commutators to zero.

1. Alternative rings and Lie multiplicative maps. Let $\mathfrak{R}$ be a unital ring, not necessarily associative or commutative, and consider the following convention for its multiplication operation: $xy \cdot z = (xy)z$ and $x \cdot yz = x(yz)$ for $x, y, z \in \mathfrak{R}$, to reduce the number of parentheses. We denote the *associator* of $\mathfrak{R}$ by $(x, y, z) = xy \cdot z - x \cdot yz$ for $x, y, z \in \mathfrak{R}$. And $[x, y] = xy - yx$ is the usual Lie product of x and y, with $x, y \in \mathfrak{R}$.

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two rings and $\varphi : \mathfrak{R} \to \mathfrak{R}'$ a map. We call φ a *Lie multiplicative map* of $\mathfrak{R}$ into $\mathfrak{R}'$ if for all $x, y \in \mathfrak{R}$,

$$\varphi\big([x, y]\big) = [\varphi(x), \varphi(y)],$$

multiplicative if $\varphi(xy) = \varphi(x)\varphi(y)$, and a *multiplicative Lie derivation* if $\mathfrak{R} = \mathfrak{R}'$ and $\varphi\big([x, y]\big) = [\varphi(x), y] + [x, \varphi(y)]$. When φ is an additive and bijective map we call φ a *Lie isomorphism*. The study of Lie isomorphisms of rings was originally inspired by the work of I. N. Herstein on generalizing classical theorems on the Lie structure of total matrix rings to results on the Lie structure of arbitrary simple rings. In [M63], W. S. Martindale studied Lie isomorphisms between primitive rings $\mathfrak{R}$, $\mathfrak{R}'$, where he assumed that the characteristic of $\mathfrak{R}$ is different from 2 and 3 and that $\mathfrak{R}$ contains three nonzero orthogonal idempotents whose sum was the identity. A few years later [M69], he studied Lie isomorphisms between two simple rings $\mathfrak{R}, \mathfrak{R}'$. In a recent work [FGF20], the authors studied a characterization of multiplicative Lie derivations.

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A ring $\mathfrak{R}$ is said to be *alternative* if

$$(x,x,y)=0 \quad \text{and} \quad (y,x,x)=0 \qquad \text{for all } x,y\in\mathfrak{R}.$$

One easily sees that any associative ring is an alternative ring.

A ring $\mathfrak{R}$ is called *k-torsion free* if $kx=0$ implies $x=0$, for any $x\in\mathfrak{R}$, where $k\in\mathbb{Z}_{k>0}$ and *prime* if $\mathfrak{A}\mathfrak{B}\neq 0$ for any two nonzero ideals $\mathfrak{A},\mathfrak{B}\subseteq\mathfrak{R}$. The *commutative centre* of a ring $\mathfrak{R}$ is defined by

$$\mathcal{Z}(\mathfrak{R})=\{r\in\mathfrak{R}\mid [r,x]=0 \text{ for all } x\in\mathfrak{R}\}.$$

The next result can be found in [FF18].

THEOREM 1.1. *Let $\mathfrak{R}$ be a 3-torsion free alternative ring. The ring $\mathfrak{R}$ is prime if and only if $a\mathfrak{R}\cdot b=0$ (or $a\cdot\mathfrak{R}b=0$) implies $a=0$ or $b=0$ for $a,b\in\mathfrak{R}$.*

DEFINITION 1.2. A ring $\mathfrak{R}$ is said to be *flexible* if

$$xy\cdot x=x\cdot yx \qquad \text{for all } x,y\in\mathfrak{R}.$$

It is known that all alternative rings are flexible.

A nonzero element $e_1\in\mathfrak{R}$ is called an *idempotent* if $e_1e_1=e_1$ and a *nontrivial idempotent* if it is an idempotent different from the multiplicative identity element of $\mathfrak{R}$. Let us consider an alternative ring $\mathfrak{R}$ and fix a nontrivial idempotent $e_1\in\mathfrak{R}$. Let $e_2\colon\mathfrak{R}\to\mathfrak{R}$ and $e'_2\colon\mathfrak{R}\to\mathfrak{R}$ be linear operators given by $e_2(a)=a-e_1a$ and $e'_2(a)=a-ae_1$. Clearly $e_2^2=e_2$, $(e'_2)^2=e'_2$ and we note that if $\mathfrak{R}$ has a unity, then we can consider $e_2=1-e_1\in\mathfrak{R}$. Let us denote $e_2(a)$ by e_2a and $e'_2(a)$ by ae_2. It is easy to see that $e_ia\cdot e_j=e_i\cdot ae_j$ $(i,j=1,2)$ for all $a\in\mathfrak{R}$. Then $\mathfrak{R}$ has a Peirce decomposition $\mathfrak{R}=\mathfrak{R}_{11}\oplus\mathfrak{R}_{12}\oplus\mathfrak{R}_{21}\oplus\mathfrak{R}_{22}$, where $\mathfrak{R}_{ij}=e_i\mathfrak{R}e_j$ $(i,j=1,2)$ [HKS80], satisfying the following multiplicative relations:

(i) $\mathfrak{R}_{ij}\mathfrak{R}_{jl}\subseteq\mathfrak{R}_{il}$ $(i,j,l=1,2)$;
(ii) $\mathfrak{R}_{ij}\mathfrak{R}_{ij}\subseteq\mathfrak{R}_{ji}$ $(i,j=1,2)$;
(iii) $\mathfrak{R}_{ij}\mathfrak{R}_{kl}=0$ if $j\neq k$ and $(i,j)\neq(k,l)$ $(i,j,k,l=1,2)$;
(iv.a) $x_{ij}^2=0$ for all $x_{ij}\in\mathfrak{R}_{ij}$ $(i,j=1,2;\ i\neq j)$;
(iv.b) $x_{ij}y_{ij}=-y_{ij}x_{ij}$ for all $x_{ij},y_{ij}\in\mathfrak{R}_{ij}$ $(i,j=1,2;\ i\neq j)$.

We will say that a map $\varphi:\mathfrak{R}\to\mathfrak{R}'$ *preserves idempotents* if

$$r-\lambda s\in I(\mathfrak{R}) \quad \text{if and only if} \quad \varphi(r)-\lambda\varphi(s)\in I(\mathfrak{R}')$$

where $I(\mathfrak{R})$ and $I(\mathfrak{R}')$ are the sets of all idempotents in $\mathfrak{R}$ and $\mathfrak{R}'$ respectively, and $\lambda\in\mathbb{Q}$. And φ is said to be *almost additive* if $\varphi(a+b)-\varphi(a)-\varphi(b)\in\mathcal{Z}(\mathfrak{R})$ for all $a,b\in\mathfrak{R}$.

The first result about the additivity of mappings on rings was given by Martindale III in [M63], who established a condition on a ring $\mathfrak{R}$ for every multiplicative isomorphism on $\mathfrak{R}$ to be additive. In [LC17, LFLW14], Li and his coauthors also considered the almost additivity of maps for the case of Lie multiplicative mappings. They proved

THEOREM 1.3. *Let $\mathfrak{R}$ be an associative ring containing a nontrivial idempotent e_1 and satisfying the following condition:*

$$(\mathbb{Q}) A_{11}B_{12} = B_{12}A_{22} \quad \textit{for all } B_{12} \in \mathfrak{R}_{12}.$$

Then $A_{11} + A_{22} \in \mathcal{Z}(\mathfrak{R})$. Let $\mathfrak{R}'$ be another ring. Suppose that a bijective map $\Phi\colon \mathfrak{R} \to \mathfrak{R}'$ satisfies

$$\Phi([A, B]) = [\Phi(A), \Phi(B)]$$

for all $A, B \in \mathfrak{R}$. Then $\Phi(A + B) = \Phi(A) + \Phi(B) + Z'_{A,B}$ for all $A, B \in \mathfrak{R}$, where $Z'_{A,B}$ is an element in the commutative centre $\mathcal{Z}(\mathfrak{R}')$ of $\mathfrak{R}'$ depending on A and B.

In [FG20], Ferreira and Guzzo investigated the additivity of Lie multiplicative maps. They obtained the following result.

THEOREM 1.4. *Let $\mathfrak{R}$ and $\mathfrak{R}'$ be alternative rings. Suppose that $\mathfrak{R}$ is a ring containing a nontrivial idempotent e_1 which satisfies*

(i) *If $[a_{11} + a_{22}, \mathfrak{R}_{12}] = 0$, then $a_{11} + a_{22} \in \mathcal{Z}(\mathfrak{R})$.*
(ii) *If $[a_{11} + a_{22}, \mathfrak{R}_{21}] = 0$, then $a_{11} + a_{22} \in \mathcal{Z}(\mathfrak{R})$.*

Then every Lie multiplicative bijection φ of $\mathfrak{R}$ onto an arbitrary alternative ring $\mathfrak{R}'$ is almost additive.

In a recent paper, Ferreira and Guzzo [FG19] studied the characterization of multiplicative Lie derivations on alternative rings. They obtained the following result.

THEOREM 1.5. *Let $\mathfrak{R}$ be a unital 2,3-torsion free alternative ring with nontrivial idempotents e_1, e_2 and with associated Peirce decomposition $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$. Suppose that $\mathfrak{R}$ satisfies the following conditions:*

(1) *If $x_{ij}\mathfrak{R}_{ji} = 0$, then $x_{ij} = 0$ $(i \neq j)$.*
(2) *If $x_{11}\mathfrak{R}_{12} = 0$ or $\mathfrak{R}_{21}x_{11} = 0$, then $x_{11} = 0$.*
(3) *If $\mathfrak{R}_{12}x_{22} = 0$ or $x_{22}\mathfrak{R}_{21} = 0$, then $x_{22} = 0$.*
(4) *If $z \in \mathcal{Z}(\mathfrak{R})$ with $z \neq 0$, then $z\mathfrak{R} = \mathfrak{R}$.*

Let $\mathfrak{D}\colon \mathfrak{R} \to \mathfrak{R}$ be a multiplicative Lie derivation of $\mathfrak{R}$. Then $\mathfrak{D}$ is of the form $\delta + \tau$, where δ is an additive derivation of $\mathfrak{R}$ and τ is a mapping from $\mathfrak{R}$ into the commutative centre $\mathcal{Z}(\mathfrak{R})$, which maps commutators to zero if and only if

(a) *$e_2\mathfrak{D}(\mathfrak{R}_{11})e_2 \subseteq \mathcal{Z}(\mathfrak{R})e_2$,*
(b) *$e_1\mathfrak{D}(\mathfrak{R}_{22})e_1 \subseteq \mathcal{Z}(\mathfrak{R})e_1$.*

Inspired by the above-mentioned results, we plan to give a result about Lie multiplicative maps on alternative rings.

REMARK 1.1. Note that prime alternative rings satisfy (1)–(3) of Theorem 1.5.

PROPOSITION 1.6. *Let $\mathfrak{R}$, $\mathfrak{R}'$ be 2-torsion free alternative rings. If $\varphi : \mathfrak{R} \to \mathfrak{R}'$ is a map that preserves idempotents, then φ is injective and $\varphi(\lambda r) = \lambda\varphi(r)$ for every $\lambda \in \mathbb{Q}$ and $r \in \mathfrak{R}$.*

Proof. First we will prove the injectivity. Suppose $\varphi(r) = \varphi(s)$ for some $r, s \in \mathfrak{R}$. Since $\varphi(r) - \varphi(s)$ is an idempotent in $\mathfrak{R}'$, it follows that $r - s$ is an idempotent in $\mathfrak{R}$. In the same way $\varphi(s) - \varphi(r)$ is an idempotent in $\mathfrak{R}'$ and therefore $s - r$ is also an idempotent in $\mathfrak{R}$. Since $r - s$ and $s - r$ are both idempotents in $\mathfrak{R}$, it follows that $r - s = (r - s)^2 = s - r$, so $r = s$ and φ is injective.

Now we will prove $\varphi(\lambda r) = \lambda\varphi(r)$ for every $\lambda \in \mathbb{Q}$ and $r \in \mathfrak{R}$. Let $r \in \mathfrak{R}$ and let $\lambda \in \mathbb{Q}$ with $\lambda \neq 0, -1$. Then $(\lambda r) - \lambda r \in I(\mathfrak{R})$ and therefore $s = \varphi(\lambda r) - \lambda\varphi(r) \in I(\mathfrak{R}')$. Similarly $r - \left(\frac{1}{\lambda}\right)(\lambda r) \in I(\mathfrak{R})$ and so $\left(\frac{-1}{\lambda}\right)s = \varphi(r) - \left(\frac{1}{\lambda}\right)\varphi(\lambda r) \in I(\mathfrak{R}')$. It follows that $-\frac{1}{\lambda}s = \frac{1}{\lambda^2}s^2 = \frac{1}{\lambda^2}s$ and so $s\left(1 + \frac{1}{\lambda}\right) = 0$. Therefore $\varphi(\lambda r) = \lambda\varphi(r)$ for $\lambda \neq 0, -1$. For $\lambda = -1$ we have $\varphi(-r) = -2\varphi\left(\frac{1}{2}r\right) = -\varphi(r)$. And for $\lambda = 0$ follows from $\varphi(0) = \varphi([0, 0]) = [\varphi(0), \varphi(0)] = 0$. ∎

PROPOSITION 1.7. *Let $\mathfrak{R}$ be a 2, 3-torsion free alternative ring satisfying (1), (2), (3).*

(♠) *If $[a_{11} + a_{22}, \mathfrak{R}_{12}] = 0$, then $a_{11} + a_{22} \in \mathcal{Z}(\mathfrak{R})$,*
(♣) *If $[a_{11} + a_{22}, \mathfrak{R}_{21}] = 0$, then $a_{11} + a_{22} \in \mathcal{Z}(\mathfrak{R})$.*

The reader can find the proof of these results in [FG19].

REMARK 1.2. Let $\mathfrak{R}$ be a 2,3-torsion free alternative ring, let $\mathfrak{R}'$ be another alternative ring and let $\varphi : \mathfrak{R} \to \mathfrak{R}'$ be a surjective Lie multiplicative map that preserves idempotents. Note that $\varphi(e_1) = f_1$ is a nontrivial idempotent in $\mathfrak{R}'$ because φ is a bijective map that preserves idempotents. Therefore $\mathfrak{R}'$ has a Peirce decomposition $\mathfrak{R}' = \mathfrak{R}'_{11} \oplus \mathfrak{R}'_{12} \oplus \mathfrak{R}'_{21} \oplus \mathfrak{R}'_{22}$ associated to the nontrivial idempotent f_1.

PROPOSITION 1.8. *If $\mathfrak{R}$ satisfies the conditions (2) and (3) of Theorem 1.5, then*

$$\mathcal{Z}(\mathfrak{R}) = \big\{z_{11} + z_{22} : z_{11} \in \mathfrak{R}_{11}, z_{22} \in \mathfrak{R}_{22},\\ [z_{11} + z_{22}, \mathfrak{R}_{12}] = [z_{11} + z_{22}, \mathfrak{R}_{21}] = \{0\}\big\}.$$

Proof. On the one hand, assume that $z = z_{11} + z_{12} + z_{21} + z_{22} \in \mathcal{Z}(\mathfrak{R})$. Then $ze_1 = e_1z$ implies $z_{12} = z_{21} = 0$. Furthermore, for any $x_{12} \in \mathfrak{R}_{12}$ and $x_{21} \in \mathfrak{R}_{21}$, it follows from $zx_{12} = x_{12}z$ and $zx_{21} = x_{21}z$ that

$$[z_{11} + z_{22}, \mathfrak{R}_{12}] = [z_{11} + z_{22}, \mathfrak{R}_{21}] = \{0\}.$$

On the other hand, assume that $z_{11} \in \mathfrak{R}_{11}, z_{22} \in \mathfrak{R}_{22}$, and the above equality holds. To prove $z_{11} + z_{22} \in \mathcal{Z}(\mathfrak{R})$, one only needs to check that $z_{ii} \in \mathcal{Z}(\mathfrak{R}_{ii})$, $i = 1, 2$. In fact, for any $r_{11} \in \mathfrak{R}_{11}$ and any $r_{12} \in \mathfrak{R}_{12}$, we have

$$\begin{aligned}(z_{11}r_{11} - r_{11}z_{11})r_{12} &= (z_{11}r_{11})r_{12} - (r_{11}z_{11})r_{12} = z_{11}(r_{11}r_{12}) - r_{11}(z_{11}r_{12}) \\ &= (r_{11}r_{12})z_{22} - r_{11}(r_{12}z_{22}) \\ &= r_{11}(r_{12}z_{22}) - r_{11}(r_{12}z_{22}) = 0,\end{aligned}$$

where we use the flexibility of $\mathfrak{R}$. Hence $(z_{11}r_{11} - r_{11}z_{11})\mathfrak{R}_{12} = 0$. Therefore $z_{11} \in \mathcal{Z}(\mathfrak{R}_{11})$. Similarly, we can check $z_{22} \in \mathcal{Z}(\mathfrak{R}_{22})$. ■

DEFINITION 1.9. We say that a map φ *almost preserves the Peirce spaces order* if $\varphi(\mathfrak{R}_{ii}) \subseteq \mathfrak{R}'_{ii} + \mathcal{Z}(\mathfrak{R}')$, and φ *almost reverses the Peirce spaces order* if $\varphi(\mathfrak{R}_{ii}) \subseteq \mathfrak{R}'_{jj} + \mathcal{Z}(\mathfrak{R}')$, $i \neq j$, $i, j \in \{1, 2\}$.

2. Main theorem. The main result below tells us when a surjective Lie multiplicative map can be written as a sum of an additive isomorphism (resp. the negative of an additive anti-isomorphism) and a center valued map killing all commutators. For this we provide two conditions ensuring that the map almost preserves the Peirce spaces order or almost reverses the Peirce spaces order.

THEOREM 2.1. *Let $\mathfrak{R}$ be a unital 2,3-torsion free (prime ring, simple ring, simple associative ring) alternative ring, let $\mathfrak{R}'$ be another (prime ring, simple ring, simple associative ring) alternative ring and let $\varphi : \mathfrak{R} \to \mathfrak{R}'$ be a surjective Lie multiplicative map that preserves idempotents. Assume that $\mathfrak{R}$ has a nontrivial idempotent e_1 with associated Peirce decomposition $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$, such that*

(1) *If $x_{ij}\mathfrak{R}_{ji} = 0$, then $x_{ij} = 0$ $(i \neq j)$.*
(2) *If $x_{11}\mathfrak{R}_{12} = 0$ or $\mathfrak{R}_{21}x_{11} = 0$, then $x_{11} = 0$.*
(3) *If $\mathfrak{R}_{12}x_{22} = 0$ or $x_{22}\mathfrak{R}_{21} = 0$, then $x_{22} = 0$.*
(4) *If $z \in \mathcal{Z}(\mathfrak{R})$ with $z \neq 0$, then $z\mathfrak{R} = \mathfrak{R}$.*

If

(†) $f_i\varphi(\mathfrak{R}_{jj})f_i \subseteq \mathcal{Z}(\mathfrak{R}')f_i$,

then φ is of the form $\psi + \tau$, where ψ is an additive isomorphism between $\mathfrak{R}$ and $\mathfrak{R}'$ and τ is a map from $\mathfrak{R}$ to $\mathcal{Z}(\mathfrak{R}')$ which maps commutators to zero.

If

(††) $f_i\varphi(\mathfrak{R}_{ii})f_i \subseteq \mathcal{Z}(\mathfrak{R}')f_i$,

then φ is of the form $\psi + \tau$, where $-\psi$ is an additive anti-isomorphism between $\mathfrak{R}$ and $\mathfrak{R}'$ and τ is a map from $\mathfrak{R}$ to $\mathcal{Z}(\mathfrak{R}')$ which maps commutators to zero. Observe $f_i = \varphi(e_i)$ and $f_j = 1_{\mathfrak{R}'} - f_i$, $i \neq j$.

Note that according to Proposition 1.8 condition (†) means $f_i\varphi(\mathfrak{R}_{jj})f_i \subseteq \mathcal{Z}(\mathfrak{R}') \cap \mathfrak{R}'_{ii}$ and condition (††) means $f_i\varphi(\mathfrak{R}_{ii})f_i \subseteq \mathcal{Z}(\mathfrak{R}') \cap \mathfrak{R}'_{ii}$.

The following lemmas are stated under the assumptions of Theorem 2.1 and we need these lemmas for the proof of the theorem.

LEMMA 2.2. *Let $i, j \in \{1, 2\}$ with $i \neq j$. Then $\varphi(\mathfrak{R}_{ij}) = \mathfrak{R}'_{ij}$.*

Proof. We show just the case $i = 1$ and $j = 2$ because the other case is similar. Let $a_{12} \in \mathfrak{R}_{12}$. Because $a_{12} = [e_1, a_{12}]$, it follows that

$$\varphi(a_{12}) = [\varphi(e_1), \varphi(a_{12})] = \varphi(a_{12})_{12} - \varphi(a_{12})_{21}.$$

This implies that $\varphi(a_{12}) = \varphi(a_{12})_{12}$ for all $a_{12} \in \mathfrak{R}_{12}$. Thus $\varphi(a_{12}) \in \mathfrak{R}'_{12}$. Applying the same argument to φ^{-1} (which exists by Proposition 1.6) we can obtain the reverse inclusion, and equality follows. ■

LEMMA 2.3. *The map φ is almost additive, that is, for every $a, b \in \mathfrak{R}$ we have $\varphi(a + b) - \varphi(a) - \varphi(b) \in \mathcal{Z}(\mathfrak{R}')$.*

Proof. Since $\mathfrak{R}$ is a $2, 3$-torsion free alternative ring satisfying (1)–(3), $\mathfrak{R}$ satisfies (♠) and (♣) by Proposition 1.7. Now using Theorem 1.4 we find that φ is an almost additive map. ■

REMARK 2.1. Regarding Peirce decomposition relative to the idempotent f_1, the assumptions (2) and (3) imply:

(2′) If $x'_{11}\mathfrak{R}'_{12} = 0$ or $\mathfrak{R}'_{21}x'_{11} = 0$, then $x'_{11} = 0$.
(3′) If $\mathfrak{R}'_{12}x'_{22} = 0$ or $x'_{22}\mathfrak{R}'_{21} = 0$, then $x'_{22} = 0$.

Indeed, just use the bijectivity of φ.

2.1. First part of Theorem 2.1. Throughout this subsection we assume that (†) holds and e_1 is a nontrivial idempotent of $\mathfrak{R}$. It is easy to see that $\varphi(\mathfrak{R}_{ii}) \subseteq \mathfrak{R}'_{11} + \mathfrak{R}'_{22}$. Let us prove that if φ satisfies (†), then it almost preserves the Peirce spaces order.

LEMMA 2.4. $\varphi(\mathfrak{R}_{ii}) \subseteq \mathfrak{R}'_{ii} + \mathcal{Z}(\mathfrak{R}')$ $(i = 1, 2)$.

Proof. We only consider the case $i = 1$ because the other case can be treated similarly. For every $a_{11} \in \mathfrak{R}_{11}$ with $\varphi(a_{11}) = b_{11} + b_{12} + b_{21} + b_{22}$ we get

$$0 = \varphi([a_{11}, e_1]) = [\varphi(a_{11}), f_1].$$

From this we have $b_{12} = b_{21} = 0$. By (†), we have

$$\varphi(a_{11}) = b_{11} + f_2\varphi(a_{11})f_2 = b_{11} + zf_2 = b_{11} - f_1z + z \in \mathfrak{R}'_{11} + \mathcal{Z}(\mathfrak{R}'). \ ■$$

By Lemmas 2.2 and 2.4 we have

(A) if $a_{ij} \in \mathfrak{R}_{ij}$, $i \neq j$, then $\varphi(a_{ij}) = b_{ij} \in \mathfrak{R}'_{ij}$;
(B) if $a_{ii} \in \mathfrak{R}_{ii}$, then $\varphi(a_{ii}) = b_{ii} + z$, $b_{ii} \in \mathfrak{R}'_{ii}$, $z \in \mathcal{Z}(\mathfrak{R}')$.

We note that in (B), the elements b_{ii} and z are uniquely determined. Now we define a map ψ of $\mathfrak{R}$ into $\mathfrak{R}'$ by $\psi(a_{ij}) = b_{ij}$ for $a_{ij} \in \mathfrak{R}_{ij}$, $i, j = 1, 2$. For every $a = a_{11} + a_{12} + a_{21} + a_{22} \in \mathfrak{R}$, define $\psi(a) = \sum \psi(a_{ij})$. Using Lemma 2.3 and (B) we can define a map τ of $\mathfrak{R}$ into $\mathcal{Z}(\mathfrak{R}')$ by

$$\begin{aligned}
\tau(a) &= \varphi(a) - \psi(a) \\
&= \varphi(a) - (\psi(a_{11}) + \psi(a_{12}) + \psi(a_{21}) + \psi(a_{22})) \\
&= \varphi(a) - (b_{11} + b_{12} + b_{21} + b_{22}) \\
&= \varphi(a) - (\varphi(a_{11}) - z_{a_{11}} + \varphi(a_{12}) + \varphi(a_{21}) + \varphi(a_{22}) - z_{a_{22}}) \\
&= \varphi(a) - (\varphi(a_{11}) + \varphi(a_{12}) + \varphi(a_{21}) + \varphi(a_{22})) + (z_{a_{11}} + z_{a_{22}}).
\end{aligned}$$

We remark that $\psi(x) \in \mathcal{Z}(\mathfrak{R}')$ if and only if $x \in \mathcal{Z}(\mathfrak{R})$. Now we need to prove that ψ and τ satisfy the conditions of Theorem 2.1.

LEMMA 2.5. *ψ is an additive map.*

Proof. We only need to show that ψ is additive on $\mathfrak{R}_{ii}$ because by [FG20, Lemma 3.3] we have $\varphi(a_{ij} + b_{ij}) = \varphi(a_{ij}) + \varphi(b_{ij})$ for $i \neq j$. For $a_{ii}, b_{ii} \in \mathfrak{R}_{ii}$ we have

$$\begin{aligned}
\psi(a_{ii} + b_{ii}) - \psi(a_{ii}) - \psi(b_{ii}) = \varphi(a_{ii} + b_{ii}) - \tau(a_{ii} + b_{ii}) - \varphi(a_{ii}) \\
+ \tau(a_{ii}) - \varphi(b_{ii}) + \tau(b_{ii}).
\end{aligned}$$

Thus, $\psi(a_{ii} + b_{ii}) - \psi(a_{ii}) - \psi(b_{ii}) \in \mathcal{Z}(\mathfrak{R}') \cap \mathfrak{R}'_{ii} = \{0\}$. ∎

Now we show that $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in \mathfrak{R}$.

LEMMA 2.6. *For all $a_{ii}, b_{ii} \in \mathfrak{R}_{ii}$, $a_{ij}, b_{ij} \in \mathfrak{R}_{ij}$, $b_{ji} \in \mathfrak{R}_{ji}$ and $b_{jj} \in \mathfrak{R}_{jj}$ with $i \neq j$ we have*

(I) $\psi(a_{ii}b_{ij}) = \psi(a_{ii})\psi(b_{ij})$,
(II) $\psi(a_{ij}b_{jj}) = \psi(a_{ij})\psi(b_{jj})$,
(III) $\psi(a_{ii}b_{ii}) = \psi(a_{ii})\psi(b_{ii})$,
(IV) $\psi(a_{ij}b_{ij}) = \psi(a_{ij})\psi(b_{ij})$,
(V) $\psi(a_{ij}b_{ji}) = \psi(a_{ij})\psi(b_{ji})$.

Proof. Let us start with (I):

$$\begin{aligned}
\psi(a_{ii}b_{ij}) &= \varphi(a_{ii}b_{ij}) = \varphi([a_{ii}, b_{ij}]) = [\varphi(a_{ii}), \varphi(b_{ij})] \\
&= [\psi(a_{ii}), \psi(b_{ij})] = \psi(a_{ii})\psi(b_{ij}).
\end{aligned}$$

Next, consider (II):

$$\begin{aligned}
\psi(a_{ij}b_{jj}) &= \varphi(a_{ij}b_{jj}) = \varphi([a_{ij}, b_{jj}]) = [\varphi(a_{ij}), \varphi(b_{jj})] \\
&= [\psi(a_{ij}), \psi(b_{jj})] = \psi(a_{ij})\psi(b_{jj}).
\end{aligned}$$

Now we show (III). By (I) we get

$$\psi((a_{ii}b_{ii})r_{ij}) = \psi(a_{ii}b_{ii})\psi(r_{ij}).$$

On the other hand,

$$\psi(a_{ii}(b_{ii}r_{ij})) = \psi(a_{ii})\psi(b_{ii}r_{ij}) = \psi(a_{ii})(\psi(b_{ii})\psi(r_{ij})).$$

Since $\mathfrak{R}$ is flexible, we have $(a_{ii}b_{ii})r_{ij} = a_{ii}(b_{ii}r_{ij})$ and $(\psi(a_{ii})\psi(b_{ii}))\psi(r_{ij}) = \psi(a_{ii})(\psi(b_{ii})\psi(r_{ij}))$, so we obtain

$$(\psi(a_{ii}b_{ii}) - \psi(a_{ii})\psi(b_{ii}))\psi(r_{ij}) = 0$$

for all $\psi(r_{ij}) \in \mathfrak{R}'_{ij}$. So $\psi(a_{ii}b_{ii}) = \psi(a_{ii})\psi(b_{ii})$ by Remark 2.1.

Next, consider (IV). By additivity of ψ and (iv.b) of Peirce relations, we have

$$\begin{aligned}2\psi(a_{ij}b_{ij}) &= \psi(2a_{ij}b_{ij}) = \varphi(2a_{ij}b_{ij}) = \varphi([a_{ij}, b_{ij}]) = [\varphi(a_{ij}), \varphi(b_{ij})]\\ &= [\psi(a_{ij}), \psi(b_{ij})] = \psi(a_{ij})\psi(b_{ij}) - \psi(b_{ij})\psi(a_{ij}) = 2\psi(a_{ij})\psi(b_{ij}).\end{aligned}$$

As $\mathfrak{R}'$ is 2-torsion free, it follows that $\psi(a_{ij}b_{ij}) = \psi(a_{ij})\psi(b_{ij})$.

Finally we show (V). We have

$$\begin{aligned}\tau([a_{ij}, b_{ji}]) &= \varphi([a_{ij}, b_{ji}]) - \psi([a_{ij}, b_{ji}]) = [\varphi(a_{ij}), \varphi(b_{ji})] - \psi(a_{ij}b_{ji} - b_{ji}a_{ij})\\ &= [\psi(a_{ij}), \psi(b_{ji})] - \psi(a_{ij}b_{ji}) + \psi(b_{ji}a_{ij})\\ &= \psi(a_{ij})\psi(b_{ji}) - \psi(b_{ji})\psi(a_{ij}) - \psi(a_{ij}b_{ji}) + \psi(b_{ji}a_{ij}),\end{aligned}$$

which implies

$$[\psi(a_{ij})\psi(b_{ji}) - \psi(a_{ij}b_{ji})] + [\psi(b_{ji}a_{ij}) - \psi(b_{ji})\psi(a_{ij})] = z' \in \mathcal{Z}(\mathfrak{R}').$$

If $z' = 0$, then $\psi(a_{ij}b_{ji}) = \psi(a_{ij})\psi(b_{ji})$. If $z' \neq 0$, then multiplying by $\psi(a_{ij})$ on the left we get

$$\psi(a_{ij})\psi(b_{ji}a_{ij}) - \psi(a_{ij})\psi(b_{ji})\psi(a_{ij}) = \psi(a_{ij})z'.$$

By (II) we have

$$\psi(a_{ij}b_{ji}a_{ij}) - \psi(a_{ij})\psi(b_{ji})\psi(a_{ij}) = \psi(a_{ij})z'. \tag{2.1}$$

Now we observe that $\psi(a_{ij}b_{ji}a_{ij}) = \psi(a_{ij})\psi(b_{ji})\psi(a_{ij})$. Indeed, observe that $[[a_{ij}, b_{ji}], a_{ij}] = 2a_{ij}b_{ji}a_{ij}$. Then

$$\begin{aligned}2\psi(a_{ij}b_{ji}a_{ij}) &= \psi(2a_{ij}b_{ji}a_{ij}) = \varphi([[a_{ij}, b_{ji}], a_{ij}]) = [[\varphi(a_{ij}), \varphi(b_{ji})], \varphi(a_{ij})]\\ &= [[\psi(a_{ij}), \psi(b_{ji})], \psi(a_{ij})] = 2\psi(a_{ij})\psi(b_{ji})\psi(a_{ij}).\end{aligned}$$

Since $\mathfrak{R}'$ is 2-torsion free, we get $\psi(a_{ij}b_{ji}a_{ij}) = \psi(a_{ij})\psi(b_{ji})\psi(a_{ij})$. So $\psi(a_{ij})z' = 0$ implies that $a_{ij}z = 0$ with $z \in \mathcal{Z}(\mathfrak{R})$.

Now, we will consider the case $i = 1$ and $j = 2$; the other case is similar. By Theorem 2.1(4) there exists $h \in \mathfrak{R}$ such that $zh = e_1 + e_2$. Hence, by flexibility of $\mathfrak{R}$ we have

$$\begin{aligned}a_{12}(z_{22}h) &= a_{12}(z_{22}h_{21}) + a_{12}(z_{22}h_{22})\\ &= (a_{12}z_{22})h_{21} + (a_{12}z_{22})h_{22} = (a_{12}z)h_{21} + (a_{12}z)h_{22} = 0.\end{aligned}$$

Now, $z_{22}h = 1_{\mathfrak{R}} - z_{11}h$ gives $a_{12}(z_{22}h) = a_{12} - a_{12}(z_{11}h)$, and it follows that $a_{12} = 0$. Together with the other case, we have $a_{ij} = 0$, which is a contradiction. Therefore $\psi(a_{ij}b_{ji}) = \psi(a_{ij})\psi(b_{ji})$. ∎

Thus, we have proved the following result.

Lemma 2.7. *ψ is a homomorphism.*

Lemma 2.8. *τ sends commutators to zero.*

Proof. We have

$$\begin{aligned}\tau([a,b]) &= \varphi([a,b]) - \psi([a,b]) = [\varphi(a),\varphi(b)] - \psi([a,b]) \\ &= [\psi(a),\psi(b)] - \psi([a,b]) = 0. \ \blacksquare\end{aligned}$$

The first part of Theorem 2.1 is proved.

2.2. Second part of Theorem 2.1. Throughout this subsection we assume that (††) holds and also let e_1 be a nontrivial idempotent of $\mathfrak{R}$. In this case φ almost reverses the Peirce spaces order.

Lemma 2.9. $\varphi(\mathfrak{R}_{ii}) \subseteq \mathfrak{R}'_{jj} + \mathcal{Z}(\mathfrak{R}')$ $(i \neq j)$.

Proof. We just show the case $i = 1$ and $j = 2$ because the other case can be treated similarly. For every $a_{11} \in \mathfrak{R}_{11}$ with $\varphi(a_{11}) = b_{11}+b_{12}+b_{21}+b_{22}$ we get

$$0 = \varphi([a_{11}, e_1]) = [\varphi(a_{11}), f_1].$$

Therefore $b_{12} = b_{21} = 0$. By item (††) of Theorem 2.1, we have

$$\varphi(a_{11}) = f_1\varphi(a_{11})f_1 + b_{22} = zf_1 + b_{22} = b_{22} - zf_2 + z \in \mathfrak{R}'_{22} + \mathcal{Z}(\mathfrak{R}'). \ \blacksquare$$

Let us define the mappings ψ and τ. By Lemmas 2.2 and 2.9 we have

(A′) if $a_{ij} \in \mathfrak{R}_{ij}$, $i \neq j$, then $\varphi(a_{ij}) = b_{ij} \in \mathfrak{R}'_{ij}$;
(B′) if $a_{ii} \in \mathfrak{R}_{ii}$, then $\varphi(a_{ii}) = b_{jj} + z$, $b_{jj} \in \mathfrak{R}'_{jj}$, $z \in \mathcal{Z}(\mathfrak{R}')$.

We again observe that in (B′), the elements b_{jj} and z are uniquely determined. Now we define a map ψ of $\mathfrak{R}$ into $\mathfrak{R}'$ by $\psi(a_{ij}) = b_{ij}, a_{ij} \in \mathfrak{R}_{ij}$ and $\psi(a_{ii}) = b_{jj}$, $a_{ii} \in \mathfrak{R}_{ii}$ with $i \neq j$. For every $a = a_{11} + a_{12} + a_{21} + a_{22} \in \mathfrak{R}$, define $\psi(a) = \sum \psi(a_{ij})$. A map τ of $\mathfrak{R}$ into $\mathcal{Z}(\mathfrak{R}')$ is then defined by

$$\tau(a) = \varphi(a) - \psi(a).$$

We again remark that $\psi(x) \in \mathcal{Z}(\mathfrak{R}')$ if and only if $x \in \mathcal{Z}(\mathfrak{R})$. Now we need to prove that ψ and τ satisfy the conditions of Theorem 2.1.

Lemma 2.10. *ψ is an additive map.*

Proof. Similar to the proof of Lemma 2.5. ∎

Now we show that $\psi(ab) = -\psi(b)\psi(a)$ for all $a, b \in \mathfrak{R}$. For this purpose we prove the following lemma whose proof is similar to that of Lemma 2.6, but we provide it for clarity.

LEMMA 2.11. *For all $a_{ii}, b_{ii} \in \mathfrak{R}_{ii}$, $a_{ij}, b_{ij} \in \mathfrak{R}_{ij}$, $b_{ji} \in \mathfrak{R}_{ji}$ and $b_{jj} \in \mathfrak{R}_{jj}$ with $i \neq j$ we have*

(I′) $\psi(a_{ii}b_{ij}) = -\psi(b_{ij})\psi(a_{ii})$,
(II′) $\psi(a_{ij}b_{jj}) = -\psi(b_{jj})\psi(a_{ij})$,
(III′) $\psi(a_{ii}b_{ii}) = -\psi(b_{ii})\psi(a_{ii})$,
(IV′) $\psi(a_{ij}b_{ij}) = -\psi(b_{ij})\psi(a_{ij})$,
(V′) $\psi(a_{ij}b_{ji}) = -\psi(b_{ji})\psi(a_{ij})$.

Proof. Let us start with (I′):

$$\begin{aligned}\psi(a_{ii}b_{ij}) &= \varphi(a_{ii}b_{ij}) = \varphi([a_{ii}, b_{ij}]) = [\varphi(a_{ii}), \varphi(b_{ij})]\\ &= [\psi(a_{ii}), \psi(b_{ij})] = -\psi(b_{ij})\psi(a_{ii}).\end{aligned}$$

Next, consider (II′):

$$\begin{aligned}\psi(a_{ij}b_{jj}) &= \varphi(a_{ij}b_{jj}) = \varphi([a_{ij}, b_{jj}]) = [\varphi(a_{ij}), \varphi(b_{jj})]\\ &= [\psi(a_{ij}), \psi(b_{jj})] = -\psi(b_{jj})\psi(a_{ij}).\end{aligned}$$

Now we show (III′). By (I) we get

$$\psi((a_{ii}b_{ii})r_{ij}) = -\psi(r_{ij})\psi(a_{ii}b_{ii}).$$

On the other hand,

$$\begin{aligned}\psi(a_{ii}(b_{ii}r_{ij})) &= -\psi(b_{ii}r_{ij})\psi(a_{ii})\\ &= -(-\psi(r_{ij})\psi(b_{ii}))\psi(a_{ii}) = \psi(r_{ij})(\psi(b_{ii})\psi(a_{ii})).\end{aligned}$$

Since $\mathfrak{R}$ is flexible we have $(a_{ii}b_{ii})r_{ij} = a_{ii}(b_{ii}r_{ij})$, and since $-\psi(r_{ij})\psi(a_{ii}b_{ii}) = \psi(r_{ij})(\psi(b_{ii})\psi(a_{ii}))$ we obtain

$$\psi(r_{ij})(\psi(a_{ii}b_{ii}) + \psi(b_{ii})\psi(a_{ii})) = 0$$

for all $\psi(r_{ij}) \in \mathfrak{R}'_{ij}$. So $\psi(a_{ii}b_{ii}) = -\psi(b_{ii})\psi(a_{ii})$ by Remark 2.1.

Next, consider (IV′):

$$\begin{aligned}2\psi(a_{ij}b_{ij}) &= \psi(2a_{ij}b_{ij}) = \varphi(2a_{ij}b_{ij}) = \varphi([a_{ij}, b_{ij}])\\ &= [\varphi(a_{ij}), \varphi(b_{ij})] = [\psi(a_{ij}), \psi(b_{ij})]\\ &= \psi(a_{ij})\psi(b_{ij}) - \psi(b_{ij})\psi(a_{ij}) = -2\psi(b_{ij})\psi(a_{ij})\end{aligned}$$

As $\mathfrak{R}'$ is 2-torsion free it follows that $\psi(a_{ij}b_{ij}) = -\psi(b_{ij})\psi(a_{ij})$.

Finally, we show (V′). We have

$$\begin{aligned}\tau([a_{ij}, b_{ji}]) &= \varphi([a_{ij}, b_{ji}]) - \psi([a_{ij}, b_{ji}])\\ &= [\varphi(a_{ij}), \varphi(b_{ji})] - \psi(a_{ij}b_{ji} - b_{ji}a_{ij})\\ &= [\psi(a_{ij}), \psi(b_{ji})] - \psi(a_{ij}b_{ji}) + \psi(b_{ji}a_{ij})\\ &= \psi(a_{ij})\psi(b_{ji}) - \psi(b_{ji})\psi(a_{ij}) - \psi(a_{ij}b_{ji}) + \psi(b_{ji}a_{ij}),\end{aligned}$$

which implies

$$[-\psi(a_{ij}b_{ji}) - \psi(b_{ji})\psi(a_{ij})] + [\psi(a_{ij})\psi(b_{ji}) + \psi(b_{ji}a_{ij})] = z' \in \mathcal{Z}(\mathfrak{R}').$$

If $z' = 0$, then $\psi(a_{ij}b_{ji}) = -\psi(b_{ji})\psi(a_{ij})$. If $z' \neq 0$, multiplying by $\psi(a_{ij})$ on the right we get

$$\psi(a_{ij})\psi(b_{ji})\psi(a_{ij}) + \psi(b_{ji}a_{ij})\psi(a_{ij}) = \psi(a_{ij})z'.$$

By (II′) we have

$$-\psi(a_{ij}b_{ji}a_{ij}) + \psi(a_{ij})\psi(b_{ji})\psi(a_{ij}) = \psi(a_{ij})z'. \tag{2.2}$$

As in the proof of the first part we get $\psi(a_{ij}b_{ji}) = -\psi(b_{ji})\psi(a_{ij})$. ∎

LEMMA 2.12. *ψ is the negative of a homomorphism.*

LEMMA 2.13. *τ sends commutators to zero.*

Proof. The proof is identical to the proof of Lemma 2.8. ∎

The second part of Theorem 2.1 is proved.

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