COLLOQUIUM MATHEMATICUM

Online First version

THE COMPLETION OF THE HYPERSPACE OF FINITE SUBSETS, ENDOWED WITH THE ℓ^1-METRIC

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Abstract. For a metric space X, let $\mathsf{F}X$ be the space of all non-empty finite subsets of X endowed with the largest metric $d^1_{\mathsf{F}X}$ such that for every $n \in \mathbb{N}$ the map $X^n \to \mathsf{F}X$, $(x_1,\dots,x_n) \mapsto \{x_1,\dots,x_n\}$, is non-expanding with respect to the ℓ^1-metric on X^n. We study the completion of the metric space $\mathsf{F}^1X = (\mathsf{F}X, d^1_{\mathsf{F}X})$ and prove that it coincides with the space Z^1X of non-empty compact subsets of X that have zero length (defined with the help of graphs). We prove that each subset of zero length in a metric space has 1-dimensional Hausdorff measure zero. A subset A of the real line has zero length if and only if its closure is compact and has Lebesgue measure zero. On the other hand, for every $n \geq 2$ the Euclidean space $\mathbb{R}^n$ contains a compact subset of 1-dimensional Hausdorff measure zero that fails to have zero length.

1. Preliminaries. Given a metric space X with metric d_X, denote by $\mathsf{K}X$ the space of all non-empty compact subsets of X, endowed with the Hausdorff metric $d_{\mathsf{K}X}$ defined by the formula

$$d_{\mathsf{K}X}(A,B) = \max\Big\{\max_{a\in A}\min_{b\in B} d_X(a,b), \max_{b\in B}\min_{a\in A} d_X(b,a)\Big\}.$$

The metric space $\mathsf{K}X$ is called the *hyperspace* of X. It plays an important role in General Topology [3, §3.2], [8, 4.5.23] and Theory of Fractals [7, §2.5], [9, §9.1]. It is well-known [8, 4.5.23] that for any complete (and compact) metric space X its hyperspace $\mathsf{K}X$ is complete (and compact). The hyperspace $\mathsf{K}X$ contains an important dense subspace $\mathsf{F}X$ consisting of non-empty finite subsets of X. The density of $\mathsf{F}X$ in $\mathsf{K}X$ implies that for a complete metric space X, the hyperspace $\mathsf{K}X$ is a completion of the hyperspace $\mathsf{F}X$.

In [2, §30] it was shown that the Hausdorff metric $d_{\mathsf{F}X}$ on $\mathsf{F}X$ coincides with the largest metric on $\mathsf{F}X$ such that for every $n \in \mathbb{N}$ the map $X^n \to \mathsf{F}X$, $x \mapsto x[n] := \{x(i) : i \in n\}$, is non-expanding, where X^n is endowed with the

2020 *Mathematics Subject Classification*: Primary 54B20, 54E35; Secondary 54E50, 54F45, 05C90.
Key words and phrases: hyperspace of finite subsets, completion, ℓ^1-metric, set of zero length.
Received 7 April 2020; revised 29 October 2020.
Published online 22 April 2021.

DOI: 10.4064/cm8226-11-2020

ℓ^∞-metric

$$d^\infty_{X^n}(x,y)=\max_{i\in n} d_X(x(i),y(i)).$$

Here we identify the natural number n with the set $\{0,\dots,n-1\}$ and think of elements of X^n as functions $x:n\to X$.

Let us recall that a function $f:Y\to Z$ between metric spaces (Y,d_Y) and (Z,d_Z) is *non-expanding* if $d_Z(f(y),f(y'))\le d_Y(y,y')$ for any $y,y'\in Y$.

It is well-known that the ℓ^∞-metric $d^\infty_{X^n}$ on X^n is the limit as $p\to\infty$ of the ℓ^p-metrics $d^p_{X^n}$ on X^n, defined by the formula

$$d^p_{X^n}(x,y)=\Big(\sum_{i=1}^n d_X\big(x(i),y(i)\big)^p\Big)^{1/p}\quad\text{for } x,y\in X^n.$$

Given any metric space (X,d) and any number $p\in[1,\infty]$, let $d^p_{\mathsf{F}X}$ be the largest metric on the set $\mathsf{F}X$ such that for every $n\in\mathbb{N}$ the map $X^n\to\mathsf{F}X$, $x\mapsto x[n]$, is non-expanding with respect to the ℓ^p-metric $d^p_{X^n}$ on X^n. The metric $d^p_{\mathsf{F}X}$ was introduced in [2], where it was shown that $d^p_{\mathsf{F}X}$ is a well-defined metric on $\mathsf{F}X$ such that

$$d_{\mathsf{F}X}=d^\infty_{\mathsf{F}X}\le d^p_{\mathsf{F}X}\le d^1_{\mathsf{F}X},$$

where $d_{\mathsf{F}X}$ stands for the Hausdorff metric on $\mathsf{F}X$.

We denote by F^pX the metric space $(\mathsf{F}X,d^p_{\mathsf{F}X})$. So, $\mathsf{F}^\infty X$ coincides with the hyperspace $\mathsf{F}X$ endowed with the Hausdorff metric.

As we already know, for any complete metric space X, the completion $\hat{\mathsf{F}}^\infty X$ of the metric space $\mathsf{F}^\infty X$ can be identified with the hyperspace $\mathsf{K}X$ endowed with the Hausdorff metric. In this paper we study the completion $\hat{\mathsf{F}}^1X$ of the metric space $\mathsf{F}^1X=(\mathsf{F}X,d^1_{\mathsf{F}X})$ and show that it can be identified with the space Z^1X of non-empty compact subsets of zero length in X. Sets of zero length are defined with the help of graphs.

By a *graph* we understand a pair $\Gamma=(V,E)$ consisting of a set V of vertices and a set E of edges. Each edge $e\in E$ is a non-empty subset of V of cardinality $|e|\le 2$. A graph (V,E) is *finite* if its set of vertices V is finite. In this case the set E of edges is finite, too.

For a graph $\Gamma=(V,E)$, a subset $C\subseteq V$ is *connected* if for any vertices $x,y\in C$ there exists a sequence of vertices $c_0,\dots,c_n\in C$ such that $c_0=x$, $c_n=y$ and $\{c_{i-1},c_i\}\in E$ for every $i\in\{1,\dots,n\}$. The maximal connected subsets of V are called the *connected components* of the graph Γ. It is easy to see that two connected components of Γ either coincide or are disjoint. For a vertex $x\in V$ we denote by $\Gamma(x)$ the unique connected component of Γ that contains x.

By a *graph in a metric space* (X, d_X) we understand any graph $\Gamma = (V, E)$ with $V \subseteq X$. In this case we can define the *total length* $\ell(\Gamma)$ of Γ by

$$\ell(\Gamma) = \sum_{\{x,y\}\in E} d_X(x, y).$$

If E is infinite, then the sum is understood to be the (finite or infinite) number

$$\sup_{E'\in \mathsf{F}E} \sum_{\{x,y\}\in E'} d_X(x, y).$$

If a graph has finite total length, then it has at most countably many nondegenerate edges.

For a subset $C \subseteq X$ we denote by $\overline{C}$ the closure of C in the metric space (X, d_X).

Given a subset A of a metric space X, we denote by $\boldsymbol{\Gamma}_X(A)$ the family of graphs $\Gamma = (V, E)$ with finitely many connected components such that $V \subseteq X$ and $A \subseteq \overline{V}$. Observe that $\boldsymbol{\Gamma}_X(A)$ contains the complete graph on the set A and hence $\boldsymbol{\Gamma}_X(A)$ is not empty.

We define the set A to have *zero length* in X if for any $\varepsilon > 0$ there exists a graph $\Gamma \in \boldsymbol{\Gamma}_X(A)$ with $\ell(A) < \varepsilon$.

Since $\boldsymbol{\Gamma}_X(A) = \boldsymbol{\Gamma}_X(\bar{A})$, a subset A of a metric space X has zero length if and only if its closure has zero length.

2. Results

PROPOSITION 2.1. *A subset A of a metric space X has zero length if and only if the family $\boldsymbol{\Gamma}_X(A)$ contains a graph Γ of finite total length.*

Proof. The “only if” part is trivial. To prove the “if” part, assume that $\boldsymbol{\Gamma}_X(A)$ contains a graph $\Gamma = (V, E)$ of finite total length $\ell(\Gamma)$. The definition of $\ell(\Gamma)$ implies that for any $\varepsilon > 0$ there exists a finite subset $F \subseteq E$ such that $\sum_{\{x,y\}\in E\setminus F} d_X(x, y) < \varepsilon$. Then the graph $\Gamma' = (V, E \setminus F)$ still belongs to $\boldsymbol{\Gamma}_X(A)$ and has total length $\ell(\Gamma') < \varepsilon$, witnessing that A has zero length. ∎

In Proposition 2.12 we shall prove that each set A of zero length in a metric space X is totally bounded and has 1-dimensional Hausdorff measure equal to zero.

For a metric space X, denote by $\mathsf{Z}X$ the family of non-empty compact subsets of zero length in X. It is clear that each finite subset of X has zero length, so $\mathsf{F}X \subseteq \mathsf{Z}X \subseteq \mathsf{K}X$.

Now we define the metric $d^1_{\mathsf{Z}X}$ on the set $\mathsf{Z}X$. Given two compact sets $A, B \in \mathsf{Z}X$, let $\Gamma_X(A, B)$ be the family of graphs $\Gamma = (V, E)$ in X such that

(i) $A \cup B \subseteq \overline{V}$;
(ii) Γ has finitely many connected components;
(iii) for every connected component C of Γ we have $A \cap \overline{C} \neq \emptyset \neq B \cap \overline{C}$.

The conditions (i) and (ii) imply that $A\cup B\subseteq\overline{V}=\bigcup_{x\in V}\overline{\Gamma(x)}$.

Observe that the family $\boldsymbol{\Gamma}_X(A,B)$ contains the complete graph on the set $A\cup B$ and hence is not empty.

For compact subsets $A,B\in\mathsf{Z}X$, let

$$d^1_{\mathsf{Z}X}(A,B):=\inf_{\Gamma\in\boldsymbol{\Gamma}_X(A,B)}\ell(\Gamma).$$

By a *completion* of a metric space X we understand any complete metric space containing X as a dense subspace.

The following theorem is the main result of this paper.

THEOREM 2.2. *Let X be a metric space with metric d_X.*

(a) *The function $d^1_{\mathsf{Z}X}$ is a well-defined metric on $\mathsf{Z}X$.*
(b) $d_{\mathsf{K}X}(A,B)\le d^1_{\mathsf{Z}X}(A,B)$ *for any $A,B\in\mathsf{Z}X$.*
(c) $d^1_{\mathsf{Z}X}(A,B)=d^1_{\mathsf{F}X}(A,B)$ *for any finite sets $A,B\in\mathsf{F}X$.*
(d) *The map $X\to\mathsf{Z}^1X$, $x\mapsto\{x\}$, is an isometric embedding.*
(e) $\mathsf{F}X$ *is a dense subset in the metric space $\mathsf{Z}^1X:=(\mathsf{Z}X,d^1_{\mathsf{Z}X})$.*
(f) *If X is complete, then so is $\mathsf{Z}^1X=(\mathsf{Z}X,d^1_{\mathsf{Z}X})$.*
(g) *If Y is a dense subspace in X, then $d^1_{\mathsf{Z}Y}(A,B)=d^1_{\mathsf{Z}X}(A,B)$ for any $A,B\in\mathsf{Z}Y$ and the set $\mathsf{Z}Y$ is dense in Z^1X.*
(h) *If $\bar X$ is a completion of X, then $\mathsf{Z}^1\bar X$ is a completion of F^1X.*

The proof of Theorem 2.2 is divided into eight lemmas.

LEMMA 2.3. $d_{\mathsf{K}X}(A,B)\le d^1_{\mathsf{Z}X}(A,B)$ *for any $A,B\in\mathsf{Z}X$.*

Proof. To derive a contradiction, assume that $d_{\mathsf{K}X}(A,B)>d^1_{\mathsf{Z}X}(A,B)$ for some compact sets $A,B\in\mathsf{Z}X$. By the definition of $d^1_{\mathsf{Z}X}$, there exists a graph $\Gamma\in\boldsymbol{\Gamma}_X(A,B)$ such that $\ell(\Gamma)<d_{\mathsf{K}X}(A,B)$. Choose $\varepsilon>0$ such that $\ell(\Gamma)+2\varepsilon<d_{\mathsf{K}X}(A,B)$. Since Γ has finitely many connected components and $A\cup B\subseteq\overline{V}$, for any $a\in A$ there exists a connected component C of Γ such that $a\in\overline{C}$. By the definition of $\boldsymbol{\Gamma}_X(A,B)$, the intersection $\overline{C}\cap B$ contains some point $b'\in B$. Since $a,b'\in\overline{C}$, there are points $c,c'\in C$ such that $d_X(a,c)<\varepsilon$ and $d_X(b',c')<\varepsilon$. Since the set C is connected in the graph $\Gamma=(V,E)$, there exists a sequence $c_0,\dots,c_n\in C$ of pairwise distinct points such that $c_0=c$, $c_n=c'$, and $\{c_{i-1},c_i\}\in E$ for all $i\in\{1,\dots,n\}$. Since $c_0,\dots,c_n$ are pairwise distinct, the edges $\{c_0,c_1\},\{c_1,c_2\},\dots,\{c_{n-1},c_n\}$ of Γ are pairwise distinct and then

$$d_X(a,b')\le d_X(a,c_0)+\sum_{i=1}^n d_X(c_{i-1},c_i)+d_X(c_n,c')<\varepsilon+\ell(\Gamma)+\varepsilon.$$

So $\min_{b\in B}d_X(a,b)\le d_X(a,b')<2\varepsilon+\ell(\Gamma)$ and $\max_{a\in A}\min_{b\in B}d_X(a,b)<2\varepsilon+\ell(\Gamma)$. By analogy we can prove that $\max_{b\in B}\min_{a\in A}d_X(b,a)<2\varepsilon+\ell(\Gamma)$.

Then

$$d_{\mathsf{K}X}(A,B) = \max\Big\{\max_{a\in A}\min_{b\in B} d(a,b), \max_{b\in B}\min_{a\in A} d(b,a)\Big\}$$
$$< 2\varepsilon + \ell(\Gamma) < d_{\mathsf{K}X}(A,B),$$

a contradiction completing the proof. ■

LEMMA 2.4. *$d^1_{\mathsf{Z}X}$ is a well-defined metric on $\mathsf{Z}X$.*

Proof. Given any sets $A, B, C \in \mathsf{Z}X$, we need to verify the three axioms of metric:

(i) $0 \le d^1_{\mathsf{Z}X}(A,B) < \infty$, and $d^1_{\mathsf{Z}X}(A,B) = 0$ iff $A = B$,
(ii) $d^1_{\mathsf{Z}X}(A,B) = d^1_{\mathsf{Z}X}(B,A)$,
(iii) $d^1_{\mathsf{Z}X}(A,B) \le d^1_{\mathsf{Z}X}(A,C) + d^1_{\mathsf{Z}X}(C,B)$.

(i) First we show that $d^1_{\mathsf{Z}X}(A,A) = 0$ for any $A \in \mathsf{Z}X$. Since A has zero length, for any $\varepsilon > 0$ there exists a graph $\Gamma = (V,E)$ in X with finitely many connected components such that $A \subseteq \overline{V}$ and $\ell(\Gamma) < \varepsilon$. Replacing Γ by a suitable subgraph, we can assume that the closure of each connected component of Γ intersects A. Then $A \in \boldsymbol{\Gamma}_X(A,A)$ and hence

$$d^1_{\mathsf{Z}X}(A,A) \le \ell(\Gamma) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $d^1_{\mathsf{Z}X}(A,A) = 0$.

If $A, B \in \mathsf{Z}X$ are distinct, then by Lemma 2.3,

$$d^1_{\mathsf{Z}X}(A,B) \ge d_{\mathsf{K}X}(A,B) > 0$$

(as the Hausdorff metric $d_{\mathsf{K}X}$ is a metric).

The proof of the first axiom of metric will be complete as soon as we check that $d^1_{\mathsf{Z}X}(A,B) < \infty$ for any $A, B \in \mathsf{Z}X$. Since A, B have zero length, there exist graphs $\Gamma_A = (V_A, E_A)$ and $\Gamma_B = (V_B, E_B)$ with finitely many connected components such that $A \subseteq \overline{V}_A$, $B \subseteq \overline{V}_B$ and $\ell(\Gamma_A) + \ell(\Gamma_B) < 1$. Let D be a finite subset of $V_A \cup V_B$ intersecting every connected component of Γ_A and Γ_B. Consider the graph $\Gamma = (V,E)$ where $V = V_A \cup V_B$ and $E = E_A \cup E_B \cup E_D$ with $E_D := \{e \subseteq D : |e| = 2\}$. It is easy to see that Γ is connected and belongs to the family $\boldsymbol{\Gamma}_X(A,B)$. Then

$$d^1_{\mathsf{Z}X}(A,B) \le \ell(\Gamma) \le \ell(\Gamma_A) + \ell(\Gamma_B) + \sum_{\{x,y\}\in E_D} d_X(x,y) < \infty.$$

(ii) follows from the definition of $d^1_{\mathsf{Z}X}$.

(iii) Finally, we check the triangle inequality. Given any $A, B, C \in \mathsf{Z}X$ and $\varepsilon > 0$, it suffices to show that

$$d^1_{\mathsf{Z}X}(A,C) \le d^1_{\mathsf{Z}X}(A,B) + d^1_{\mathsf{Z}X}(B,C) + 6\varepsilon.$$

By the definition of $d^1_{\mathsf{Z}X}(A,B)$ and $d^1_{\mathsf{Z}X}(B,C)$, there exist $\Gamma \in \boldsymbol{\Gamma}_X(A,B)$ and $\Gamma' \in \boldsymbol{\Gamma}_X(B,C)$ such that $\ell(\Gamma) < d^1_{\mathsf{Z}X}(A,B) + \varepsilon$ and $\ell(\Gamma') < d^1_{\mathsf{Z}X}(B,C) + \varepsilon$.

By the definition of $\boldsymbol{\Gamma}_X(A,B)$ and $\Gamma_X(B,C)$, the graphs $\Gamma=(V,E)$ and $\Gamma'=(V',E')$ have finitely many connected components and their closures meet the sets A,B and B,C, respectively.

Fix a finite set $D\subseteq V$ intersecting all connected components of Γ and a finite set $D'\subseteq V'$ intersecting all connected components of Γ'. Fix a function $f:D\to B$ assigning to each $x\in D$ a point $f(x)\in B\cap\overline{\Gamma(x)}$. As $f(x)\in\overline{\Gamma(x)}$, there exists a point $g(x)\in\Gamma(x)\subseteq V$ such that $d_X(g(x),f(x))<\varepsilon/|D|$. Since $f(x)\in B\subseteq\overline{V'}$, there exists a vertex $h(x)\in V'$ with $d_X(f(x),h(x))<\varepsilon/|D|$. Then

$$d_X(g(x),h(x))\le d_X(g(x),f(x))+d_X(f(x),h(x))<2\varepsilon/|D|.$$

Next, do the same for Γ': choose $f':D'\to B$ with $f'(x)\in B\cap\overline{\Gamma'(x)}$ for every $x\in D'$, and $g':D'\to V'$ and $h':D'\to V$ such that for every $x\in D'$ we have $g'(x)\in\Gamma'(x)$ and

$$\max\{d_X(g'(x),f'(x)),d_X(h'(x),f'(x))\}<\varepsilon/|D'|.$$

Consider the graph $\Gamma''=(V'',E'')$ where $V''=V\cup V'$ and

$$E''=E\cup E'\cup\big\{\{g(x),h(x)\}:x\in D\big\}\cup\big\{\{g'(x),h'(x))\}:x\in D'\big\}$$

in which every connected component of Γ is connected to a connected component of Γ' and vice versa. It can be shown that $\Gamma''\in\boldsymbol{\Gamma}_X(A,C)$, hence

$$\begin{aligned}
d^1_{\mathsf{Z}X}(A,C)&\le\ell(\Gamma'')\\
&\le\ell(\Gamma)+\ell(\Gamma')+\sum_{x\in D}d(g(x),h(x))+\sum_{x\in D'}d(g'(x),h'(x))\\
&<(d^1_{\mathsf{Z}X}(A,B)+\varepsilon)+(d^1_{\mathsf{Z}X}(B,C)+\varepsilon)+|D|\cdot\frac{2\varepsilon}{|D|}+|D'|\cdot\frac{2\varepsilon}{|D'|}\\
&=d^1_{\mathsf{Z}X}(A,B)+d^1_{\mathsf{Z}X}(B,C)+6\varepsilon.\ \blacksquare
\end{aligned}$$

Given any finite sets $A,B\in\mathsf{F}X$, let $\boldsymbol{\Gamma}^{\mathsf{f}}_X(A,B)$ be the subfamily of finite graphs in $\boldsymbol{\Gamma}_X(A,B)$. Then $A\cup B\subseteq\overline{V}=V$ for any $(V,E)\in\boldsymbol{\Gamma}^{\mathsf{f}}_X(A,B)$.

LEMMA 2.5. *We have*

$$d^1_{\mathsf{Z}X}(A,B)=d^1_{\mathsf{F}X}(A,B)=\inf_{\Gamma\in\boldsymbol{\Gamma}^{\mathsf{f}}_X(A,B)}\ell(\Gamma)$$

for all $A,B\in\mathsf{F}X$.

Proof. Fix any $A,B\in\mathsf{F}X$ and put

$$I=\inf_{\Gamma\in\boldsymbol{\Gamma}_X(A,B)}\ell(\Gamma)\quad\text{and}\quad I_{\mathsf{f}}=\inf_{\Gamma\in\boldsymbol{\Gamma}^{\mathsf{f}}_X(A,B)}\ell(\Gamma).$$

The equality $d^1_{\mathsf{F}X}(A,B)=I_{\mathsf{f}}$ was proved in [2, Theorem 30.4]. So, it remains to show that $I=I_{\mathsf{f}}$. The inequality $I\le I_{\mathsf{f}}$ is trivial and follows from $\boldsymbol{\Gamma}^{\mathsf{f}}_X(A,B)\subseteq\boldsymbol{\Gamma}_X(A,B)$. The inequality $I_{\mathsf{f}}\le I$ will follow as soon as we show that $I_{\mathsf{f}} \le I + 5\varepsilon$ for any $\varepsilon > 0$. Given any $\varepsilon > 0$, find a graph $\Gamma \in \Gamma_X(A, B)$ such that $\ell(\Gamma) < I + \varepsilon$.

Since $\Gamma \in \overline{\Gamma_X(A, B)}$, for every $a \in A$ we can find a point $v(a) \in V$ such that $a \in \overline{\Gamma(v(a))}$ and $B \cap \overline{\Gamma(v(a))}$ contains some point $\beta(a)$. Since $\beta(a) \in \overline{\Gamma(v(a))}$, there exists $u(a) \in \Gamma(v(a))$ such that

$$d_X(u(a), \beta(a)) < \varepsilon/|A|.$$

Since $a \in \overline{\Gamma(f(x))}$, we can replace $v(a)$ by a suitable point in the connected component $\Gamma(v(a))$ and additionally assume that

$$d_X(a, v(a)) < \varepsilon/|A|.$$

Since $v(a), u(a)$ belong to the same connected component of Γ, there exist $n_a \in \mathbb{N}$ and $v_0(a), \ldots, v_{n_a}(a) \in V$ such that $v_0(a) = v(a)$, $v_{n_a}(a) = u(a)$ and $\{v_{i-1}(a), v_i(a)\} \in E$ for every $i \in \{1, \ldots, n_a\}$.

Now do the same with the set B: for every point $b \in B$ choose $\alpha(b) \in A$ and $v'(b), u'(b) \in V$ such that $b \in \overline{\Gamma(v'(b))}$, $\alpha(b) \in A \cap \overline{\Gamma(v'(b))}$,

$$d_X(b, v'(b)) < \varepsilon/|B|,$$

$u'(b) \in \Gamma(v'(b))$, and

$$d_X(\alpha(b), u'(b)) < \varepsilon/|B|.$$

Since $v'(b), u'(b)$ belong to the same connected component of Γ, there exist $m_a \in \mathbb{N}$ and $v'_0(b), \ldots, v'_{m_b}(b) \in V$ such that $v'_0(b) = v'(b)$, $v'_{m_b}(b) = u'(b)$ and $\{v'_{i-1}(b), v'_i(b)\} \in E$ for every $i \in \{1, \ldots, m_a\}$.

Now consider the finite graph $\Gamma' = (V', E')$ with the set of vertices

$$V' = A \cup B \cup \bigcup_{a\in A} \{v_i(a) : 1 \le i \le n_a\} \cup \bigcup_{b\in B} \{v'_i(b) : 1 \le i \le m_a\}$$

and the set of edges

$$\begin{aligned} E' = & \bigcup_{a\in A} \big\{\{a, v(a)\}, \{u(a), \beta(a)\}, \{v_{i-1}(a), v_i(a)\} : 1 \le i \le n_a\big\} \\ & \cup \bigcup_{b\in B} \big\{\{b, v'(b)\}, \{u'(b), \alpha(b)\}, \{v'_{i-1}(b), v'_i(b)\} : 1 \le i \le m_a\big\}. \end{aligned}$$

It is easy to see that $\Gamma' \in \Gamma^{\mathsf{f}}_X(A, B)$ and hence

$$\begin{aligned} I_{\mathsf{f}} \le \ell(\Gamma') \le \ell(\Gamma) &+ \sum_{a\in A} \big(d_X(a, v(a)) + d_X(u(a), \beta(a))\big) \\ &+ \sum_{b\in B} \big(d_X(b, v'(b)) + d_X(\alpha(b), u'(b))\big) \\ &< I + \varepsilon + 2\varepsilon + 2\varepsilon = I + 5\varepsilon. \ \blacksquare \end{aligned}$$

Lemma 2.3 and the definition of the metric $d^1_{\mathsf{Z}X}$ imply the following.

LEMMA 2.6. *The map $X \to \mathsf{Z}^1 X$, $x \mapsto \{x\}$, is an isometric embedding.*

LEMMA 2.7. *For any dense subset $Y \subseteq X$, the set $\mathsf{F}Y$ is dense in the metric space $\mathsf{Z}^1X = (\mathsf{Z}X, d^1_{\mathsf{Z}X})$.*

Proof. Given any $A \in \mathsf{Z}X$ and $\varepsilon > 0$, it suffices to find a set $B \in \mathsf{F}Y$ such that $d^1_{\mathsf{Z}X}(A,B) < 2\varepsilon$. Since $\ell(A) = 0$, there exists a graph $\Gamma = (V,E)$ in X such that Γ has finitely many connected components, $A \subseteq \overline{V}$ and $\ell(A) < \varepsilon$. Replacing Γ by a smaller subgraph, we can assume that the closure of each connected component of Γ meets A. Choose a finite set $B' \subseteq V$ that meets each connected component of Γ. It is easy to see that $\Gamma \in \boldsymbol{\Gamma}_X(A,B')$ and hence $d^1_{\mathsf{Z}X}(A,B') \le \ell(\Gamma) < \varepsilon$.

Using the density of Y in X, choose a finite set $B \subseteq Y$ and a surjective function $f : B' \to B$ such that $d_X(x, f(x)) < \varepsilon/|B'|$ for all $x \in B'$. Consider the graph $\Gamma' = (V', E')$ with $V' = B' \cup f(B')$ and $E' = \{\{x, f(x)\} : x \in B'\}$. Observe that $\Gamma' \in \boldsymbol{\Gamma}_X(B', B)$ and hence

$$d^1_{\mathsf{Z}X}(B,B') \le \ell(\Gamma') \le \sum_{x \in B'} d_X(x, f(x)) < \varepsilon.$$

Therefore

$$d^1_{\mathsf{Z}X}(A,B) \le d^1_{\mathsf{Z}X}(A,B') + d^1_{\mathsf{Z}X}(B',B) < \varepsilon + \varepsilon = 2\varepsilon. \ \blacksquare$$

LEMMA 2.8. *If the metric space X is complete, then so is the metric space Z^1X.*

Proof. We need to prove that each Cauchy sequence in the space Z^1X is convergent. Since the space F^1X is dense in Z^1X (see Lemmas 2.5, 2.7), it suffices to prove that each Cauchy sequence in F^1X converges to some set $A \in \mathsf{Z}X$. So, fix a Cauchy sequence $\{A_n\}_{n\in\omega} \subseteq \mathsf{F}^1X$. As $d_{\mathsf{F}X} = d^\infty_{\mathsf{F}X} \le d^1_{\mathsf{F}X}$, the sequence $(A_n)_{n\in\omega}$ remains Cauchy in the Hausdorff metric $d_{\mathsf{F}X}$. By the completeness of the hyperspace $\mathsf{K}X$, the sequence $(A_n)_{\in\omega}$ converges (in the Hausdorff metric $d_{\mathsf{K}X}$) to some non-empty compact set $A \in \mathsf{K}X$. It remains to show that $A \in \mathsf{Z}X$ and $(A_n)_{n\in\omega}$ converges to A in Z^1X.

Given any $\varepsilon > 0$, use the Cauchy property of $(A_n)_{n\in\omega}$ and find an increasing number sequence $(n_k)_{k\in\omega}$ such that

$$d^1_{\mathsf{Z}X}(A_{n_k}, A_i) < \frac{\varepsilon}{2^{k+1}}$$

for any $k \in \omega$ and $i \ge n_k$. By Lemma 2.5, for every $k \in \omega$ there exists a graph $\Gamma_k \in \boldsymbol{\Gamma}^{\mathsf{f}}_X(A_{n_k}, A_{n_{k+1}})$ such that $\ell(\Gamma_k) < \varepsilon/2^{k+1}$. Now consider the graph $\Gamma = (V,E)$ with $V = \bigcup_{k\in\omega} V_k$ and $E = \bigcup_{k\in\omega} E_k$ and observe that each connected component of Γ meets the finite set A_{n_0}, which implies that Γ has finitely many connected components. Taking into account that A is the limit of $(A_{n_k})_{k\in\omega}$ in the Hausdorff metric, we conclude that $A \subseteq \overline{\bigcup_{k\in\omega} A_{n_k}} \subseteq \overline{V}$ and the closure of each connected component of Γ meets A. Then $\Gamma \in \boldsymbol{\Gamma}_X(A)$ and

$$\ell(A) \le \ell(\Gamma) \le \sum_{k\in\omega} \ell(\Gamma_k) < \sum_{k\in\omega} \frac{\varepsilon}{2^{k+1}} = \varepsilon.$$

This shows that $\ell(A) = 0$ and $A \in \mathsf{Z}X$.

It remains to show that $(A_n)_{n\in\omega}$ converges to A in Z^1X. Since the sequence is Cauchy, it suffices to show that the subsequence $(A_{n_k})_{k\in\omega}$ converges to A. For every $k \in \omega$, consider the graph $\widetilde{\Gamma}_k = (\widetilde{V}_k, \widetilde{E}_k)$ with $\widetilde{V}_k = \bigcup_{i=k}^{\infty} V_i$ and $\widetilde{E}_k = \bigcup_{i=k}^{\infty} E_i$. It can be shown that $\widetilde{\Gamma}_k \in \boldsymbol{\Gamma}_X(A, A_{n_k})$ and hence

$$d^1_{\mathsf{Z}X}(A, A_{n_k}) \le \ell(\widetilde{\Gamma}_k) \le \sum_{i=k}^{\infty} \ell(\Gamma_i) < \sum_{i=k}^{\infty} \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2^k} \xrightarrow[k\to\infty]{} 0,$$

which means that $(A_{n_k})_{k\in\omega}$ converges to A in Z^1X. ∎

Lemma 2.9. *If Y is a dense subspace of X, then*

$$d^1_{\mathsf{Z}X}(A, B) = d^1_{\mathsf{Z}Y}(A, B)$$

for any $A, B \in \mathsf{Z}Y$.

Proof. The inequality $d^1_{\mathsf{Z}X}(A, B) \le d^1_{\mathsf{Z}Y}(A, B)$ is trivial and follows from the inclusion $\boldsymbol{\Gamma}_Y(A, B) \subseteq \boldsymbol{\Gamma}_X(A, B)$.

Assuming $d^1_{\mathsf{Z}X}(A, B) < d^1_{\mathsf{Z}Y}(A, B)$, find $\varepsilon > 0$ such that

$$d^1_{\mathsf{Z}X}(A, B) + 7\varepsilon < d^1_{\mathsf{Z}Y}(A, B).$$

Using Lemma 2.7, choose finite sets $A', B' \in \mathsf{F}Y$ such that $d^1_{\mathsf{Z}Y}(A, A') < \varepsilon$ and $d^1_{\mathsf{Z}Y}(B, B') < \varepsilon$. Then also

$$d^1_{\mathsf{Z}X}(A, A') \le d^1_{\mathsf{Z}Y}(A, A') < \varepsilon \quad \text{and} \quad d^1_{\mathsf{Z}X}(B, B' \le d^1_{\mathsf{Z}Y}(B, B') < \varepsilon.$$

By the triangle inequality,

$$\begin{aligned} d^1_{\mathsf{Z}X}(A', B') &< d^1_{\mathsf{Z}X}(A', A) + d^1_{\mathsf{Z}X}(A, B) + d^1_{\mathsf{Z}X}(B, B') \\ &\le 2\varepsilon + d^1_{\mathsf{Z}X}(A, B) < 2\varepsilon + d^1_{\mathsf{Z}Y}(A, B) - 7\varepsilon \\ &\le d^1_{\mathsf{Z}Y}(A, A') + d^1_{\mathsf{Z}Y}(A', B') + d^1_{\mathsf{Z}Y}(B', B) - 5\varepsilon \\ &< \varepsilon + d^1_{\mathsf{Z}Y}(A', B') + \varepsilon - 5\varepsilon = d^1_{\mathsf{Z}Y}(A', B') - 3\varepsilon. \end{aligned}$$

By Lemma 2.5, there exists a finite graph $\Gamma = (V, E) \in \boldsymbol{\Gamma}^{\mathsf{f}}_X(A', B')$ such that

$$\ell(\Gamma) < d^1_{\mathsf{Z}X}(A', B') + \varepsilon.$$

Since Y is dense in X, we can find a function $f : V \to Y$ such that $f(x) = x$ if $x \in Y$ and $d_X(f(x), x) < \varepsilon/|E|$ if $x \in V \setminus Y$. Consider the graph $\Gamma' = (V', E')$ with $V' = f(V)$ and $E' = \{\{f(x), f(y)\} : \{x, y\} \in E\}$. Observe that

$\Gamma' \in \boldsymbol{\Gamma}^{\mathsf{f}}_Y(A', B')$ and hence

$$\begin{aligned} d^1_{\mathsf{Z}Y}(A', B') &\le \ell(\Gamma') = \sum_{\{x',y'\}\in E'} d_X(x', y') \le \sum_{\{x,y\}\in E} d_X(f(x), f(y)) \\ &\le \sum_{\{x,y\}\in E} \big(d_X(f(x), x) + d_X(x, y) + d_X(y, f(y))\big) \\ &< \sum_{\{x,y\}\in E} \left(\frac{\varepsilon}{|E|} + d_X(x, y) + \frac{\varepsilon}{|E|}\right) < 2\varepsilon + \sum_{\{x,y\}\in E} d_X(x, y) \\ &= 2\varepsilon + \ell(\Gamma) < 2\varepsilon + d^1_{\mathsf{Z}X}(A', B') + \varepsilon < d^1_{\mathsf{Z}Y}(A', B'), \end{aligned}$$

a contradiction showing that $d^1_{\mathsf{Z}X}(A, B) = d^1_{\mathsf{Z}Y}(A, B)$. ■

REMARK 2.10. The density of Y in X is essential in Lemma 2.9. Indeed, for $Y = \{z \in \mathbb{C} : z^3 = 1\}$ and $X = Y \cup \{0\}$ in the complex plane we have

$$d^1_{\mathsf{Z}Y}(A, B) = 2\sqrt{3} > 3 = d^1_{\mathsf{Z}X}(A, B)$$

where $A = \{1\}$ and $B = Y \setminus A$.

LEMMA 2.11. *If $\bar{X}$ is a completion of X, then $\mathsf{Z}^1\bar{X}$ is a completion of F^1X.*

Proof. By Lemma 2.8, the metric space $\mathsf{Z}^1\bar{X}$ is complete. By Lemmas 2.5 and 2.9, for any $A, B \in \mathsf{F}X$ we have

$$d^1_{\mathsf{F}X}(A, B) = d^1_{\mathsf{Z}X}(A, B) = d^1_{\mathsf{Z}\bar{X}}(A, B),$$

so the metric space F^1X is a subspace of the complete metric space $\mathsf{Z}^1\bar{X}$. By Lemma 2.7, it is a dense subspace. ■

Now we discuss the interplay between zero length and 1-dimensional Hausdorff measure. We recall that a subset A of a metric space X has 1-*dimensional Hausdorff measure zero* if for any $\varepsilon > 0$ there exists a countable set $C \subseteq X$ and a function $\epsilon : C \to (0, 1]$ such that $\sum_{c\in C} \epsilon(c) < \varepsilon$ and $A \subseteq \bigcup_{c\in C} B[c, \epsilon(c)]$. Here and further on, by

$$B(x, \delta) = \{y \in X : d_X(x, y) < \delta\} \quad \text{and} \quad B[x, \delta] = \{y \in X : d_X(x, y) \le \delta\}$$

we denote respectively the open and closed balls of radius δ around a point x in the metric space (X, d_X).

PROPOSITION 2.12. *If a subset A of a metric space (X, d_X) has zero length, then it is totally bounded and has 1-dimensional Hausdorff measure zero.*

Proof. If A has zero length, then for every $\varepsilon > 0$ there exists a graph $\Gamma = (V, E)$ in X with finitely many connected components such that $\ell(\Gamma) < \varepsilon$ and $A \subseteq \overline{V}$. To see that A has 1-dimensional Hausdorff measure zero, choose a finite set $D \subseteq V$ that meets each connected component of V in a single point. Then $\{\Gamma(x)\}_{x\in D}$ is a finite disjoint cover of V. For every $x \in D$ let $\epsilon(x) := \sup_{y\in\Gamma(x)} d_X(x,y)$ and observe that $A \subseteq \overline{V} \subseteq \bigcup_{x\in D} B[x,\epsilon(x)]$. The connectedness of $\Gamma(x)$ implies that $\epsilon(x) \le \ell(\Gamma(x))$ and $\sum_{x\in D}\epsilon(x) \le \ell(\Gamma) < \varepsilon$. Since $A \subseteq \bigcup_{x\in D} B[x,\epsilon(x)]$, the set A is totally bounded and has 1-dimensional Hausdorff measure zero. ■

To characterize sets of zero length in the real line, we shall need the following lemma.

Lemma 2.13. *Let $a_1 < b_1 < \cdots < a_n < b_n$ be real numbers and $A \subseteq \bigcup_{i=1}^n (a_i,b_i)$ be a compact subset of Lebesgue measure zero in $X = \mathbb{R}$ such that $A \cap (a_i,b_i) \neq \emptyset$ for all $i \in \{1,\dots,n\}$. There exists a graph $\Gamma \in \boldsymbol{\Gamma}_X(A,\{a_i\}_{i=1}^n)$ of total length $\ell(\Gamma) < 4\sum_{i=1}^n (b_i - a_i)$ witnessing that $d^1_{\mathsf{Z}X}(A,\{a_i\}_{i=1}^n) < 4\sum_{i=1}^n(b_i-a_i)$.*

Proof. Let $\varepsilon = \sum_{i=1}^n (b_i - a_i)$, $U_0 = \bigcup_{i=1}^n (b_i - a_i)$, $n_0 = n$ and $a_{i,0} = a_i$, $b_{i,0} = b_i$ for $i \in \{1,\dots,n_0\}$.

Using the compactness of A and the regularity of the Lebesgue measure, construct inductively a decreasing sequence $(U_k)_{k\in\omega}$ of bounded open neighborhoods of A such that for every $k \in \mathbb{N}$ the following conditions are satisfied:

- $\overline{U}_k \subset U_{k-1}$;
- $\lambda(U_k) < \varepsilon/4^k$;
- $U_k = \bigcup_{i=1}^{n_k} (a_{i,k}, b_{i,k})$ for some $n_k \in \mathbb{N}$ and $a_{1,k} < b_{1,k} \le \cdots \le a_{n_k,k} < b_{n_k,k}$ such that $A \cap (a_{i,k}, b_{i,k}) \neq \emptyset$ for every $i \in \{1,\dots,n_k\}$.

For every $k \in \omega$ let

$$a'_{i,k} := \min\big\{a_{j,k+1} : j \in \{1,\dots,n_{k+1}\},\ a_{i,k} < a_{j,k+1}\big\}$$

and observe that $a'_{i,k} \le \min(\bar{A} \cap (a_{i,k}, b_{i,k}))$, hence $|a_{i,k} - a'_{i,k}| \le |a_{i,k} - b_{i,k}|$. For every $k \in \mathbb{N}$, let

$$\Omega_k = \big\{i \in \{1,\dots,n_k - 1\} : \\ \exists j \in \{1,\dots,n_{k-1}\}\ (b_{i,k}, a_{i+1,k}) \subseteq (a_{j,k-1}, b_{j,k-1})\big\}.$$

Consider the graph $\Gamma = (V,E)$ with the set of vertices

$$V = \bigcup_{k\in\omega} \{a_{i,k}, b_{i,k} : 1 \le i \le n_k\}$$

and the set of edges

$$E = \big\{\{a_{i,k}, b_{i,k}\}, \{a_{i,k}, a'_{i,k}\} : k \in \omega,\ i \in \{1,\dots,n_k\}\big\} \\ \cup \big\{\{b_{i,k}, a_{i+1,k}\} : k \in \mathbb{N},\ i \in \Omega_k\big\}.$$

It is easy to see that $A \subseteq \overline{V}$ and each connected component of Γ intersects the sets A and $\{a_{i,0}\}_{i=1}^{n_0} = \{a_i\}_{i=1}^n$. Therefore, Γ has finitely many connected components and $\varGamma \in \boldsymbol{\varGamma}_X(A, \{a_i\}_{i=1}^n)$. Also

$$\begin{aligned}\ell(\varGamma) &\le \sum_{k=0}^{\infty}\sum_{i=1}^{n_k}(|b_{i,k}-a_{i,k}|+|a'_{i,k}-a_{i,k}|)+\sum_{k=1}^{\infty}\sum_{i\in\varOmega_k}|a_{i+1,k}-b_{i,k}|\\ &< 2\sum_{k=0}^{\infty}\sum_{i=1}^{n_k}|b_{i,k}-a_{i,k}|+\sum_{k=1}^{\infty}\sum_{j=1}^{n_{k-1}}|b_{j,k-1}-a_{j,k-1}|\\ &= 3\sum_{k=0}^{\infty}\sum_{i=1}^{n_k}|b_{i,k}-a_{i,k}| \le 3\sum_{k=0}^{\infty}\lambda(U_k) < 3\sum_{k=0}^{\infty}\frac{\varepsilon}{4^k} = 4\varepsilon.\end{aligned}$$

Therefore

$$d^1_{\mathsf{Z}X}(A, \{a_i\}_{i=1}^n) \le \ell(\varGamma) < 4\varepsilon = 4\sum_{i=1}^{n}(b_i - a_i). \ \blacksquare$$

Now we are ready to prove a characterization of sets of zero length in the real line.

PROPOSITION 2.14. *For a subset A of the real line the following conditions are equivalent:*

(1) *A has zero length;*
(2) *$\bar{A}$ is compact and has zero length;*
(3) *$\bar{A}$ is compact and has 1-dimensional Hausdorff measure zero;*
(4) *$\bar{A}$ is compact and has Lebesgue measure zero.*

Proof. The implications (1)⇒(2)⇒(3) follow from Proposition 2.12 and the observation that the closure of a set of zero length has zero length. The implication (3)⇒(4) follows from the definition of the Lebesgue measure (as the 1-dimensional Hausdorff measure) on the real line, and (4)⇒(1) follows from Lemma 2.13. ■

PROPOSITION 2.15. *For $X = \mathbb{R}$, the identity inclusion $\mathsf{Z}^1X \to \mathsf{K}X$ is a topological embedding.*

Proof. Because of Lemma 2.3, it suffices to prove that for every $A \in \mathsf{Z}X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for any $B \in \mathsf{Z}X$ the inequality $d_{\mathsf{K}X}(A, B) < \delta$ implies $d^1_{\mathsf{Z}X}(A, B) < \varepsilon$.

By Proposition 2.14, the compact set A has Lebesgue measure zero. By the regularity of the Lebesgue measure on the real line, there exists an open neighborhood U of A in $\mathbb{R}$ such that $U = \bigcup_{i=1}^n (a_i, b_i)$ for some sequence $a_1 < b_1 < \cdots < a_n < b_n$ with $\sum_{i=1}^n |b_i - a_i| < \frac{1}{8}\varepsilon$. Find $\delta > 0$ such that every set $B \in \mathsf{K}X$ with $d_{\mathsf{K}X}(A, B) < \delta$ is contained in U. Given any compact set $B \in \mathsf{Z}X$ with $d_{\mathsf{K}X}(A, B) < \delta$, apply Lemmas 2.4 and 2.13 to conclude that

$$d^1_{\mathsf{Z}X}(A,B) \le d^1_{\mathsf{Z}X}(A,\{a_i\}_{i=1}^n) + d^1_{\mathsf{Z}X}(B,\{a_i\}_{i=1}^n) < 8\sum_{i=1}^n (b_i - a_i) < \varepsilon. \blacksquare$$

Proposition 2.15 is specific to the real line and does not hold for higher-dimensional Euclidean spaces. To prove this, let us recall the definition of the upper box-counting dimension $\overline{\dim}_B(X)$ of a metric space X. Given any $\varepsilon > 0$, denote by $N_\varepsilon(X)$ the smallest cardinality of a cover of X by subsets of diameter $\le \varepsilon$. Observe that a metric space X is totally bounded iff $N_\varepsilon(X)$ is finite for every $\varepsilon > 0$. If X is not totally bounded, then put $\overline{\dim}_B(X) = \infty$. If X is totally bounded, then let

$$\overline{\dim}_B(X) := \limsup_{\varepsilon\to+0} \frac{\ln N_\varepsilon(X)}{\ln(1/\varepsilon)} \in [0,\infty].$$

By [9, §3.2], for every $n \in \mathbb{N}$, every bounded set $X \subseteq \mathbb{R}^n$ with non-empty interior has $\overline{\dim}_B(X) = n$.

Proposition 2.16. *Let X be a metric space and $Y \subseteq X$ be a subspace of X such that $\overline{\dim}_B(Y) > 1$. Then for any $l \in \mathbb{N}$ there exists a non-empty finite subset $A \subseteq Y$ such that $d^1_{\mathsf{F}X}(A,\{x\}) \ge l$ for any singleton $\{x\} \subseteq X$.*

Proof. To derive a contradiction, assume that there exists $l \in \mathbb{N}$ such that for any finite set $A \subseteq Y$ there exists $x \in X$ such that $d^1_{\mathsf{F}X}(A,\{x\}) < l$.

We are going to show that $N_{2\varepsilon}(Y) \le (2l+1)/\varepsilon$ for every $\varepsilon \in (0,1]$. Given any $\varepsilon \in (0,1]$, use the Kuratowski–Zorn Lemma to find a maximal subset M in Y which is 2ε-separated in the sense that $d_X(y,z) \ge 2\varepsilon$ for any distinct $y,z \in M$. The maximality implies that $Y \subseteq \bigcup_{y\in M} B(y,2\varepsilon)$.

We claim that $|M| \le (1+2l)/\varepsilon$. To derive a contradiction, assume that $|M| > (1+2l)/\varepsilon$. Then we can find a finite subset $A \subseteq M$ with $|A| > (1+2l)/\varepsilon$. The choice of l ensures that $d^1_{\mathsf{F}X}(A,\{x\}) < l$ for some $x \in X$. By Lemma 2.5, there exists a finite graph $\Gamma \in \Gamma_X(\{x\},A)$ such that $\ell(\Gamma) < l$. Since each connected component of Γ meets $\{x\}$, the graph $\Gamma = (V,E)$ is connected. Replacing Γ by a minimal connected subgraph, we can assume that Γ is a tree.

By Lemma 2.17 (proved below), there exists a sequence $v_0,\dots,v_n \in V$ such that

(i) $V = \{v_0,\dots,v_n\}$;
(ii) $\{\{v_{i-1},v_i\} : 1 \le i \le n\} \subseteq E$;
(iii) for every $e \in E$ the set $\{i \in \{1,\dots,n\} : \{v_{i-1},v_i\} = e\}$ contains at most two elements.

Choose a sequence of real numbers $t_0,\dots,t_n$ with $t_0 = 0$ and $t_i - t_{i-1} = d_X(v_i,v_{i-1})$ for every $i \in \{1,\dots,n\}$. Condition (iii) implies $t_n \le 2\ell(\Gamma) < 2l$.

Then the set $T = \{t_0, \dots, t_n\}$ has

$$N_\varepsilon(T) < 1 + \frac{t_n}{\varepsilon} < 1 + \frac{2l}{\varepsilon} \le \frac{1+2l}{\varepsilon}.$$

Taking into account that the map $T \to V$, $t_i \mapsto v_i$, is non-expanding, we conclude that $N_\varepsilon(A) \le N_\varepsilon(V) \le N_\varepsilon(T) < (1+2l)/\varepsilon$. Since A is 2ε-separated, we have $|A| = N_\varepsilon(A) < (1+2l)/\varepsilon$, which contradicts the choice of A.

This contradiction shows that $|M| \le (1+2l)/\varepsilon$ and then $N_{2\varepsilon}(Y) \le |M| \le (1+2l)/\varepsilon$ for any $\varepsilon > 0$. Taking the upper limit as $\varepsilon \to +0$, we obtain the upper bound

$$\begin{aligned}\overline{\dim}_B(Y) &= \limsup_{\varepsilon\to+0} \frac{\ln N_\varepsilon(Y)}{\ln(1/\varepsilon)} = \limsup_{\varepsilon\to+0} \frac{\ln N_{2\varepsilon}(Y)}{\ln(1/(2\varepsilon))} \\ &\le \limsup_{\varepsilon\to+0} \frac{\ln((1+2l)/\varepsilon)}{\ln(1/(2\varepsilon))} = 1,\end{aligned}$$

which contradicts our assumption. ∎

LEMMA 2.17. *For any finite tree $\Gamma = (V, E)$, there exists a sequence $v_0, \dots, v_n \in V$ such that*

- (i) $V = \{v_0, \dots, v_n\}$;
- (ii) $\{\{v_{i-1}, v_i\} : 1 \le i \le n\} = E$;
- (iii) *for every edge $e \in E$ the set $\{i \in \{1, \dots, n\} : \{v_{i-1}, v_i\} = e\}$ contains at most two elements.*

Proof. We use induction on $|V|$. If $|V| = 1$, then let v_0 be the unique vertex of V and observe that the sequence v_0 has the properties (i)–(iii). Assume that for some $k \ge 2$ the lemma has been proved for all trees on $<k$ vertices. Let $\Gamma = (V, E)$ be any tree with $|V| = k$. By [6, 1.5.1], the tree Γ has exactly $k-1$ edges. Consequently, there exists a vertex $v \in V$ having a unique neighbor $u \in V \setminus \{v\}$ in (V, E). Put $V' = V \setminus \{v\}$, $E' = E \setminus \{\{u, v\}\}$ and observe that (V', E') is a tree on $k-1$ vertices. By the inductive assumption, there exists a sequence $v'_0, \dots, v'_n \in V'$ such that $V' = \{v'_0, \dots, v'_n\}$, $\{\{v'_{i-1}, v'_i\} : i \in \{1, \dots, n\}\} = E'$, and for every $e \in E'$ the set $\{i \in \{1, \dots, n\} : \{v'_{i-1}, v'_i\} = e\}$ contains at most two elements.

Find an index $j \in \{0, \dots, n\}$ such that $v'_j = u$ and consider the sequence $v_0, \dots, v_{n+2}$ where $v_i = v'_i$ for $i \le j$, $v_{j+1} = v$, and $v_i = v'_{i-2}$ for $i \in \{j+2, \dots, n+2\}$. It is easy to see that this sequence satisfies (i)–(iii). ∎

Proposition 2.16 implies the following corollary, in which $\mathsf{F}X$ denotes the hyperspace of non-empty finite subsets of X, endowed with the Hausdorff metric.

COROLLARY 2.18. *Let X be a metric space. If for some point $x \in X$ the identity map $\mathsf{F}X \to \mathsf{Z}^1X$ is continuous at $\{x\}$, then the point x has a neighborhood $O_x \subseteq X$ with $\overline{\dim}_B(O_x) \le 1$.*

Proof. Assuming that the identity map $\mathsf{F}X \to \mathsf{Z}^1X$ is continuous at $\{x\}$, we can find $\delta > 0$ such that for any set $A \in \mathsf{F}X$ with $d_{\mathsf{F}X}(A, \{x\}) < \delta$ we have $d^1_{\mathsf{F}X}(A, \{x\}) < 1$. Let $O_x := B(x, \delta)$. Assuming that $\overline{\dim}_B(O_x) > 1$, we can apply Proposition 2.16 and find a finite set $A \subseteq O_x$ such that $d^1_{\mathsf{F}X}(A, \{x\}) > 1$. On the other hand, the inclusion $A \subseteq O_x = B(x, \delta)$ implies that $d_{\mathsf{F}X}(A, \{x\}) < \delta$ and hence $d^1_{\mathsf{F}X}(A, \{x\}) < 1$ by the choice of δ. This contradiction shows that $\overline{\dim}_B(O_x) \leq 1$. ■

Finally, we present an example showing that the equivalence (2)⇔(3) in Proposition 2.14 does not hold for higher-dimensional Euclidean spaces.

EXAMPLE 2.19. Assume that a metric space X contains a point $x \in X$ such that $\overline{\dim}_B(O_x) > 1$ for any neighborhood $O_x \subseteq X$ of x. Then X contains a compact subset A with unique non-isolated point x such that A has 1-dimensional Hausdorff measure zero and A fails to have zero length.

Proof. Using Proposition 2.16, for every $n \in \omega$ find a finite set $A_n \subset B(x, 2^{-n})$ with $d^1_{\mathsf{F}X}(A_n, \{x\}) > n$. Then $A = \{x\} \cup \bigcup_{n\in\omega} A_n$ is a required compact set with unique non-isolated point x such that A has 1-dimensional Hausdorff measure zero and A fails to have zero length. ■

REMARK 2.20. There are interesting algorithmic problems related to efficient calculation of the distance $d^1_{\mathsf{F}X}(A, B)$ between non-empty finite subsets A, B of a metric space.

For a non-empty finite subset A of the Euclidean plane $\mathbb{R}^2$ and a singleton $B = \{x\} \subset \mathbb{R}^2$, the problem of calculating the distance $d^1_{\mathsf{F}X}(A, B)$ reduces to the classical Steiner's problem [5] of finding a tree of the smallest length that contains the set $A \cup B$. This problem is known [4] to be computationally very difficult.

On the other hand, for non-empty finite subsets of the real line, there exists an efficient algorithm [1] of complexity $O(n \ln n)$ calculating the distance $d^1_{\mathsf{F}\mathbb{R}}(A, B)$ between two sets $A, B \in \mathsf{F}\mathbb{R}$ of cardinality $|A| + |B| \leq n$. Also there exists an algorithm of the same complexity $O(n \ln n)$ calculating the Hausdorff distance $d_{\mathsf{F}\mathbb{R}}(A, B)$ between the sets A, B.

Finally, let us remark that the evident brute force algorithm for calculating the Hausdorff distance $d_{\mathsf{F}X}(A, B)$ between non-empty finite subsets of an arbitrary metric space (X, d_X) has complexity $O(|A| \cdot |B|)$. Here we assume that calculating the distance between points requires a constant amount of time.

Acknowledgements. The authors express sincere thanks to the anonymous referee for a careful reading of the manuscript and many useful suggestions that have essentially improved the presentation.

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