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EQUIVARIANT MAPPINGS AND INVARIANT SETS ON MINKOWSKI SPACE

BY

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Abstract. We start a systematic study of invariant functions and equivariant mappings defined on Minkowski space under the action of the Lorentz group. We adapt some known results from the orthogonal group acting on Euclidean space to the Lorentz group acting on Minkowski space. In addition, an algorithm is given to compute generators of the ring of functions that are invariant under an important class of Lorentz subgroups, namely those generated by involutions, which is also useful to compute equivariants. Furthermore, general results on invariant subspaces of Minkowski space are presented, with a characterization of invariant lines and planes in the two lowest dimensions.

1. Introduction. The interest of studying the effect of symmetries in a mathematical model is common sense. They frequently appear in the applications, in general as a result of the geometry of the configuration domain, the modeling assumptions, the method to reducing to normal forms and so on. In dynamical systems, certain observations are typically related to patterns of symmetry, namely degeneracy of solutions, high codimension bifurcations, unexpected stabilities, phase relations, synchrony in coupled systems and periodicity of solutions. These phenomena, not expected in absence of symmetries, can be predicted once the inherent symmetries are taken into account in the model formulation. This observation has stimulated the great achievements in the theory of equivariant dynamical systems during the last decades. The set-up is based on group representation theory, once the set formed by these symmetries has a group structure. We mention for example [1, 2, 3, 4] where algebraic invariant theory is used to derive the general form of equivariant mappings that define vector fields on real or complex Euclidean n-dimensional vector spaces under the action of subgroups of the orthogonal group $\mathbf{O}(n)$, $n \geq 2$. A numerical analysis of such systems through computational programming has also been done, after the pioneering works in [10, 11, 20]. Steady-state bifurcation, Hopf bifurcation and classification

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of equivariant singularities are among the subjects that have also been extensively studied in isolated dynamical systems or in networks of dynamical systems; see [2, 3, 7, 8, 12, 15, 16, 19] and the references therein.

The results in the present work go in a related but distinct direction. We use tools from group representation theory and from invariant theory to start a systematic study of symmetries of problems defined on the Minkowski space $\mathbb{R}^{n+1}_1$, $n \geq 1$. This is the $(n+1)$-dimensional real vector space $\mathbb{R}^{n+1}$ endowed with the pseudo inner product $\langle\, ,\rangle : \mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1 \to \mathbb{R}$ of signature $(n, 1)$,

$$\langle x, y\rangle = x^t J y = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}, \tag{1.1}$$

where

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$$

(also denoted by $I_{n,1}$) with I_n denoting the identity matrix of order n. This generalizes the spacetime $\mathbb{R}^4_1$, which is the mathematical structure on which Einstein's theory of relativity is formulated. In Minkowski space, the spatio-temporal coordinates of different observers are related by Lorentz transformations, so any laws for systems in Minkowski spacetime must be Lorentz invariant [14]. The group of symmetries is assumed to be a subgroup of the Lorentz group

$$\mathbf{O}(n, 1) = \{A \in \mathrm{GL}(n+1, \mathbb{R}) : A^t J A = J\},$$

which is closed with respect to inversion and to transposition and formed by isometries of $\mathbb{R}^{n+1}_1$; here the superscript t denotes transposition. We consider the standard action of (subgroups of) $\mathbf{O}(n, 1)$ on $\mathbb{R}^{n+1}_1$, given by matrix multiplication. We use algebraic invariant theory and group representation theory to deduce basic general results for the systematic study of mappings with symmetries defined on Minkowski space. We refer e.g. to [5] and [6] for potential applications of our results.

Many previous results for the Euclidean case adapt to our context in a natural way. The first to mention is a way to obtain equivariant mappings on $\mathbb{R}^{n+1}_1$ under the action of a subgroup $\Gamma < \mathbf{O}(n, 1)$ from invariant functions under the diagonal action of the same group on the cartesian product $\mathbb{R}^{n+1}_1 \times \mathbb{R}^{n+1}_1$ (Theorem 2.2). We also adapt and extend [3, Theorem 3.2], regarding general forms of functions invariant under a class of Lorentz subgroups, given by Algorithm 2.4, using Reynolds operators.

There are however two new structures here, namely the pseudo inner product and the distinct group structure of the set of symmetries, leading to new facts. For example, for an arbitrary Lorentz subgroup Γ, it is not necessarily the case that for every Minkowski subspace $W \subset \mathbb{R}_1^{n+1}$ its orthogonal subspace $W^\perp$ (with respect to the pseudo inner product) is an invariant complement under the action of Γ; for the Euclidean case see [12, Proposition XII.2.1]. In fact, we prove that this holds, that is, $W \oplus W^\perp = \mathbb{R}_1^{n+1}$ with both Γ-invariant, if and only if W is Γ-invariant and nondegenerate with respect to the pseudo inner product (Proposition 3.2). In addition, we give a sufficient condition for a degenerate subspace to admit an invariant complement (Proposition 3.3). In Section 3 we characterize the invariant subspaces for the lowest dimensions $n = 1, 2$ using, in the second case, the singular value decomposition of elements of the Lorentz group. This classification is directly related to the structure of the Lorentz group: Let $\mathbf{SO}_0(n, 1)$ denote the connected component of the identity and consider the elements (we follow the usual notation, as in [9] for example)

$$\Lambda^p = \begin{pmatrix} I_{n-1,1} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Lambda^t = J.$$

We use the decomposition of $\mathbf{O}(n, 1)$ as a semidirect product,

$$\mathbf{O}(n,1) = \mathbf{SO}_0(n,1) \rtimes (\mathbb{Z}_2(\Lambda^t) \times \mathbb{Z}_2(\Lambda^p)), \tag{1.2}$$

or, in other words, as the disjoint union

$$\mathbf{O}(n,1) = \mathbf{SO}_0(n,1) \,\dot\cup\, \Lambda^p\mathbf{SO}_0(n,1) \,\dot\cup\, \Lambda^t\mathbf{SO}_0(n,1) \,\dot\cup\, \Lambda^{pt}\mathbf{SO}_0(n,1), \tag{1.3}$$

where $\Lambda^{pt} = \Lambda^p\Lambda^t$. We then use this decomposition to identify conjugacy classes of Lorentz subgroups, since subgroups in distinct connected components are nonconjugate. We also give the type of each possible invariant subspace, namely as spacelike, timelike or lightlike subspaces, and recognize which are fixed-point subspaces of Lorentz subgroups.

Here is what comes in the following sections. Section 2 is devoted to invariant functions and equivariant mappings under subgroups of $\mathbf{O}(n, 1)$. The two main results are Theorem 2.2, used to compute equivariants, and the algorithm in Subsection 2.1, used to compute invariants under a class of groups generated by involutions. An example in Subsection 2.2 illustrates both methods. In Section 3 we present general results about invariant subspaces and their orthogonal complements, with special attention to fixed-point subspaces. The subjects of Subsections 3.1 and 3.2 are the invariant subspaces in the Minkowski plane and Minkowski 3-dimensional space.

2. Invariant functions and equivariant mappings. The aim of this section is to give results on the construction of invariant functions and equivariant mappings under the action of a Lorentz subgroup Γ. We adapt some results from invariant theory on Euclidean space to Minkowski space. We point out that the results are algebraic in nature, so they hold for functions and mappings defined on any subspace of $W \subseteq \mathbb{R}_1^{n+1}$ as long as it is *Γ-invariant*, that is, $\gamma x \in W$ for all $\gamma \in \Gamma$, $x \in W$. For simplification, from now on we assume the domain to be the whole $\mathbb{R}_1^{n+1}$.

A function $f : \mathbb{R}_1^{n+1} \to \mathbb{R}$ is called *Γ-invariant* if

$$f(\gamma x) = f(x), \quad \forall \gamma \in \Gamma, \forall x \in \mathbb{R}_1^{n+1}.$$

The set of Γ-invariant functions is a ring, which we denote $\mathcal{I}(\Gamma)$. A map $g : \mathbb{R}_1^{n+1} \to \mathbb{R}_1^{n+1}$ is *Γ-equivariant* if it commutes with the action of Γ, that is,

$$g(\gamma x) = \gamma g(x), \quad \forall \gamma \in \Gamma, \forall x \in \mathbb{R}_1^{n+1}.$$

The set of Γ-equivariant maps is a module over the ring $\mathcal{I}(\Gamma)$, denoted here by $\mathcal{M}(\Gamma)$.

In the Euclidean context it is well-known that if $f : \mathbb{R}^n \to \mathbb{R}$ is invariant under the action of a subgroup of the orthogonal group $\mathbf{O}(n)$, then the gradient $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is an equivariant map under that action. The corresponding result for the Lorentz group $\mathbf{O}(n,1)$ is given in Proposition 2.1, which can be used as a starting point to find generators for the module of equivariants if generators of the ring of invariants are known:

PROPOSITION 2.1. *Let Γ be a subgroup of $\mathbf{O}(n,1)$ and let $f : \mathbb{R}_1^{n+1} \to \mathbb{R}$ be a Γ-invariant function. Then $J\nabla f$ is a Γ-equivariant map.*

Proof. The equality $f(\gamma x) = f(x)$ implies $\gamma^t \nabla(f(\gamma x)) = \nabla(f(x))$. Now just multiply both sides by γJ and use $\gamma J \gamma^t = J$. ∎

The following result shows that a Γ-equivariant mapping $\mathbb{R}_1^{n+1} \to \mathbb{R}_1^{n+1}$ can be constructed from a Γ-invariant function $\mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1} \to \mathbb{R}$, and vice versa. The proof is constructive and follows the idea of [12, proof of Theorem XII.6.8], providing an appropriate formula.

THEOREM 2.2. *Let Γ be a subgroup of the Lorentz group $\mathbf{O}(n,1)$. There is a one-to-one correspondence between Γ-equivariant mappings $\mathbb{R}_1^{n+1} \to \mathbb{R}_1^{n+1}$ and Γ-invariant functions $\mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1} \to \mathbb{R}$ under the diagonal action.*

Proof. Given $g : \mathbb{R}_1^{n+1} \to \mathbb{R}_1^{n+1}$, take $f : \mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1} \to \mathbb{R}$ defined by

$$f(x,y) = \langle g(x), y \rangle, \tag{2.1}$$

where $\langle\ ,\ \rangle$ is the pseudo inner product in (1.1). This implies that

$$g(x) = J\,(d_y f)^t_{(x,0)}\,. \tag{2.2}$$

If g is Γ-equivariant, then for the diagonal action of Γ on $\mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1}$ we have, for all $\gamma \in \Gamma$,

$$f(\gamma(x,y)) = \langle g(\gamma x), \gamma y \rangle = \langle \gamma g(x), \gamma y \rangle = f(x,y).$$

Conversely, for any $\gamma \in \Gamma$, if f is Γ-invariant, then differentiating $f(\gamma x, \gamma y) = f(x, y)$ with respect to y at $(x, 0)$, we have

$$(d_y f)_{(\gamma x, 0)} \gamma = (d_y f)_{(x,0)} .$$

Taking the transpose and using (2.1), we get

$$g(\gamma x) = J(\gamma^t)^{-1} J g(x),$$

so the result holds, since $J(\gamma^t)^{-1}J = \gamma$. ∎

The next result is our method to find a set of generators for the module of equivariant mappings:

COROLLARY 2.3. *If* $\{u_i : i = 1, \dots, s\}$ *is a set of generators of the ring of* Γ*-invariant functions* $\mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1} \to \mathbb{R}$ *under the diagonal action, then*

$$J(d_y u_i)^t_{(x,0)}, \qquad i = 1, \dots, s,$$

form a set of generators of the module $\mathcal{M}(\Gamma)$ *over* $\mathcal{I}(\Gamma)$.

Proof. For $g \in \mathcal{M}(\Gamma)$, we take f as in (2.1). By hypothesis,

$$f(x, y) = h(u_1(x, y), \dots, u_s(x, y))$$

for some $h : \mathbb{R}^s \to \mathbb{R}$. This implies, using (2.2), that

$$g(x) = \sum_{i=0}^{s} \frac{\partial h}{\partial X_i}(u_1, \dots, u_s)\bigg|_{y=0} \cdot J(d_y u_i)^t_{(x,0)},$$

where $X = (X_1, \dots, X_s) \in \mathbb{R}^s$. So the result follows, since clearly we have $\partial h/\partial X_i(u_1(x, 0), \dots, u_s(x, 0)) \in \mathcal{I}(\Gamma)$, $i = 1, \dots, s$. ∎

This result is illustrated with an example, presented in Subsection 2.2.

2.1. An algorithm to compute invariants under subgroups containing involutions. Recall that an *involution* is an invertible map which is its own inverse. Involutions in $\mathbf{O}(n, 1)$ are order-2 matrices, also called generalized reflections. The aim here is to present Algorithm 2.4 which gives generators of the ring of invariant functions under the class of Lorentz subgroups $\Gamma < \mathbf{O}(n, 1)$ that are generated by a finite set of involutions. More generally, we consider

$$\Gamma = \Sigma \rtimes \Delta, \tag{2.3}$$

where $\Sigma < \Gamma$ is a subgroup such that $\mathcal{I}(\Sigma)$ is finitely generated, and $\Delta = [\delta_1, \dots, \delta_m] < \Gamma$ is generated by involutions, that is, $\delta_i^2 = I$, $i = 1, \dots, m$, $m \geq 1$. More precisely, we deduce generators of the invariants under the whole group from the generators under the subgroup Σ, using the proof of [3, Theorem 3.2]. Recall that the two *Reynolds operators* $R, S : \mathcal{I}(\Sigma) \to \mathcal{I}(\Sigma)$ are given by

$$R(f)(x) = \tfrac{1}{2}(f(x) + f(\delta x)), \qquad S(f)(x) = \tfrac{1}{2}(f(x) - f(\delta x)). \tag{2.4}$$

In [3, Theorem 3.2] it is proved that if $\{u_1, \ldots, u_s\}$ is a set of generators of the ring $\mathcal{I}(\Sigma)$, then the set

$$\{R(u_i), S(u_i)S(u_j) : 1 \leq i, j \leq s\} \tag{2.5}$$

generates the ring $\mathcal{I}(\Sigma \rtimes \mathbb{Z}_2(\delta))$. The idea here is to compute the generators recursively, imposing the invariance under each δ_i, $i = 1, \ldots, m$, at each step. This follows from the fact that the equality

$$\mathcal{I}(\Sigma \rtimes \mathbb{Z}_2(\delta)) = \mathcal{I}(\Sigma) \cap \mathcal{I}(\mathbb{Z}_2(\delta)),$$

which is the foundation to obtain (2.5), generalizes to

$$\mathcal{I}(\Sigma \rtimes [\delta_1, \ldots, \delta_m]) = \mathcal{I}(\Sigma) \cap \bigcap_{i=1}^{m} \mathcal{I}(\mathbb{Z}_2(\delta_i)).$$

Obviously the equalities above for the semidirect product hold for a direct product as well. We then have:

ALGORITHM 2.4.

INPUT:
- *a set* $\{u_1, \ldots, u_s\}$ *of generators of the ring* $\mathcal{I}(\Sigma)$
- *a set of involutions* δ_i, $i = 1, \ldots, m$

OUTPUT: *Generators of* $\mathcal{I}(\Gamma)$, *where* $\Gamma = \Sigma \rtimes [\delta_1, \ldots, \delta_m]$ *or* $\Sigma \times [\delta_1, \ldots, \delta_m]$

PROCEDURE:

- $\Sigma_0 := \Sigma$
- $u_{0i} := u_i$ *for* $i = 1, \ldots, s$
- $s_0 := s$

for k *from* 1 *to* m *do*
  $\Sigma_k := \Sigma_{k-1} \rtimes \mathbb{Z}_2(\delta_k)$
  for δ *in* (2.4), *do* $\delta := \delta_k$
    compute the set in (2.5) *from* $\{u_{k-1,j} : j = 1, \ldots, s_{k-1}\}$
    return generators of $\mathcal{I}(\Sigma_k)$: $\{u_{k1}, \ldots, u_{ks_k}\}$
  end
end
return generators of $\mathcal{I}(\Gamma)$: $\{u_{m1}, \ldots, u_{ms_m}\}$

2.2. An example. Consider the Lorentz subgroup $\Gamma < \mathbf{O}(3,1)$, where

$$\Gamma = \widetilde{\mathbf{SO}}_0(3,1) \rtimes (\mathbb{Z}_2(\delta_1) \times \mathbb{Z}_2(\delta_2)),$$

with

$$\widetilde{\mathbf{SO}}_0(3,1) = \left\{ \xi = \begin{pmatrix} \tilde{\xi} & 0 \\ 0 & I_2 \end{pmatrix} : \tilde{\xi} \in \mathbf{SO}(2) \right\}, \tag{2.6}$$

and for an arbitrary (but fixed) $\theta \in \mathbb{R},\ \theta \neq 0$,

$$\delta_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\theta & \sinh\theta \\ 0 & 0 & -\sinh\theta & -\cosh\theta \end{pmatrix}, \quad \delta_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\cosh\theta & -\sinh\theta \\ 0 & 0 & \sinh\theta & \cosh\theta \end{pmatrix}.$$

We obtain a set of generators for the ring $\mathcal{I}(\Gamma)$ of Γ-invariant functions $\mathbb{R}^4_1 \to \mathbb{R}$. The representation of $\widetilde{\mathbf{SO}}_0(3,1)$ is isomorphic to the 4-dimensional standard representation of $\mathbf{SO}(2) \times \{I_2\}$; it then follows trivially that in coordinates (x_1, x_2, x_3, x_4) of $\mathbb{R}^4_1$ the set $\{u_1 := x_1^2 + x_2^2, u_2 := x_3, u_3 := x_4\}$ generates the ring $\mathcal{I}(\widetilde{\mathbf{SO}}_0(3,1))$. We now apply Algorithm 2.4: for $k = 1$, $\Sigma_1 = \widetilde{\mathbf{SO}}_0(3,1) \rtimes \mathbb{Z}_2(\delta_1)$ and the Reynolds operators in (2.4) are taken for $\delta = \delta_1$, which are denoted here by R_{δ_1}, S_{δ_1}. The computation of the set (2.5) of generators of $\mathcal{I}(\Sigma_1)$ gives

$$\begin{aligned} R_{\delta_1}(u_{01}) &= x_1^2 + x_2^2, \\ R_{\delta_1}(u_{02}) &= \tfrac{1}{2}((\cosh\theta + 1)x_3 + \sinh\theta x_4), \\ R_{\delta_1}(u_{03}) &= -\tfrac{1}{2}((\cosh\theta - 1)x_4 + \sinh\theta x_3), \end{aligned}$$

and

$$\begin{aligned} S_{\delta_1}(u_{01}) &= 0, \\ S_{\delta_1}(u_{02}) &= -\tfrac{1}{2}((\cosh\theta - 1)x_3 + \sinh\theta x_4), \\ S_{\delta_1}(u_{03}) &= \tfrac{1}{2}((\cosh\theta + 1)x_4 + \sinh\theta x_3). \end{aligned}$$

Observing that

$$\begin{aligned} R_{\delta_1}(u_{03}) &= ((1 - \cosh\theta)/\sinh\theta) R_{\delta_1}(u_{02}), \\ S_{\delta_1}(u_{02}) &= ((1 - \cosh\theta)/\sinh\theta) S_{\delta_1}(u_{03}), \end{aligned}$$

it follows that $\{R_{\delta_1}(u_{01}), R_{\delta_1}(u_{02}), S_{\delta_1}(u_{02})^2\}$ generates $\mathcal{I}(\widetilde{\mathbf{SO}}_0(3,1)) \rtimes \mathbb{Z}_2(\delta_1))$. Alternatively, manipulating these generators, another generating set for this ring is given by $\{u_{11} := R_{\delta_1}(u_{01}),\ u_{12} := R_{\delta_1}(u_{03}),\ u_{13} := x_3^2 - x_4^2\}$, since

$$\frac{2}{1 - \cosh\theta}(S_{\delta_1}(u_{02})^2 - R_{\delta_1}(u_{03})^2) = x_3^2 - x_4^2.$$

For $k = 2$, $\Sigma_2 = \Sigma_1 \rtimes \mathbb{Z}_2(\delta_2)$, and the operators in (2.4) are taken for $\delta = \delta_2$, which are denoted here by R_{δ_2}, S_{δ_2}. The computation of the set (2.5) of generators of $\mathcal{I}(\Sigma_2)$ gives

$$\begin{aligned} &R_{\delta_2}(u_{11}) = x_1^2 + x_2^2, \quad R_{\delta_2}(u_{12}) = 0, \quad R_{\delta_2}(u_{13}) = x_3^2 - x_4^2, \\ &S_{\delta_2}(u_{11}) = 0, \quad S_{\delta_2}(u_{12}) = -((\cosh\theta - 1)x_4 + \sinh\theta x_3), \quad S_{\delta_2}(u_{13}) = 0. \end{aligned}$$

It then follows that

$$u_{21} := x_1^2 + x_2^2, \quad u_{22} := x_3^2 - x_4^2, \quad u_{23} := ((\cosh\theta - 1)x_4 + \sinh\theta x_3)^2$$

generate $\mathcal{I}(\Gamma)$.

We now compute a set of generators of the module $\mathcal{M}(\Gamma)$ of Γ-equivariant mappings over the ring $\mathcal{I}(\Gamma)$. We use Corollary 2.3.

For $(x, y) = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in \mathbb{R}_1^4 \times \mathbb{R}_1^4$, consider the diagonal action $\xi(x, y) = (\xi x, \xi y)$ for $\xi \in \widetilde{\mathbf{SO}}_0(3, 1)$. The invariant ring under this subgroup is generated by the polynomials

(2.7)
$$u_{01} := x_1^2 + x_2^2, \;\; u_{02} := x_3, \;\; u_{03} := x_4, \;\; v_{01} := y_1^2 + y_2^2, \;\; v_{02} := y_3, \;\; v_{03} := y_4,$$

(2.8)
$$w_{01} := x_1 y_2 - x_2 y_1, \quad w_{02} = x_1 y_1 + x_2 y_2.$$

The generators in (2.7) are the obvious ones. In fact, the computations above trivially give the generators of the ring of invariant functions on $\mathbb{R}_1^4 \times \mathbb{R}_1^4$ under $\widetilde{\mathbf{SO}}_0(3, 1)$, except the new ones coming from w_1, w_2 given in (2.8), for which $R_{\delta_1}(w_{01}) = w_{01}$, $R_{\delta_1}(w_{02}) = w_{02}$, and therefore $S_{\delta_1}(w_{01}) = 0$, $S_{\delta_1}(w_{02}) = 0$. It follows that

$$\begin{aligned} &u_{11} := x_1^2 + x_2^2, && u_{12} := -((\cosh\theta - 1)x_4 + \sinh\theta x_3), && u_{13} := x_3^2 - x_4^2, \\ &v_{11} := y_1^2 + y_2^2, && v_{12} := -((\cosh\theta - 1)y_4 + \sinh\theta y_3), && v_{13} := y_3^2 - y_4^2, \\ &w_{11} := x_1 y_2 - x_2 y_1, && w_{12} := x_1 y_1 + x_2 y_2 && \end{aligned}$$

are generators of $\mathcal{I}(\widetilde{\mathbf{SO}}_0(3, 1) \rtimes \mathbb{Z}_2(\delta_1))$. Also, $R_{\delta_2}(w_{11}) = w_{11}$ and $R_{\delta_2}(w_{12}) = w_{12}$, and therefore $S_{\delta_2}(w_{11}) = S_{\delta_2}(w_{12}) = 0$. Finally, $S_{\delta_2}(u_{12}) = u_{12}$ and $S_{\delta_2}(v_{12}) = v_{12}$, so $u_{12}v_{12}$ is an element in (2.5). Therefore, the generators of the ring of Γ-invariants $\mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1} \to \mathbb{R}$ are

$$\{x_1^2 + x_2^2, x_3^2 - x_4^2, y_1^2 + y_2^2, y_3^2 - y_4^2, \omega_{11}, \omega_{12}, u_{12}v_{12}\}.$$

We now use (2.2) of Theorem 2.2 which produces the zero map except that

$$\begin{aligned} J(d_y w_{11})^t_{(x,0)} &= (-x_2, x_1, 0, 0), \\ J(d_y w_{12})^t_{(x,0)} &= (x_1, x_2, 0, 0), \\ J(d_y u_{12} v_{12})^t_{(x,0)} &= (0, 0, \sinh\theta, \cosh\theta - 1)((\cosh\theta - 1)x_4 + \sinh\theta x_3), \end{aligned}$$

forming a system of generators of $\mathcal{M}(\Gamma)$ over $\mathcal{I}(\Gamma)$.

3. Invariant subspaces. In this section we discuss subspaces of $\mathbb{R}_1^{n+1}$ that are invariant under the action of a group $\Gamma < \mathbf{O}(n, 1)$, and we discuss existence of their invariant complement. For a given subspace $W \subseteq \mathbb{R}_1^{n+1}$, we shall say that a subspace is its complement if their direct sum is $\mathbb{R}_1^{n+1}$. We start with some general results on invariant subspaces. In Subsections 3.1

and 3.2 we characterize them for the lowest dimensions, in $\mathbb{R}^2_1$ and $\mathbb{R}^3_1$ respectively, and classify according to their type, as space-, time- or lightlike subspaces.

Recall that a nonzero vector $x \in \mathbb{R}^{n+1}_1$ is called *spacelike*, *timelike* or *lightlike* if $\langle x, x\rangle > 0$, $\langle x, x\rangle < 0$ or $\langle x, x\rangle = 0$, respectively. A vector subspace of $\mathbb{R}^{n+1}_1$ is called *spacelike* if all of its nonzero vectors are spacelike; it is *timelike* if it has a timelike vector; and it is *lightlike* if none of the above conditions are satisfied [18]. In Figure 1 we illustrate each case with planes in $\mathbb{R}^3_1$ with respect to their position with respect to the *lightcone*

$$LC = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = 0\}.$$

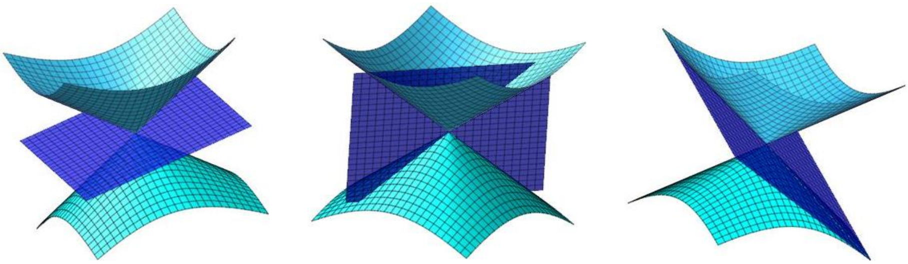

Fig. 1. From left to right: a spacelike, a timelike and a lightlike plane in $\mathbb{R}^3_1$

We start with an important lemma whose proof is immediate:

LEMMA 3.1. *Let W be a subspace of $\mathbb{R}^{n+1}_1$.*

(a) *For any $\gamma \in \mathbf{O}(n,1)$, W and γW are subspaces of the same type (spacelike, timelike or lightlike).*
(b) *If $\Sigma_1, \Sigma_2 < \mathbf{O}(n,1)$ are conjugate subgroups, that is, $\Sigma_2 = \gamma\Sigma_1\gamma^{-1}$ for some $\gamma \in \mathbf{O}(n,1)$, then W is Σ_1-invariant if and only if γW is Σ_2-invariant.*

A subspace is *nondegenerate* if the pseudo inner product restricted to it is a nondegenerate bilinear form. Hence, by definition, a subspace is lightlike if and only if it is degenerate. For

$$W^\perp = \{u \in \mathbb{R}^{n+1}_1 : \langle u, w\rangle = 0, \forall w \in W\},$$

it is well-known that W is nondegenerate if and only if $W \cap W^\perp$ is trivial (see [17]). We show that an invariant subspace admits $W^\perp$ as an invariant orthogonal complement if and only if it is nondegenerate (Proposition 3.2). If the subspace is degenerate (lightlike) we shall see that it still admits a complement, but it is not orthogonal, and its invariance is attained under a condition imposed on the group (Proposition 3.3).

PROPOSITION 3.2. *For any subspace $W \subseteq \mathbb{R}^{n+1}_1$,*

(a) $\dim W + \dim W^\perp = n + 1$.
(b) *For a group $\Gamma < \mathbf{O}(n, 1)$, W is nondegenerate and Γ-invariant if and only if its orthogonal subspace $W^\perp$ is a Γ-invariant complement.*

Proof. (a) Consider an operator $\varphi : \mathbb{R}^{n+1}_1 \to \mathbb{R}^{n+1}_1$, $\varphi(x) = AJx$, where A is a square matrix whose lines are formed by basic vectors of W and by lines with all entries equal to 0. We have $\dim \operatorname{Im} \varphi = \operatorname{rank}(A) = \dim W$. Thus, $\dim \ker \varphi + \dim W = n + 1$. But $\ker \varphi = W^\perp$, so the result follows.

(b) Let $u \in W^\perp$ and $\gamma \in \Gamma$. From the Γ-invariance of W, for all $w \in W$ we have

$$\langle w, \gamma u\rangle = \langle \gamma^{-1} w, u\rangle = 0.$$

The "if" part is immediate since $(W^\perp)^\perp = W$. Now, W is nondegenerate if and only if $W \cap W^\perp$ is trivial, which together with (a) gives the result. ∎

The following proposition gives a sufficient condition for an invariant subspace to admit an invariant complement, relaxing the "nondegeneracy" condition and therefore including lightlike subspaces:

PROPOSITION 3.3. *Let $\Gamma < \mathbf{O}(n, 1)$ be such that $\gamma^t \in \Gamma$ for all $\gamma \in \Gamma$. If W is a Γ-invariant subspace, then $JW^\perp$ is a Γ-invariant complement of W.*

Proof. First, notice that $JW^\perp$ is Γ-invariant: for $u = Jv \in JW^\perp$,

$$\gamma u = \gamma J v = J(\gamma^t)^{-1} v,$$

which belongs to $JW^\perp$, since $W^\perp$ is Γ-invariant and $\gamma^t \in \Gamma$ by hypothesis. Also, if $w = Jv \in W \cap JW^\perp$, then $\langle w, v\rangle = 0$, and so $\langle Jv, v\rangle = 0$, implying that $v = 0$. Hence, $W \cap JW^\perp$ is trivial, Now, it follows from Proposition 3.2(a) that $\dim W + \dim(JW^\perp) = n+1$. Therefore, $W \oplus JW^\perp = \mathbb{R}^{n+1}_1$. ∎

The two propositions above provide the general way to decompose $\mathbb{R}^{n+1}_1$ as a direct sum of an invariant subspace and its complement, depending on its type: if U is a spacelike or a timelike subspace and W is a lightlike subspace, then

$$\mathbb{R}^{n+1}_1 = U \oplus U^\perp = W \oplus JW^\perp.$$

In the presence of symmetries, one important class of invariant subspaces is the class of fixed-point subspaces. For a given group $\Gamma < \mathbf{O}(n, 1)$, recall that the fixed-point subspace of a subgroup $\Sigma < \Gamma$ is the subspace

$$\operatorname{Fix}(\Sigma) = \{v \in V : \sigma v = v, \ \forall \sigma \in \Sigma\}.$$

There are two basic facts that are the main motivations for the results presented in this section, in view of the applications. First, we recall that for a Γ-equivariant mapping g,

$$g(\operatorname{Fix}(\Sigma)) \subset \operatorname{Fix}(\Sigma). \tag{3.1}$$

Also, fixed-point subspaces of conjugate subgroups are related by

$$\gamma \mathrm{Fix}(\Sigma) \subseteq \mathrm{Fix}(\gamma\Sigma\gamma^{-1}). \tag{3.2}$$

For example, if dynamics is ruled by a Γ-equivariant mapping g (defining a vector field, for instance), then (3.1) holds for all $\Sigma < \Gamma$. Therefore, these are subspaces on which the dynamics must remain invariant. In another direction, we mention the study of the geometry of surfaces which are given as the inverse image $f^{-1}(c)$, $c \in \mathbb{R}$, for some Γ-invariant function f. By construction, the whole group leaves such a surface setwise invariant, and in addition the whole space is foliated by such surfaces in a symmetric way. Now, recall that the normalizer $\mathrm{N}(\Sigma)$ is the symmetry group of the set $\mathrm{Fix}(\Sigma)$, in the sense that it is the largest subgroup of Γ that leaves $\mathrm{Fix}(\Sigma)$ setwise invariant. Hence, we can use that structure to understand the surface "in pieces" preserving their symmetries, since $\mathrm{N}(\Sigma)$ gives the symmetries of $f^{-1}(c) \cap \mathrm{Fix}(\Sigma)$. It now follows from (3.2) that two such pieces associated with conjugate subgroups are related by

$$f^{-1}(c) \cap \mathrm{Fix}(\gamma\Sigma\gamma^{-1}) \subseteq \gamma(f^{-1}(c) \cap \mathrm{Fix}(\Sigma)),$$

which allows one to analyse only selected representatives of conjugacy classes of subgroups of Γ.

Notice that in $\mathbb{R}^3_1$ the orthogonal subspace to a lightlike line W is a plane tangent to the lightcone which contains W. Likewise, if W is a lightlike plane then its orthogonal subspace is a lightlike line contained in W. This follows directly from the fact that if W in $\mathbb{R}^{n+1}_1$ is lightlike then $\dim W \cap W^\perp = 1$ (see [13] for example) and also from the fact that W is spacelike if and only if $W^\perp$ is timelike (see [18] for example).

3.1. Invariant lines in $\mathbb{R}^2_1$. The standard 2-dimensional representation of the Lorentz group is

$$\mathbf{O}(1,1) = \mathbf{SO}_0(1,1) \,\dot\cup\, \Lambda^p\mathbf{SO}_0(1,1) \,\dot\cup\, \Lambda^t\mathbf{SO}_0(1,1) \,\dot\cup\, -I\mathbf{SO}_0(1,1), \tag{3.3}$$

where $\mathbf{SO}_0(1,1)$ is the group of *hyperbolic rotations,*

$$H_\theta := \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{3.4}$$

for $\theta \in \mathbb{R}$, and

$$\Lambda^p = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Lambda^t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Lambda^{pt} = -I.$$

- For an arbitrary element in $\mathbf{SO}_0(1,1)$ or in $-I\mathbf{SO}_0(1,1)$, it is direct that the invariant lines under this element are the two light lines of the light-cone,

$$W_1 = \{(x,y) \in \mathbb{R}^2_1 : y = x\}, \qquad W_2 = \{(x,y) \in \mathbb{R}^2_1 : y = -x\}.$$

These are degenerate subspaces and one is the complement of the other as in Proposition 3.3, since $JW_1^{\perp} = W_2$ and $JW_2^{\perp} = W_1$. Clearly, neither of them is a fixed-point subspace.

- The invariant lines under each Λ^p and Λ^t are the x-axis (a space line) and the y-axis (a time line), and one is the orthogonal complement of the other. The x-axis is $\mathrm{Fix}(\mathbb{Z}_2(\Lambda^t))$ and the y-axis is $\mathrm{Fix}(\mathbb{Z}_2(\Lambda^p))$. Any other involution in $\mathbf{O}(1,1)$ belongs to $\Lambda^p\mathbf{SO}_0(1,1)$ or to $\Lambda^t\mathbf{SO}_0(1,1)$, and it is conjugate to either Λ^p, if it is of the form $\Lambda^p H_\theta$, or to Λ^t, if it is of the form $\Lambda^t H_\theta$, the conjugacy matrix for both cases being $\gamma = H_{\theta/2}$. Up to conjugacy, there are therefore only two classes of involutions. For any $\theta \in \mathbb{R}$, we can use Lemma 3.1 for this γ to obtain explicitly all the invariant lines under the involutions of $\mathbf{O}(1,1)$.
- Any subgroup Σ generated by ≥ 2 elements has nontrivial invariant subspaces only if $\Sigma < \mathbf{SO}_0(1,1) \times \mathbb{Z}_2(-I)$, and these are W_1 and W_2 above.

In Table 1 we summarize these results. The straight lines in the second column are the lines invariant under the subgroups given in the first column. Their type is given in the third column and the last column shows their complement subspaces.

Table 1. Invariant subspaces of $\mathbb{R}^2_1$

Subgroup of $\mathbf{O}(1,1)$	Invariant subspaces	Type	Complement subspace
$[H_\theta]$	$W_1 = \{(x,y) \in \mathbb{R}^2_1 : y = x\}$ $W_2 = \{(x,y) \in \mathbb{R}^2_1 : y = -x\}$	light	$JW_1^{\perp} = W_2$ $JW_2^{\perp} = W_1$
$[-H_\theta]$	W_1 W_2	light	$JW_1^{\perp} = W_2$ $JW_2^{\perp} = W_1$
$\mathbb{Z}_2(\Lambda^t)$	$\mathrm{Fix}(\mathbb{Z}_2(\Lambda^t)) = x$-axis y-axis	space time	$\mathrm{Fix}(\mathbb{Z}_2(\Lambda^t))^{\perp} = y$-axis $\mathrm{Fix}(\mathbb{Z}_2(\Lambda^p))^{\perp} = x$-axis
$\mathbb{Z}_2(\Lambda^p)$	$\mathrm{Fix}(\mathbb{Z}_2(\Lambda^p)) = y$-axis x-axis	time space	x-axis y-axis

3.2. Invariant lines and invariant planes in $\mathbb{R}^3_1$. The standard 3-dimensional representation of the Lorentz group is

$$\mathbf{O}(2,1) = \mathbf{SO}_0(2,1) \,\dot\cup\, \Lambda^p\mathbf{SO}_0(2,1) \,\dot\cup\, \Lambda^t\mathbf{SO}_0(2,1) \,\dot\cup\, \Lambda^{pt}\mathbf{SO}_0(2,1), \tag{3.5}$$

where $\mathbf{SO}_0(2,1)$ is the group of *hyperbolic rotations* H^+, which, using singular value decomposition [9], are written as

$$H^+ = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.6}$$

for $\varphi, \theta, \phi \in \mathbb{R}$ and $\epsilon = 1$, and

$$\Lambda^p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda^t = J. \tag{3.7}$$

Matrices in the component $\Lambda^{pt}\mathbf{SO}_0(2,1)$ are of the form

(3.8)

$$H^- = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ \sin\varphi & -\cos\varphi & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $\varphi, \theta, \phi \in \mathbb{R}$ and $\epsilon = -1$. Matrices in $\Lambda^t\mathbf{SO}_0(2,1)$ are of the form (3.6) for $\epsilon = -1$, and those in $\Lambda^p\mathbf{SO}_0(2,1)$ are of the form (3.8) for $\epsilon = 1$.

Below we consider invariant subspaces under an element in each of the connected components of $\mathbf{O}(2,1)$. We recall that elements in distinct components are not conjugate. We assume $\theta \neq 0$; otherwise, the elements are in the orthogonal group $\mathbf{O}(3)$ for which the results are well-known.

- For $H^+ \in \mathbf{SO}_0(2,1)$ given in (3.6) with $\epsilon = 1$, if $\varphi + \phi = \pi$ then it is conjugate to $-\Lambda^t$ with the conjugacy matrix

$$\begin{pmatrix} -\cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi)\cosh\left(\frac{\theta}{2}\right) & -\cos(\phi)\cosh\left(\frac{\theta}{2}\right) & -\sinh\left(\frac{\theta}{2}\right) \\ \sin(\phi)\sinh\left(\frac{\theta}{2}\right) & \cos(\phi)\sinh\left(\frac{\theta}{2}\right) & \cosh\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{3.9}$$

  The invariant lines under $-\Lambda^t$ are the time line given by the z-axis, which is $\mathrm{Fix}(\mathbb{Z}_2(-\Lambda^t))$, and the space lines in the plane $z = 0$. The invariant planes are the space plane $z = 0$ and all the time planes containing the z-axis.
- For $H^+ \in \mathbf{SO}_0(2,1)$ given in (3.6) with $\epsilon = 1$, if $\varphi + \phi \neq \pi$ then invariant lines and planes can be of any type, depending on the values of φ, θ, ϕ. The subgroup $[H^+]$ generated by H^+ is noncompact, and $\mathrm{Fix}([H^+])$ is the line generated by the vector

$$\left(1, \frac{\sin\varphi - \sin\phi}{\cos\phi + \cos\varphi}, \frac{(\cos\varphi\sin\phi + \sin\varphi\cos\phi)\sinh\theta}{(1-\cosh\theta)(\cos\phi + \cos\varphi)}\right),$$

  which can also be of any type.
- For $\Lambda^t H^+ \in \Lambda^t\mathbf{SO}_0(2,1)$ given as in (3.6) for $\epsilon = -1$, if $\varphi = -\phi$ then it is conjugate to Λ^t with conjugacy matrix (3.9). The invariant lines under Λ^t are the time line given by the z-axis, and the space lines in the plane $z = 0$. The invariant planes are the space plane $z = 0$, which is $\mathrm{Fix}(\mathbb{Z}_2(\Lambda^t))$, and all the time planes containing the z-axis.

- For $\Lambda^t H^+ \in \Lambda^t \mathbf{SO}_0(2,1)$ given as in (3.6) for $\epsilon = -1$, and for $\varphi \neq -\phi$, $\mathrm{Fix}([\Lambda^t H^+])$ is trivial. There are no invariant lines or planes under the action of the group $[\Lambda^t H^+]$.
- For $\Lambda^p H^+ \in \Lambda^p \mathbf{SO}_0(2,1)$ given as in (3.8) with $\epsilon = 1$, if $\varphi = -\phi$ then it is conjugate to Λ^p with conjugacy matrix (3.9). The invariant lines under Λ^p are the space line given by the y-axis, and the time lines in the plane $y = 0$. The invariant planes are the time plane $y = 0$, which is $\mathrm{Fix}(\mathbb{Z}_2(\Lambda^p))$, and all the time planes containing the y-axis.
- For $\Lambda^p H^+ \in \Lambda^p \mathbf{SO}_0(2,1)$ given as in (3.8) with $\epsilon = 1$, if $\varphi \neq -\phi$ then $\mathrm{Fix}([\Lambda^p H^+])$ is trivial. There are no invariant lines or planes under the action of the group $[\Lambda^p H^+]$.
- For $H^- \in \Lambda^{pt} \mathbf{SO}_0(2,1)$ given in (3.8) with $\epsilon = -1$, if $\varphi + \phi = \pi$ then it is conjugate to $-\Lambda^p$ with conjugacy matrix (3.9). The invariant lines under $-\Lambda^p$ are the space line given by the y-axis, which is $\mathrm{Fix}(\mathbb{Z}_2(-\Lambda^p))$, and all the lines in the plane $y = 0$, whose types are time-, light- and spacelike. The invariant planes are the time plane $y = 0$ and all the planes containing the y-axis, which are the two light planes $z \pm x = 0$ (tangent to the lightcone) and all space and time planes containing the y-axis.
- For $H^- \in \Lambda^{pt} \mathbf{SO}_0(2,1)$ given in (3.8) with $\epsilon = -1$, if $\varphi + \phi \neq \pi$, invariant lines and planes can be of any type, depending on the values of φ, θ, ϕ. The subgroup $[\Lambda^{pt} H^+]$ generated by $\Lambda^{pt} H^+$ is noncompact, and $\mathrm{Fix}([\Lambda^{pt} H^+])$ is the line generated by

$$\left(1, -\frac{\sin\phi - \sin\varphi}{\cos\phi + \cos\varphi}, -\frac{(\cos\varphi \sin\phi + \sin\varphi \cos\phi)\sinh\theta}{(\cosh\theta + 1)(\cos\phi + \cos\varphi)}\right),$$

  which can also be of any type.

Finally, if a subgroup $\Sigma < \mathbf{O}(2,1)$ is generated by more than one element, then these appear in the list above and so $\mathrm{Fix}(\Sigma)$ is computed directly as the intersection of the fixed-point subspaces of each of its generators.

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