COLLOQUIUM MATHEMATICUM

Online First version

$W_0^{1,1}(\Omega)$ SOLUTIONS FOR SOME NONLINEAR ELLIPTIC EQUATIONS

BY

HAYDAR ABDELHAMID (Tabuk)

Abstract. We prove the existence of renormalized solutions belonging to $W_0^{1,1}(\Omega)$ of a class of strongly nonlinear elliptic p-Laplace type problems when $1 < p < 2 - 1/N$ with data of poor summability in the Lebesgue space $L^m(\Omega)$, $m = N/((p-1)N + 1)$. Our method consists in applying Schaefer's classical fixed point theorem relying on new estimates of the gradient of the unique renormalized solution of problems with datum in $L^m(\Omega)$.

1. Introduction. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. We consider the following class of elliptic equations:

$$\begin{cases} -\mathrm{div}(\mathcal{A}(x, \nabla u)) = H(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathcal{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfies, for almost every $x \in \Omega$ and for all ξ, $\xi' \in \mathbb{R}^N$, the following conditions:

$$\begin{cases} \mathcal{A}(x, \xi) \cdot \xi \geq \alpha|\xi|^p, \\ |\mathcal{A}(x, \xi)| \leq \beta|\xi|^{p-1}, \\ (\mathcal{A}(x, \xi) - \mathcal{A}(x, \xi')) \cdot (\xi - \xi') > 0 \quad \forall \xi \neq \xi', \end{cases} \tag{1.2}$$

for some $p > 1$, $\alpha > 0$ and $\beta > 0$. A typical example of such $\mathcal{A}$ is given by $\mathcal{A}(x, \xi) = |\xi|^{p-2}\xi$ which gives rise to the p-Laplace operator $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$.

The nonlinearity $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfies, for almost every $x \in \Omega$, every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}^N$, the following growth condition:

$$|H(x, s, \xi)| \leq \gamma|s|^q|\xi|^l + f(x), \tag{1.3}$$

where $\gamma, q, l \geq 0$ and for now $f \in L^m(\Omega)$ with $m \geq 1$.

2020 *Mathematics Subject Classification*: Primary 35A01, 35J60, 35J66, 35J92.
Key words and phrases: $W_0^{1,1}$ solutions, renormalized solutions, elliptic equations.
Received 3 August 2020; revised 3 December 2020.
Published online 12 May 2021.

DOI: 10.4064/cm8364-12-2020

From the theory of elliptic equations, it is well known that it is necessary to go out of the framework of Sobolev spaces $W_0^{1,q}(\Omega)$ in order to obtain solutions for p close to 1 and nonregular right hand side. The existence of $W_0^{1,1}(\Omega)$ solutions for elliptic equations is not so usual when the datum is of poor summability in Lebesgue spaces. For this kind of problems, Boccardo and Gallouët [BG1] proved the existence of $W_0^{1,1}(\Omega)$ distributional solutions for p-Laplace type equations in some borderline cases of Calderón–Zygmund theory. Boccardo, Croce and Orsina [BCO1], [BCO2], [BCO3] studied the existence of this type of solutions to elliptic scalar problems with degenerate coercivity (see also Boccardo and Croce [BC]). Some results have been extended to elliptic systems with degenerate coercivity by Boccardo, Croce and Tanteri [BCT]. From [BG1, Theorem 2.2], we recall that there exists a distributional solution $u \in W_0^{1,1}(\Omega)$ for the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\mathcal{A}(x, \nabla u)) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

under the hypothesis

$$1 < p < 2 - \frac{1}{N}, \qquad f \in L^m(\Omega), \quad m = \frac{N}{(p-1)N+1}. \tag{1.5}$$

In the present paper we prove the existence of a $W_0^{1,1}(\Omega)$ solution for problems of type (1.1) in the renormalized sense assuming (1.5) and

$$0 \le l < p - 1, \qquad 0 \le q < p - 1 - l. \tag{1.6}$$

For the notion of renormalized solution we refer to [DMOP], [LM] and [M]. Our main result is the following.

THEOREM 1.1. *Assume* (1.2), (1.3), (1.5) *and* (1.6). *Then there exists a renormalized solution* $u \in W_0^{1,1}(\Omega)$ *of* (1.1).

To prove our existence result we apply in a convenient way the classical Schaefer fixed point theorem. Our proof relies on our regularity result and estimates of the following lemma. In this lemma, we prove that, under the assumption (1.5), the unique renormalized solution of problem (1.4) belongs to $W_0^{1,1}(\Omega)$, giving precise estimates of u and its gradient. The proof follows some ideas of [BG1, Theorem 2.2] and it is given in Section 4. Our regularity result is optimal in the sense that $m = \frac{N}{(p-1)N+1}$ is the smallest exponent for which the hypothesis $f \in L^m(\Omega)$ guarantees $u \in W_0^{1,1}(\Omega)$; see Remark 4.1.

LEMMA 1.2. *Assume* (1.2) *and* (1.5). *Then the renormalized solution* u *of* (1.4) *satisfies the following:*

(i) $u \in L^{\frac{N}{N-1}}(\Omega)$ *with*

$$\|u\|_{L^{\frac{N}{N-1}}(\Omega)} \le C_1 \|f\|_{L^m(\Omega)}^{\frac{1}{p-1}}, \tag{1.7}$$

(ii) $|u|^{\frac{(1-p)N}{p(N-1)}} |\nabla u| \in L^p(\Omega)$ *with*

$$\int_\Omega |u|^{\frac{N(1-p)}{N-1}} |\nabla u|^p \le C_2 \|f\|_{L^m(\Omega)}^{\frac{N-p}{(p-1)(N-1)}}, \tag{1.8}$$

(iii) $u \in W_0^{1,1}(\Omega)$ *with*

$$\int_\Omega |\nabla u| \le C_3 \|f\|_{L^m(\Omega)}^{\frac{1}{p-1}}, \tag{1.9}$$

$$\int_{\{|u|\ge k\}} |\nabla u| \le C_4 \Big(\int_{\{|u|\ge k\}} |f|^m \Big)^{\frac{1}{m}}, \tag{1.10}$$

where C_1, C_2, C_3, C_4 are positive constants that only depend on p and N.

The plan of this paper is as follows. The next section is devoted to recalling some notations, definitions and basic tools which will be used in the proof of the main existence result. In Section 3, we prove our existence result. In the last section we prove our regularity result.

2. Preliminaries. In this section we recall some of basic tools and useful results. From [DMOP], we recall the definition and some properties of a renormalized solution for the problem

$$\begin{cases} -\operatorname{div}(\mathcal{A}(x, \nabla u)) = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

in the case when $g \in L^1(\Omega)$. For every $k > 0$ and every $s \in \mathbb{R}$, the truncation is defined by:

$$T_k(s) = \max(-k, \min(k, s)).$$

In the following definition, we recall the notion of the gradient of a function whose truncations belong to $W_0^{1,p}(\Omega)$, introduced in [BBGGPV, Lemma 2.1].

Definition 2.1. Let u be a measurable function defined on Ω which is finite almost everywhere and such that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. Then there exists a measurable function $v : \Omega \to \mathbb{R}^N$ such that

$$T_k(u) = v\chi_{\{|u|<k\}} \quad \text{almost everywhere in } \Omega, \text{ for every } k > 0,$$

which is unique in the almost everywhere sense. We define the gradient ∇u of u as this function v, and denote $\nabla u = v$.

Remark 2.2. In Definition 2.1 the function u does not necessarily belong to $L^1_{\mathrm{loc}}(\Omega)$ and the gradient is not in general the gradient used to define

Sobolev spaces. However, if $v \in (L^1_{\mathrm{loc}}(\Omega))^N$ then $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and v is the distributional gradient of u (see [DMOP, Remark 2.10]). If v is moreover assumed to belong to $(L^q(\Omega))^N$ for some $1 \le q \le p$, then $u \in W^{1,q}_0(\Omega)$.

Now we recall the definition of a *renormalized solution* of (2.1) from [DMOP, Definition 2.13].

Definition 2.3. A function u is a renormalized solution of problem (2.1) if the following conditions hold:

(a) The function u is measurable and finite almost everywhere in Ω, and $T_k(u) \in W^{1,p}_0(\Omega)$ for every $k > 0$.
(b) The gradient ∇u of u satisfies $|\nabla u|^{p-1} \in L^q(\Omega)$ for every $1 \le q < \frac{N}{N-1}$.
(c) The identity

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla w = \int_\Omega g\, w, \tag{2.2}$$

holds for every $w \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ such that there exist $k > 0$ and w^+ and w^- in $W^{1,r}_0(\Omega) \cap L^\infty(\Omega)$, with $r > N$, such that $w = w^+$ a.e. on $\{u > k\}$ and $w = w^-$ a.e. on $\{u < -k\}$.

It is well known that problem (2.1) has a unique renormalized solution u when $g \in L^1(\Omega)$. Moreover, using $T_k(u)$ in (2.2) and the assumption (1.2), we obtain

$$\alpha \int_\Omega |\nabla T_k(u)|^p \le k \int_\Omega |g|. \tag{2.3}$$

A key tool in our proof of the main result is the stability of renormalized solutions of

$$\begin{cases} -\operatorname{div}(\mathcal{A}(x, \nabla u_n)) = g_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $g_n, g \in L^1(\Omega)$ are such that $\{g_n\}$ converges to g weakly in $L^1(\Omega)$. In [DMOP, Theorem 3.4], stability results for general Radon measure data are proved. Theorem 2.4 and Lemma 2.5 are taken from [DMOP, Theorem 3.4] and its proof in the particular case when the right hand sides belong to $L^1(\Omega)$ instead of being general Radon measures. Those statements in this particular case do not involve lower order terms. It is worthwhile to clarify that we apply those results to prove continuity and compactness of the operator defined in the proof of Theorem 1.1. To apply those results, we do not need to state them for problems with lower order terms. Indeed, we apply Theorem 2.4 and Lemma 2.5 to deduce convergence properties of sequences of renormalized solutions u_n of

$$\begin{cases} -\operatorname{div}(\mathcal{A}(x, \nabla u_n)) = H(x, v_n, \nabla v_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

for some sequences v_n with convergence properties. We point out that since the sequence $g_n(x) = H(x, v_n, \nabla v_n)$ does not depend on u_n, Theorem 2.4 and Lemma 2.5 may be applied to deduce convergence properties of u_n when g_n satisfies the theorem and lemma's hypotheses.

THEOREM 2.4. *Let $g_n, g \in L^1(\Omega)$ be such that $\{g_n\}$ converges to g weakly in $L^1(\Omega)$. Let u_n be the renormalized solution of* (2.4)*. There exists a subsequence $\{u_{n_i}\}$ such that $u_{n_i} \to u$ and $\nabla u_{n_i} \to \nabla u$ almost everywhere in Ω, where u is the renormalized solution of problem* (2.1)*.*

The almost everywhere convergence of the gradients obtained in the stability result holds under a weaker assumption on the right hand side. If we assume that $\{g_n\}$ is bounded in $L^1(\Omega)$, the following result is well known: see the proof of [DMOP, Theorem 3.4]. See also [BG2] or [BM].

LEMMA 2.5. *Assume that $\{g_n\}$ is a bounded sequence in $L^1(\Omega)$. Let u_n be the renormalized solution of* (2.4)*. There exists a subsequence $\{u_{n_i}\}$ and a measurable function u satisfying $T_k(u) \in W_0^{1,p}(\Omega)$ and $|\nabla u|^{p-1} \in L^\sigma(\Omega)$ for every $\sigma \in \left[1, \frac{N}{N-1}\right)$ such that ∇u_{n_i} converges to ∇u almost everywhere in Ω.*

In order to prove our main existence result, we will use the classical Schaefer fixed point theorem (see [E, Theorem 4, p. 541]).

THEOREM 2.6 (Schaefer's fixed point theorem). *Let S be a continuous and compact mapping of a Banach space X into itself such that the set*

$$\{x \in X : \exists \lambda \in [0,1] \text{ such that } x = \lambda S x\}$$

is bounded. Then S has a fixed point.

Finally, Vitali's convergence theorem will be used in our proof (see for example [Z17, Theorem 6.30]).

THEOREM 2.7. *Let $\{f_n\}$ be a sequence in $L^p(\Omega)$ with $1 \le p < \infty$. Assume that:*

(a) *The sequence $\{f_n\}$ is equi-integrable: $\forall \epsilon > 0\ \exists \delta > 0$ such that $\int_E |f_n|^p < \epsilon$ for all n and all $E \subset \Omega$ measurable with $|E| < \delta$.*
(b) *$f_n \to f$ a.e. in Ω.*
(c) *Ω is bounded.*

Then $f \in L^p(\Omega)$ and $f_n \to f$ in $L^p(\Omega)$.

3. Proof of the existence result. In this section we prove our existence result of Theorem 1.1.

Proof of Theorem 1.1. First, we formulate our problem (1.1) as a fixed point problem in $W_0^{1,1}(\Omega)$. Let $v \in W_0^{1,1}(\Omega)$ be fixed. We see that $F(x) = |v|^q|\nabla v|^l \in L^m(\Omega)$. Indeed, let $\theta = \frac{1}{lm}$. Note that $\theta > 1$ since $l < p-1$. Let $s = qm\theta' = \frac{qm}{1-lm}$. By using Sobolev's embedding we conclude that $v \in L^s(\Omega)$ since $1 \le s \le 1^* = \frac{N}{N-1}$, thanks to the assumption (1.6). Using Hölder's inequality and then Sobolev's inequality, we conclude that $F \in L^m(\Omega)$ with

$$\|F\|_{L^m(\Omega)} \le \|\nabla v\|^l_{L^1(\Omega)}\|v\|^q_{L^s(\Omega)} \le C\|\nabla v\|^{l+q}_{L^1(\Omega)}$$

for some $C = C(N) > 0$. So, using the assumption (1.3), we conclude that $H(x,v,\nabla v) \in L^m(\Omega)$ and

$$\|H(x,v,\nabla v)\|_{L^m(\Omega)} \le (\gamma C\|v\|^{l+q}_{W_0^{1,1}(\Omega)} + \|f\|_{L^m(\Omega)}). \tag{3.1}$$

Now, let us define u to be the unique renormalized solution of the problem

$$\begin{cases} -\operatorname{div}(\mathcal{A}(x,\nabla u)) = H(x,v,\nabla v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

By Lemma 1.2, we have $u \in W_0^{1,1}(\Omega)$ since the right hand side belongs to $L^m(\Omega)$. Hence we can define the operator $S : W_0^{1,1}(\Omega) \to W_0^{1,1}(\Omega)$, $S(v) = u$. Next we prove that S satisfies the conditions of Schaefer's fixed point theorem (Theorem 2.6), in several steps.

Step 1: *The set $Z = \{v \in W_0^{1,1}(\Omega) : \exists\lambda \in [0,1]$ such that $v = \lambda S(v)\}$ is bounded.* Let $v \in Z$. Taking into account that $0 \le \lambda \le 1$ and $p > 1$ and using (3.1) and (1.9) we get

$$\begin{aligned} \|v\|^{p-1}_{W_0^{1,1}(\Omega)} = \lambda^{p-1}\|S(v)\|^{p-1}_{W_0^{1,1}(\Omega)} &\le C_3^{p-1}\|H(x,v,\nabla v)\|_{L^m(\Omega)} \\ &\le C_3^{p-1}(\gamma C\|\nabla v\|^{l+q}_{L^1(\Omega)} + \|f\|_{L^m(\Omega)}). \end{aligned}$$

Therefore, for every $v \in Z$, we have

$$\|v\|^{p-1}_{W_0^{1,1}(\Omega)} - A\|v\|^{l+q}_{W_0^{1,1}(\Omega)} - B \le 0$$

for some positive constants A and B. So, given that $q+l < p-1$, we conclude that Z is bounded.

Step 2: *S is continuous.* Let $v_n \to v$ in $W_0^{1,1}(\Omega)$. We want to prove that $\nabla S(v_n)$ converges to $\nabla S(v)$ in $L^1(\Omega)$. In what follows, we denote $g_n(x) = H(x,v_n,\nabla v_n)$ and $g(x) = H(x,v,\nabla v)$.

By strong convergence in $W_0^{1,1}(\Omega)$, we can extract a subsequence still denoted $\{v_n\}$ such that v_n converges to v and ∇v_n converges to ∇v a.e. in Ω. Thus g_n converges to g a.e. in Ω.

On the other hand, from (3.1) and the boundedness of v_n in $W_0^{1,1}(\Omega)$, we conclude that $\|g_n\|_{L^m(\Omega)}$ is uniformly bounded. Since $m > 1$, we conclude that g_n converges to g in $L^1(\Omega)$ by using Vitali's convergence theorem.

Now, by Theorem 2.4, there exists a subsequence still denoted $\{S(v_n)\}$ such that $\nabla S(v_n)$ converges to $\nabla S(v)$ a.e. in Ω. For every measurable subset E of Ω and $k > 0$, using Hölder's inequality, (1.10) and (2.3), we have

$$\begin{aligned}\int_E |\nabla S(v_n)| &\le \int_E |\nabla T_k(S(v_n))| + \int_{\{|S(v_n)|\ge k\}} |\nabla S(v_n)| \\ &\le \operatorname{meas}(E)^{1/p'} \|\nabla T_k(S(v_n))\|_{L^p(\Omega)} + C_4\Big(\int_{\{|S(v_n)|\ge k\}} |g_n|^m\Big)^{1/m} \\ &\le \operatorname{meas}(E)^{1/p'} (k\alpha^{-1}\|g_n\|_{L^1})^{1/p} + C_4\Big(\int_{\{|S(v_n)|\ge k\}} |g_n|^m\Big)^{1/m}.\end{aligned}$$

Using the assumption (1.3) and Hölder's inequality, we get

$$\begin{aligned}\int_E |\nabla S(v_n)| \le{}& \operatorname{meas}(E)^{1/p'} (k\alpha^{-1} \operatorname{meas}(\Omega)^{1/m'} \|g_n\|_{L^m(\Omega)})^{1/p} \\ &+ \bigg(\Big(\int_{\{|S(v_n)|\ge k\}} |v_n|^s\Big)^{1-lm} \|\nabla v_n\|^{lm}_{L^1(\Omega)} + \int_{\{|S(v_n)|\ge k\}} |f|^m\bigg)^{1/m},\end{aligned}$$

where $s = \frac{qm}{1-lm}$.

Recalling that $\{g_n\}$ is bounded in $L^m(\Omega)$ and $\{v_n\}$ is bounded in $W_0^{1,1}(\Omega)$, we deduce from the last inequality that

$$\begin{aligned}&\int_E |\nabla S(v_n)| \\ &\le D_1 \operatorname{meas}(E)^{1/p'} + \bigg(D_2\Big(\int_{\{|S(v_n)|\ge k\}} |v_n|^s\Big)^{1-lm} + \int_{\{|S(v_n)|\ge k\}} |f|^m\bigg)^{1/m}, \end{aligned} \tag{3.3}$$

where D_i are positive constants not depending on n.

Now, from (1.7), we deduce that there exists a positive constant D_3 not depending on k or n such that

$$\operatorname{meas}\{|S(v_n)| \ge k\} \le D_3 k^{-\frac{N}{N-1}}.$$

Thus

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} (\operatorname{meas}\{|S(v_n)| \ge k\}) = 0. \tag{3.4}$$

On the other hand, using (1.9) and (3.1), one can see that $\{S(v_n)\}$ is bounded in $W_0^{1,1}(\Omega)$. So, since $s < \frac{N}{N-1}$, by the compact Sobolev injection we can extract a subsequence $S(v_{n_i})$ such that $S(v_{n_i}) \to S(v)$ a.e. in Ω and strongly in $L^s(\Omega)$. Thanks to the strong convergence in $L^s(\Omega)$ and (3.4), the estimate (3.3) implies that $\{|\nabla S(v_{n_i})|\}$ is equi-integrable.

By Vitali's convergence theorem we conclude that $\nabla S(v_{n_i})$ converges to $\nabla S(v)$ in $L^1(\Omega)$. Since the limit does not depend on the subsequence, we deduce that the whole sequence $\nabla S(v_n)$ converges to $\nabla S(v)$ in $L^1(\Omega)$. Hence, the continuity of S is proved.

STEP 3: *S is compact.* Let v_n be a bounded sequence in $W_0^{1,1}(\Omega)$. We want to prove that, up to a subsequence, $S(v_n)$ converges to some function ∇w in $W_0^{1,1}(\Omega)$. As before, we can conclude that $S(v_n)$ is bounded in $W_0^{1,1}(\Omega)$ and $H(x, v_n, \nabla v_n)$ is bounded in $L^m(\Omega)$. By Lemma 2.5, there exists a subsequence $\{S(v_{n_i})\}$ and a function w such that $\nabla S(v_{n_i})$ converges to ∇w a.e. in Ω. Following the technique used in Step 2, by Vitali's convergence theorem we conclude that $\nabla S(v_{n_i})$ converges to ∇w in $L^1(\Omega)$. Therefore $w \in W_0^{1,1}(\Omega)$ and $S(v_{n_i}) \to w$ strongly in $W_0^{1,1}(\Omega)$. ∎

4. Proof of the regularity result. In this section we prove our regularity results and estimates of Lemma 1.2. We follow some details of the proof of [BG1, Theorem 2.2].

Proof of Lemma 1.2. Let us consider $\lambda = \frac{(2-p)N-1}{N-1}$. We have $\lambda > 0$ since $p < 2 - \frac{1}{N}$. For fixed $\epsilon > 0$, we consider the function

$$\phi(s) = ((\epsilon + |s|)^\lambda - \epsilon^\lambda)\,\mathrm{sign}(s).$$

We observe that $\phi(T_k(u))$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\nabla\phi(T_k(u)) = \lambda(\epsilon + |T_k(u)|)^{\lambda-1}\nabla T_k(u)$. Take $\phi(T_k(u))$ as a test function in (2.2) to get

$$\lambda \int_\Omega (\epsilon + |T_k(u)|)^{\lambda-1}|\nabla T_k(u)|^p = \int_\Omega f\phi(T_k(u)),$$

and then, by Hölder's inequality,

$$\lambda \int_\Omega (\epsilon + |T_k(u)|)^{\lambda-1}|\nabla T_k(u)|^p \le \|f\|_{L^m(\Omega)}\|\phi(T_k(u))\|_{L^{m'}(\Omega)}. \tag{4.1}$$

Let us set $\gamma = \frac{\lambda-1}{p} + 1 = \frac{N-p}{p(N-1)}$ and

$$\psi(s) = ((\epsilon + |s|)^\gamma - \epsilon^\gamma)\,\mathrm{sign}(s).$$

Taking into account that $\nabla\psi(T_k(u)) = \gamma(\epsilon + |T_k(u)|)^{\frac{\lambda-1}{p}}\nabla T_k(u)$, we deduce from (4.1) that

$$\frac{\lambda}{|\gamma|^p}\int_\Omega |\nabla\psi(T_k(u))|^p \le \|f\|_{L^m(\Omega)}\|\phi(T_k(u))\|_{L^{m'}(\Omega)},$$

which, by the Sobolev embedding $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, yields

$$\frac{\lambda}{C|\gamma|^p}\Big(\int_\Omega |\psi(T_k(u))|^{p^*}\Big)^{p/p^*} \le \|f\|_{L^m(\Omega)}\|\phi(T_k(u))\|_{L^{m'}(\Omega)}.$$

Then Fatou's lemma for $\epsilon \to 0$ implies

$$\frac{\lambda}{C|\gamma|^p}\Big(\int_\Omega |T_k(u)|^{\gamma p^*}\Big)^{p/p^*} \le \|f\|_{L^m(\Omega)}\||T_k(u)|^\lambda\|_{L^{m'}(\Omega)}.$$

It follows from $\gamma p^* = \lambda m' = \frac{N}{N-1}$ that

$$\frac{\lambda}{C|\gamma|^p}\Big(\int_\Omega |T_k(u)|^{\frac{N}{N-1}}\Big)^{p/p^*} \le \|f\|_{L^m(\Omega)}\Big(\int_\Omega |T_k(u)|^{\frac{N}{N-1}}\Big)^{1/m'},$$

and then we deduce

$$\Big(\int_\Omega |T_k(u)|^{\frac{N}{N-1}}\Big)^{\frac{(p-1)(N-1)}{N}} \le C_1\|f\|_{L^m(\Omega)} \tag{4.2}$$

from $\frac{p}{p^*} - \frac{1}{m'} = \frac{(p-1)(N-1)}{N}$.

Now, by Fatou's lemma for $\epsilon \to 0$, the inequality (4.1) implies

$$\lambda\int_\Omega |T_k(u)|^{\lambda-1}|\nabla T_k(u)|^p \le \|f\|_{L^m(\Omega)}\big\||T_k(u)|^\lambda\big\|_{L^{m'}(\Omega)}.$$

Using $\lambda m' = \frac{N}{N-1}$ and the estimate (4.2) we obtain

$$\int_\Omega |T_k(u)|^{\lambda-1}|\nabla T_k(u)|^p \le C_2\|f\|_{L^m(\Omega)}^{1+\frac{N}{(p-1)(N-1)m'}}.$$

Then

$$\int_\Omega |T_k(u)|^{\frac{N(1-p)}{N-1}}|\nabla T_k(u)|^p \le C_2\|f\|_{L^m(\Omega)}^{\frac{N-p}{(p-1)(N-1)}} \tag{4.3}$$

from $\lambda - 1 = \frac{N(1-p)}{N-1}$ and $1 + \frac{N}{(p-1)(N-1)m'} = \frac{N-p}{(p-1)(N-1)}$.

From (4.2) and (4.3), using Fatou's lemma for $k \to \infty$, we deduce (i) and (ii) of Lemma 1.2.

Finally, using Hölder's inequality, (1.7) and (1.8) we obtain

$$\begin{aligned}\int_\Omega |\nabla u| &= \int_\Omega |u|^{\frac{N(1-p)}{p(N-1)}}|\nabla u|\,|u|^{\frac{N(p-1)}{p(N-1)}} \le \Big(\int_\Omega |u|^{\frac{N(1-p)}{N-1}}|\nabla u|^p\Big)^{1/p}\Big(\int_\Omega |u|^{\frac{N}{N-1}}\Big)^{1/p'}\\ &\le C_3\|f\|_{L^m(\Omega)}^{\frac{N-p}{p(p-1)(N-1)}}\|f\|_{L^m(\Omega)}^{\frac{N}{p(N-1)}} = C_3\|f\|_{L^m(\Omega)}^{\frac{1}{p-1}}.\end{aligned}$$

Therefore we obtain the estimate (1.9) and we conclude that $u \in W_0^{1,1}(\Omega)$; see Remark 2.2. For fixed $k > 0$, we consider the function

$$\varphi_k(s) = ((|s|)^\lambda - k^\lambda)^+\,\mathrm{sign}(s).$$

Let $h > k$. Taking $\varphi_k(T_h(u))$ as a test function in (2.2), using Hölder's inequality and (4.2), we get

$$\begin{aligned}\lambda \int\limits_{\{|u|\geq k\}} &|T_h(u)|^{\lambda-1}|\nabla T_h(u)|^p\\ &\leq \Big(\int\limits_{\{|u|\geq k\}} |f|^m\Big)^{1/m}\Big(\int\limits_{\{|u|\geq k\}} (|T_h(u)|^\lambda - k^\lambda)^{m'}\Big)^{1/m'_c}\\ &\leq \Big(\int\limits_{\{|u|\geq k\}} |f|^m\Big)^{1/m}\Big(\int\limits_{\{|u|\geq k\}} |T_h(u)|^{\lambda m'}\Big)^{1/m'_c} \leq C_4\Big(\int\limits_{\{|u|\geq k\}} |f|^m\Big)^{1/m}.\end{aligned}$$

Using Hölder's inequality as before, we obtain

$$\int\limits_{\{|u|\geq k\}} |\nabla T_h(u)| \leq C_4\Big(\int\limits_{\{|u|\geq k\}} |f|^m\Big)^{1/m}.$$

By Fatou's lemma we deduce (1.10). ∎

Remark 4.1. The statment (iii) of Lemma 1.2 is optimal in the sense that we can find a datum f which belongs to $L^m(\Omega)$ for every $1 \leq m < \frac{N}{(p-1)N+1}$ but not for $m = \frac{N}{(p-1)N+1}$ such that the renormalized solution of (1.4) does not belong to $W^{1,1}_{\mathrm{loc}}(\Omega)$. Let $\Omega = B(0,1) = \{x \in \mathbb{R}^N : |x| < 1\}$ and $\alpha = (p-1)N + 1$. Consider the boundary value problem

$$\begin{cases} -\Delta_p u = f(x) = |x|^{-\alpha} & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

The datum $f(x) = |x|^{-\alpha}$ belongs to $L^m(\Omega)$ for every $1 \leq m < \frac{N}{(p-1)N+1}$ but does not belong to $L^m(\Omega)$ for $m = \frac{N}{(p-1)N+1}$. Consider the function $u(x) = \beta(|x|^{1-N} - 1)$ with $\beta = (N-1)^{-1}(N-\alpha)^{\frac{1}{1-p}} = (N-1)^{-1}((2-p)N-1)^{\frac{1}{1-p}}$. We have $\beta > 0$ since $N \geq 2$ and $p < 2 - \frac{1}{N}$. In what follows, we prove that u is the (positive) renormalized solution of (4.4) but $u \notin W^{1,1}_{\mathrm{loc}}(\Omega)$. For each $k > 0$, the truncation

$$T_k(u) = \begin{cases} \beta(|x|^{1-N} - 1) & \text{if } (1 + k\beta^{-1})^{\frac{1}{1-N}} \leq |x| \leq 1,\\ k & \text{if } |x| < (1 + k\beta^{-1})^{\frac{1}{1-N}} \end{cases}$$

is Lipschitz continuous and zero on the boundary of Ω, therefore it belongs to $W^{1,p}_0(\Omega)$. Thus (a) is satisfied.

By Definition 2.1 of the gradient we have

$$\nabla u = -(N-\alpha)^{\frac{1}{1-p}} \frac{x}{|x|^{N+1}},$$

which implies that $|\nabla u|^{p-1}$ equals $(N-\alpha)^{-1}|x|^{-(p-1)N}$ and belongs to $L^q(\Omega)$ for every $1 \leq q < \frac{1}{p-1}$. Note that $\frac{1}{p-1} > \frac{N}{N-1}$ since $p < 2 - \frac{1}{N}$. Thus (b) is satisfied.

Now, let $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be such that there exist $k > 0$ and w^+ in $W_0^{1,r}(\Omega) \cap L^\infty(\Omega)$, with $r > N$, such that $w = w^+$ a.e. on $\Omega_k = \{u > k\} = \{|x| < (1 + k\beta^{-1})^{\frac{1}{1-N}}\}$. Taking into account that

$$|\nabla u|^{p-2}\nabla u = -(N-\alpha)^{-1}\frac{x}{|x|^\alpha}$$

and integrating by parts on $\Omega_\epsilon = B(0,1) \setminus B(0,\epsilon)$ for small ϵ, we obtain

$$I(\epsilon) = \int_{\Omega_\epsilon} |\nabla u|^{p-2}\nabla u \cdot \nabla w\, dx = -(N-\alpha)^{-1}(I_1(\epsilon) + I_2 + I_3(\epsilon)), \tag{4.5}$$

where

$$I_1(\epsilon) = \int_{\Omega_\epsilon} -\operatorname{div}\left(\frac{x}{|x|^\alpha}\right) w\, dx = -(N-\alpha)\int_{\Omega_\epsilon} |x|^{-\alpha} w\, dx$$

by a direct calculation,

$$I_2 = \int_{\{|x|=1\}} \frac{x}{|x|^\alpha} \cdot \frac{x}{|x|} w(x)\, d\sigma_1 = 0$$

since $w = 0$ almost everywhere on $\{|x| = 1\}$, and

$$\begin{aligned} I_3(\epsilon) &= \int_{\{|x|=\epsilon\}} \frac{x}{|x|^\alpha} \cdot \left(-\frac{x}{|x|}\right) w\, d\sigma_\epsilon = -\epsilon^{1-\alpha} \int_{\{|x|=\epsilon\}} w(x)\, w\, d\sigma_\epsilon \\ &= -\epsilon^{1-\alpha} \int_{\{|\xi|=1\}} w(\epsilon\xi)\epsilon^{N-1}\, d\sigma_1 = -\epsilon^{(2-p)N-1} \int_{\{|\xi|=1\}} w(\epsilon\xi)\, d\sigma_1. \end{aligned}$$

Now, we will take the limit in (4.5) as $\epsilon \to 0$. For what concerns $I(\epsilon)$, it is easy to see that the term

$$|\nabla u|^{p-2}\nabla u \cdot \nabla w = -(N-\alpha)^{-1}\frac{x}{|x|^\alpha} \cdot \nabla w$$

belongs to $L^1(\Omega)$ since in the neighbourhood of the origin Ω_k we have $w = w^+ \in (L^r(\Omega_k))^N$ with $r > N$. Therefore, by the dominated convergence theorem, we deduce that $I(\epsilon)$ converges to $\int_{B(0,1)} |\nabla u|^{p-2}\nabla u \cdot \nabla w\, dx$. On the other hand, $I_1(\epsilon)$ goes to $-(N-\alpha)\int_{B(0,1)} |x|^{-\alpha}\, w\, dx$. Using the continuity of $w = w^+$ near the origin, we see that $I_3(\epsilon)$ goes to 0 since $(2-p)N - 1 > 0$ (which is equivalent to $p < 2 - \frac{1}{N}$). Finally, we conclude that

$$\int_{B(0,1)} |\nabla u|^{p-2}\nabla u \cdot \nabla w\, dx = \int_{B(0,1)} |x|^{-\alpha} w\, dx.$$

Thus (c) is satisfied and u is a renormalized solution of (4.4). However, one can see that $|\nabla u| = (N-\alpha)^{\frac{1}{p-1}}|x|^{-N}$ does not belong to $L^1(\Omega)$. Thus $u \notin W_{\mathrm{loc}}^{1,1}(\Omega)$.

Acknowledgements. The author would like to express his gratitude to the anonymous referee for his/her comments and suggestions that contributed to improving the final version of the manuscript.

REFERENCES

[BBGGPV] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. L. Vazquez, *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 22 (1995), 241–273.

[BC] L. Boccardo and G. Croce, *$W_0^{1,1}$ solutions in some borderline cases of elliptic equations with degenerate coercivity*, in: Analysis and Topology in Nonlinear Differential Equations, Progr. Nonlinear Differential Equations Appl. 85, Birkhäuser/Springer, 2014, 135–143.

[BCO1] L. Boccardo, G. Croce and L. Orsina, *$W_0^{1,1}$ minima of noncoercive functionals*, Rend. Lincei Mat. Appl. 22 (2011), 513–523.

[BCO2] L. Boccardo, G. Croce and L. Orsina, *A semilinear problem with a $W_0^{1,1}(\Omega)$ solution*, Rend. Lincei Mat. Appl. 23 (2012), 97–103.

[BCO3] L. Boccardo, G. Croce and L. Orsina, *Nonlinear degenerate elliptic problems with $W_0^{1,1}(\Omega)$ solutions*, Manuscripta Math. 137 (2012), 419–439.

[BCT] L. Boccardo, G. Croce and C. Tanteri, *An elliptic system with degenerate coercivity*, Rend. Mat. Appl. (7) 36 (2015), 1–9.

[BG1] L. Boccardo and T. Gallouët, *$W_0^{1,1}$ solutions in some borderline cases of Calderón–Zygmunt theory*, J. Differential Equations 253 (2012), 2698–2714.

[BG2] L. Boccardo and T. Gallouët, *Nonlinear elliptic equations with right-hand side measures*, Comm. Partial Differential Equations 17 (1992), 641–655.

[BM] L. Boccardo and F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal. 19 (1992), 581–597.

[DMOP] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28 (1999), 741–808.

[E] L. C. Evans, *Partial Differential Equations*, 2nd ed., Amer. Math. Soc., 2010.

[LM] P.-L. Lions et F. Murat, *Solutions renormalisées d'équations elliptiques non linéaires*, manuscript.

[M] F. Murat, *Soluciones renormalizadas de EDP elipticas no lineales*, preprint 93023, Lab. Anal. Numér. Univ. Paris VI, 1993.

[Z17] W. P. Ziemer, *Modern Real Analysis*, 2nd ed., Grad. Texts in Math. 278, Springer, 2017.

Haydar Abdelhamid
Faculty of Sciences and Humanities
Fahad Bin Sultan University
P.O. Box 15700
Tabuk 71454, Saudi Arabia
E-mail: haydarabdelhamid@gmail.com