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Existence results for quasilinear Schrödinger equations under a general critical growth term

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Abstract. We study the existence of solutions for the following quasilinear Schrödinger equation:

$$-\Delta u - \Delta(u^2)u = |u|^{2\cdot 2^*-2}u + g(u), \quad x \in \mathbb{R}^N,$$

where $N \geq 3$ and g satisfies very weak growth conditions. The method is to analyze the behavior of solutions for subcritical problems from Colin and Jeanjean's work [Nonlinear Anal. 56 (2004), 213–226] and to take the limit as the exponent approaches the critical exponent.

1. Introduction and main result. The existence of solutions for the quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = h(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

which models several physical phenomena, has been the subject of extensive study in recent years. The main mathematical difficulty with problem (1.1) is caused by the second order derivatives $\Delta(u^2)u$ the natural functional corresponding to problem (1.1) is not well defined for all $u \in H^1(\mathbb{R}^N)$ if $N \geq 2$. To overcome this difficulty, various arguments have been developed, such as a constrained minimization argument (see [LWW, LW, PSW, RS]), the perturbation method (see [LLW1, LLW3, LLW4]) and a change of variables (see [DMS, F, LXT, XT1, XT2]).

Here, we consider a special case of the form

$$-\Delta u - \Delta(u^2)u = |u|^{2\cdot 2^*-2}u + g(u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $N \geq 3$ and the nonlinear term reaches the critical growth. The Ambrosetti–Rabinowitz condition is not required.

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In particular, for the subcritical growth case, we recall that Colin and Jeanjean [CJ] proved the existence of a solution for the problem

$$-\Delta u - \Delta(u^2)u = k(u), \tag{1.3}$$

where k satisfies the following conditions:

(k_0) $k(s)$ is locally Hölder continuous on $[0, \infty)$.
(k_1) $-\infty < \liminf_{s\to 0^+} k(s)/s \le \limsup_{s\to 0^+} k(s)/s = -\nu < 0$.
(k_2) $\lim_{s\to\infty} |k(s)|/s^{2\cdot 2^*-1} = 0$.
(k_3) There exists $\zeta_0 > 0$ such that $K(\zeta_0) = \int_0^{\zeta_0} k(s)\, ds > 0$.

Under the above conditions, they obtained the following existence result.

Theorem A. *Suppose that $N \ge 3$ and (k_0)–(k_3) are satisfied. Then problem* (1.3) *admits a positive radially symmetric solution.*

This theorem can be regarded as the Berestycki–Lions theorem for the subcritical case of the quasilinear Schrödinger equation (1.3). For other results in subcritical cases, see [ASW, XLT]. To our best knowledge, there is no work on the existence of solutions for equation (1.3) where $k(u)$ stands for the Berestycki–Lions condition with critical growth.

The main purpose of this paper is to extend Theorem A to the case of critical exponents.

Set $G(s) := \int_0^s g(t)\, dt$, where the function $g \in C(\mathbb{R}^+, \mathbb{R})$ satisfies:

(g_1) $-\infty < \liminf_{s\to 0^+} g(s)/s \le \limsup_{s\to 0^+} g(s)/s < 0$.
(g_2) $\lim_{s\to +\infty} g(s)/s^{2\cdot 2^*-1} = 0$.
(g_3) (i) $\lim_{s\to +\infty} G(s)/s^{2\cdot 2^*-1} = +\infty$ when $3 \le N \le 10$.
(ii) $\lim_{s\to +\infty} G(s)/s^4 = +\infty$ when $N > 10$.

Now we state our main result:

Theorem 1.1. *Suppose that $N \ge 3$ and (g_1)–(g_3) are satisfied. Then problem* (1.2) *possesses a positive solution.*

Remark 1.2. To prove our result we borrow an idea from [BN, LLW2, LLT]. The method is to analyze the behavior of solutions for subcritical problems and to take the limit as the exponent approaches the critical exponent.

Remark 1.3. It is worth pointing out that the use of the Pohozaev manifold was shown very effective when treating nonlinearities which do not satisfy the Ambrosetti–Rabinowitz condition and the monotonicity condition [AP, CLM, JT, LMR, LM]. We will use the Pohozaev manifold in our proof.

NOTATION. In this paper, we use the following notations:

- $E := H^1(\mathbb{R}^N)$ is the usual Hilbert space endowed with the norm
$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx.$$
- $L^s(\mathbb{R}^N)$ is the usual Banach space endowed with the norm
$$\|u\|_s^s = \int_{\mathbb{R}^N} |u|^s\, dx, \qquad \forall s \in [1, +\infty).$$
- $B_r(y) := \{x \in \mathbb{R}^N : |x - y| < r\}$.
- $C, C_1, C_2, \ldots$ denote various positive (possibly different) constants.

2. Some preliminary results. We observe that formally problem (1.2) is the Euler–Lagrange equation associated with the energy functional
$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} [(1 + 2u^2)|\nabla u|^2]\, dx - \frac{1}{2 \cdot 2^*} \int_{\mathbb{R}^N} |u|^{2 \cdot 2^*}\, dx - \int_{\mathbb{R}^N} G(u)\, dx.$$
Variational methods cannot be applied directly to find weak solutions of problem (1.2), since the associated functional $J(u)$ is not defined for all u in $H^1(\mathbb{R}^N)$ unless $N = 1$. Hence we employ an argument developed in [CJ] to introduce a variational framework associated with problem (1.2). We make a change of variables $v := f^{-1}(u)$, where f is defined by
$$\begin{cases} f'(t) = \frac{1}{(1+2f^2(t))^{1/2}}, & t \in [0, +\infty), \\ f(t) = -f(-t), & t \in (-\infty, 0]. \end{cases} \tag{2.1}$$
After the change of variables, from J we obtain the following functional:
$$I(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2\, dx - \frac{1}{2 \cdot 2^*} \int_{\mathbb{R}^N} |f(v)|^{2 \cdot 2^*}\, dx - \int_{\mathbb{R}^N} G(f(v))\, dx.$$
Then $I(v) = J(u) = J(f(v))$ is well defined on E, and $I \in C^1(E, \mathbb{R})$ under the hypotheses (g_1)–(g_3). Moreover, we observe that if v is a critical point of the functional I, then the function $u = f(v)$ is a solution of problem (1.2) (see [CJ]).

We summarize the properties of f, proved in [CJ, DS, SV2].

LEMMA 2.1. *The function f satisfies the following properties:*

(1) *f is uniquely defined, C^∞ and invertible.*
(2) *$|f'(t)| \le 1$ for all $t \in \mathbb{R}$.*
(3) *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$.*
(4) *$f(t)/t \to 1$ as $t \to 0$.*
(5) *$f(t)/\sqrt{t} \to 2^{1/4}$ as $t \to \infty$.*
(6) *$f(t)/2 \le tf'(t) \le f(t)$ for all $t > 0$.*

(7) $|f(t)| \le 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$.
(8) $f^2(t) - f(t)f'(t)t \ge 0$ *for all* $t \in \mathbb{R}$.
(9) *There exists a positive constant* C *such that* $|f(t)| \ge C|t|$ *for* $|t| \le 1$ *and* $|f(t)| \ge C|t|^{1/2}$ *for* $|t| \ge 1$.
(10) *There exist constants* $M, R > 0$ *such that, for all* $t \ge R$,

$$f^{2\cdot 2^*}(t) - 2^{2^*/2}t^{2^*} \ge -Mt^{2^*-1/2}.$$

(11) *There is a constant* $C^* > 0$ *such that*

$$f^2(t) \ge C^*(t^2 - |t|^{2^*}).$$

Set $g(s) = -g(-s)$ for $s < 0$; then $g \in C(\mathbb{R}, \mathbb{R})$ is an odd function.

LEMMA 2.2. *Suppose that* (g_1)–(g_3) *hold. Then:*

(i) *For any* $\delta > 0$ *there exists* C_δ *such that*

$$|G(s)| \le C_\delta|s|^2 + \delta|s|^{2\cdot 2^*}, \quad \forall s \in \mathbb{R}. \tag{2.2}$$

(ii) *There exist* $L, C_0 > 0$ *such that*

$$NG(s) \le -L|s|^2 + C_0|s|^{2\cdot 2^*}, \quad \forall s \ge 0. \tag{2.3}$$

(iii) *For any* $\delta > 0$ *there exist* $C_\delta > 0$ *and* $\sigma \in (2, 2^*)$ *such that*

$$|g(s)| \le \delta(|s| + |s|^{2\cdot 2^*-1}) + C_\delta|s|^{2\sigma-1}, \quad \forall s \in \mathbb{R}, \tag{2.4}$$

$$|G(s)| \le \frac{\delta}{2}|s|^2 + \frac{\delta}{2\cdot 2^*}|s|^{2\cdot 2^*} + \frac{C_\delta}{2\sigma}|s|^{2\sigma}, \quad \forall s \in \mathbb{R}. \tag{2.5}$$

(iv) *There exists* $C > 0$ *such that*

$$G(s) \ge -C|s|^2, \quad \forall s \ge 0. \tag{2.6}$$

Proof. The above inequalities follow from (g_1)–(g_3) immediately. ∎

LEMMA 2.3. *Suppose that* $\{v_n\} \subset E$ *is a bounded sequence and* $v_n \to 0$ *in* $L^\alpha(\mathbb{R}^N)$ *for* $\alpha \in (2, 2^*)$. *Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (2^{2^*/2}|v_n|^{2^*} - |f(v_n)|^{2\cdot 2^*})\, dx = 0, \tag{2.7}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \big(\tfrac{1}{2}|f(v_n)|^{2\cdot 2^*-2}f(v_n)f'(v_n)v_n - \tfrac{1}{2}2^{(2^*-2)/2}|v_n|^{2^*}\big)\, dx = 0, \tag{2.8}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(f(v_n))f'(v_n)v_n\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} G(f(v_n))\, dx = 0. \tag{2.9}$$

Proof. We first prove (2.7). On the one hand, by Lemma 2.1(5), for any $\epsilon > 0$, there is $R > 0$ such that

$$\left|1 - \left(\frac{|f(t)|}{2^{1/4}|t|^{1/2}}\right)^{2\cdot 2^*}\right| < \epsilon, \quad \forall |t| \ge R.$$

Thus

$$\limsup_{n\to\infty} \int\limits_{\{x\colon |v_n(x)|>R\}} \left|2^{2^*/2}|v_n|^{2^*} - |f(v_n)|^{2\cdot 2^*}\right| dx$$

$$\le \limsup_{n\to\infty} \int\limits_{\{x\colon |v_n(x)|>R\}} 2^{2^*/2}|v_n|^{2^*}\cdot \left|1-\left(\frac{|f(v_n)|}{2^{1/4}|v_n|^{1/2}}\right)^{2\cdot 2^*}\right| dx$$

$$\le 2^{2^*/2}\epsilon \limsup_{n\to\infty} \int\limits_{\mathbb{R}^N} |v_n|^{2^*}\, dx.$$

On the other hand, by Lemma 2.1(7), one gets

$$\int\limits_{\{x\colon |v_n(x)|\le R\}} \left|2^{2^*/2}|v_n|^{2^*} - |f(v_n)|^{2\cdot 2^*}\right| dx$$

$$\le 2^{(2^*+2)/2} \int\limits_{\{x\colon |v_n(x)|\le R\}} |v_n|^{2^*}\, dx \le 2^{(2^*+2)/2} R^{2^*-\alpha} \int\limits_{\mathbb{R}^N} |v_n|^{\alpha}\, dx.$$

Hence

$$\limsup_{n\to\infty} \int\limits_{\mathbb{R}^N} \left|2^{2^*/2}|v_n|^{2^*} - |f(v_n)|^{2\cdot 2^*}\right| dx$$

$$\le 2^{(2^*+2)/2} R^{2^*-\alpha} \int\limits_{\mathbb{R}^N} |v_n|^{\alpha}\, dx + 2^{2^*/2}\epsilon \limsup_{n\to\infty} \int\limits_{\mathbb{R}^N} |v_n|^{2^*}\, dx.$$

Since $\epsilon > 0$ is arbitrary and $v_n \to 0$ in $L^{\alpha}(\mathbb{R}^N)$ for $\alpha \in (2, 2^*)$, and $\{v_n\}$ is bounded, we get (2.7).

The proof of (2.8) is similar to the proof of (2.7). We only need to use the fact that

$$\frac{|f(t)|^{2\cdot 2^*-2} f(t) f'(t) t}{2^{(2^*-2)/2}|t|^{2^*}} = \frac{1}{\sqrt{1/(2f^2(t))+1}} \left(\frac{|f(t)|}{2^{1/4}|t|^{1/2}}\right)^{2\cdot 2^*-2} \to 1 \quad \text{as } t\to\infty.$$

Finally, we prove (2.9). It follows from (2.4), (2.5) and Lemma 2.1(3, 6, 7) that

$$\int\limits_{\mathbb{R}^N} g(f(v_n)) f'(v_n) v_n\, dx$$

$$\le \int\limits_{\mathbb{R}^N} \left(\delta|f(v_n)| + \delta|f(v_n)|^{2\cdot 2^*-1} + C_\delta |f(v_n)|^{2\sigma-1}\right) f'(v_n) v_n\, dx$$

$$\le \delta \int\limits_{\mathbb{R}^N} |v_n|^2\, dx + 2^{2^*/2}\delta \int\limits_{\mathbb{R}^N} |v_n|^{2^*}\, dx + 2^{\sigma/2} C_\delta \int\limits_{\mathbb{R}^N} |v_n|^{\sigma}\, dx$$

$$\to 0 \quad \text{as } \delta \to 0,$$

and

$$\int_{\mathbb{R}^N} G(f(v_n))\,dx \le \int_{\mathbb{R}^N} \left(\frac{\delta}{2}|f(v_n)|^2 + \frac{\delta}{2\cdot 2^*}|f(v_n)|^{2\cdot 2^*} + \frac{C_\delta}{2\sigma}|f(v_n)|^{2\sigma}\right) dx$$
$$\le \frac{\delta}{2}\int_{\mathbb{R}^N} |v_n|^2\,dx + \frac{\delta}{2\cdot 2^*}2^{2^*/2}\int_{\mathbb{R}^N} |v_n|^{2^*}\,dx$$
$$+ \frac{C_\delta}{2\sigma}2^{\sigma/2}\int_{\mathbb{R}^N} |v_n|^{\sigma}\,dx \to 0 \quad \text{as } \delta\to 0.$$

This completes the proof of (2.9). ∎

3. Proof of Theorem 1.1. Inspired by [BN, LLT, LLW2], we introduce the following equation:

$$-\Delta u - \Delta(u^2)u = |u|^{q-2}u + g(u), \quad x\in\mathbb{R}^N. \tag{3.1}$$

By using the same change of variable as in (2.1), we write (3.1) as follows:

$$-\Delta v = |f(v)|^{q-2}f(v)f'(v) + g(f(v))f'(v), \quad x\in\mathbb{R}^N. \tag{3.2}$$

The energy functional of equation (3.2) is

$$I_q(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \frac{1}{q}\int_{\mathbb{R}^N} |f(v)|^q\,dx - \int_{\mathbb{R}^N} G(f(v))\,dx,$$

where $N\ge 3$, $q\in[\theta, 2\cdot 2^*)$ with $\theta = \max\{\frac{4N^2}{2LC^*+N^2-2N}, \frac{4N}{N-1}\}$, $L>0$ is the constant in (2.3) and $C^*>0$ is the constant in Lemma 2.1(11). Set

$$\mathcal{P} := \{v\in E : v\ne 0,\ \gamma(v)=0\}, \qquad p := \inf_{v\in\mathcal{P}} I(v),$$
$$\mathcal{P}_q := \{v\in E : v\ne 0,\ \gamma_q(v)=0\}, \quad p_q := \inf_{v\in\mathcal{P}_q} I_q(v),$$

where

$$\gamma(v) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \frac{N}{2\cdot 2^*}\int_{\mathbb{R}^N} |f(v)|^{2\cdot 2^*}\,dx - N\int_{\mathbb{R}^N} G(f(v))\,dx,$$
$$\gamma_q(v) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \frac{N}{q}\int_{\mathbb{R}^N} |f(v)|^{q}\,dx - N\int_{\mathbb{R}^N} G(f(v))\,dx.$$

Suppose that $u,v\in E\setminus\{0\}$ are solutions of equations (1.2) and (3.2), respectively. By the Pohozaev equality, one has $u\in\mathcal{P}$ and $v\in\mathcal{P}_q$. Then $\mathcal{P}$ and $\mathcal{P}_q$ are good constraints. Theorem A implies that equation (3.2) has a positive solution $u_q\in E$ which is radially symmetric for each $q\in[\theta, 2\cdot 2^*)$.

LEMMA 3.1. *Suppose that $N\ge 3$ and (g_1)–(g_3) hold. Then $\mathcal{P}\ne\emptyset$.*

Proof. Define $H(s) := G(f(s)) + \frac{1}{2\cdot 2^*}|f(s)|^{2\cdot 2^*}$. It follows from (2.2) with $\delta = \frac{1}{4\cdot 2^*}$ and Lemma 2.1(5) that

$$\lim_{s\to+\infty} \frac{H(s)}{s^2} \geq -C_\delta + \frac{1}{4\cdot 2^*} \lim_{s\to+\infty} \left(\frac{f^2(s)}{s}\right)^{2^*} s^{2^*-2} = +\infty.$$

Then there is $\zeta > 0$ such that $H(\zeta) > 0$.

We claim that there exists $v^0 \in E$ such that

$$\int_{\mathbb{R}^N} H(v^0)\, dx > 0. \tag{3.3}$$

Using an idea of [BL], for $R > 1$ we define

$$w_R(x) = \begin{cases} \zeta & \text{for } |x| \leq R, \\ \zeta(R+1-r) & \text{for } r = |x| \in [R, R+1], \\ 0 & \text{for } \ |x| \geq R+1. \end{cases}$$

Then $w_R \in E$ and

$$\int_{\mathbb{R}^N} H(w_R)\, dx \geq H(\zeta)\operatorname{meas}(B_R) - \Big(\max_{0\leq s\leq\zeta} |H(s)|\Big)\operatorname{meas}(B_{R+1} - B_R).$$

Hence there exist constants $C_1, C_2 > 0$ such that

$$\int_{\mathbb{R}^N} H(w_R)\, dx \geq C_1 R^N - C_2 R^{N-1}.$$

So there exists $R_0 > 0$ large enough such that $\int_{\mathbb{R}^N} H(w_{R_0})\, dx > 0$. Let $v^0 = w_{R_0}$. Then we get the claim (3.3).

Define $v_t^0(x) := v^0(x/t)$ for $t > 0$. Set $\varphi_0(t) := I(v_t^0)$. Then

$$\begin{aligned}\varphi_0(t) &= \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v^0|^2\, dx - t^N \int_{\mathbb{R}^N} G(f(v^0))\, dx - \frac{t^N}{2\cdot 2^*} \int_{\mathbb{R}^N} |f(v^0)|^{2\cdot 2^*}\, dx \\ &= \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v^0|^2\, dx - t^N \int_{\mathbb{R}^N} H(v^0)\, dx.\end{aligned}$$

Since

$$\varphi_0'(t)t = \frac{(N-2)t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v^0|^2\, dx - N t^N \int_{\mathbb{R}^N} H(v^0)\, dx,$$

we see that $\varphi_0'(t)t > 0$ for $t > 0$ small enough and $\varphi_0'(t)t < 0$ for $t > 0$ large enough. Hence there is $t_0>0$ such that $\varphi_0'(t_0)t_0=0$, namely

$$\begin{aligned}\gamma(v^0_{t_0}) &= \frac{(N-2)t_0^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla v^0|^2\,dx - Nt_0^N\int_{\mathbb{R}^N}G(f(v^0))\,dx \\ &\quad - \frac{Nt_0^N}{2\cdot 2^*}\int_{\mathbb{R}^N}|f(v^0)|^{2\cdot 2^*}\,dx \\ &= \frac{(N-2)t_0^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla v^0|^2\,dx - Nt_0^N\int_{\mathbb{R}^N}H(v^0)\,dx \\ &= \varphi_0'(t_0)t_0 = 0.\end{aligned}$$

That is, $v^0_{t_0}\in\mathcal{P}$, hence $\mathcal{P}\neq\emptyset$. ∎

Lemma 3.2. *Let $v\in E\setminus\{0\}$ be such that*

$$\int_{\mathbb{R}^N}G(f(v))\,dx + \frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N}|f(v)|^{2\cdot 2^*}\,dx > 0.$$

Then there is a unique $t_v>0$ such that $v_{t_v}\in\mathcal{P}$. Moreover, the maximum of $I(v_t)$ for $t\ge 0$ is achieved at t_v.

Proof. Define $v_t(x):=v(x/t)$ and $\varphi(t):=I(v_t)$. Let $s=t^N$ and define $\psi(s):=\varphi(s^{1/N})$. We have

$$\psi(s)=\frac{s^{(N-2)/N}}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,dx - s\bigg(\int_{\mathbb{R}^N}G(f(v))\,dx+\frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N}|f(v)|^{2\cdot 2^*}\,dx\bigg).$$

Obviously, $\psi(s)>0$ for $s>0$ small enough and $\psi(s)<0$ for $s>0$ large enough. Since

$$\frac{d^2\psi(s)}{ds^2} = -\frac{N-2}{N^2}s^{-2/N-1}\int_{\mathbb{R}^N}|\nabla v|^2\,dx<0,$$

$\psi(s)$ is a concave function. Then there exists a unique s_v such that

$$0=\psi'(s_v)=\frac{1}{N}s_v^{1/N-1}\varphi'(s_v^{1/N}),\qquad \psi(s_v)=\sup_{s>0}\psi(s).$$

Hence there exists a unique $t_v>0$ such that $t_v=s_v^{1/N}$, $\varphi'(t_v)=0$ and

$$I(v_{t_v})=\varphi(t_v)=\psi(s_v)=\sup_{s>0}\psi(s)=\sup_{t>0}\psi(t^N)=\sup_{t>0}\varphi(t)=\sup_{t>0}I(v_t).\ ∎$$

Lemma 3.3. *Suppose (g_1)–(g_3) hold. Then* $\limsup_{q\to(2\cdot 2^*)^-}p_q\le p$.

Proof. By the definition of infimum, for any $\epsilon\in(0,1/2)$ there exists $v\in\mathcal{P}$ such that $I(v)<p+\epsilon$. It follows from $\gamma(v)=0$ that

$$\frac{N-2}{2N}\int_{\mathbb{R}^N}|\nabla v|^2\,dx=\frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N}|f(v)|^{2\cdot 2^*}\,dx+\int_{\mathbb{R}^N}G(f(v))\,dx>0.$$

Then there is $T > 0$ large enough such that

$$I(v_T) = \frac{T^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx - T^N \frac{1}{2 \cdot 2^*} \int_{\mathbb{R}^N} |f(v)|^{2 \cdot 2^*} \, dx - T^N \int_{\mathbb{R}^N} G(f(v)) \, dx \le -1.$$

Since $\frac{t^N}{q} \int_{\mathbb{R}^N} |f(v)|^q \, dx$ is continuous in $(t, q) \in [0, T] \times [\theta, 2 \cdot 2^*]$, there is $\eta > 0$ such that for all $2 \cdot 2^* - \eta < q < 2 \cdot 2^*$ and $0 \le t \le T$,

$$|I_q(v_t) - I(v_t)| = \left| \frac{t^N}{q} \int_{\mathbb{R}^N} |f(v)|^q \, dx - \frac{t^N}{2 \cdot 2^*} \int_{\mathbb{R}^N} |f(v)|^{2 \cdot 2^*} \, dx \right| < \epsilon.$$

Hence $I_q(v_T) \le -1/2$ for all $2 \cdot 2^* - \eta < q < 2 \cdot 2^*$. Since

$$I_q(v_t) = \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx - t^N \left(\frac{1}{q} \int_{\mathbb{R}^N} |f(v)|^q \, dx + \int_{\mathbb{R}^N} G(f(v)) \, dx \right),$$

one gets $I_q(v_t) > 0$ for $t > 0$ small enough. Then there exists $t_q \in (0, T)$ such that $\frac{d}{dt} I_q(v_t) \big|_{t = t_q} = 0$ and $v_{t_q} \in \mathcal{P}_q$. It follows from Lemma 3.2 that $I(v_{t_q}) \le I(v)$. Hence

$$p_q \le I_q(v_{t_q}) \le I(v_{t_q}) + \epsilon \le I(v) + \epsilon < p + 2\epsilon$$

for $2 \cdot 2^* - \eta < q < 2 \cdot 2^*$, which completes the proof. ∎

It follows from Theorem A that for $q_n \in [\theta, 2 \cdot 2^*)$ with $q_n \to (2 \cdot 2^*)^-$, there exists a positive and radially symmetric sequence $\{v_n\} \subset E$ such that

$$I'_{q_n}(v_n) = 0, \quad I_{q_n}(v_n) = p_{q_n}. \tag{3.4}$$

LEMMA 3.4. *Suppose that a positive and radially symmetric sequence $\{v_n\} \subset E$ satisfies* (3.4). *Then $\{v_n\}$ is a bounded sequence in E. Also* $\liminf_{n \to \infty} p_{q_n} > 0$.

Proof. Since $v_n \in \mathcal{P}_q$, it follows from Lemma 3.3 that for n large enough,

$$p + 1 \ge p_{q_n} = I_{q_n}(v_n) - \frac{1}{N} \gamma_{q_n}(v_n) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx.$$

Hence there is a constant $C_3 > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx < C_3. \tag{3.5}$$

Thanks to the fact $q_n \ge \theta = \max\{\frac{4N^2}{2LC^* + N^2 - 2N}, \frac{4N}{N-1}\}$, we can deduce that

$$\frac{2N}{q_n} \frac{2 \cdot 2^* - q_n}{2 \cdot 2^* - 4} \le \frac{LC^*}{2}. \tag{3.6}$$

By (2.3), (3.6), Lemma 2.1(7, 11), and the Young and Sobolev inequalities, we have

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx &= \frac{N}{q_n}\int_{\mathbb{R}^N}|f(v_n)|^{q_n}\,dx + N\int_{\mathbb{R}^N}G(f(v_n))\,dx\\ &\le \frac{N}{q_n}\int_{\mathbb{R}^N}|f(v_n)|^{\frac{8\cdot 2^*-4q_n}{2\cdot 2^*-4}}|f(v_n)|^{\frac{2\cdot 2^*(q_n-4)}{2\cdot 2^*-4}}\,dx\\ &\quad+\int_{\mathbb{R}^N}(-L|f(v_n)|^2+C_0|f(v_n)|^{2\cdot 2^*})\,dx\\ &\le \frac{N}{q_n}\frac{2\cdot 2^*-q_n}{2\cdot 2^*-4}\int_{\mathbb{R}^N}|f(v_n)|^4\,dx+\frac{N}{q_n}\frac{q_n-4}{2\cdot 2^*-4}\int_{\mathbb{R}^N}|f(v_n)|^{2\cdot 2^*}\,dx\\ &\quad- LC^*\int_{\mathbb{R}^N}(v_n^2-|v_n|^{2^*})\,dx+2^{2^*/2}C_0\int_{\mathbb{R}^N}|v_n|^{2^*}\,dx\\ &\le -\frac{LC^*}{2}\int_{\mathbb{R}^N}v_n^2\,dx+C_4\int_{\mathbb{R}^N}|v_n|^{2^*}\,dx\\ &\le -\frac{LC^*}{2}\int_{\mathbb{R}^N}v_n^2\,dx+C_4S^{-2^*/2}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx\right)^{2^*/2},\end{aligned}$$

where $C_4=\frac{N}{q_n}\frac{q_n-4}{2\cdot 2^*-4}2^{2^*/2}+LC^*+2^{2^*/2}C_0$. Combining the above inequality with (3.5), we obtain the boundedness of $\int_{\mathbb{R}^N}v_n^2\,dx$, hence $\{v_n\}$ is bounded in E. From the above inequality, we can also infer that there exists $\mu>0$ such that $\int_{\mathbb{R}^N}|\nabla v|^2\,dx\ge\mu$. Thus

$$p_{q_n}=I_{q_n}(v_n)-\frac{1}{N}\gamma_{q_n}(v_n)=\frac{1}{N}\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx\ge\frac{1}{N}\mu.$$

Hence $\liminf_{n\to\infty}p_{q_n}\ge\frac{1}{N}\mu>0$. ∎

Given $\epsilon>0$, we consider the function $w_\epsilon:\mathbb{R}^N\to\mathbb{R}$ defined by

$$w_\epsilon(x)=C(N)\frac{\epsilon^{(N-2)/2}}{(\epsilon^2+|x|^2)^{(N-2)/2}},$$

where

$$C(N)=[N(N-2)]^{(N-2)/4}.$$

We observe (see [BN]) that $\{w_\epsilon\}$ is a family of functions on which the infimum that defines the best constant S for the Sobolev embedding $D^{1,2}(\mathbb{R}^N)\hookrightarrow L^{2^*}(\mathbb{R}^N)$ is attained. Let $\phi\in C_0^\infty(\mathbb{R}^N,[0,1])$ be a cut-off function satisfying $\phi\equiv 1$ in $B_1(0)$, $\phi\equiv 0$ in $\mathbb{R}^N\setminus B_2(0)$. Define

$$u_\epsilon=\phi w_\epsilon,\quad v_\epsilon=\frac{u_\epsilon}{(\int_{\mathbb{R}^N}u_\epsilon^{2^*}\,dx)^{1/2^*}}.$$

Then there exist positive constants $k_1,\ k_2$ and ϵ_0 such that

$$\|v_\epsilon\|_{2^*} = 1, \tag{3.7}$$

$$k_1 < \int_{\mathbb{R}^N} u_\epsilon^{2^*}\,dx < k_2 \quad \text{for all } 0<\epsilon<\epsilon_0, \tag{3.8}$$

$$\int_{\mathbb{R}^N} |\nabla v_\epsilon|^2\,dx \le S + O(\epsilon^{N-2}) \quad \text{as } \epsilon \to 0^+. \tag{3.9}$$

As $\epsilon \to 0$, we have

$$\|v_\epsilon\|_2^2 = \begin{cases} O(\epsilon) & \text{if } N=3,\\ O(\epsilon^2|\ln\epsilon|) & \text{if } N=4,\\ O(\epsilon^2) & \text{if } N\ge 5.\end{cases} \tag{3.10}$$

$$\|v_\epsilon\|_{2^*-1/2}^{2^*-1/2} = O(\epsilon^{(N-2)/4}) \ \text{ if } \ N\ge 3. \tag{3.11}$$

The standard proofs of (3.7)–(3.11) are omitted.

Lemma 3.5. *There is a constant $C>0$ such that*

$$\frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N} |f(v_\epsilon)|^{2\cdot 2^*}\,dx \ge \frac{1}{2^*}2^{2^*/2-1} - C\int_{\mathbb{R}^N} v_\epsilon^{2^*-1/2}\,dx.$$

Proof. Note that there exists $C_1>0$ such that if $0\le |s| < R$, then $|s|^{2^*} \le C_1|s|^{2^*-1/2}$. Hence

$$\int_{\{x:\,v_\epsilon(x)<R\}} v_\epsilon^{2^*}\,dx \le C_1 \int_{\{x:\,v_\epsilon(x)<R\}} v_\epsilon^{2^*-1/2}\,dx \le C_1\int_{\mathbb{R}^N} v_\epsilon^{2^*-1/2}\,dx.$$

Combining the above inequality with Lemma 2.1(7), one has

$$\begin{aligned}\frac{1}{2\cdot 2^*}&\int_{\{x:\,v_\epsilon(x)<R\}} \big||f(v_\epsilon)|^{2\cdot 2^*} - 2^{2^*/2}v_\epsilon^{2^*}\big|\,dx \\ &\le \frac{1}{2\cdot 2^*}\int_{\{x:\,v_\epsilon(x)<R\}} \big(|f(v_\epsilon)|^{2\cdot 2^*} + |2^{2^*/2}v_\epsilon^{2^*}|\big)\,dx \\ &\le \frac{1}{2^*}2^{2^*/2}\int_{\{x:\,v_\epsilon(x)<R\}} v_\epsilon^{2^*}\,dx \\ &\le \frac{1}{2^*}2^{2^*/2}C_1\int_{\mathbb{R}^N} v_\epsilon^{2^*-1/2}\,dx.\end{aligned} \tag{3.12}$$

Invoking (3.12), Lemma 2.1(10) and (3.7), we have

$$\begin{aligned}\frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N}|f(v_\epsilon)|^{2\cdot 2^*}\,dx &= \frac{1}{2\cdot 2^*}\int_{\{x:\,v_\epsilon(x)<R\}}\big[|f(v_\epsilon)|^{2\cdot 2^*}-2^{2^*/2}v_\epsilon^{2^*}\big]\,dx\\ &\quad+\frac{1}{2\cdot 2^*}\int_{\{x:\,v_\epsilon(x)\ge R\}}\big[|f(v_\epsilon)|^{2\cdot 2^*}-2^{2^*/2}v_\epsilon^{2^*}\big]\,dx\\ &\quad+\frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N}2^{2^*/2}v_\epsilon^{2^*}\,dx\\ &\ge -\frac{1}{2^*}2^{2^*/2}C_1\int_{\mathbb{R}^N}v_\epsilon^{2^*-1/2}\,dx-\frac{M}{2\cdot 2^*}\int_{\mathbb{R}^N}v_\epsilon^{2^*-1/2}\,dx\\ &\quad+\frac{1}{2^*}2^{2^*/2-1}\int_{\mathbb{R}^N}v_\epsilon^{2^*}\,dx\\ &=\frac{1}{2^*}2^{2^*/2-1}-C\int_{\mathbb{R}^N}(v_\epsilon)^{2^*-1/2}\,dx,\end{aligned}$$

where $C=\frac{1}{2^*}2^{2^*/2}C_1+\frac{M}{2\cdot 2^*}$. ■

LEMMA 3.6. *Suppose $N\ge 3$ and (g_1)–(g_3) are satisfied. Then there exists a unique $t_\epsilon>0$ such that $(v_\epsilon)_{t_\epsilon}\in\mathcal{P}$ and there are constants T_1, T_2 such that $0<T_1<t_\epsilon<T_2$.*

Proof. By (2.2) with $\delta=\frac{1}{4\cdot 2^*}$, and Lemmas 3.5 and 2.1(3), we have

$$\begin{aligned}&\int_{\mathbb{R}^N}G(f(v_\epsilon))\,dx+\frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N}|f(v_\epsilon)|^{2\cdot 2^*}\,dx\\ &\quad\ge\int_{\mathbb{R}^N}\big(-C_\delta|f(v_\epsilon)|^2-\delta|f(v_\epsilon)|^{2\cdot 2^*}\big)\,dx+\frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N}|f(v_\epsilon)|^{2\cdot 2^*}\,dx\\ &\quad\ge\frac{1}{4\cdot 2^*}\int_{\mathbb{R}^N}|f(v_\epsilon)|^{2\cdot 2^*}\,dx-C_\delta\int_{\mathbb{R}^N}|f(v_\epsilon)|^2\,dx\\ &\quad\ge\frac{1}{2^*}2^{2^*/2-2}-\frac{C}{2}\int_{\mathbb{R}^N}v_\epsilon^{2^*-1/2}\,dx-C_\delta\int_{\mathbb{R}^N}|v_\epsilon|^2\,dx.\end{aligned}$$

Combining the above inequality with (3.10), (3.11), one can deduce that there exists $\delta_0>0$ such that

$$\int_{\mathbb{R}^N}G(f(v_\epsilon))\,dx+\frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N}|f(v_\epsilon)|^{2\cdot 2^*}\,dx\ge\delta_0. \tag{3.13}$$

It follows from Lemma 3.2 that there exists a unique $t_\epsilon>0$ such that $(v_\epsilon)_{t_\epsilon}\in\mathcal{P}$ and $I((v_\epsilon)_{t_\epsilon})=\max_{t>0}I((v_\epsilon)_t)$.

On the one hand, if $t_\epsilon \to 0$ as $\epsilon \to 0$, then $p \le I((v_\epsilon)_{t_\epsilon}) \to 0$, a contradiction. Hence there is $T_1 > 0$ such that $t_\epsilon \ge T_1$ for ϵ small enough. On the other hand, if $t_\epsilon \to +\infty$ as $\epsilon \to 0$, then it follows from (3.13) that $p \le I((v_\epsilon)_{t_\epsilon}) \to -\infty$, a contradiction. Hence there is $T_2 > 0$ such that $t_\epsilon \le T_2$ for ϵ small enough. ∎

LEMMA 3.7. *Suppose that $N \ge 3$ and (g_1)–(g_3) are satisfied. Then*

$$p < \frac{1}{2N} S^{N/2}.$$

Proof. Denote

$$l(t) := \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v_\epsilon|^2\,dx - \frac{t^N}{2^*} 2^{2^*/2-1}.$$

It is standard to show that $l(t)$ achieves its maximum at

$$t_0 = 2^{-(2^*-2)/4} \Big[\int_{\mathbb{R}^N} |\nabla v_\epsilon|^2\,dx \Big]^{1/2},$$

and

$$l(t_0) = \frac{1}{2N} \Big[\int_{\mathbb{R}^N} |\nabla v_\epsilon|^2\,dx \Big]^{N/2}. \tag{3.14}$$

It follows from Lemma 3.5, (3.14) and Lemma 3.6 that

$$\begin{aligned} I((v_\epsilon)_{t_\epsilon}) &= \frac{t_\epsilon^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v_\epsilon|^2\,dx - \frac{t_\epsilon^N}{2\cdot 2^*} \int_{\mathbb{R}^N} |f(v_\epsilon)|^{2\cdot 2^*}\,dx \\ &\quad - t_\epsilon^N \int_{\mathbb{R}^N} G(f(v_\epsilon))\,dx \\ &\le \frac{t_\epsilon^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v_\epsilon|^2\,dx - t_\epsilon^N \frac{1}{2^*} 2^{2^*/2-1} + CT_2^N \int_{\mathbb{R}^N} v_\epsilon^{2^*-1/2}\,dx \\ &\quad - t_\epsilon^N \int_{\mathbb{R}^N} G(f(v_\epsilon))\,dx \\ &\le l(t_0) + CT_2^N \|v_\epsilon\|_{2^*-1/2}^{2^*-1/2} - t_\epsilon^N \int_{\mathbb{R}^N} G(f(v_\epsilon))\,dx. \end{aligned}$$

Applying (3.9), (3.11) and the inequality

$$(b+c)^\zeta \le b^\zeta + \zeta(b+c)^{\zeta-1}c, \qquad b, c \ge 0,\ \zeta \ge 1,$$

we have

$$\begin{aligned} p &\le I((v_\epsilon)_{t_\epsilon}) \\ &\le \frac{1}{2N} S^{N/2} + O(\epsilon^{(N-2)/4}) - t_\epsilon^N \int\limits_{\mathbb{R}^N} G(f(v_\epsilon))\,dx \\ &= \frac{1}{2N} S^{N/2} + \epsilon^{(N-2)/4} \left(C_5 - t_\epsilon^N \frac{\int_{\mathbb{R}^N} G(f(v_\epsilon))\,dx}{\epsilon^{(N-2)/4}} \right), \end{aligned}$$

where $C_5 > 0$ is a constant. In order to prove Lemma 3.7, we only need to verify that

$$\lim_{\epsilon \to 0^+} \frac{\int_{\mathbb{R}^N} G(f(t_\epsilon v_\epsilon))\,dx}{\epsilon^{(N-2)/4}} > \frac{C_5}{T_1^N}. \tag{3.15}$$

We analyze two cases.

CASE 1: $3 \le N \le 10$. From (g_3), for any given $A_0 > 0$, there exists $R = R(A_0) > 0$ such that, for $s \ge R$,

$$G(s) \ge A_0 s^{2\cdot 2^*-1}. \tag{3.16}$$

We consider the function $\eta_\epsilon : [0, \infty) \to \mathbb{R}$ defined by

$$\eta_\epsilon(r) = \frac{\epsilon^{(N-2)/2}}{(\epsilon^2 + r^2)^{(N-2)/2}}.$$

Since $\phi \equiv 1$ in $B_1(0)$, in view of (3.8) we can find a constant $C_6 > 0$ such that $v_\epsilon(x) \ge C_6 \eta_\epsilon(|x|)$ for $|x| < 1$. Notice that η_ϵ is decreasing and f is increasing, so there is a positive constant α such that, for $|x| < \epsilon$,

$$f(v_\epsilon(x)) \ge f(C_6 \eta_\epsilon(|x|)) \ge f(C_6 \eta_\epsilon(\epsilon)) \ge f(\alpha \epsilon^{(2-N)/2}).$$

Then we can choose $\epsilon_1 > 0$ such that

$$\alpha \epsilon^{(2-N)/2} \ge 1, \quad f(v_\epsilon(x)) \ge f(\alpha \epsilon^{(2-N)/2}) \ge R, \tag{3.17}$$

for $|x| < \epsilon$, $0 < \epsilon < \epsilon_1$. It follows from (3.16), (3.17) and Lemma 2.1(9) that, for any $x \in B_\epsilon(0)$ and $0 < \epsilon < \epsilon_1$,

$$\begin{aligned} G(f(v_\epsilon)) &\ge A_0 f^{2\cdot 2^*-1}(v_\epsilon) \ge A_0 f^{2\cdot 2^*-1}(\alpha \epsilon^{(2-N)/2}) \\ &\ge A_0 C^{2\cdot 2^*-1} \alpha^{(2\cdot 2^*-1)/2} \epsilon^{(2-N)(2\cdot 2^*-1)/4} \\ &= A_0 C^{2\cdot 2^*-1} \alpha^{(2\cdot 2^*-1)/2} \epsilon^{(-3N-2)/4}. \end{aligned} \tag{3.18}$$

By (3.18), (2.6) and Lemma 2.1(3), for $0 < \epsilon < \epsilon_1$ we have

$$
\begin{aligned}
(3.19)\qquad \int_{\mathbb{R}^N} G(f(v_\epsilon))\,dx &= \int_{B_\epsilon(0)} G(f(v_\epsilon))\,dx + \int_{\mathbb{R}^N\setminus B_\epsilon(0)} G(f(v_\epsilon))\,dx \\
&\geq A_0 C^{2\cdot 2^*-1}\alpha^{(2\cdot 2^*-1)/2}\epsilon^{(-3N-2)/4}\,\mathrm{meas}(B_\epsilon(0)) \\
&\quad - C\int_{\mathbb{R}^N\setminus B_\epsilon(0)} f^2(v_\epsilon)\,dx \\
&\geq A_0 C^{2\cdot 2^*-1}\alpha^{(2\cdot 2^*-1)/2}\epsilon^{(N-2)/4}\omega_N - C\|v_\epsilon\|_2^2.
\end{aligned}
$$

It follows from (3.10) and (3.19) that

$$
(3.20)\qquad \frac{\int_{\mathbb{R}^N} G(f(v_\epsilon))\,dx}{\epsilon^{(N-2)/4}} \geq A_0 C^{2\cdot 2^*-1}\alpha^{(2\cdot 2^*-1)/2}\omega_N - \Pi(\epsilon),
$$

where

$$
\Pi(\epsilon) = \begin{cases} O(\epsilon^{3/4}) & \text{for } N = 3, \\ O(\epsilon^{3/2}|\ln\epsilon|) & \text{for } N = 4, \\ O(\epsilon^{(10-N)/4}) & \text{for } 5 \leq N \leq 10. \end{cases}
$$

Choosing $A_0 > 0$ sufficiently large, (3.20) establishes (3.15).

CASE 2: $N > 10$. By (g_3), (3.17) and Lemma 2.1(9), one has

$$
\begin{aligned}
\int_{B_\epsilon(0)} G(f(v_\epsilon)) &\geq A_0 \int_{B_\epsilon(0)} |f(v_\epsilon)|^4\,dx \\
&\geq A_0 \int_{B_\epsilon(0)} |f(\alpha\epsilon^{(2-N)/2})|^4\,dx \\
&\geq A_0 \int_{B_\epsilon(0)} C^4\alpha^2\epsilon^{2-N}\,dx = A_0 C^4\alpha^2\epsilon^2\omega_N
\end{aligned}
$$

for $0 < \epsilon < \epsilon_1$. Hence, letting $A_0 \to +\infty$, we obtain

$$
(3.21)\qquad \lim_{\epsilon\to 0^+} \frac{\int_{B_\epsilon(0)} G(f(v_\epsilon))\,dx}{\epsilon^2} = +\infty.
$$

For $N > 10$, one has

$$
(3.22)\qquad \lim_{\epsilon\to 0^+} \frac{O(\epsilon^{(N-2)/4})}{\epsilon^2} = 0.
$$

Invoking (3.10), (3.11), (2.6) and Lemma 2.1(3), we get

$$\begin{aligned} I((v_\epsilon)_{t_\epsilon}) &\le \frac{1}{2N}S^{N/2} + O(\epsilon^{(N-2)/4}) - t_\epsilon^N \int\limits_{B_\epsilon(0)} G(f(v_\epsilon))\,dx \\ &\quad - t_\epsilon^N \int\limits_{\mathbb{R}^N\setminus B_\epsilon(0)} G(f(v_\epsilon))\,dx \\ &\le \frac{1}{2N}S^{N/2} + O(\epsilon^{(N-2)/4}) - t_\epsilon^N \int\limits_{B_\epsilon(0)} G(f(v_\epsilon))\,dx + T_2^N C \int\limits_{\mathbb{R}^N} v_\epsilon^2\,dx \\ &= \frac{1}{2N}S^{N/2} - \epsilon^2\Bigg(\frac{t_\epsilon^N \int_{B_\epsilon(0)} G(f(v_\epsilon))\,dx}{\epsilon^2} - \frac{O(\epsilon^{(N-2)/4})}{\epsilon^2} - T_2^N C\Bigg). \end{aligned}$$

Combining the above inequality with (3.21) and (3.22), we obtain

$$p \le I((v_\epsilon)_{t_\epsilon}) < \frac{1}{2N}S^{N/2}. \ \blacksquare$$

Proof of Theorem 1.1. Firstly, we invoke Theorem A to find a positive and radially symmetric sequence $\{v_n\} \subset E$ that satisfies (3.4), namely

$$I'_{q_n}(v_n) = 0, \quad I_{q_n}(v_n) = p_{q_n},$$

where $q_n \in [\theta, 2\cdot 2^*)$ with $q \to (2\cdot 2^*)^-$. Applying Lemma 3.4, we may get, up to a subsequence, $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^\alpha(\mathbb{R}^N)$ for all $\alpha \in (2, 2^*)$, and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. For any $\phi \in C_0^\infty(\mathbb{R}^N)$, one has

$$\begin{aligned} 0 &= \int\limits_{\mathbb{R}^N} \nabla v_n \cdot \nabla\phi\,dx - \int\limits_{\mathbb{R}^N} |f(v_n)|^{q_n-1} f'(v_n)\phi\,dx - \int\limits_{\mathbb{R}^N} g(f(v_n))f'(v_n)\phi\,dx \\ &\quad + o_n(1) \\ &= \int\limits_{\mathbb{R}^N} \nabla v \cdot \nabla\phi\,dx - \int\limits_{\mathbb{R}^N} |f(v)|^{2\cdot 2^*-1} f'(v)\phi\,dx - \int\limits_{\mathbb{R}^N} g(f(v))f'(v)\phi\,dx, \end{aligned}$$

i.e. v is a weak solution of problem (1.2).

Secondly, we claim that $v \ne 0$. Indeed, suppose that $v = 0$; then $v_n \to 0$ in $L^\alpha(\mathbb{R}^N)$ for $\alpha \in (2, 2^*)$. It follows from Lemma 2.3 that

$$\begin{aligned} p_{q_n} &= I_{q_n}(v_n) - \tfrac{1}{2}\langle I'_{q_n}(v_n), v_n\rangle \\ &= \frac{1}{2}\int\limits_{\mathbb{R}^N} g(f(v_n))f'(v_n)v_n\,dx - \int\limits_{\mathbb{R}^N} G(f(v_n))\,dx \\ &\quad + \frac{1}{2}\int\limits_{\mathbb{R}^N} |f(v_n)|^{q_n-2} f(v_n)f'(v_n)v_n\,dx - \frac{1}{q_n}\int\limits_{\mathbb{R}^N} |f(v_n)|^{q_n}\,dx \\ &= o(1) + \frac{1}{2}\int\limits_{\mathbb{R}^N} |f(v_n)|^{2\cdot 2^*-2} f(v_n)f'(v_n)v_n\,dx - \frac{1}{2\cdot 2^*}\int\limits_{\mathbb{R}^N} |f(v_n)|^{2\cdot 2^*}\,dx \end{aligned}$$

$$= o(1) + \int_{\mathbb{R}^N} \left(\frac{1}{2} |f(v_n)|^{2\cdot 2^*-2} f(v_n) f'(v_n) v_n - \frac{1}{2} 2^{(2^*-2)/2} |v_n|^{2^*} \right) dx$$
$$+ \int_{\mathbb{R}^N} \frac{1}{2} 2^{(2^*-2)/2} |v_n|^{2^*}\, dx - \frac{1}{2\cdot 2^*} \int_{\mathbb{R}^N} 2^{2^*/2} |v_n|^{2^*}\, dx$$
$$+ \frac{1}{2\cdot 2^*} \int_{\mathbb{R}^N} (2^{2^*/2}|v_n|^{2^*} - |f(v_n)|^{2\cdot 2^*})\, dx$$
$$= \int_{\mathbb{R}^N} |v_n|^{2^*} \left[\frac{1}{2} 2^{(2^*-2)/2} - \frac{1}{2^*} 2^{(2^*-2)/2} \right] dx + o(1)$$
$$= 2^{(2^*-2)/2} \frac{1}{N} \int_{\mathbb{R}^N} |v_n|^{2^*}\, dx + o(1).$$

Hence,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{2^*}\, dx = 2^{-(2^*-2)/2} N p_{q_n}. \tag{3.23}$$

Thanks to (2.8), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |f(v_n)|^{2\cdot 2^*-2} f(v_n) f'(v_n) v_n\, dx = N p_{q_n}. \tag{3.24}$$

It follows from (2.9) that

$$o(1) = \langle I'(v_n), v_n \rangle$$
$$= \int_{\mathbb{R}^N} |\nabla v_n|^2\, dx - \int_{\mathbb{R}^N} |f(v_n)|^{2\cdot 2^*-2} f(v_n) f'(v_n) v_n\, dx$$
$$- \int_{\mathbb{R}^N} g(f(v_n)) f'(v_n) v_n\, dx$$
$$= \int_{\mathbb{R}^N} |\nabla v_n|^2\, dx - \int_{\mathbb{R}^N} |f(v_n)|^{2\cdot 2^*-2} f(v_n) f'(v_n) v_n\, dx + o(1).$$

Combining the above equality with (3.24), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2\, dx = N p_{q_n}. \tag{3.25}$$

By (3.23), (3.25) and the Sobolev inequality, one has

$$2^{-(2^*-2)/2} N p_{q_n} = \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{2^*}\, dx$$
$$\le S^{-2^*/2} \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |\nabla v_n|^2\, dx \right)^{2^*/2} = S^{-2^*/2} (N p_{q_n})^{2^*/2}.$$

Therefore

$$p_{q_n} \geq \frac{1}{2N} S^{N/2}.$$

Combining the above inequality with Lemma 3.3, one gets

$$p \geq \limsup_{q\to(2\cdot 2^*)^-} p_q \geq p_{q_n} \geq \frac{1}{2N} S^{N/2},$$

which contradicts Lemma 3.7.

Thirdly, we claim the existence of a solution which is a minimizer of I on the Pohozaev manifold $\mathcal{P}$. Since v is a weak solution of problem (1.2), $\gamma(v) = 0$. It follows from Lemma 3.3 that

$$\begin{aligned} p &\leq I(v) = I(v) - \frac{1}{N}\gamma(v) \\ &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2\,dx \leq \liminf_{n\to\infty} \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx \\ &= \liminf_{n\to\infty}\left(I_{q_n}(v_n) - \frac{1}{N}\gamma_{q_n}(v_n)\right) = \liminf_{n\to\infty} p_{q_n} \\ &\leq \limsup_{n\to\infty} p_{q_n} \leq p. \end{aligned}$$

Hence $I(v) = p$, that is, v is a solution of problem (1.2).

Finally, the strong maximum principle implies $v > 0$, and Theorem 1.1 is proved. ∎

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