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BEHAVIOR OF RULED REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM FROM THE VIEWPOINT OF THE CURVATURE TENSOR

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Abstract. In this paper, we investigate ruled real hypersurfaces in a nonflat complex space form from the viewpoint of ϕ-invariances of curvature tensors of real hypersurfaces. Furthermore we give a new characterization of these real hypersurfaces.

1. Introduction. The theory of real hypersurfaces in a nonflat complex space form (namely a complex projective space $\mathbb{C}P^n(c)$ or a complex hyperbolic space $\mathbb{C}H^n(c)$) is one of the active research fields of the theory of submanifolds. In particular, *the class of ruled real hypersurfaces* in a nonflat complex space form is a nice example (for details, see [CR15], [NR98]).

The purpose of this paper is to characterize the class of ruled real hypersurfaces from the point of view of almost contact metric geometry. It is well-known that real hypersurfaces in a nonflat complex space form admit an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure J of the ambient space. It is useful to investigate real hypersurfaces in a nonflat complex space form.

In almost contact metric geometry, the class of *normal* almost contact metric manifolds plays an important role. It is well-known that Sasakian manifolds, cosymplectic manifolds and Kenmotsu manifolds are normal. Many geometers have studied these manifolds from various viewpoints (see [B10]).

To characterize the class of ruled real hypersurfaces, it is natural to compare them with normal almost contact metric manifolds. We remark that ruled real hypersurfaces in a nonflat complex space form are *not* normal (see [KMH14]). However, if a ruled real hypersurface displays a behavior regarding almost contact metric geometry similar to a certain normal almost contact metric manifold, then we can see that this real hypersurface possesses

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many geometric properties of interest within almost contact metric manifold geometry. In this paper, we focus on a relationship between ruled real hypersurfaces and cosymplectic manifolds. Specifically, we find that ruled real hypersurfaces behave similarly to cosymplectic manifolds (Lemma 3.3).

In particular, we concentrate on the following property which cosymplectic manifolds have:

$$R(X,Y) = R(\phi X, \phi Y) \tag{1.1}$$

for all tangent vector fields X and Y on a real hypersurface M^{2n-1}, where R is the curvature tensor of M^{2n-1}. The curvature tensor is called *strongly ϕ-invariant* if it satisfies (1.1).

However, this condition is too strong for real hypersurfaces in a nonflat complex space form. Indeed, ruled real hypersurfaces do not fulfill this condition (see Remark 5.5). Instead, we consider the following weaker condition:

$$R(X,Y) = R(\phi X, \phi Y) \tag{1.2}$$

for all tangent vector fields X and Y orthogonal to the characteristic vector field ξ on M^{2n-1}. The curvature tensor R is called *weakly ϕ-invariant* if it satisfies (1.2). Ruled real hypersurfaces satisfy this condition (Proposition 4.2). What is more, we can prove the following:

Main Theorem. *Let M^{2n-1} be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geq 3)$. Then M^{2n-1} is locally congruent to a ruled real hypersurface in $\widetilde{M}_n(c)$ if and only if M^{2n-1} has the property that*

$$R(X,Y) = R(\phi X, \phi Y)$$

for any tangent vector fields X and Y orthogonal to the characteristic vector field ξ on M^{2n-1}.

2. Almost contact metric manifolds.

First we recall a definition and several facts concerning almost contact metric manifolds.

Let M^{2n-1} be an odd-dimensional differentiable manifold. It has an *almost contact structure* (ϕ, ξ, η) if it possesses a tensor ϕ of type $(1,1)$, a vector field ξ and a 1-form η such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where I denotes the identity map of the tangent bundle TM. We call $(M^{2n-1}, \phi, \xi, \eta)$ an *almost contact manifold*. An almost contact manifold M^{2n-1} has a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any tangent vector fields X and Y on M^{2n-1}. Then $(M^{2n-1}, \phi, \xi, \eta, g)$ is called an *almost contact metric manifold*. Moreover, we call ϕ, ξ and η the *structure tensor*, the *characteristic vector field* (or the *Reeb vector field*) and the *contact form*, respectively. Almost contact metric manifolds M^{2n-1} satisfy the following conditions:

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi).$$

It is well-known that on the product manifold $M^{2n-1} \times \mathbb{R}$ we can define an almost complex structure J. Then M^{2n-1} is said to be *normal* if the almost complex structure J is integrable. Furthermore, an almost contact metric manifold M^{2n-1} is normal if and only if

$$[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0,$$

where

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y].$$

Next we define the fundamental 2-form $\varPhi$ by

$$\varPhi(X, Y) = g(\phi X, Y)$$

for any vectors X and Y on M^{2n-1}. An almost contact metric manifold M^{2n-1} is a *cosymplectic manifold* if it is normal and both the contact form η and the fundamental 2-form $\varPhi$ are closed, that is, $d\eta = 0$ and $d\varPhi = 0$. The following lemma gives a characterization of cosymplectic manifolds:

LEMMA 2.1 ([B10]). *Let M^{2n-1} be an almost contact metric manifold. Then M^{2n-1} is a cosymplectic manifold if and only if the structure tensor ϕ is parallel.*

3. Real hypersurfaces in a nonflat complex space form. We denote by $\widetilde{M}_n(c)$ $(n \geq 2)$ a complex n-dimensional nonflat complex space form. Namely, $\widetilde{M}_n(c)$ is congruent to either an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c > 0$ or an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c < 0$. Let M^{2n-1} be a real hypersurface in $\widetilde{M}_n(c)$ with a unit normal local vector field $\mathcal{N}$. That is, M^{2n-1} is an isometrically immersed submanifold of real codimension 1. *The Levi-Civita connections* $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M^{2n-1} are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{3.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{3.2}$$

for vector fields X and Y tangent to M^{2n-1}, where g denotes the induced metric from the standard Riemannian metric of $\widetilde{M}_n(c)$ and A is the shape operator of M^{2n-1} in $\widetilde{M}_n(c)$. (3.1) is called *Gauss's formula*, and (3.2) is *Weingarten's formula*. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal vectors* of M^{2n-1} in $\widetilde{M}_n(c)$, respectively.

It is known that M^{2n-1} admits the almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure J of $\widetilde{M}_n(c)$. The characteristic vector field ξ of M^{2n-1} and the structure tensor ϕ are defined by $\xi = -J\mathcal{N}$ and $\phi X = JX - \eta(X)\mathcal{N}$, respectively. The following equations are fundamental in the theory of real hypersurfaces in $\widetilde{M}_n(c)$:

$$\nabla_X \xi = \phi AX, \tag{3.3}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{3.4}$$

for any X and Y tangent to M^{2n-1}.

The *Codazzi equation* reads

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{3.5}$$

for any tangent vector fields X and Y on M^{2n-1}. Let R be the curvature tensor of M^{2n-1} in $\widetilde{M}_n(c)$:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

We have the *equation of Gauss*, stating that

$$\begin{aligned} R(X, Y)Z = (c/4)\{&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \\ &- g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{3.6}$$

for all tangent vector fields X, Y and Z on M^{2n-1}.

The Ricci tensor Q of type $(1, 1)$ for the Levi-Civita connection of an arbitrary real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ $(n \geq 2)$ is expressed as

$$QX = \frac{c}{4}((2n+1)X - 3\eta(X)\xi) + (\operatorname{Trace} A)AX - A^2X \tag{3.7}$$

for any X tangent to M^{2n-1}.

A real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ is said to be a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M^{2n-1}. It is known that every tube of sufficiently small constant radius around each Kähler submanifold of $\widetilde{M}_n(c)$ is a Hopf hypersurface (see [CR82], [Mo85]). The following lemma lists some useful properties of Hopf hypersurfaces in $\widetilde{M}_n(c)$:

LEMMA 3.1 ([KS90], [Ma76], [NR98]). *Let M^{2n-1} be a Hopf hypersurface in $\widetilde{M}_n(c)$ with principal curvature α corresponding to the characteristic vector field ξ. Then we have the following:*

(1) *α is locally constant on M^{2n-1}.*
(2) *If X is a tangent vector field to M^{2n-1} perpendicular to ξ with $AX = \lambda X$, then $(2\lambda - \alpha)A\phi X = (\alpha\lambda + c/2)\phi X$.*

Next, we give the definition of ruled real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$. It is known that ruled real hypersurfaces are examples of non-Hopf hypersurfaces in $\widetilde{M}_n(c)$. A real hypersurface M^{2n-1} is called a *ruled real hypersurface* in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geq 2)$ if the holomorphic distribution $T^0M = \{X \in TM \mid \eta(X) = 0\}$ is integrable and each of its maximal integral manifolds is a totally geodesic complex hypersurface $\widetilde{M}_{n-1}(c)$ in $\widetilde{M}_n(c)$. A ruled real hypersurface is constructed in the following way. Given an arbitrary regular real smooth curve γ in $\widetilde{M}_n(c)$ defined on an interval I we have at each point $\gamma(t)$ $(t \in I)$ a totally geodesic complex hypersurface $\widetilde{M}^{(t)}_{n-1}(c)$ that is orthogonal to the plane spanned by $\{\dot{\gamma}(t), J\dot{\gamma}(t)\}$. We thus obtain a ruled real hypersurface $M^{2n-1} = \bigcup_{t\in I} \widetilde{M}^{(t)}_{n-1}(c)$ in $\widetilde{M}_n(c)$.

The following lemma characterizes ruled real hypersurfaces in terms of the shape operator A (see [K87], [NR98]):

LEMMA 3.2 ([K87], [NR98]). *Let M^{2n-1} be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geq 2)$. Then the following three conditions are mutually equivalent:*

(1) *M^{2n-1} is a ruled real hypersurface.*
(2) *The shape operator A of M^{2n-1} satisfies the following equalities on the open dense subset $M_1 = \{x \in M^{2n-1} \mid \beta(x) \neq 0\}$ with a unit vector field U orthogonal to ξ:*

$$A\xi = \alpha\xi + \beta U, \qquad AU = \beta\xi, \qquad AX = 0$$

for an arbitrary tangent vector field X orthogonal to ξ and U, where α, β are the differentiable functions on M_1 defined by $\alpha = g(A\xi, \xi)$ and $\beta = \|A\xi - \alpha\xi\|$.
(3) *The shape operator A of M^{2n-1} satisfies the condition $g(AX, Y) = 0$ for arbitrary vector fields $X, Y \in T^0M$.*

By using the above lemma and (3.4), we can easily prove the following lemma:

LEMMA 3.3. *Let M^{2n-1} be a real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geq 2)$. Then M^{2n-1} is locally congruent to a ruled real hypersurface if and only if*

$$\nabla_X \phi = 0$$

for any tangent vector field X orthogonal to the characteristic vector field ξ.

Comparing this lemma with Lemma 2.1, we get the motivation for this paper.

4. ϕ-Invariance of curvature tensors of real hypersurfaces in a nonflat complex space form. In this section, we investigate real hypersurfaces in $\widetilde{M}_n(c)$ with ϕ-invariant curvature tensors. First, we study the case of Hopf hypersurfaces.

Proposition 4.1. *There do not exist Hopf hypersurfaces in $\widetilde{M}_n(c)$ $(n\geq 3)$ with weakly ϕ-invariant curvature tensor.*

Proof. From (1.2) and (3.6), we know that weak ϕ-invariance of the curvature tensor of a real hypersurface M^{2n-1} is equivalent to the relation

(4.1)
$$g(A\phi Y,Z)A\phi X-g(A\phi X,Z)A\phi Y-g(AY,Z)AX+g(AX,Z)AY=0$$

for all tangent vector fields X and Y on M^{2n-1} orthogonal to the characteristic vector ξ.

We suppose that M^{2n-1} is a Hopf hypersurface (with $A\xi=\alpha\xi$) in $\widetilde{M}_n(c)$ having a weakly ϕ-invariant curvature tensor. Now we choose a local field of orthonormal frames $\{e_1,\dots,e_{n-1},\phi e_1,\dots,\phi e_{n-1},\xi\}$ such that

$$Ae_i=\lambda_i e_i.$$

We fix $i\in\{1,\dots,n-1\}$. Putting $X=e_j$, $Y=e_i$ and $Z=e_i$ $(i\neq j)$ in (4.1) we get

$$\lambda_i\lambda_j=0$$

for any j $(\neq i)$. This implies that $\lambda_i=0$ or $\lambda_j=0$. Now we assume $\lambda_i\neq 0$ and put $X=\phi e_j$, $Y=e_i$ and $Z=e_i$ in (4.1). Then we have $\lambda_i A\phi e_j=0$. By Lemma 3.1(2), we can see that

$$\lambda_i(2\lambda_j-\alpha)A\phi e_j=\lambda_i(\alpha\lambda_j+c/2)\phi e_j=0.$$

Since $\lambda_i\neq 0$ and $\lambda_j=0$, we obtain $c=0$, which is a contradiction. Similarly, the case of $\lambda_i=0$ and $\lambda_j\neq 0$ does not hold.

If $\lambda_i=0$ and $\lambda_j=0$, we set $X=\phi e_j$, $Y=\phi e_i$ and $Z=\phi e_i$ $(i\neq j)$ in (4.1). Again, by Lemma 3.1(2), we get

$$(\alpha\lambda_i+c/2)(\alpha\lambda_j+c/2)=0.$$

Since $\lambda_i=0$ and $\lambda_j=0$, we have $c^2=0$, which is a contradiction. Hence M^{2n-1} does not admit a weakly ϕ-invariant curvature tensor. ∎

Next we consider the case of non-Hopf hypersurfaces. Note that there exist non-Hopf hypersurfaces M^{2n-1} in $\widetilde{M}_n(c)$ having a weakly ϕ-invariant curvature tensor.

Proposition 4.2. *Every ruled hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ has a weakly ϕ-invariant curvature tensor.*

Proof. Let M^{2n-1} be a ruled real hypersurface in $\widetilde{M}_n(c)$. We shall show that M^{2n-1} satisfies (4.1). Obviously, if $Z\perp\xi$, it follows from Lemma 3.2(2, 3) that M^{2n-1} satisfies (4.1). Hence it suffices to check the case $Z=\xi$. On the left-hand side of (4.1), we have

$$\begin{aligned}(4.2)\qquad & g(A\phi Y,\xi)A\phi X-g(A\phi X,\xi)A\phi Y-g(AY,\xi)AX+g(AX,\xi)AY\\ &=\nu\{g(\phi Y,U)A\phi X-g(\phi X,U)A\phi Y-g(Y,U)AX+g(X,U)AY\}.\end{aligned}$$

Since $\nu \neq 0$, we shall check that M^{2n-1} satisfies

$$g(\phi Y, U)A\phi X - g(\phi X, U)A\phi Y - g(Y, U)AX + g(X, U)AY = 0 \tag{4.3}$$

for all tangent vector fields X and Y orthogonal to ξ. Our discussion is divided into the following three cases:

(1) $X = Y = U$;
(2) $X = U$ and $Y \perp U, \xi$;
(3) $X, Y \perp U, \xi$.

By using Lemma 3.2, we can easily check cases (1) and (2). Thus we only need to prove our claim in case (3). For any tangent vector fields X and Y which are orthogonal to both U and ξ,

$$\begin{aligned} g(\phi Y, U)A\phi X - g(\phi X, U)A\phi Y - g(Y, U)AX + g(X, U)AY \\ = g(\phi Y, U)A\phi X - g(\phi X, U)A\phi Y. \end{aligned} \tag{4.4}$$

Take an inner product of (4.4) with vectors ξ, U and any vector $V(\perp U, \xi)$, respectively. Then we have

$$\begin{aligned} g(\phi Y, U)g(A\phi X, \xi) - g(\phi X, U)g(A\phi Y, \xi) \\ = \nu\{g(\phi Y, U)g(\phi X, U) - g(\phi X, U)g(\phi Y, U)\} = 0, \end{aligned} \tag{4.5}$$

$$\begin{aligned} g(\phi Y, U)g(A\phi X, U) - g(\phi X, U)g(A\phi Y, U) \\ = g(\phi Y, U)g(\phi Y, \nu\xi) - g(\phi X, U)g(\phi Y, \nu\xi) = 0, \end{aligned} \tag{4.6}$$

$$\begin{aligned} g(\phi Y, U)g(A\phi X, V) - g(\phi X, U)g(A\phi Y, V) \\ = g(\phi Y, U)g(\phi X, AV) - g(\phi X, U)g(\phi Y, AV) = 0 \end{aligned} \tag{4.7}$$

for all tangent vector fields X and Y on M^{2n-1} orthogonal to both U and ξ. Equations (4.5)–(4.7) imply that

$$g(\phi Y, U)A\phi X - g(\phi X, U)A\phi Y = 0$$

for all tangent vector fields X and Y orthogonal to both U and ξ.

Therefore every ruled real hypersurface in $\widetilde{M}_n(c)$ has a weakly ϕ-invariant curvature tensor. ∎

5. Proof of the main theorem. In this section, we shall prove the main theorem. To do this, we begin with some lemmas concerning almost contact metric manifolds:

LEMMA 5.1. *Let M^{2n-1} be an almost contact metric manifold ($n \geq 2$). Suppose M^{2n-1} has weakly ϕ-invariant curvature tensor R. Then*

$$g(R(X, Y)Z, W) = g(R(\phi X, \phi Y)\phi Z, \phi W) \tag{5.1}$$

for any vectors X, Y, Z, W orthogonal to the characteristic vector field ξ.

Proof. Using algebraic properties of the curvature tensor, we see that

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) &= g(R(X, Y)\phi Z, \phi W) = g(R(\phi Z, \phi W)X, Y) \\ &= g(R(Z, W)X, Y) = g(R(X, Y)Z, W) \end{aligned}$$

for any $X, Y, Z, W \in T^0M$. ∎

REMARK 5.2. An almost contact metric manifold is called a *ϕ-RK manifold* if it satisfies condition (5.1) (see [BY16]).

Now we recall the following lemma about ϕ-RK manifolds (see [BY16, Lemma 4.1]):

LEMMA 5.3 ([BY16]). *Let M^{2n-1} be an almost contact metric manifold $(n \geq 3)$. Suppose that M^{2n-1} is a ϕ-RK manifold. Then*

$$QX = R(X, \xi)\xi - \phi(Q\phi X - R(\phi X, \xi)\xi) \tag{5.2}$$

for any tangent vector field X on M^{2n-1} orthogonal to the characteristic vector ξ.

Now we shall prove the main theorem.

THEOREM 5.4. *Let M^{2n-1} be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geq 3)$. Then M^{2n-1} is locally congruent to a ruled real hypersurface in $\widetilde{M}_n(c)$ if and only if*

$$R(X, Y) = R(\phi X, \phi Y)$$

for any tangent vector fields X and Y orthogonal to the characteristic vector field ξ on M^{2n-1}.

Proof. It suffices to prove the "if" part. By Proposition 4.1, this is reduced to the case of non-Hopf hypersurfaces. Thus, the shape operator A of M^{2n-1} satisfies $A\xi = \alpha\xi + \beta U$, where α and β $(\neq 0)$ are smooth functions on M^{2n-1}, and U is a unit vector field with $U \perp \xi$. Now let M^{2n-1} satisfy (1.2).

Putting $Z = \xi$ in (4.1), we have

$$\beta\{g(\phi Y, U)A\phi X - g(\phi X, U)A\phi Y - g(Y, U)AX + g(X, U)AY\} = 0.$$

Since $\beta \neq 0$, we get

$$g(\phi Y, U)A\phi X - g(\phi X, U)A\phi Y - g(Y, U)AX + g(X, U)AY = 0. \tag{5.3}$$

Setting $Y = U$ in (5.3), we obtain

$$AX = g(X, U)AU + g(X, \phi U)A\phi U \tag{5.4}$$

for any X orthogonal to ξ. Then equation (5.4) implies that

$$AX = 0 \tag{5.5}$$

for any X orthogonal to ξ, U and ϕU.

From Lemma 5.3, (3.6) and (3.7) we have

$$QX = (c/4)(2n+1)X + g(A\xi,\xi)AX - g(AX,\xi)A\xi$$
$$- (\mathrm{Trace}\, A)\phi A\phi X + \phi A^2\phi X + g(A\xi,\xi)\phi A\phi X - g(A\phi X,\xi)\phi A\xi$$

for any vector field X orthogonal to ξ. From this, together with (3.7), we get

$$(\mathrm{Trace}\, A - \eta(A\xi))(AX + \phi A\phi X) - A^2X - \phi A^2\phi X + \eta(AX)A\xi + \eta(A\phi X)\phi A\xi = 0 \tag{5.6}$$

for any tangent vector field X orthogonal to ξ.

Taking the inner product of (5.6) and ξ, we see that

$$(\mathrm{Trace}\, A - \alpha)\beta g(X,U) = \beta g(AX,U).$$

Since $\beta \neq 0$, we get

$$(\mathrm{Trace}\, A - \alpha)g(X,U) = g(AX,U) \tag{5.7}$$

for any X orthogonal to ξ. Putting $X = U$ in (5.7), we have

$$g(AU,U) = \mathrm{Trace}\, A - \alpha. \tag{5.8}$$

Moreover, (5.7) means that

$$g(AX,U) = 0 \tag{5.9}$$

for any X orthogonal to ξ and U.

We take a local field of orthonormal frames $\{\xi, U, \phi U, e_1, \ldots, e_{2n-4}\}$ on M^{2n-1}. Then $\mathrm{Trace}\, A$ is expressed as

$$\mathrm{Trace}\, A = \alpha + g(AU,U) + g(A\phi U,\phi U) + \sum_{i=1}^{2n-4} g(Ae_i,e_i)$$
$$= \alpha + g(AU,U) + g(A\phi U,\phi U) \quad \text{(from (5.5))}.$$

This, combined with (5.8), yields

$$g(A\phi U,\phi U) = 0. \tag{5.10}$$

By using (5.9), we obtain

$$g(AU,\phi U) = g(U,A\phi U) = 0. \tag{5.11}$$

Obviously, $g(A\phi U,\xi) = 0$. This, together with (5.5), (5.10) and (5.11), implies

$$A\phi U = 0.$$

It follows from (5.9) that

$$AU = \beta\xi + \gamma U$$

for the function $\gamma = g(AU,U)$. By using the Codazzi equation (3.5), we have

$$(\nabla_U A)V - (\nabla_V A)U = 0 \quad \text{for } V \in T^0M \cap \mathrm{span}\{U,\phi U\}^{\perp}.$$

On the other hand, by using (3.3), we see that

$$\begin{aligned}(\nabla_U A)V - (\nabla_V A)U &= \nabla_U(AV) - A\nabla_U V - \nabla_V(AU) + A\nabla_V U \\ &= -A\nabla_U V - \nabla_V(\beta\xi + \gamma U) + A\nabla_V U \\ &= -A\nabla_U V - (V\beta)\xi - \beta\phi AV - (V\gamma)U \\ &\quad - \gamma\nabla_V U + A\nabla_V U \\ &= -A\nabla_U V - (V\beta)\xi - (V\gamma)U - \gamma\nabla_V U + A\nabla_V U.\end{aligned}$$

These two equations yield

$$\gamma g(\nabla_V U, \phi V) = 0.$$

Now we shall show that $\gamma = 0$. Indeed, if $\gamma \neq 0$ then

$$g(\nabla_V U, \phi V) = 0.$$

Again, by using the Codazzi equation (3.5), we have

$$(\nabla_\xi A)V - (\nabla_V A)\xi = (c/4)\phi V \quad \text{for } V \in T^0M \cap \operatorname{span}\{U, \phi U\}^\perp.$$

On the other hand,

$$\begin{aligned}(\nabla_\xi A)V - (\nabla_V A)\xi &= \nabla_\xi(AZ) - A\nabla_\xi V - \nabla_V(A\xi) + A\nabla_V\xi \\ &= -A\nabla_\xi V - \nabla_V(\alpha\xi + \beta U) + A\phi AV \\ &= -A\nabla_\xi V - (V\alpha)\xi - (V\beta)U - \beta\nabla_V U.\end{aligned}$$

Here we have used (3.3). These equations imply

$$-\beta g(\nabla_V U, \phi V) = c/4.$$

Hence we have $c = 0$, which is a contradiction. Thus we can see that $\gamma = 0$. Then the shape operator A of M^{2n-1} satisfies

$$A\xi = \alpha\xi + \beta U, \quad AU = \beta\xi, \quad AX = 0$$

for any tangent vector field X orthogonal to ξ and U on M^{2n-1}. By Lemma 3.2, M^{2n-1} is locally congruent to a ruled real hypersurface in $\widetilde{M}_n(c)$. ■

REMARK 5.5. It is easy to check that ruled real hypersurfaces in $\widetilde{M}_n(c)$ do not have strongly ϕ-invariant curvature tensor. Indeed, if a ruled real hyperusrface M^{2n-1} has strongly ϕ-invariant curvature tensor, then

$$R(\xi, Y)\xi = -(c/4)Y \neq 0 \quad \text{for } Y \perp \xi, U.$$

On the other hand, we can see that

$$R(\phi\xi, \phi Y)\xi = 0,$$

which is a contradiction. This implies that there does not exist a real hypersurface in $\widetilde{M}_n(c)$ with strongly ϕ-invariant curvature tensor.

In almost contact metric geometry, there exist examples with strongly ϕ-invariant curvature tensor. In fact, cosymplectic manifolds have strongly ϕ-invariant curvature tensor (see [GY69]). On the other hand, there does not exist a cosymplectic real hypersurface in $\widetilde{M}_n(c)$ (see [C15], [KMH14]). However, by our main theorem, ruled real hypersurfaces display a behavior similar to cosymplectic manifolds.

REMARK 5.6. In [Ku98], H. Kurihara investigated real hypersurfaces satisfying the condition

$$R(X,Y)\phi Z = \phi R(X,Y)Z \tag{5.12}$$

for arbitrary tangent vector fields X, Y, Z orthogonal to ξ on M^{2n-1}.

Using this condition, he gave a characterization of ruled real hypersurfaces in $\widetilde{M}_n(c)$ $(n \geq 3)$. Here, we emphasize that this condition is different from condition (1.2) on almost contact metric manifolds. However we note that cosymplectic manifolds satisfy both (1.2) and (5.12) (see [CNY13]).

REMARK 5.7. In [O16], the author studied the ϕ-invariance of Ricci tensor on real hypersurfaces in $\widetilde{M}_n(c)$. The Ricci tensor S of type $(0,2)$ of M^{2n-1} is called *weakly ϕ-invariant* if

$$S(X,Y) = S(\phi X, \phi Y)$$

for arbitrary tangent vector fields X and Y orthogonal to the characteristic vector field ξ on M^{2n-1}.

Cosymplectic manifolds satisfy this condition but ruled real hypersurfaces do not (cf. [CNY13], [O16]). Hence there exists a gap of behavior in terms of ϕ-invariance of the Ricci tensor between cosymplectic manifolds and ruled real hypersurfaces.

REMARK 5.8. The author does not know whether for $n = 2$ the same results hold or not.

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