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GROUPS WITH FRAMES OF TRANSLATES

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Abstract. Let G be a locally compact group with left regular representation λ_G. We say that G admits a frame of translates if there exist a countable set $\Gamma \subset G$ and $\varphi \in L^2(G)$ such that $(\lambda_G(x)\varphi)_{x\in\Gamma}$ is a frame for $L^2(G)$. The present work aims to characterize locally compact groups having frames of translates, and to this end, we derive necessary and/or sufficient conditions for the existence of such frames. Additionally, we exhibit surprisingly large classes of Lie groups admitting frames of translates.

1. Introduction. Throughout this paper, G denotes a second countable locally compact group G with Haar measure μ_G. We denote the associated L^2-space by $L^2(G)$. The group G acts on this space unitarily via the left regular representation, which we denote by λ_G.

We briefly recall the definitions of *frames* and *Bessel sequence*: A system $(\eta_i)_{i\in I}$ of vectors in a Hilbert space is called a *frame* if there exist constants $0 < A \leq B < \infty$ such that, for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, \eta_i\rangle|^2 \leq B\|f\|^2.$$

The constants A, B are called *frame bounds*. We refer to A as a *lower frame bound* and to B as an *upper frame bound*. If $(\eta_i)_{i\in I}$ only admits an upper frame bound then we say that $(\eta_i)_{i\in I}$ is a *Bessel family*.

An often studied setup for frame construction, which can be traced back to the origins of frame theory in [9], uses group representations. Here, a unitary representation $(\pi, \mathcal{H}_\pi)$ of a locally compact group G is considered, and one asks for frames of the type $(\pi(x)\varphi)_{x\in\Gamma}$ of the representation space $\mathcal{H}_\pi$, for a suitable subset Γ and a generator φ. Investigations of constructions of this type abound in wavelet theory and its generalizations (see e.g. [8, 1, 2]), as well as in time-frequency analysis [4, 21]. Further relevant sources can be found in the references of the cited books [2, 21]. The problem of constructing frames from group actions is also central to coorbit theory [12],

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which is currently actively pursued [7, 18]. Outside of generalized wavelet and time-frequency analysis, the problem of frame construction via group representation has recently been studied in the realm of solvable and nilpotent Lie groups [27, 22], and in the setting of finite groups [29, 28].

In this paper, we study frame construction for the left regular representation λ_G of the group G, which is the unitary action of G on $L^2(G)$ by left translations. We are interested in groups of the following type:

DEFINITION 1.1. *G admits a frame of translates*, or *is an FT group*, if there exists a family $\Gamma \subset G$ and $\varphi \in L^2(G)$ such that the family $(\lambda_G(x)\varphi)_{x\in\Gamma} \subset L^2(G)$ is a frame, i.e., there exist constants $0 < A \le B < \infty$ such that, for all $g \in L^2(G)$,

$$A\|g\|_2^2 \le \sum_{x\in\Gamma} |\langle g, \lambda_G(x)\varphi\rangle|^2 \le B\|g\|_2^2.$$

The next remark lists some known or expected results. Part (b) is [4, Theorem 1.2], and the proof given in [4] relies on fairly technical concepts such as Beurling density.

REMARK 1.2. (a) If G is discrete, then $(\lambda_G(x)\delta_e)_{x\in G}$ is an orthonormal basis of $L^2(G) = \ell^2(G)$, where δ_e denotes the Kronecker delta at the neutral element. Hence G is an FT group.

(b) $G = \mathbb{R}^d$ is not an FT group.

The main objective of the present work is to investigate solutions to the following question.

QUESTION 1.3. Which nondiscrete locally compact groups have the FT property?

Intuitively, one may not expect many positive answers to Question 1.3 outside the discrete case. One source of skepticism in this regard is provided by Theorem 2.11 below, describing a *universal procedure for frame construction*, for a large class of representations covering all of the above-mentioned settings from wavelet and time-frequency analysis. This procedure is applicable once the underlying group is recognized to have the FT property. However, our results will show that FT groups are surprisingly not rare.

REMARK 1.4. Instead of using subsets $\Gamma \subset G$, one may pose the central question of our paper with reference to families $(x_i)_{i\in I} \subset G$ and associated families of vectors $(\lambda_G(x_i)\varphi)_{i\in I}$. The difference between the two is that the latter allows repetitions of elements. However, it is easy to see that $(\lambda_G(x_i)\varphi)_{i\in I}$ is a frame if and only if $(\lambda_G(x)\varphi)_{x\in\Gamma}$ is a frame, where $\Gamma = \{x_i : i \in I\}$, and additionally, $\sup_{y\in\Gamma} \sharp\{i \in I : y = x_i\} < \infty$. Hence exchanging subsets for families does not affect the FT property, and we will switch between the two notions if need be.

In the course of this paper, we will freely use notions from frame theory, representation theory of locally compact groups and Lie theory; our primary references for these topics are [3], [13], and [23] respectively.

2. Necessary criteria. In this section, we will consider various necessary conditions for frames of the type $(\lambda_G(x)\varphi)_{x\in\Gamma}$. These conditions will concern either the family Γ of shifts, the function φ, or the group. In our analysis, we proceed precisely in this order.

DEFINITION 2.1. Let Γ be a subset of G. We say that Γ is (*left*) *relatively separated* if $\sup_{x\in G} \sharp(\Gamma\cap xU) < \infty$ for some and hence all relatively compact neighborhoods U of the identity. Next, we say that Γ is V*-separated* if for some relatively compact neighborhood V of the identity, the family $(xV)_{x\in\Gamma}$ consists of pairwise disjoint sets; and Γ is called *separated* if it is V-separated for some suitable V.

The following result has been rediscovered several times in frame theory. We rephrase it for our setting.

LEMMA 2.2. *If $(\lambda(\gamma)\varphi)_{\gamma\in\Gamma}$ is a Bessel family then Γ must be relatively separated.*

Proof. Suppose that Γ is not relatively separated. We consider the function $x \mapsto \langle\varphi, \lambda(x)\varphi\rangle$ defined over G. Since $x \mapsto \langle\varphi, \lambda(x)\varphi\rangle$ is continuous, there exists an open set V around the identity element such that $\inf\{|\langle\varphi, \lambda(x)\varphi\rangle| : x \in V\} = \mu > 0$. Given an arbitrary natural number N, the assumption that Γ is not relatively separated provides a $y \in G$ such that yV contains at least N elements from Γ. Letting $\Gamma_N = yV \cap \Gamma$, we have

$$\begin{aligned}\sum_{\gamma\in\Gamma} |\langle\lambda(y)\varphi, \lambda(\gamma)\varphi\rangle|^2 &\geq \sum_{\gamma\in\Gamma_N} |\langle\lambda(y)\varphi, \lambda(\gamma)\varphi\rangle|^2 = \sum_{\gamma\in\Gamma_N} |\langle\varphi, \lambda(y^{-1}\gamma)\varphi\rangle|^2 \\ &= \sum_{y\alpha\in\Gamma_N} |\langle\varphi, \lambda(y^{-1}y\alpha)\varphi\rangle|^2 = \sum_{\alpha\in y^{-1}\Gamma_N} |\langle\varphi, \lambda(\alpha)\varphi\rangle|^2 \\ &\geq \sharp(y^{-1}\Gamma_N)\cdot\mu^2 \geq N\cdot\mu^2 = \frac{N\cdot\mu^2}{\|\varphi\|^2}\|\lambda(y)\varphi\|^2.\end{aligned}$$

Since N is arbitrary, this computation shows that $(\lambda(\gamma)\varphi)_{\gamma\in\Gamma}$ is not a Bessel sequence. ∎

The following result is formulated in [12, Lemma 3.3], with proof attributed to [11]. Since we need it in the following, and the argument in [11] is given for the slightly different context of admissible coverings, we include a short proof.

LEMMA 2.3. *Let $\Gamma \subset G$ denote a relatively separated set of a locally compact group G. Then Γ is the finite union of separated sets.*

Proof. Fix a relatively compact and symmetric neighborhood $V \subset G$ of the identity. Then relative separatedness of Γ yields

$$\sup_{\gamma \in \Gamma} \sharp\{\gamma' \in \Gamma : \gamma' V \cap \gamma V \neq \emptyset\} \leq \sup_{x \in G} \sharp\{\gamma' \in \Gamma : \gamma' \in xV^2\} = m < \infty.$$

Zorn's Lemma allows us to choose a subset $\Gamma_1 \subset \Gamma$ that is V-discrete and maximal with respect to inclusion. If $\Gamma_1 = \Gamma$, then Γ itself is V-discrete. We continue this procedure of choosing a maximal V-discrete $\Gamma_{s+1} \subset \Gamma \setminus \bigcup_{j \leq s} \Gamma_j$, as long as the complement is nonempty. We claim that this procedure breaks off after at most $m+1$ steps. Assuming that $\gamma_0 \in \Gamma \setminus \bigcup_{j \leq m+1} \Gamma_j$, we find for every fixed $1 \leq j \leq m+1$ that $\gamma_0 \in \Gamma \setminus \bigcup_{i<j} \Gamma_i$, and by maximality of Γ_j, $\Gamma_j \cup \{\gamma_0\}$ is not V-discrete. Hence there exists $\gamma_j \in \Gamma_j$ such that $\gamma_0 V \cap \gamma_j V \neq \emptyset$. Since the Γ_j are pairwise disjoint, this entails

$$\sharp\{\gamma' \in \Gamma : \gamma' V \cap \gamma_0 V \neq \emptyset\} \geq m+1,$$

contrary to our choice of m. ∎

Next, we derive necessary conditions on the function φ giving rise to frames of translates. Our aim is to show that for nondiscrete groups such functions must necessarily be somewhat pathological. For instance, bounded functions with compact support will not do. In fact, we will be able to exclude a substantially larger space of functions, namely a particular *Wiener amalgam space*.

We define a local maximum function as follows: Fix a compact neighborhood U of the identity. Given a measurable function f on G, we define

$$f_U^\sharp(x) = \operatorname*{ess\,sup}_{y \in xU} |f(y)|.$$

Next, given $1 \leq p \leq \infty$, we define the Wiener amalgam space $W(L^\infty, L^p)$ as the space of Borel functions f for which the norm

$$\|f\|_{W(L^\infty, L^p)} = \|f_U^\sharp\|_p$$

is finite. It is well-known that, up to equivalence, the Wiener amalgam norm does not depend on the choice of U.

Note that compactly supported bounded functions φ are in $W(L^\infty, L^p)$ for all $1 \leq p \leq \infty$. For the following result, we also need the following. The convolution of two functions f, φ on G is defined as the integral

$$(f * \varphi)(x) = \int_G f(y)\varphi(y^{-1}x)\, dy$$

and we write $\varphi^*(x) = \overline{\varphi(x^{-1})}$. Then a straightforward calculation gives $(f * \varphi^*)(x) = \langle f, \lambda_G(x)\varphi \rangle$ for all $f, \varphi \in L^2(G)$.

Now the next proposition excludes compactly supported bounded functions from frame generation.

PROPOSITION 2.4. *Let G be nondiscrete. Let $\varphi \in L^2(G)$ be such that $\varphi^* \in W(L^\infty, L^2)$. Then there does not exist a discrete set $\Gamma \subset G$ such that $(\lambda_G(x)\varphi)_{x\in\Gamma}$ is a frame of $L^2(G)$.*

Proof. Suppose for contradiction that $(\lambda_G(x)\varphi)_{x\in\Gamma}$ is a frame with lower frame bound A. Then, for all $f \in L^2(G)$,

$$\|f\|_2^2 \le A^{-1} \sum_{x\in\Gamma} |f * \varphi^*(x)|^2. \tag{2.1}$$

By Lemmas 2.2 and 2.3, we can write $\Gamma = \bigcup_{i=1}^n \Gamma_i$ disjointly, and all Γ_i are U-discrete for a suitable symmetric neighborhood U of the identity. Hence we get

$$\begin{aligned}\sum_{x\in\Gamma} |f * \varphi^*(x)|^2 &= \sum_{i=1}^n \sum_{x\in\Gamma_i} |f * \varphi^*(x)|^2 = \sum_{i=1}^n \sum_{x\in\Gamma_i} \frac{1}{|U|} \int_{xU} |f * \varphi^*(x)|^2 \, dy \\ &\le \sum_{i=1}^n \sum_{x\in\Gamma_i} \frac{1}{|U|} \int_{xU} |(f * \varphi^*)_U^\sharp(y)|^2 \, dy \le \frac{n}{|U|} \|(f * \varphi^*)_U^\sharp\|_2^2.\end{aligned}$$

Applying the well-known pointwise estimate

$$(f * g)_U^\sharp(y) \le (|f| * g_U^\sharp)(y),$$

valid for arbitrary measurable functions f, g, yields

$$\|(f * \varphi^*)_U^\sharp\|_2 \le \big\| |f| * (\varphi^*)_U^\sharp \big\|_2.$$

Coming back to (2.1), we thus obtain

$$\|f\|_2 \le \left(\frac{n}{A|U|}\right)^{1/2} \big\| |f| * (\varphi^*)_U^\sharp \big\|_2.$$

On the other hand, Young's inequality yields, for all $f \in L^1(G) \cap L^2(G)$,

$$\big\| |f| * (\varphi^*)_U^\sharp \big\|_2 \le \|f\|_1 \|(\varphi^*)_U^\sharp\|_2 = \|f\|_1 \|\varphi^*\|_{W(L^\infty, L^2)},$$

and the Wiener amalgam norm is finite by assumption.

In summary, we have shown

$$\|f\|_2 \le \left(\frac{n}{A|U|}\right)^{1/2} \|\varphi^*\|_{W(L^\infty,L^2)} \|f\|_1 \tag{2.2}$$

for all $f \in L^1(G) \cap L^2(G)$. Now replacing f with the indicator function of V_n for a sequence of nonempty open sets V_n with $\lambda_G(V_n) \to 0$ yields the desired contradiction. Note that such a sequence exists precisely when G is nondiscrete. ■

We next derive various classes of groups that are not FT.

COROLLARY 2.5. *If G is compact, then it is FT if and only if it is finite.*

Proof. By Lemma 2.2 and the fact that relatively separated subsets of compact groups are finite, $L^2(G)$ is finite-dimensional whenever G is a compact FT group. ∎

THEOREM 2.6. *Let G be nondiscrete, satisfying the following property: The inverse of any subset of G which is relatively separated is also relatively separated. Then G is not an FT group.*

Proof. Suppose by way of contradiction that the stated assumptions hold and that there exists $\varphi \in L^2(G)$ and $\Gamma \subset G$ such that $(\lambda(\gamma)\varphi)_{\gamma\in\Gamma}$ is a frame for $L^2(G)$. Thus, $(\lambda(\gamma)\varphi)_{\gamma\in\Gamma}$ is a Bessel sequence. By Lemma 2.2, Γ must be relatively separated. By assumption, Γ^{-1} is also relatively separated. Furthermore, according to Lemma 2.3, Γ^{-1} can be written as a disjoint union of separated sets. Let $\Gamma^{-1} = \bigcup_{k=1}^{s} \Psi_k$ where each collection $\{xV : x \in \Psi_k\}$ consists of essentially disjoint subsets of G. Then

$$\begin{aligned}
\sum_{\gamma\in\Gamma} |\langle \chi_V, \lambda(\gamma)\varphi\rangle|^2 &= \sum_{\gamma\in\Gamma} |\langle \chi_V, \chi_V \cdot \lambda(\gamma)\varphi\rangle|^2 \\
&\le \|\chi_V\|^2 \Big(\sum_{k=1}^{s} \sum_{\gamma\in\Psi_k} \|\chi_V \cdot \lambda(\gamma)\varphi\|^2\Big) \\
&= \|\chi_V\|^2 \Big(\sum_{k=1}^{s} \sum_{\gamma\in\Psi_k} \int_V |\varphi(\gamma^{-1}x)|^2\, dx\Big) \\
&= \|\chi_V\|^2 \sum_{k=1}^{s} \Big(\int_{\bigcup_{\gamma\in\Psi_k} \gamma^{-1}V} |\varphi(x)|^2\, dx\Big).
\end{aligned}$$

Note that since φ is square-integrable,

$$\int_{\bigcup_{\gamma\in\Psi_k} \gamma^{-1}V} |\varphi(x)|^2\, dx \le \int_G |\varphi(x)|^2\, dx = \|\varphi\|^2 < \infty.$$

By Lebesgue's Dominated Convergence Theorem, taking a nested family of relatively compact and open sets converging to the singleton containing the identity in G, we obtain

$$\lim_{V\to\{e\}} \int_{\bigcup_{\gamma\in\Psi_k} \gamma^{-1}V} |\varphi(x)|^2\, dx = 0.$$

Thus for any $\epsilon > 0$, there exists a sufficiently small open set V around the identity such that

$$\sum_{\gamma\in\Gamma} |\langle \chi_V, \lambda(\gamma)\varphi\rangle|^2 \le \epsilon \|\chi_V\|^2.$$

This violates the lower frame bounds condition and gives us the desired contradiction. ∎

This observation allows us to generalize [4, Theorem 1.2] to a larger class of groups called *[IN]-groups*, groups having a *compact neighborhood* of the identity which is invariant under all inner automorphisms.

LEMMA 2.7. *If G is an [IN]-group then every relatively separated subset of G has a relatively separated inverse. In particular, nondiscrete [IN]-groups are not FT.*

Proof. Let G be an [IN]-group. Then by definition, there exists a compact neighborhood $W \subset G$ of the identity which is conjugation-invariant. Next, let Γ be a subset of G which is relatively separated. By assumption, $\sup_{x\in G} \sharp(\Gamma \cap xW) < \infty$. On the other hand for any $x \in G$,

$$\Gamma \cap xW = \Gamma \cap (xWx^{-1})x = \Gamma \cap Wx,$$

and thus

$$(\Gamma \cap xW)^{-1} = \Gamma^{-1} \cap x^{-1}W^{-1}.$$

Consequently, since inversion on G is bijective,

$$\sup_{y\in G} \sharp(\Gamma^{-1} \cap yW^{-1}) = \sup_{x\in G} \sharp(\Gamma^{-1} \cap x^{-1}W^{-1}) = \sup_{x\in G} \sharp(\Gamma \cap xW) < \infty$$

which proves that Γ^{-1} is relatively separated. ∎

REMARK 2.8. Clearly, Lemma 2.7 applies to abelian groups, thus it directly generalizes [4, Theorem 1.2].

A result due to Iwasawa [25, Theorem 2] implies that a connected topological group G is an [IN]-group if and only if the topological commutator of G is compact.

A nonabelian group to which this applies is the reduced Weyl–Heisenberg group, which is the quotient of the simply connected, connected Heisenberg group by a central discrete subgroup. More generally, Lemma 2.7 also implies that no step-two nilpotent Lie group with compact center is FT.

It is currently open which groups have the property that inverses of relatively separated sets are relatively separated again. As the previous remark shows, some nonabelian nilpotent Lie groups do. However, *simply connected* nilpotent Lie groups generally do not:

LEMMA 2.9. *If G is a nonabelian, simply connected connected nilpotent Lie group, then there exists a separated set $\Gamma \subset G$ such that Γ^{-1} is not relatively separated.*

Proof. We start out by considering the special case where G is the simply connected and connected three-dimensional Heisenberg Lie group, with Lie algebra spanned by

$$X_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with nontrivial Lie brackets $[X_1, X_2] = X_3$. For $X = \sum_{k=1}^3 x_k X_k$ and $Y = \sum_{k=1}^3 y_k X_k$, it is easy to verify that

$$\begin{aligned} \exp(X)\exp(Y) &= \exp\left(X + Y + \tfrac{1}{2}[X,Y]\right) \\ &= \exp \begin{bmatrix} 0 & x_1 + y_1 & x_3 + y_3 + \frac{x_1 y_2 - x_2 y_1}{2} \\ 0 & 0 & x_2 + y_2 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Next, we endow the Heisenberg Lie group with the quasi-norm

$$\left\| \exp \begin{bmatrix} 0 & x_1 & x_3 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{bmatrix} \right\| = \left((x_1^2 + x_2^2)^2 + x_3^4\right)^{1/4}.$$

This induces a left-invariant quasi-metric between two arbitrary elements by

$$\begin{aligned} &d(\exp(x_1X_1 + x_2X_2 + x_3X_3), \exp(y_1X_1 + y_2X_2 + y_3X_3)) \\ &\quad = \|\exp(-x_1X_1 - x_2X_2 - x_3X_3)\exp(y_1X_1 + y_2X_2 + y_3X_3)\| \\ &\quad = \left(((y_1 - x_1)^2 + (y_2 - x_2)^2)^2 + \left(y_3 - x_3 + \frac{x_2y_1 - x_1y_2}{2}\right)^4\right)^{1/4}, \end{aligned}$$

inducing the group topology on G. For any natural number $N \in \mathbb{N}$, we define

$$U_N = \begin{bmatrix} 0 & N^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_{N,\ell} = \begin{bmatrix} 0 & N^2 & \ell/2 \\ 0 & 0 & \ell/N^2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \ell = 1, \ldots, N.$$

Let

$$\Gamma = \{\exp(-V_{M,\ell}) : M \in \mathbb{N}, \ell = 1, \ldots, M\}.$$

Note that, for all $N \in \mathbb{N}$,

$$d(\exp U_N, \exp V_{N,\ell}) = \ell/N^2 \leq 1/N.$$

In other words, the ball with center U_N and radius N^{-1} contains at least N elements of Γ^{-1}. Since the quasi-metric generates the topology, this implies that Γ^{-1} is not relatively separated. On the other hand, it is easy to verify for distinct $\exp(V_{N,\kappa}), \exp(V_{M,\ell}) \in \Gamma$ that

$$d(\exp(-V_{N,\kappa}), \exp(-V_{M,\ell})) \geq 1.$$

Thus, Γ is left-uniformly discrete.

Now, if G is a simply connected, connected nonabelian nilpotent Lie group of dimension larger than three, then by Kirillov's lemma [5], G contains a closed subgroup H that is isomorphic to the simply connected Heisenberg group. By the case of the three-dimensional Heisenberg Lie group, there exists a separated (in H) $\Gamma \subset H$ with the property that Γ^{-1} is not separated (in H). However, for subsets of H, being separated in H is the same as being separated in G. ∎

An interesting property of FT groups, which we already mentioned in the introduction, is a general method of frame constructions for strongly square-integrable representations. First some terminology:

DEFINITION 2.10. Given a unitary representation π of a locally compact group G acting on a Hilbert space $\mathcal{H}_\pi$, and $\eta \in \mathcal{H}_\pi$, we define the associated wavelet transform $V_\eta : \mathcal{H}_\pi \to C_b(G)$ as

$$V_\eta u(x) = \langle u, \pi(x)\eta \rangle.$$

The vector η is called *admissible* if $V_\eta : \mathcal{H}_\pi \to L^2(G)$ is well-defined and isometric. We call π *strongly square-integrable* if there exists an admissible vector $\eta \in \mathcal{H}_\pi$.

Strongly square-integrable representations of type I groups are discussed and characterized at length in the book [16]. See also [17] for a characterization of strong square-integrability for a large class of semidirect products $\mathbb{R}^d \rtimes H$ and the associated quasi-regular representation acting on $L^2(\mathbb{R}^d)$. The cited sources make it clear that there is a vast pool of examples for this type of representation.

It is well-known that admissible vectors associated to strongly square-integrable representations give rise to weak-sense inversion formulae

$$u = \int_G V_\eta u(x) \pi(x) \eta \, dx.$$

We can now derive a discretization result for generalized wavelet transforms over FT groups. Observe here that the sampling set is *universal*, i.e., it is picked independently of the representation π.

THEOREM 2.11. *Assume that G has a frame $(\lambda_G(x)\varphi)_{x\in\Gamma}$ of translates for $L^2(G)$, with frame bounds A, B. Let π be a strongly square-integrable representation of G, with admissible vector ψ, and define*

$$\eta = \int_G \varphi(x) \pi(x) \psi \, dx \in \mathcal{H}_\pi,$$

which converges in the weak sense. Then $(\pi(x)\eta)_{x\in\Gamma}$ is a frame of $\mathcal{H}_\pi$, with frame bounds A, B.

Proof. By assumption, V_ψ is a unitary equivalence between $\mathcal{H}_\pi$ and $\mathcal{H}_{\pi,\psi} = V_\psi(\mathcal{H}_\pi)$, and the latter is a closed, left-invariant subspace of $L^2(G)$. The orthogonal projection onto $\mathcal{H}_{\pi,\psi}$ is $P = V_\psi V_\psi^*$. It follows that $(P\lambda_G(x)\varphi)_{x\in\Gamma}$ is a frame of $\mathcal{H}_{\pi,\psi}$, with frame bounds A and B. Then, since

$V_\psi^* : \mathcal{H}_{\pi,\psi} \to \mathcal{H}_\pi$ is a unitary equivalence intertwining the group actions, we finally see that $\eta = V_\psi^* \varphi$ is as required, and it is easy to check that η is computed by the weak operator integral in the theorem. ∎

Remark 2.12. There is an alternative proof available of the fact that the reduced Weyl–Heisenberg group is not FT, which makes use of Theorem 2.11, in combination with well-known necessary density conditions for Gabor frames, as derived in [4].

For the remainder of this section we concentrate on discrete groups. For the following results, recall that a *Riesz basis* of a Hilbert space is a frame $(\eta_i)_{i\in I}$ satisfying the additional inequality

$$\sum_{i\in I} |c_i|^2 \le C \Big\| \sum_{i\in I} c_i \eta_i \Big\|^2 .$$

Theorem 2.13. *Let G denote a discrete group, $\Gamma \subset G$ and $\varphi \in \ell^2(G)$. Then the following are equivalent:*

(a) *$(\lambda_G(x)\varphi)_{x\in\Gamma}$ is a frame of $\ell^2(G)$.*
(b) *$\Gamma = G$, and $(\lambda_G(x)\varphi)_{x\in G}$ is Riesz basis of $\ell^2(G)$.*

Proof. We only need to prove (a)⇒(b). We first show that $\Gamma \subset G$ is uniformly dense. Let A denote the lower frame bound, and pick $U \subset G$ finite such that

$$\sum_{x\notin U} |\varphi(x)|^2 < A.$$

Towards a contradiction, assume that Γ is not uniformly dense. Then there exists $y \in G$ such that $yU \cap \Gamma$ is empty. We then get

$$A \le \sum_{x\in\Gamma} |\langle \mathbf{1}_{\{y\}}, \lambda_G(x)\varphi\rangle|^2 = \sum_{x\in y^{-1}\Gamma} |\varphi(x)|^2 \le \sum_{x\notin U} |\varphi(x)|^2 < A,$$

a contradiction.

Hence Γ is uniformly dense, which means that $(\lambda_G(x)\varphi)_{x\in G}$ is a frame as well. Thus the associated frame operator $S_\varphi = V_\varphi^* V_\varphi : f \mapsto f * \varphi^* * \varphi$ is a bounded, self-adjoint operator with bounded inverse, commuting with left translations on G. Hence, letting $\eta = S_\varphi^{-1/2}\varphi$ yields a tight frame generator, i.e., $f \mapsto f * \eta^*$ is an isometry, or equivalently,

$$f = V_\eta^* V_\eta f = f * \eta^* * \eta \quad \text{for all } f \in \ell^2(G).$$

In particular

$$\delta_e = \delta_e * \eta^* * \eta = \eta^* * \eta,$$

and thus $\|\eta\|^2 = (\eta^* * \eta)(e) = 1$. Consequently, $(\lambda_G(x)\eta)_{x\in G} \subset \ell^2(G)$ is a Parseval frame consisting of unit vectors; and it is well-known that such frames are actually orthonormal bases.

But then $(\lambda_G(x)\varphi)_{x\in G} = (S_\varphi^{1/2}\lambda_G(x)\eta)_{x\in G}$ is a Riesz basis, as the image of an orthonormal basis under an invertible operator. In particular, the system $(\lambda_G(x)\varphi)_{x\in \Lambda}$ is incomplete in $\ell^2(G)$ for every proper subset Λ of G. Thus $\Gamma = G$. ∎

3. Sufficient criteria. The results established so far do not seem to point towards the existence of nondiscrete FT groups. As Proposition 2.4 shows, the functions φ occurring in frames of translates are necessarily somewhat pathological, an observation that seems to raise the bar somewhat further. Nonetheless, the remainder of this paper will show that FT groups exist in abundance. The strategy for proving such a result rests on two observations. The first one is the following remarkably general discretization result, recently established by Freeman and Speegle [14, Theorem 1.3]:

THEOREM 3.1. *Let $(\eta_x)_{x\in X} \subset \mathcal{H}$ denote a family of bounded vectors in a separable Hilbert space, measurably indexed by $x \in X$, where (X,μ) is a σ-finite measure space. Assume that $(\eta_x)_{x\in X}$ is a* ***continuous frame with respect to*** *μ, i.e., there exist constants $0 < A \leq B < \infty$ such that*

$$\forall g \in \mathcal{H}: \quad A\|g\|^2 \leq \int_X |\langle g, \eta_x\rangle|^2 \, d\mu(x) \leq B\|g\|^2.$$

Then there exists a countable family $(x_i)_{i\in I}$ such that $(\eta_{x_i})_{i\in I}$ is a frame.

The main consequence of this result is the following theorem which reveals a first, large class of FT groups.

THEOREM 3.2. *Let G be type I and nonunimodular. Then G is an FT group.*

Proof. By [15, 16], λ_G has an admissible vector η, i.e., the family $(\lambda_G(x)\eta)_{x\in G}$ is a continuous frame with respect to μ_G. Theorem 3.1 yields a family $(x_i)_{i\in I} \subset G$ such that $(\lambda_G(x_i)\eta)_{i\in I}$ is a frame. ∎

The second observation enlarges the class of FT further, by considering the restriction of the regular representation to a suitable closed subgroup.

DEFINITION 3.3. Given a representation π of G, we say that π *has infinite multiplicity* if $\pi \simeq \infty \cdot \pi$.

REMARK 3.4. Let G denote a type I group. Then the Plancherel transform of G gives rise to a unique direct integral decomposition

$$\lambda_G \simeq \int_{\widehat{G}} m_\sigma \cdot \sigma \, d\nu_G(\sigma),$$

where ν_G is the *Plancherel measure* of G, and the multiplicity m_σ with which $\sigma \in \widehat{G}$ enters the Plancherel decomposition is equal to the Hilbert space dimension of $\mathcal{H}_\sigma$. In particular, λ_G has infinite multiplicity if and only if $m_\sigma = \infty$ for ν_G-almost every σ.

Important classes of groups for which this holds are the nonunimodular type I groups, and the nonabelian connected nilpotent Lie groups. In both cases, ν_G-almost all irreducible representations are induced from subgroups of infinite index: For the nonunimodular case, this was established in [10]; for the nilpotent case, it follows by Kirillov's orbit method [5]. But the representation spaces associated to these representations are then infinite-dimensional.

It is worth noting that if a representation λ_G has infinite multiplicity, then

$$\lambda_G \simeq m \cdot \lambda_G$$

as well, for all natural numbers m.

The following result exploits an idea due to Iverson [24, Theorem 3.6] for our purposes.

THEOREM 3.5. *Let $H < G$ denote a closed subgroup that has FT, and such that λ_H has infinite multiplicity. Then G is FT. In fact, there exists a vector $\varphi \in L^2(G)$ and $\Gamma \subset H$ such that $(\lambda_G(x)\varphi)_{x\in\Gamma} \subset L^2(G)$ is a frame.*

Proof. Using a measurable set of coset representatives C mod H, we have a Borel isomorphism $H \times C \ni (h,x) \mapsto hx \in G$. Furthermore, this isomorphism intertwines the left action $H \times (H \times C) \ni (h_1,(h_2,x)) \mapsto (h_1h_2,x) \in H \times C$ with the left action of H on G. By Weil's formula, we can then identify the Haar measure on G with the product of Haar measure on H and a suitable measure on the quotient. Since the latter is Borel isomorphic to C, this identification induces a unitary equivalence $L^2(G) \to L^2(H) \otimes L^2(C)$. Since the Borel isomorphism intertwines the actions on $H \times C$ and G, the unitary map intertwines $\lambda_G|_H$ with $\lambda_H \otimes \mathbf{1}$, where $\mathbf{1}$ denotes the trivial representation acting on $L^2(C)$.

Hence, denoting by $\kappa \in \mathbb{N} \cup \{\infty\}$ the Hilbert space dimension of $L^2(C)$, we find

$$\lambda_G|_H \simeq \kappa \cdot \lambda_H \simeq \lambda_H.$$

By assumption on H, there exists a frame of translates in $L^2(H)$, and the image of this frame under the intertwining operator $\lambda_H \simeq \lambda_G|_H$ has the desired properties. ∎

The simplest example of a type I nonunimodular group is the $ax+b$-group, the semidirect product $\mathbb{R} \rtimes \mathbb{R}^+$. Note that $\mathrm{SL}(2,\mathbb{R})$ contains the closed subgroup

$$\left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a > 0,\, b \in \mathbb{R} \right\},$$

which is isomorphic to the $ax + b$-group. Hence Theorem 3.5 implies the following:

COROLLARY 3.6. $\mathrm{SL}(2, \mathbb{R})$ *is an FT group.*

Note that $\mathrm{SL}(2, \mathbb{R})$ is unimodular, so unimodular FT groups exist. Moreover, let $p, q \geq 1$ be natural numbers satisfying $p + q > 2$. Next, let

$$\mathrm{SO}(p, q) = \{M \in \mathrm{GL}(p + q, \mathbb{R}) : M^{\mathrm{tr}} J(p, q) M = J(p, q)\}$$

where $J(p, q) = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ has trace $p - q$. Then $\mathrm{SO}(p, q)$ is a closed subgroup of $\mathrm{GL}(p + q, \mathbb{R})$ and the following holds true.

COROLLARY 3.7. *Given natural numbers* $p, q \geq 1$ *such that* $p + q > 2$, $\mathrm{SO}(p, q)$ *is an FT group.*

Proof. We first observe that every element X of the Lie algebra of $\mathrm{SO}(p, q)$ can be written in block form as

$$X = \begin{bmatrix} Z & S \\ S^{\mathrm{tr}} & Y \end{bmatrix} \quad \text{for some } (Z, Y) \in \mathfrak{so}(p) \times \mathfrak{so}(q).$$

Hence, if $q = 1$ then

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \in \mathfrak{so}(p, 1)$$

and $[A, X] = X$. Thus, $\exp(\mathbb{R}X) \exp(\mathbb{R}A)$ is a closed nonunimodular subgroup of $\mathrm{SO}(p, 1)$ which is isomorphic to the $ax + b$-group and it follows that $\mathrm{SO}(p, 1)$ is FT. More generally, letting

$$I = \{(1, p + 1), (p + 1, 1)\} \quad \text{and} \quad J = \{(1, p), (p, p + 1), (p + 1, p)\}$$

and defining matrices $B, Y \in \mathfrak{gl}(p + q, \mathbb{R})$ with entries satisfying

$$B_{jk} = \begin{cases} 1 & \text{if } (j, k) \in I \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad Y_{jk} = \begin{cases} 1 & \text{if } (j, k) \in J \\ -1 & \text{if } (j, k) = (p, 1) \\ 0 & \text{otherwise} \end{cases}$$

it is easy to verify that $[B, Y] = Y$. Thus, the $ax + b$-group $\exp(\mathbb{R}Y) \exp(\mathbb{R}B)$ is a closed subgroup of $\mathrm{SO}(p, q)$ and it follows immediately that $\mathrm{SO}(p, q)$ is FT as well. ∎

REMARK 3.8. Since $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$, for $n \geq 2$, contain closed isomorphic copies of $\mathrm{SL}(2, \mathbb{R})$, all of these groups are FT groups again. The same reasoning applies to the symplectic groups $\mathrm{Sp}(2n, \mathbb{R})$and $\mathrm{Sp}(2n, \mathbb{C})$:

One has $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$, and the higher-dimensional groups contain closed isomorphic copies of $\mathrm{Sp}(2,\mathbb{R})$. An identical reasoning entails the FT property for the metaplectic groups in any dimension.

REMARK 3.9. We can use previous results to construct a rather unusual shearlet frame. Consider the matrix group

$$H = \left\{ \pm \begin{bmatrix} a & b \\ 0 & a^{1/2} \end{bmatrix} : a > 0,\, b \in \mathbb{R} \right\}.$$

Let $G = \mathbb{R}^2 \rtimes H$. The natural affine action of G on $\mathbb{R}^2$ gives rise to the quasi-regular representation on $L^2(\mathbb{R}^2)$, which is known to be strongly square-integrable [6]. The generalized wavelet transform associated with this group is the so-called *shearlet transform*. Shearlet frames are typically constructed by choosing a lattice $\Gamma \subset \mathbb{R}^2$ and a discrete subset $H_d \subset H$ and considering families of the kind

$$(\pi(hx, h)\psi)_{x \in \Gamma,\, h \in H_d}$$

for suitably chosen functions ψ; see [1] for an early source using this type of construction. Note that H is a closed subgroup of G that is isomorphic to the $ax+b$-group. Hence combining Theorems 2.11 and 3.5, we can now show that there exists a frame of the type

$$(\pi(0, h)\varphi)_{h \in H_d},$$

i.e., *using only dilations!* This example generalizes easily to shearlet groups in arbitrary dimensions.

REMARK 3.10. A recent source considering a similar setup is the paper [26], which constructs wavelet systems in $L^2(\mathbb{R}^2)$ arising from translations along the x_1-axis, and dilations. One can understand this action as the restriction of the quasi-regular representation of the semidirect product $G = \mathbb{R}^2 \rtimes \mathbb{R}^+$ to the subgroup H generated by the translations along the axis and the dilations. It is well-known that the quasi-regular representation of G on $L^2(\mathbb{R}^2)$ is strongly square-integrable, and clearly H is a closed subgroup that is isomorphic to the $ax+b$-group. Hence we can once again appeal to Theorem 2.11 to show the existence of a frame arising from the action of H.

4. Lie groups with FT. Using the tools developed in the previous sections, we will now explore the FT property for a class of Lie groups known as exponential Lie groups. To present the findings of this section, we need the following. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

We say that $\mathfrak{g}$ is of *type R* if for every $X \in \mathfrak{g}$, the eigenvalues of the endomorphism $\mathrm{ad}_{\mathfrak{g}}(X)$ are purely imaginary.

The *lower central series* of $\mathfrak{g}$ is inductively defined as follows: $C^1\mathfrak{g} = \mathfrak{g}$ and $C^{j+1}\mathfrak{g} = [\mathfrak{g}, C^j\mathfrak{g}]$ for $j > 0$. Moreover, a Lie algebra is *nilpotent* if there exists a natural number k such that $C^k\mathfrak{g}$ is trivial. According to Engel's characterization, a Lie algebra $\mathfrak{g}$ is nilpotent if and only if for every $X \in \mathfrak{g}$, the endomorphism $\mathrm{ad}_{\mathfrak{g}}(X) : Y \mapsto [X, Y]$ is nilpotent (see [23, Theorem 5.2.8]). The *derived series* of $\mathfrak{g}$ is a decreasing collection $D^j\mathfrak{g}$ of ideals in $\mathfrak{g}$ defined as

$$D^0\mathfrak{g} = \mathfrak{g} \quad \text{and} \quad D^{j+1}\mathfrak{g} = [D^j\mathfrak{g}, D^j\mathfrak{g}] \quad \text{for } j \in \mathbb{N}.$$

A Lie algebra $\mathfrak{g}$ is *solvable* if its derived series reaches the trivial algebra in finitely many steps. Suppose for now that $\mathfrak{g}$ is a real solvable Lie algebra of dimension n. Then $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ is a solvable algebra of endomorphisms acting on $\mathfrak{g}$. Next, let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. As a $\mathfrak{g}_{\mathbb{C}}$-module, the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ has a *Jordan–Hölder* sequence

$$\{0\} = \mathfrak{g}_{\mathbb{C}}^{(0)} \subset \mathfrak{g}_{\mathbb{C}}^{(1)} \subset \cdots \subset \mathfrak{g}_{\mathbb{C}}^{(n-1)} \subset \mathfrak{g}_{\mathbb{C}}^{(n)} = \mathfrak{g}_{\mathbb{C}}.$$

That is, each $\mathfrak{g}_{\mathbb{C}}^{(k)}$ is an ideal and $\dim \mathfrak{g}_{\mathbb{C}}^{(k)} = k$. Moreover, the action of $\mathfrak{g}_{\mathbb{C}}$ on the vector space $\mathfrak{g}_{\mathbb{C}}^{(k)}/\mathfrak{g}_{\mathbb{C}}^{(k-1)}$ where $1 \leq k \leq n$ defines a linear form on $\mathfrak{g}_{\mathbb{C}}$ called a *root* of $\mathfrak{g}$. We say that $\mathfrak{g}$ is an *exponential solvable Lie algebra* if it has no root with a nonzero purely imaginary value. In other words, each root of $\mathfrak{g}$ is given by $X \mapsto \lambda(X)(1 + i\alpha)$ where $\lambda \in \mathfrak{g}^*$ and $\alpha \in \mathbb{R}$.

Next, let $G = \exp \mathfrak{g}$ be a simply connected and connected Lie group with solvable Lie algebra $\mathfrak{g}$. Then it is known that $\mathfrak{g}$ is exponential if and only if the exponential map determines an analytic diffeomorphism between $\mathfrak{g}$ and G. In this case, G is called an *exponential solvable Lie group*.

The following theorem summarizes our results

THEOREM 4.1. *Exponential solvable Lie groups which are not nilpotent are FT.*

It is worth noting that for all FT groups mentioned in Theorem 4.1, the set $\Gamma \subset G$ of shifts generating the frame can be chosen as a subset of a closed subgroup of dimension at most 3, *regardless of the dimension of G itself.*

The case of nonabelian simply connected, connected nilpotent Lie groups is currently open. There are several reasons why this class is intriguing: A comprehensive answer would ultimately settle the exponential solvable case. Moreover, it would help clarify the relationship between the FT property and the property that inverses of relatively separated sets are relatively separated again. Recall that the latter implies the negation of the former, and we currently have no example that the converse does not hold in general. If such a general converse holds, it implies that no nonabelian simply connected, connected nilpotent Lie groups are FT, via Lemma 2.9. However, with current knowledge, all we can do is to reduce the discussion to certain test cases. In the following proposition, $T(n, \mathbb{R})$ is the group of upper triangular matrices in $\mathrm{GL}(n, \mathbb{R})$ with ones on the diagonal.

PROPOSITION 4.2.

(a) *Every nonabelian simply connected, connected nilpotent Lie group is FT if and only if the Heisenberg group is FT.*
(b) *No nonabelian simply connected, connected nilpotent Lie group is FT if and only if for all $n \geq 3$, $T(n, \mathbb{R})$ is not FT.*

Proof. Both statements are consequences of Theorem 3.5, via the observation that if G is nonabelian, simply connected, and nilpotent, there exist embeddings

$$H \subset G \subset T(n, \mathbb{R})$$

as closed subgroups, where H is isomorphic to a Heisenberg group, and n is sufficiently large. Here the first embedding is due to Kirillov's lemma [5], the second is a well-known consequence of Engel's Theorem. ∎

There are two solvable Lie algebras, which are of crucial importance for the following. The first one is the $ax + b$ Lie algebra, which is a two-dimensional solvable Lie algebra spanned by A, X such that $[A, X] = X$. The second one, the *Grélaud algebra*, is a three-dimensional solvable Lie algebra spanned by A, Y_1, Y_2 with nontrivial Lie brackets

$$[A, Y_1] = Y_1 + \beta Y_2 \quad \text{and} \quad [A, Y_2] = -\beta Y_1 + Y_2$$

for some nonzero real β. Since the linear map $\mathrm{ad}(A) : \mathfrak{g} \to \mathfrak{g}, Y \mapsto [A, Y]$, has nonzero trace, the associated simply connected connected Lie group G is nonunimodular.

LEMMA 4.3. *If $\mathfrak{g}$ is not of type R then $\mathfrak{g}$ admits a Lie subalgebra $\mathfrak{h} < \mathfrak{g}$, which is either isomorphic to the $ax + b$-algebra or the Grélaud algebra.*

Proof. By assumption, one of the following must hold.

CASE 1: There exists $X \in \mathfrak{g}$ with a nonzero real eigenvalue λ. Then there is an eigenvector Y for $\mathrm{ad}(X)$ with eigenvalue λ satisfying $[\lambda^{-1}X, Y] = Y$ and $\mathfrak{h} = \mathbb{R}Y + \mathbb{R}X$ as desired.

CASE 2: Case 1 does not hold. Then there exists $X \in \mathfrak{g}$ such that $\alpha + i\beta$ is an eigenvalue for $\mathrm{ad}(X)$ and $\alpha \neq 0$. Thus, there exist $Y_1, Y_2 \in \mathfrak{g}$ such that $[X, Y_1] = \alpha Y_1 + \beta Y_2$ and $[X, Y_2] = -\beta Y_1 + \alpha Y_2$. Next, we claim that Y_1, Y_2 must commute. Otherwise, a straightforward application of Jacobi's identity yields $[X, [Y_1, Y_2]] = 2\alpha[Y_1, Y_2]$. However, this contradicts the fact that X does not have a nonzero real eigenvalue. Finally, setting $\mathfrak{h} = \mathbb{R}$-span$\{X, Y_1, Y_2\}$ gives the desired result. ∎

LEMMA 4.4. *Let $\mathfrak{g}$ be an n-dimensional exponential solvable Lie algebra. Then the following statements are equivalent:*

(1) *$\mathfrak{g}$ is not a nilpotent Lie algebra.*

(2) *There exists a Lie subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *such that* $\exp \mathfrak{h}$ *is a closed type I nonunimodular Lie group.*

Proof. To prove that (2)→(1), suppose that there exists a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\exp \mathfrak{h}$ is a type I nonunimodular Lie group. Since nilpotent Lie groups are unimodular, this subalgebra cannot be nilpotent. Then $\mathfrak{g}$ is not a nilpotent Lie algebra since every subalgebra of a nilpotent Lie algebra is nilpotent (see [5, Proposition 1.1.6]).

Conversely, suppose that $\mathfrak{g}$ is an exponential solvable Lie group which is not nilpotent. By Engel's Theorem, there exists $X \in \mathfrak{g}$ such that $\mathrm{ad}_{\mathfrak{g}}(X)$ is not nilpotent. The fact that $\mathfrak{g}$ has no purely imaginary roots together with Lemma 4.3 implies that G admits a subalgebra isomorphic to the $ax + b$ Lie algebra or Grelaud's algebra. Since G is exponential, the associated Lie subgroup is simply connected and closed, and since it is exponential solvable, it is of type I. Therefore (1)⇒(2) holds. ■

Proof of Theorem 4.1. The statement is a direct consequence of Lemma 4.4 and Theorem 3.5. ■

Concluding remarks. Our paper's primary purpose was to demonstrate the extent to which group-based frame constructions can be pushed. As illustrated by Theorem 2.11 and its consequences, the existence of frames for the regular representation implies the existence of frames in many unexpected settings.

It should be stressed that the price to pay for this degree of generality is explicitness, in particular with regard to the generators. Note that the only general result about generators in the nondiscrete setting is a negative one (Proposition 2.4), indicating the potential difficulty of obtaining such generators. Most of our examples of FT groups are based on the existence of a closed subgroup H that is isomorphic to the $ax + b$-group, which was our main ally in this whole endeavour. While our results allow us to conclude in this case that there exists a generator φ and a subset $\Gamma \subset H$ such that $(\lambda_G(x)\varphi)_{x\in\Gamma}$ is a frame of $L^2(G)$, the proof does not really provide a viable recipe for constructing generators: It requires first constructing a frame generator for the regular representation of the $ax+b$-group, and then computing the image of that generator under a unitary equivalence $\lambda_G|_H \simeq \lambda_H$. Both steps are quite hard to carry out explicitly.

Our results also provide an interesting contrast to the more standard work on frame construction, which usually relies on notions of density, as in [12, 19]. These sources study the discretization of continuous inversion formulae, and impose additional conditions on the generators φ, which are expressed in terms of Wiener amalgam norms or oscillation estimates. In this context, typical results show that for all sufficiently nice generators φ, any relatively separated set Γ gives rise to a frame $(\pi(x)\varphi)_{x\in\Gamma}$, as long as Γ is sufficiently dense. Here, the density criterion is *uniform density*, meaning that $\Gamma U = G$, for a relatively compact open set $U \subset G$ that typically depends on φ. Conversely, more quantitative *necessary* criteria for Γ, involving a version of *Beurling density*, have been found more recently [20], under suitable assumptions on G and φ.

By contrast, for all the examples of FT groups G which we obtained in this paper by spotting a closed subgroup H isomorphic to the $ax+b$-group, the quotient G/H was not compact. As a consequence, the subset $\Gamma \subset H$ underlying the construction of a frame of translates for $L^2(G)$ is *never* dense, in any of the above-mentioned senses.

Hence the frames obtained by our methods are typically very distinct from the frames obtained by more standard approaches, and they exist in wider contexts. As already mentioned above, this comes at the price of significantly less control over the properties of the generators.

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