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A note on approximation and homotopy in $C(X, S^n)$, $n = 1, 3, 7$

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Abstract. For any compact space X and a certain class of $\mathbb{R}$-subalgebras $R \subseteq C(X, \mathbb{R})$ we study the subsets $S^n(R) \subseteq C(X, S^n)$ of maps the coordinate functions of which belong to R. The space $C(X, S^n)$ is endowed with the compact-open topology. A classical result of Eilenberg on nullhomotopic maps $X \to S^1$ is extended to maps $X \to S^n$, $n = 3, 7$, by using the multiplication on S^n. In case $n = 1, 3, 7$, we make a detailed study of the closure of $S^n(R)$, the homotopy classes of maps in $S^n(R)$ and the interrelation between approximation and homotopy. As an application, some previous results on compact real algebraic sets can be supplemented.

1. Introduction. Let the set $C(X, Y)$ of continuous maps between two topological spaces be endowed with the compact-open topology. It is an important task to identify interesting subsets, study their closure and the homotopy classes represented by their elements. This paper contributes to the study of $C(X, S^n)$ with the main focus on the dimension $n = 1, 3, 7$ where X is any compact space.

The starting point is a lemma of Eilenberg on nullhomotopic maps $X \to S^1$. It will be extended, with uniform proofs, to a characterization of homotopic pairs of maps $f, g \in C(X, S^n)$, $n = 1, 3, 7$, by using the algebraic properties of the composition algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ (see Theorem 1).

We work with distinguished $\mathbb{R}$-algebras R of functions inside $C(X, \mathbb{R})$ and consider the subsets $S^n(R)$ of continuous maps $X \to S^n$ the coordinate functions of which belong to R. We study their closure $\overline{S^n(R)}$ in $C(X, S^n)$ and the homotopy classes they induce, the latter forming a subset $\pi^n_R(X)$ of the full cohomotopy set $\pi^n(X)$.

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If $n = 1, 3, 7$ the algebraic operations of the composition algebras induce multiplications on $C(X, S^n)$, $\overline{S^n(R)}$, $\pi^n(X)$ and $\pi^n_R(X)$ which may be non-associative but are still manageable. A first consequence is Theorem 2: the closure $\overline{S^n(R)}$ equals the union of the homotopy classes of the elements in $S^n(R)$. Secondly, the relationship between these four structures can be precisely described by relating them to the group of units of R (see Theorem 3).

In case X and Y are real algebraic sets one looks at geometrically defined subalgebras R. This special topic will be addressed in the last section. The results above allow one to supplement earlier results of J. Bochnak and W. Kucharz on real algebraic sets X and $Y = S^p$, p arbitrary (see Theorem 4 e.g.). Further applications to real function fields are briefly mentioned, including the topic of positive units of those subalgebras and the role and the properties of the real holomorphy ring H of F and its associated space of real places.

2. The general setting. Let X denote any compact space. The compact-open topology on the space $C(X, \mathbb{R}^d)$ coincides with the topology of uniform convergence with respect to the Euclidean norm $\|\cdot\|_2$ which induces the maximum norm

$$\|f\| = \max_{x \in X} \|f(x)\|_2$$

on this space.

Next, consider any $n \in \mathbb{N}$. Then

$$C(X, S^n) = \{f \in C(X, \mathbb{R}^{n+1}) \mid \|f(x)\|_2 = 1 \ \forall x \in X\}$$

is a closed subspace of $C(X, \mathbb{R}^{n+1})$.

We will study continuous maps

$$f = (f_0, \ldots, f_n) : X \to S^n$$

with all components f_i belonging to certain prescribed $\mathbb{R}$-algebras $R \subseteq C(X, \mathbb{R})$ specified below.

Any $\mathbb{R}$-algebra is assumed to contain the unit element of $C(X, \mathbb{R})$. Let R^* denote the group of units of R and, accordingly, $C(X, \mathbb{R})^* = \{f \mid f(x) \neq 0 \ \forall x \in X\}$ the group of units of $C(X, \mathbb{R})$.

The $\mathbb{R}$-algebras R we will be dealing with have to meet the following two conditions:

$$\text{(I) } R \textit{ is dense in } C(X, \mathbb{R}), \quad \text{(II) } R^* = R \cap C(X, \mathbb{R})^*.$$

As an ad hoc terminology these algebras are called *admissible.*

If R is admissible then the subset

$$R_+ := \{f \in R \mid f(x) > 0 \ \forall x \in X\}$$

is a subgroup of the group of units R^* and is called the group of *positive units* of R.

Admissible algebras belong to the class of rings studied by R. G. Swan [16]. In fact, the inclusion $R \hookrightarrow C(X,\mathbb{R})$ satisfies the conditions (1), (2), (3′) and $C(X,\mathbb{R})$ satisfies the conditions $(SBI_n)(a)$ of [16].

Assume that functions $f_i \in R$ are given. Then, clearly,

$$f = (f_0, \dots, f_n) : X \to \mathbb{R}^{n+1}$$

is a continuous map. The assumption $f(X) \subseteq S^n$ is equivalent to $\sum_{i=0}^n f_i^2 = 1$. This latter condition describes the n-sphere $S^n(R)$ over R. So, we are interested in maps $f \in S^n(R)$.

As already said in the introduction, this paper is concerned with the study of the n-sphere $S^n(R)$, its closure $\overline{S^n(R)}$ in $C(X,S^n)$ and the impact of these subspaces on approximation and homotopy in $C(X,S^n)$, with main focus on the cases $n = 1,3,7$. The maps in $\overline{S^n(R)}$ are exactly the maps which can be approximated by means of functions from the admissible algebra R.

Throughout the paper we will make use of a continuous *group action* of the orthogonal group $O(n+1,R)$ on $C(X,S^n)$: if $A = (a_{i,j}) \in O(n+1,R)$, $f = (f_0,\dots,f_n) \in C(X,S^n)$ then

$$Af := (g_0,\dots,g_n) \quad \text{where} \quad g_i = \sum_{j=0}^n a_{i,j} f_j \text{ for } i = 0,\dots,n.$$

Indeed, for every $x \in X$ the matrix $A(x) := (a_{i,j}(x))$ is an orthogonal matrix over $\mathbb{R}$, so A maps $C(X,S^n)$ into itself. Also, $S^n(R)$ is invariant under this group action.

From $\|Af - Ag\| = \|f - g\|$ whenever $A \in O(n+1,R)$, $f,g \in C(X,\mathbb{R}^{n+1})$, one derives that the map $f \mapsto Af$ is a homeomorphism of $C(X,S^n)$ onto itself.

In the proof of the following proposition we will apply the stereographic projection st with center $Z = (1,0,\dots,0)$ which maps $S^n \setminus Z$ homeomorphically onto $\mathbb{R}^n$:

$$st(a_0,a_1,\dots,a_n) = \frac{1}{1-a_0}(a_1,\dots,a_n).$$

The inverse of st, denoted by ι, is

$$\iota(b_1,\dots,b_n) = \frac{1}{1+\sum_{i=1}^n b_i^2}\Big(-1 + \sum_{i=1}^n b_i^2, 2b_1,\dots,2b_n\Big).$$

It induces a (topological) embedding $\iota_* : C(X,\mathbb{R}^n) \to C(X,S^n)$, $f \mapsto \iota \circ f$, with image $C(X,S^n \setminus Z)$. A map $g = (g_1,\dots,g_n)$ is mapped under ι_* onto a map $(f_0, \ldots, f_n)$ with

$$f_0 = \frac{-1 + \sum_{i=1}^n g_i^2}{1 + \sum_{i=1}^n g_i^2}, \qquad f_j = \frac{2g_j}{1 + \sum_{i=1}^n g_i^2} \qquad \text{for } j = 1, \ldots, n.$$

Now, the assumptions on the algebra R imply the following statements:

(a) $R^n = \{(g_1, \ldots, g_n) \mid g_1, \ldots, g_n \in R\}$ is dense in $C(X, \mathbb{R}^n)$ and mapped onto a dense subset of $C(X, S^n \setminus Z)$,
(b) $\iota_*(R^n) \subseteq S^n(R)$.

Proposition 1.

(1) $\overline{S^n(R)}$ *is invariant under* $O(n+1, R)$,
(2) $\overline{S^n(R)}$ *contains all non-surjective maps* $X \to S^n$,
(3) $\overline{S^n(R)}$ *is contained in the union of the homotopy classes of the maps in* $S^n(R)$.

Proof. (1) This follows from the invariance of $S^n(R)$ and the continuity of the group action.

(2) Let $f = (f_0, \ldots, f_n) : X \to S^n$ be a non-surjective map. Assume first that the center Z is not in $f(X)$. Then the properties of ι_*, shown above, prove that $f \in \overline{S^n(R)}$. Next pick any $a \in S^n \setminus f(X)$. It is well known that the orthogonal group $O(n+1, \mathbb{R})$ acts transitively on S^n. So we find an orthogonal matrix $A \in O(n+1, \mathbb{R}) \subseteq O(n+1, R)$ with $Aa = Z$. Then the continuous map Af omits the point Z, and we find that Af lies in the closure of $S^n(R)$. Using the inverse of A and applying (1), we conclude that f itself is contained in the closure.

(3) follows from the well known statement: if two continuous maps $f, g : X \to S^n$ satisfy $\|f - g\| < 2$ then they are homotopic. ∎

It is one of the main results of this paper, Theorem 2 in Section 4, that the union of the homotopy classes of the maps in $S^n(R)$ coincides with the closure $\overline{S^n(R)}$ if $n = 1, 3, 7$.

There are quite a few examples where $S^n(R)$ is not dense in $C(X, S^n)$ (see the last section). Yet, the algebra R can be used to construct a dense set of maps.

Given $f \in C(X, \mathbb{R}^{n+1})$ we consider its "absolute value"

$$|f| : X \to \mathbb{R}, \qquad x \mapsto \|f(x)\|_2.$$

Obviously, $C(X, S^n) = \{f \in C(X, \mathbb{R}^{n+1}) \mid |f| = 1\}$ and $|(f_0, \ldots, f_n)|^2 = \sum_i f_i^2$.

We will use the set

$$\mathbb{E}_{n+1} = \Big\{ f = (f_0, \ldots, f_n) \in R^{n+1} \,\Big|\, \sum_{i=0}^n f_i^2 \in R^* \Big\}.$$

By our assumption on R we know that $f \in \mathbb{E}_{n+1}$ if and only if $|f|$ has no zero on X, equivalently, if the functions $f_0, \dots, f_n$ have no common zero on X. Any $f \in \mathbb{E}_{n+1}$ gives rise to the continuous map

$$\frac{1}{|f|} f : X \to S^n.$$

PROPOSITION 2. *The family* $\left(\frac{1}{|f|} f\right)$, $f \in \mathbb{E}_{n+1}$, *forms a dense subset and a set of representatives of the homotopy classes in* $C(X, S^n)$.

Proof. Our assumption on R implies that R^{n+1} is a dense subset of $C(X, \mathbb{R}^{n+1})$. Hence, using the compactness of X one derives the first statement. Finally, we again use the fact that if two continuous maps $f, g : X \to S^n$ satisfy $\|f - g\| < 2$ then they are homotopic. ∎

3. Algebraic structures on $C(X, \mathbb{R}^{n+1})$ and $C(X, S^n)$, $n = 1, 3, 7$. In this preparatory section the number n is always equal to 1, 3 or 7. It is a classical result that in these cases the sphere S^n carries a continuous multiplication which will be used to define a continuous multiplication on $C(X, S^n)$. It is this multiplication that leads to the results of this paper.

Multiplication on S^n originates from the field $\mathbb{C}$ if $n = 1$, from the skew field of quaternions $\mathbb{H}$ if $n = 3$, and from the non-associative, alternative division algebra of octonions $\mathbb{O}$ if $n = 7$. We assume familiarity with the definition and properties of these algebras referred to as *composition algebras*. One may consult the following references: [6], [13, III] or [15, Chapter 1: Appendix, Excercises]; the paper [1] by J. C. Baez is a stunning report on the relevance of these algebras for topology.

Let A denote any of these composition algebras. The underlying vector space of A is $\mathbb{R}^{n+1} = \{x = (x_0, \dots, x_n) \mid \text{all } x_i \in \mathbb{R}\}$, endowed with the anisotropic quadratic form $q(x) = \sum_{i=0}^{n} x_i^2$, referred to as the *norm form*, and the linear involution

$$^{-} : A \to A, \qquad x = (x_0, \dots, x_n) \mapsto \bar{x} := (x_0, -x_1, \dots, -x_s, \dots, -x_n).$$

The usual unit vectors are denoted by e_r, $r = 0, \dots, n$. We recall the following basic facts:

(1) e_0 is the multiplicative unit, for any r, s we have $e_r \cdot e_s = \sigma(r, s) e_t$ for some t, $\sigma(r, s) = \pm 1$, $e_r^2 = -e_r$ if $r \geq 1$,

(2) $x\bar{x} = q(x) = q(\bar{x})$, $q(xy) = q(x)q(y)$.

We have $S^n = \{x \in A \mid q(x) = 1\}$, hence S^n is closed under multiplication, also closed under involution and taking the inverse of an element: if $x \in S^n$ then $x^{-1} = \bar{x}$.

We transfer the algebraic structure of the composition algebras to the $C(X,\mathbb{R})$-module

$$\tilde{A} := C(X,\mathbb{R}^{n+1}) = C(X,\mathbb{R})^{n+1}$$

and to $C(X,S^n)$, $n = 1,3,7$, which turns $\tilde{A}$ into an alternative $C(X,\mathbb{R})$-algebra with continuous algebraic operations and $C(X,S^n)$ into an alternative subloop under multiplication.

In what follows, the components of f resp. $g \in \tilde{A}$ are denoted by f_i resp. g_i, $i = 0,\dots,n$.

Let $\widetilde{e_i} = (\dots,0,1,0,\dots)$ denote the constant map $X \to \mathbb{R}^{n+1}$, $x \mapsto e_i$, for $i = 0,\dots,n$. Then $\tilde{A}$ is a free $C(X,\mathbb{R})$-module with basis $\widetilde{e_i}$, $i = 0,\dots,n$, which is actually an orthonormal basis for the norm form $\tilde{q} : \tilde{A} \to C(X,\mathbb{R})$, $f \mapsto \sum_{i=0}^n f_i^2$. There is the $C(X,\mathbb{R})$-linear involution

$$^{-} : \tilde{A} \to \tilde{A}, \qquad f \mapsto \bar{f} = (f_0, -f_1, \dots, -f_s, \dots, -f_n),$$

satisfying $\bar{f}(x) = \overline{f(x)}$ for any $x \in X$.

Using multiplication in the composition algebra $\mathbb{R}^{n+1}$, multiplication on $\tilde{A}$ will be defined pointwise:

$$(f \cdot g)(x) := f(x) \cdot g(x) \qquad \forall x \in X.$$

One verifies the following properties where $f, g \in \tilde{A}$:

(1) $\widetilde{e_0}$ is the multiplicative unit,

$$(*) \qquad f \cdot g = \sum_{t=0}^{n} \Big(\sum_{r,s:\, e_r \cdot e_s = e_t} \sigma(r,s) f_r g_s \Big) \widetilde{e_t},$$

(2) multiplication on $\tilde{A}$ satisfies the *alternative* law:

$$f(fg) = f^2 g, \qquad f(gf) = (fg)f, \qquad (fg)g = fg^2 \qquad \forall f, g \in \tilde{A},$$

(3) we have

$$\overline{fg} = \bar{g}\bar{f}, \qquad f\bar{f} = \tilde{q}(f) = \bar{f}f, \qquad \tilde{q}(fg) = \tilde{q}(f)\tilde{q}(g),$$
$$\bar{f}(fg) = (gf)\bar{f} = \tilde{q}(f)g \qquad \forall f, g \in \tilde{A}.$$

From (1) one sees that $\tilde{A}$ is a $C(X,\mathbb{R})$-algebra with continuous addition and multiplication.

Let R be an admissible $\mathbb{R}$-subalgebra of $C(X,\mathbb{R})$. Then $R^{n+1} \subseteq C(X,\mathbb{R}^{n+1})$ is also a free R-module with basis $\widetilde{e_i}$, $i = 0,\dots,n$, and, in view of $(*)$, an R-subalgebra of $C(X,\mathbb{R}^{n+1})$, closed under involution.

Expressed differently, we see that R^{n+1} is nothing but the R-algebra $R \otimes_{\mathbb{R}} A$ where A denotes the composition algebra of dimension $n+1$ over $\mathbb{R}$.

As the norm form is multiplicative, the image of the restriction of the norm form to R^{n+1} is closed under multiplication. One finds that

$$\tilde{q}(R^{n+1}) = \Big\{ \sum_{i=0}^{n} f_i^2 \;\Big|\; f_0, \dots, f_n \in R \Big\}$$

is the subset of R consisting of all sums of $n+1$ squares in R.

An element $f \in \tilde{A}$ is called *invertible* or a *unit* if there exists $g \in \tilde{A}$ with $fg = gf = 1$. We denote by $\tilde{A}^*$ the set of invertible elements. Using the multiplicativity of the norm form one derives

$$f \text{ is invertible } \iff \tilde{q}(f) \in C(X, \mathbb{R})^* \iff f \in C(X, \mathbb{R}^{n+1} \setminus \{0\}).$$

$\tilde{A}^*$ is closed under multiplication and involution. Multiplication satisfies the alternative law, and each $f \in \tilde{A}^*$ admits a unique two-sided inverse which is $f^{-1} = \frac{1}{\tilde{q}(f)} \bar{f}$ and which satisfies the rule

$$f^{-1}(fg) = (gf)f^{-1} = g \quad \forall g \in \tilde{A}.$$

$\tilde{A}^*$ with multiplication is an example of an alternative loop.

In our situation, $\tilde{A}^*$ is called the *loop of units* of $\tilde{A} = C(X, \mathbb{R}^{n+1})$; it is an abelian group if $n = 1$, a non-abelian group if $n = 3$, and just an alternative loop if $n = 7$. The norm form induces an epimorphism $\tilde{A}^* \to C_+(X, \mathbb{R})$ where $C_+(X, \mathbb{R})$ denotes the subgroup of strictly positive maps $X \to \mathbb{R}$. Obviously, $C(X, S^n) = \{f \in \tilde{A} \mid \tilde{q}(f) = 1\}$ is the kernel of q, which implies that $C(X, S^n)$ is a subloop, meaning in this case that it is closed under multiplication, with unit element $1 = \widetilde{e_0}$, and with each $f \in C(X, S^n)$ admitting an inverse $f^{-1} = \bar{f}$.

In the case of the subalgebra R^{n+1} we find that an element f is invertible in R^{n+1} if and only if f is invertible in $\tilde{A}$, if and only if $\tilde{q}(f) \in R^*$. This means that the loop of units $(R^{n+1})^*$ coincides with the set $\mathbb{E}_{n+1}$ of Section 2. Of course, an invertible element $f \in R^{n+1}$ has inverse $f^{-1} = \frac{1}{\tilde{q}(f)} \bar{f} \in R^{n+1}$.

The norm from Section 1 on the Banach space $C(X, \mathbb{R}^{n+1})$ can be expressed as

$$\|f\| = \max_X \sqrt{\tilde{q}(f)} \quad \text{since} \quad |f| = \sqrt{\tilde{q}(f)}.$$

The norm is well behaved with respect to addition and multiplication:

$$\|f + g\| \le \|f\| + \|g\|, \quad \|f \cdot g\| \le \|f\| \cdot \|g\|, \quad \|h \cdot g\| = \|g\|$$

for $f, g \in C(X, \mathbb{R}^{n+1})$ and $h \in C(X, S^n)$. The first inequality is clear, the second one follows from the multiplicativity of the norm form. The last equality uses the second inequality and additionally the following facts: $\|h\| = \|\bar{h}\| = 1$ and $g = \bar{h}(hg)$.

4. Approximation and homotopy. Again we are dealing with any compact space X and the dimensions $n = 1, 3, 7$.

The first issue we address is homotopy in $C(X, S^n)$. The notation $f \simeq g$ means that f and g are homotopic; we set $I = [0,1] \subseteq \mathbb{R}$.

Homotopy is compatible with multiplication in $C(X, S^n)$:

LEMMA 1. *If $f_1 \simeq g_1$ and $f_2 \simeq g_2$ then $f_1 \cdot f_2 \simeq g_1 \cdot g_2$.*

Proof. If $H_i : X \times I \to S^n$ is a homotopy between f_i and g_i then $H_1 \cdot H_2 \in C(X \times I, S^n)$ is a homotopy between $f_1 \cdot f_2$ and $g_1 \cdot g_2$. ∎

We denote by $\pi^n(X)$ the cohomotopy set of (free) homotopy classes $[f]$ of continuous maps $f : X \to S^n$. The lemma implies that multiplication in $C(X, S^n)$ induces a multiplication in $\pi^n(X)$: $[f][g] = [fg]$. So, $\pi^n(X)$ is an alternative loop with unit element being the class of the constant map $1 : X \to S^n, x \mapsto e_0$. There are the following rules:

$$[f]^{-1} = [\bar{f}], \qquad [f]^{-1}([f][g]) = ([g][f])[f]^{-1} = [g].$$

This loop is a group if $n = 3$, and an abelian group if $n = 1$.

On the other hand, if X is any CW-complex of dimension $\leq 2m - 2$, a general procedure defines the structure of an abelian group on $\pi^m(X)$. In our cases, both structures coincide for a CW-complex meeting the constraint on the dimension. A proof is given in [9, Prop. 5.3]. In fact, the arguments presented are also valid in the case $n = 7$, which is not treated there.

It is unknown whether in our setting of an arbitrary compact space X the cohomotopy loops $\pi^n(X), n = 3, 7$, are always abelian groups.

Nevertheless, we will make ample use of the algebraic structure on the cohomotopy sets.

To describe homotopic pairs f, g we extend an idea of S. Eilenberg. In [7, p. 361] one finds (for metric spaces) the following characterizations of nullhomotopic maps into S^1.

PROPOSITION 3. *Given a compact space X and a continuous map $f : X \to S^1$, the following statements are equivalent:*

(1) *f is nullhomotopic,*
(2) *f is a product of finitely many non-surjective continuous maps of X into S^1,*
(3) *there is a continuous function $\phi : X \to \mathbb{R}$ such that $f(x) = \exp(i\phi(x))$.*

The idea of the proof can be extended to our situation.

THEOREM 1. *Given continuous maps $f, g : X \to S^n$ where $n = 1, 3, 7$, the following statements are equivalent:*

(1) *$f \simeq g$,*
(2) *$g = h \cdot f$ where h is a product of finitely many non-surjective continuous maps $X \to S^n$,*

(3) $g = f \cdot h$ *where* h *is a product of finitely many non-surjective continuous maps* $X \to S^n$.

Proof. To prove that (2) or (3) implies (1) one uses the following facts: a non-surjective map in $C(X, S^n)$ is homotopic to a constant map, the homotopy is compatible with multiplication by the previous lemma, and each map in $C(X, S^n)$ admits an inverse. Concerning the implications (1)⇒(2) and (1)⇒(3), we pass to the nullhomotopic functions $g \cdot \bar{f}$ in the first instance and $\bar{f} \cdot g$ in the second one. In fact, the previous lemma shows that these maps are homotopic to the constant map $1 : X \to S^n$ as $f^{-1} = \bar{f}$. In the next step we are going to prove that a nullhomotopic map h in $C(X, S^n)$ is a product of non-surjective continuous maps $X \to S^n$. Having done this we get $g = (g \cdot \bar{f}) \cdot f = h \cdot f$, and $g = f \cdot h$ in the other case.

So, let $h \in C(X, S^n)$ be nullhomotopic and consider a homotopy $H : X \times I \to S^n$ with $H(\cdot, 0) = 1, H(\cdot, 1) = h$. Clearly, H is uniformly continuous, so there exists $\delta > 0$ such that

$$\|H(x,t) - H(x,t')\|_2 < 2 \quad \text{for all } x \in X \text{ and } t, t' \in I \text{ with } |t - t'| < \delta.$$

We choose a subdivision $0 = t_0 < t_1 < \cdots < t_m = 1$ of I such that $|t_{r+1} - t_r| < \delta$ and set $h_r := H(\cdot, t_r)$. So, $1 = h_0$, $h = h_m$. Then each h_r is a continuous function $X \to S^n$ and we have $\|h_r - h_{r-1}\| < 2$. Set $g_r := \overline{h_{r-1}} \cdot h_r$, $r = 1, \ldots, m$. Then

$$h_{r-1} \cdot g_r = h_r, \qquad \|g_r - 1\| = \|h_{r-1} \cdot (g_r - 1)\| = \|h_r - h_{r-1}\| < 2,$$

and each g_r turns out to be non-surjective. By recursion one proves that $(\ldots(g_1 g_2) g_3 \ldots) g_m = h$. ■

There are two immediate applications.

Corollary 1. *A map* $f \in C(X, S^n)$, $n = 1, 3, 7$, *is nullhomotopic if and only if it is the product of finitely many non-surjective continuous maps* $X \to S^n$.

The second application can be understood as a variant of the Stone–Weierstraß Theorem.

Proposition 4. *Suppose a subset* $\mathcal{F} \subseteq C(X, S^n)$, $n = 1, 3, 7$, *is closed under multiplication and its closure contains all non-surjective maps* $X \to S^n$. *Then* $\overline{\mathcal{F}}$ *is the union of the homotopy classes of the maps in* $\mathcal{F}$.

Proof. First note that the closure is again closed under multiplication. Now, Theorem 1 tells us that the homotopy classes in question are contained in the closure. The other inclusion follows by standard arguments. ■

The foregoing results will be applied to an *admissible* $\mathbb{R}$-algebra $R \subseteq C(X, \mathbb{R})$.

As shown at the end of the previous section, R^{n+1} is an R-subalgebra of the alternative algebra $C(X,\mathbb{R}^{n+1})$, it is closed under involution and admits the multiplicative norm form

$$\tilde{q} : R^{n+1} \to R, \qquad f = (f_0,\dots,f_n) \mapsto f\bar{f} = \sum_{i=0}^{n} f_i^2.$$

The loop of units of R^{n+1} was shown to be

$$(R^{n+1})^* = \{f \in R^{n+1} \mid \tilde{q}(f) \in R^*\} = \mathbb{E}_{n+1}.$$

The homomorphic norm form maps $\mathbb{E}_{n+1}$ onto its image

$$\tilde{q}(\mathbb{E}_{n+1}) = \Big\{ a \in R^* \;\Big|\; a = \sum_{i=0}^{n} a_i^2 \text{ for some } a_0,\dots,a_n \in R \Big\},$$

which shows that the latter set is a subgroup of the group R_+ defined in Section 2.

The loop of units $\mathbb{E}_{n+1}$ contains the set $(R_+)^{n+1}$ which is not a subloop. Surprisingly, as will be shown in the proof of Theorem 3 below, the image under the norm form, i.e.

$$\tilde{q}((R_+)^{n+1})) = \Big\{ a \in R^* \;\Big|\; a = \sum_{i=0}^{n} a_i^2 \text{ for some } a_0,\dots,a_n \in R_+ \Big\},$$

is also a subgroup of R_+, obviously contained in $\tilde{q}(\mathbb{E}_{n+1})$.

The kernel of the norm form $\tilde{q} : (R^{n+1})^* \to R^*$ is just the n-sphere $S^n(R) = \{f \in R^{n+1} \mid \sum_i f_i^2 = 1\}$, which turns out to be a subloop of $C(X,S^n)$ where the two-sided inverse of $f \in S^n(R)$ equals $\bar{f} \in S^n(R)$. The closure $\overline{S^n(R)}$ is a subloop as well since multiplication is continuous.

As a subloop, the set $S^n(R)$ is closed under multiplication. In addition, its closure also contains all non-surjective maps $X \to S^n$, due to Proposition 1. Hence, the last proposition leads to the following result.

THEOREM 2. *If $f \in C(X,S^n)$ is homotopic to some map $g \in S^n(R)$ then $f \in \overline{S^n(R)}$.*

Let $\pi^n_R(X) \subseteq \pi^n(X)$ denote the subset consisting of the homotopy classes of maps in $S^n(R)$. The epimorphism $p : C(X,S^n) \to \pi^n(X)$, $f \mapsto [f]$, maps $S^n(R)$ onto $\pi^n_R(X)$, so this last set is an alternative subloop of the cohomotopy loop. From Theorem 2 we deduce

$$p^{-1}(\pi^n_R(X)) = \overline{S^n(R)}.$$

In what follows, we will construct an invariant, i.e. an epimorphism $i : \pi^n(X) \to G$, G an abelian group, such that

$$\ker(i) = \pi^n_R(X).$$

All this will be summarized in the following theorem.

Theorem 3. *If $n = 1, 3, 7$ and R is any admissible $\mathbb{R}$-algebra then:*

(1) $\tilde{q}\big((R_+)^{n+1}\big)$ *is a subgroup of the group* $\tilde{q}(\mathbb{E}_{n+1})$,
(2) *there are isomorphisms, induced by p and i resp.,*

$$C(X, S^n)/\overline{S^n(R)} \xrightarrow{\sim} \pi^n(X)/\pi^n_R(X) \xrightarrow{\sim} \tilde{q}(\mathbb{E}_{n+1})/\tilde{q}\big((R_+)^{n+1}\big).$$

Before entering the proof, a remark on factor structures of loops modulo a subloop is in order. As we will see below, we are in a favorable situation. In fact, we always deal with an epimorphism $\tau : L \to L'$ of alternative loops, both satisfying $f^{-1}(fg) = (gf)f^{-1} = g$, and the kernel $N := \ker(\tau)$. One obtains a factor loop $L/N := \{aN \mid a \in L\}$ with multiplication $aN \cdot bN = (ab)N$ where we have $aN = Na = \{b \in L \mid \tau(b) = \tau(a)\}$, and $aN = bN \Leftrightarrow ab^{-1} \in N$. In addition, the usual homomorphism theorem can be extended from group theory and states that τ induces an isomorphism between L/N and L'. By the way, an additional example is provided by the isomorphism $C(X, S^n)/N \xrightarrow{\sim} \pi^n(X)$ with $N = \{\text{finite products of non-surjective maps}\}$ (cf. Corollary 1).

Proof of Theorem 3. The proof makes use of the loop structures on $C(X, S^n)$, $\pi^n(X), S^n(R)$ and Proposition 2 which states that the maps $\phi(f) := \frac{1}{|f|}f$, $f \in \mathbb{E}_{n+1}$ form a set of representatives of the homotopy classes in $C(X, S^n)$. In the present situation, the assignment $\phi : f \mapsto \phi(f)$ is a homomorphism from the loop $\mathbb{E}_{n+1} = (R^{n+1})^*$ to $C(X, S^n)$, as $|fg| = |f| \cdot |g|$.

We first claim that

$$\phi(f) \in \overline{S^n(R)} \iff \tilde{q}(f) \in \tilde{q}((R_+)^{n+1}).$$

Assume $\phi(f) \in \overline{S^n(R)}$. Then we find $g \in S^n(R)$ with $\|\phi(f) - g\| < \frac{1}{\sqrt{n+1}}$. The constant function $h = (\frac{1}{\sqrt{n+1}}, \ldots, \frac{1}{\sqrt{n+1}}) : X \to S^n$ also lies in the loop $S^n(R)$. So, there is $a \in S^n(R)$ subject to $ag = h$. Therefore,

$$\|\phi(f) - g\| = \|a(\phi(f) - g)\| = \|\phi(af) - h\| < \frac{1}{\sqrt{n+1}},$$

by the remark at the end of Section 3 and $\phi(a) = a$.

Set $af = (u_0, \ldots, u_n) \in R^{n+1}$. We derive that $\left|\frac{1}{|f|}u_i - \frac{1}{\sqrt{n+1}}\right| < \frac{1}{\sqrt{n+1}}$ for each i. This implies that each u_i is a strictly positive function, i.e. $u_i \in R_+$. Hence, $\tilde{q}(f) = \tilde{q}(af) = \sum_{i=0}^n u_i^2 \in \tilde{q}((R_+)^{n+1})$.

Conversely, assume $\tilde{q}(f) = \sum_{i=0}^n u_i^2$ with each u_i in R_+. Then set $g = (u_0, \ldots, u_n)$. From $\tilde{q}(f) = \tilde{q}(g) \in R^*$ we deduce $\tilde{q}(g^{-1}f) = 1$, hence $f = gh$ with $h \in S^n(R)$ and $\phi(f) = \phi(g)h$. Clearly, $\phi(g)$ is a non-surjective map because all its components are positive functions. So, $\phi(g)$ lies in $\overline{S^n(R)}$, and hence so does $\phi(f)$ as well.

As a consequence, $\tilde{q}((R_+)^{n+1})$ is a group under multiplication. In fact, if $u = \sum_i u_i^2$ and $v = \sum_i v_i^2$, with all u_i, v_i in R_+, set $f = (u_0, \ldots, u_n)$ and

$g = (v_0, \ldots, v_n)$. Then $f, g \in \mathbb{E}_{n+1}$ and $\phi(fg) = \phi(f)\phi(g) \in \overline{S^n(R)}$ as both $\phi(f), \phi(g)$ are surely non-surjective functions. This leads to $uv = \tilde{q}(f)\tilde{q}(g) = \tilde{q}(fg) \in \tilde{q}((R_+)^{n+1})$. Finally, we see $u^{-1} = \sum_i (u_i/u)^2 \in \tilde{q}((R_+)^{n+1})$.

The invariant $i : \pi^n(X) \to G$ is now defined as follows:

$$G := \tilde{q}(\mathbb{E}_{n+1})/\tilde{q}((R_+)^{n+1}), \qquad i([\phi(f)]) = \text{ class of } \tilde{q}(f) \text{ in } G.$$

This assignment produces a well-defined map. In fact, if $[\phi(f)] = [\phi(g)]$ then $[\phi(fg^{-1})] = 1$, which firstly implies that $\phi(fg^{-1}) \in \overline{S^n(R)}$ (cf. Corollary 1), then that the class of $\tilde{q}(f)(\tilde{q}(g))^{-1}$ is trivial in G, i.e. f and g give rise to the same image in G. As every homotopy class is of the form $[\phi(f)]$, the invariant is now fully defined.

Since the norm form $\tilde{q}$ is homomorphic, the invariant i is an epimorphism with kernel $\ker(i) = \pi^n_R(X)$. This yields the isomorphism on the right hand side in the theorem. The isomorphism on the left hand side is induced by the epimorphism $C(X, S^n) \to \pi^n(X)/\pi^n_R(X)$, $f \mapsto$ class of $[f]$, since its kernel equals $\overline{S^n(R)}$.

The proof of Theorem 3 is complete. ∎

It should be noted that Theorem 3 can be read in either way: to determine $\overline{S^n(R)}$ resp. $\pi^n_R(X)$ inside $C(X, S^n)$ resp. $\pi^n(X)$ via algebra or, vice versa, to determine $\tilde{q}\big((R_+)^{n+1}\big)$ inside $\tilde{q}(\mathbb{E}_{n+1})$ via topology. We single out an interesting case.

Corollary 2. *If $n = 1, 3, 7$ the following statements are equivalent:*

(1) *$S^n(R)$ is dense in $C(X, S^n)$,*
(2) $\pi^n_R(X) = \pi^n(X)$,
(3) $\tilde{q}\big((R_+)^{n+1}\big) = \tilde{q}(\mathbb{E}_{n+1})$.

5. Real algebraic geometry. The book [3, Chapter 11 ff.] and the papers [4, 8, 10, 11, 12] provide an overview of the broad area of research for which results on homotopy and approximation in $C(X, Y)$ are most relevant. The special case of $Y = S^p$, p any number, is still of great interest. As samples, we are going to describe a small segment of problems and results appearing in real algebraic geometry and related areas.

We start with a compact real algebraic subset $X \subseteq \mathbb{R}^k$ and consider two $\mathbb{R}$-algebras associated with X, naturally embedded in $C(X, \mathbb{R})$: the algebra $\mathcal{P}(X)$ of polynomial functions $X \to \mathbb{R}$ and the algebra $\mathcal{R}(X)$ of (real) regular functions (see [3, Chapter 3]). Here $\mathcal{P}(X)$ is an affine $\mathbb{R}$-algebra, and using [3, 4.1.9] one obtains

$$\mathcal{R}(X) = \mathcal{P}(X)_S \quad \text{with} \quad S = \Big\{1 + \sum_{i=1}^{k} f_i^2 \;\Big|\; k \in \mathbb{N},\, f_1, \ldots, f_k \in \mathcal{P}(X)\Big\}.$$

Due to the Stone–Weierstraß Theorem both algebras are dense in $C(X,\mathbb{R})$; the algebra $\mathcal{R}(X)$ of regular functions is admissible, but $\mathcal{P}(X)$ is not, except for special cases. So, we continue with the algebra of regular functions. By the way, it is the smallest admissible subalgebra of $C(X,\mathbb{R})$ which contains $\mathcal{P}(X)$.

In the literature, the notation $\mathcal{R}(X,Y)$ is used for the set of regular maps between two real algebraic sets, i.e. those maps where all components are regular functions on X. Recall that $\mathcal{R}(X,S^p)$ is just $S^p(\mathcal{R}(X))$ in our notation.

Section 4 of [12], entitled "Homotopy and Approximation", provides a comprehensive account of the current state of the art. The following two questions were posed for a continuous map $f : X \to S^p$:

(i) Is f homotopic to a map in $\mathcal{R}(X,S^p)$?
(ii) Can f be approximated by maps from $\mathcal{R}(X,S^p)$?

It is conjectured that both questions are equivalent. This was proven for $p = 1,2,4$ in [4]. Now, incorporating Theorem 2 above, we can settle two more cases and get:

THEOREM 4. *For X as above and a continuous map $X \to S^p$, $p = 1,2,3,4,7$, the following statements are equivalent:*

(1) *f is homotopic to a regular map,*
(2) *f can be approximated by regular maps.*

REMARKS. Possibly, Proposition 4 above may help to identify other situations where homotopy and approximation are closely linked. Also, Theorem 2 is valid for any admissible algebra, so possible applications are not confined to the algebra of regular functions.

We next turn to the question whether $S^p(R)$ is dense in $C(X,S^p)$ and impose the assumption that X is a *compact, smooth* (= *non-singular*) *irreducible real algebraic set* in some $\mathbb{R}^k$. Under this condition, kept fixed for the rest of this section, the algebra $\mathcal{P}(X)$ is an integral affine $\mathbb{R}$-algebra. Its quotient field, denoted by $\mathbb{R}(X)$, is called the *function field* of X. Its elements are the rational functions on X which give rise to functions on non-empty Zariski-open subsets of X. Under the present assumption, non-empty Zariski-open subsets are dense in X with respect to the Euclidean topology.

The smooth algebraic set X is a differentiable submanifold of $\mathbb{R}^k$, and its topological and algebraic dimensions are equal. Clearly, $\mathcal{R}(X) \subseteq C^1(X,\mathbb{R})$. Using the fact that differentiable maps into a manifold of higher dimension are never surjective, as well as Propositions 1 and 2, one obtains the following well-known result.

Proposition 5. *If* $\dim(X) < p$ *then* $\mathcal{R}(X, S^p)$ *is dense in* $C(X, S^p)$.

Consider the case $\dim(X) = p$. If $\dim(X) = 1$, then each continuous map $X \to S^1$ can be approximated by regular maps [12, 4.4]. But there are counterexamples for $p > 1$: [3, Chapter 13], [11, 12]. However, by combining Theorem 2 with [5, Corollary 2.5] we get further cases where $S^p(R)$ is dense in $C(X, S^p)$.

Proposition 6. *Assume* $p = 3, 7$, *and let* X *be a connected and oriented* C^∞*-manifold with* $\dim(X) = p$. *If* X *admits a regular map to* S^p *of odd topological degree then* $\mathcal{R}(X, S^p)$ *is dense in* $C(X, S^p)$.

Proof. From [5] we know that in this situation every continuous map $X \to S^p$ is homotopic to a regular map. Now apply Theorem 2. ∎

Clearly, S^n satisfies the condition of the last proposition. Previous results state that $\mathcal{R}(S^n, S^p)$ is dense in $C(S^n, S^p)$ if either $n < p$ or n arbitrary but $p = 1, 2, 4$ [4, 12]. As a consequence of the last theorem, we can add two more cases:

Corollary 3. $\mathcal{R}(S^p, S^p)$ *is dense in* $C(S^p, S^p)$ *if* $p = 3, 7$.

About 15 years ago W. Kucharz drew attention to the so-called "continuous rational functions" (or "régulues" [8], "regulous" [12]). They are defined as those rational functions $f \in \mathbb{R}(X)$ which admit a continuous extension to the whole algebraic set X, beyond their domain $\operatorname{dom}(f)$ of definition. The extension is unique since $\operatorname{dom}(f)$ is dense in X. The continuous rational functions form an $\mathbb{R}$-algebra, denoted by $\mathcal{R}^0(X)$, which is naturally embedded in $C(X, \mathbb{R})$.

So far we have introduced three algebras inside the function field which are also dense subalgebras of $C(X, \mathbb{R})$:

$$\mathcal{P}(X) \subseteq \mathcal{R}(X) \subseteq \mathcal{R}^0(X).$$

One checks that, additionally, $\mathcal{R}^0(X)$ is an admissible algebra.

The larger algebra of continuous rational functions $\mathcal{R}^0(X)$ allows density results even in cases where the algebra of regular functions is too small. E.g., the following was proven in [12, Section 4]:

Proposition 7. *Assume* $\dim(X) = p$. *Then:*

(1) $\mathcal{R}^0(X, S^p) := S^p(\mathcal{R}^0(X))$ *is dense in* $C(X, S^p)$,
(2) $\mathcal{R}^0(S^n, S^p)$ *is dense in* $C(S^n, S^p)$ *for every* n.

Corollary 2 above allows one to draw consequences for the units of a given admissible algebra R provided density can be proven. The author is not aware of any purely algebraic proof which yields the arithmetic statement below.

Summarizing the results above we obtain two lists of favorable cases *where always* $p = 1, 3, 7$:

(I) "$\dim(X) < p$" or "$\dim(X) = p$ and X satisfies the conditions of Proposition 6" or "$X = S^p$"; $R = \mathcal{R}(X)$,
(II) "$\dim(X) \le p$" or "$X = S^n$, any n"; $R = \mathcal{R}^0(X)$.

Proposition 8. *In each of the cases of the lists* (I) *and* (II), *every unit in* R *which is a sum of* $p+1$ *squares in* R *is already a sum of* $p+1$ *squares of positive units.*

We conclude this paper by a few remarks concerning the impact of the algebras of continuous rational functions on the theory of the so-called *real holomorphy ring* H of the function field $F := \mathbb{R}(X)$, X a compact, smooth, irreducible real algebraic set. This ring is defined as

$$H = \Big\{ f \in F \;\Big|\; r \pm f \in \sum F^2 \text{ for some } r \in \mathbb{N} \Big\};$$

it plays a role in the birational geometry of the function field F, as far as real points on smooth models of F are concerned, and in the study of sums of powers in F [2, 10, 14]. Here, the notation "$\sum F^2$" stands for the subset consisting of all sums of squares in F.

One first realizes the inclusion $\mathcal{R}^0(X) \subseteq H$. In fact, a rational function f has an extension to a continuous function $\bar{f} : X \to \mathbb{R}$. As X is compact we find a bound $r \in \mathbb{N}$ such that $r \pm f \ge 0$ on $\operatorname{dom}(f)$. Using Artin's solution of the 17th problem of Hilbert one gets $r \pm f \in \sum F^2$, i.e. $f \in H$.

The real holomorphy ring H admits a natural injective representation $H \to C(M, \mathbb{R})$ where M denotes the compact space of real places of F. The image of H is an admissible algebra. As explained in [14], from Hironaka's theory of the resolution of singularities one derives geometric interpretations for H and M. By including all iterated blowing ups of X (or $\operatorname{Spec}(\mathcal{P}(X))$) one finally obtains a directed family $(Y_\alpha)_\alpha$ of compact, smooth, irreducible real algebraic sets and the directed family of algebras $\mathcal{R}(Y_\alpha)$ such that

$$H = \varinjlim \mathcal{R}(Y_\alpha) \quad \text{and} \quad M = \varprojlim Y_\alpha.$$

This powerful result implies:

- a given continuous rational function on X becomes a regular function on some iterated blowing up Y of X,
- $S^p(H)$ is dense in $C(M, S^p)$ if $p \ge \dim(X)$.

The proof of the latter statement in case $p > \dim(X)$ is easy and applies Proposition 5; the case $p = \dim(X)$ requires the much deeper result presented in Proposition 7 making use of the rings of continuous rational functions. Invoking only rings of regular functions would not work.

The features of the real holomorphy ring allow one to go beyond the scope of Theorems 2 and 3 above.

Firstly, for any dimension n and any map $f : M \to S^n$ which is homotopic to some $g \in S^n(H)$ one can show that $f \in \overline{S^n(H)}$.

Secondly, it can be proven for all $p \geq \dim(X)$ that units of H which are sums of $p + 1$ squares are already sums of $p + 1$ squares of positive units. The latter result provides sharper bounds for the so-called higher Pythagoras numbers than those presented in [2].

Proofs for these special statements on real holomorphy rings and the application to higher Pythagoras numbers rings will be published elsewhere.

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