COLLOQUIUM MATHEMATICUM

Online First version

ON GEODESIC MAPPINGS IN A PARTICULAR CLASS OF ROTER SPACES

BY

RYSZARD DESZCZ and MARIAN HOTLOŚ (Wrocław)

Dedicated to the memory of Professor Witold Roter

Abstract. We determine a particular class of Roter warped product spaces. We show that every manifold of that class admits a non-trivial geodesic mapping onto some Roter warped product space. Moreover, both geodesically related manifolds are pseudosymmetric of constant type.

1. Introduction. Let (M, g) and $(\overline{M}, \overline{g})$ be n-dimensional semi-Riemannian manifolds. A diffeomorphism $h : M \to \overline{M}$ which maps geodesic lines into geodesic lines is called a *geodesic transformation*, or a *geodesic mapping*, or a *projective mapping*.

The well-known result of Beltrami is presented in [40, Theorem 10] as follows:

THEOREM (Beltrami). *The real space forms constitute the projective class of the locally Euclidean spaces, or, still, by applying geodesic transformations to locally Euclidean spaces one obtains spaces of constant curvature and the class of the spaces of constant curvature is closed under geodesic transformations.*

Manifolds satisfying various curvature conditions and admitting geodesic transformations were investigated by several authors (see, e.g., [8, 9, 23, 24, 25, 41, 42, 46, 47, 51, 54]). In particular, we have the following extension of Beltrami's theorem [40, Theorem 19].

THEOREM (Sinyukov, Mikeš, Venzi, Defever and Deszcz). *If a semisymmetric Riemannian space admits a geodesic transformation onto some other Riemannian manifold, then this latter manifold must itself be pseudosym-*

2020 *Mathematics Subject Classification*: Primary 53B20, 53B21.
Key words and phrases: geodesic mapping, warped product manifold, Einstein manifold, quasi-Einstein manifold, Roter space, pseudosymmetric manifold, pseudosymmetry type curvature condition.
Received 17 December 2018; revised 28 January 2021.
Published online 17 May 2021.

metric, and, if a pseudosymmetric Riemannian space admits a geodesic transformation onto some other Riemannian manifold, then this latter manifold must itself also be pseudosymmetric.

Thus, among the three classes mentioned in the above theorems, the class of pseudosymmetric manifolds is the widest class of manifolds which is closed with respect to geodesic mappings. It is known that the curvature tensor of certain non-conformally-flat and non-quasi-Einstein pseudosymmetric manifolds of dimension ≥ 4 is a linear combination of some Kulkarni–Nomizu products involving the Ricci tensor and the metric tensor of the manifolds in question. A semi-Riemannian manifold with curvature tensor having this property is called a *Roter space*. Evidently, every Roter space is pseudosymmetric. The converse is not true. It seems that the Roter spaces form an important and interesting class of manifolds. In particular, we can consider the following questions related to geodesic mappings of these manifolds:

(i) Can a Roter space admit a non-trivial geodesic mapping?
(ii) If a Roter space (M, g) admits a non-trivial geodesic mapping onto some manifold $(\overline{M}, \overline{g})$, then in view of the above-mentioned theorem, $\overline{M}$ is pseudosymmetric. Therefore, it is natural to ask: is $\overline{M}$ also a Roter space?

In this paper we answer these questions. First of all, we construct warped product manifolds, with a 2-dimensional base and with fiber of constant curvature, which are Roter spaces and admit non-trivial geodesic mappings. Moreover, we prove that manifolds geodesically related to these warped products are also Roter spaces. Furthermore, we derive some curvature conditions of pseudosymmetry type which are satisfied by the manifolds constructed here.

Continuing the study of geodesic mappings in Roter spaces, we have also obtained some new results in [27].

2. Preliminary results. Let (M, g), $n = \dim M \geq 3$, be a semi-Riemannian manifold. We denote by ∇, R, S, κ and C the Levi-Civita connection, the Riemann–Christoffel curvature tensor, the Ricci tensor, the scalar curvature and the Weyl conformal curvature tensor of (M, g), respectively. Throughout this paper all manifolds are assumed to be connected paracompact manifolds of class C^∞.

Let $\varXi(M)$ be the Lie algebra of vector fields on M. We define the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X, Y)$ of $\varXi(M)$ by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y,$$
$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

respectively, where A is a symmetric $(0, 2)$-tensor on M and $X, Y, Z \in \varXi(M)$.

The Ricci tensor S, the Ricci operator $\mathcal{S}$, the tensor S^2 and the scalar curvature κ of (M,g) are defined by $S(X,Y) = \operatorname{tr}\{Z \to \mathcal{R}(Z,X)Y\}$, $g(\mathcal{S}X,Y) = S(X,Y)$, $S^2(X,Y) = S(\mathcal{S}X,Y)$ and $\kappa = \operatorname{tr}\mathcal{S}$, respectively. The endomorphism $\mathcal{C}(X,Y)$ of (M,g), $n \geq 3$, is defined by

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\right)Z.$$

The $(0,4)$-tensor G, the Riemann–Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) are defined by

$$\begin{aligned} G(X_1,X_2,X_3,X_4) &= g((X_1 \wedge_g X_2)X_3, X_4), \\ R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3, X_4), \\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3, X_4), \end{aligned}$$

respectively, where $X_1, X_2, X_3, X_4 \in \Xi(M)$.

Let $\mathcal{B}$ be a tensor field sending any $X, Y \in \Xi(M)$ to a skew-symmetric endomorphism $\mathcal{B}(X,Y)$ and let B be the $(0,4)$-tensor associated with $\mathcal{B}$ by

$$B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3, X_4). \tag{2.1}$$

The tensor B is said to be a *generalized curvature tensor* if the following conditions are satisfied: $B(X_1,X_2,X_3,X_4) = B(X_3,X_4,X_1,X_2)$ and

$$B(X_1,X_2,X_3,X_4) + B(X_3,X_1,X_2,X_4) + B(X_2,X_3,X_1,X_4) = 0.$$

For $\mathcal{B}$ as above, let B be again defined by (2.1). We extend the endomorphism $\mathcal{B}(X,Y)$ to a derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X,Y)\cdot f = 0$ for any smooth function f on M.

For a $(0,k)$-tensor field T, $k \geq 1$, we can define the $(0,k+2)$-tensor $B\cdot T$ by

$$\begin{aligned} (B\cdot T)(X_1,\ldots,X_k,X,Y) &= (\mathcal{B}(X,Y)\cdot T)(X_1,\ldots,X_k) \\ &= -T(\mathcal{B}(X,Y)X_1, X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},\mathcal{B}(X,Y)X_k). \end{aligned}$$

If A is a symmetric $(0,2)$-tensor, we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$\begin{aligned} Q(A,T)(X_1,\ldots,X_k,X,Y) &= (X \wedge_A Y \cdot T)(X_1,\ldots,X_k) \\ &= -T((X \wedge_A Y)X_1, X_2,\ldots,X_k) - \cdots - T(X_1,\ldots,X_{k-1},(X \wedge_A Y)X_k). \end{aligned}$$

The tensor $Q(A,T)$ is called the *Tachibana tensor* of the tensors A and T (see, e.g., [13, 21, 22, 28]). Thus we have the $(0,6)$-tensors $R\cdot R$, $R\cdot C$, $C\cdot R$, $C\cdot C$, $Q(g,R)$, $Q(S,R)$, $Q(g,C)$ and $Q(S,C)$, as well as the $(0,4)$-tensors $R\cdot S$, $C\cdot S$ and $Q(g,S)$. For symmetric $(0,2)$-tensors A and B we define their *Kulkarni–Nomizu product* $A \wedge B$ by (see, e.g., [13, 21])

$$\begin{aligned} (A\wedge B)(X_1,X_2,X_3,X_4) &= A(X_1,X_4)B(X_2,X_3) + A(X_2,X_3)B(X_1,X_4) \\ &\quad - A(X_1,X_3)B(X_2,X_4) - A(X_2,X_4)B(X_1,X_3). \end{aligned}$$

A semi-Riemannian manifold (M,g), $n \geq 3$, is said to be an *Einstein manifold* (see, e.g., [1]) if at every point of M its Ricci tensor S is proportional to the metric tensor g, i.e., on M we have

$$S = \frac{\kappa}{n} g. \tag{2.2}$$

In [1, p. 432], (2.2) is called the *Einstein metric condition.* The Einstein manifolds form a natural subclass of several classes of semi-Riemannian manifolds which are determined by curvature conditions imposed on their Ricci tensor [1, Table, pp. 432–433]. These conditions are named *generalized Einstein metric conditions* [1, Chapter XVI].

A semi-Riemannian manifold (M,g), $n \geq 3$, is locally symmetric if

$$\nabla R = 0 \tag{2.3}$$

on M (see, e.g., [45, Section 1.5]). Indecomposable locally symmetric manifolds are Einstein manifolds. Equation (2.3) implies the integrability condition $\mathcal{R}(X,Y) \cdot R = 0$ or, briefly,

$$R \cdot R = 0. \tag{2.4}$$

A semi-Riemannian manifold satisfying (2.4) is called *semisymmetric* (see, e.g., [3, Section 8.5.3], [45, Section 1.6], [53, 55]). The semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds. A semi-Riemannian manifold (M,g), $n \geq 3$, is said to be *pseudosymmetric* if the tensors $R \cdot R$ and $Q(g,R)$ are linearly dependent at every point of M [23, 25] (see also [3, Section 8.5.3], [22, Section 6], [45, Section 12.4], [47, Section 7], [39, 40, 55]). This is equivalent to

$$R \cdot R = L_R Q(g,R) \tag{2.5}$$

on $\mathcal{U}_R = \{x \in M \mid R - (\kappa/(n-1)n)G \neq 0 \text{ at } x\}$, where L_R is some function on this set. Examples of non-semisymmetric pseudosymmetric manifolds are presented in, among others, [30, 33].

Let $\mathcal{U}_S$ be the set of all points of a semi-Riemannian manifold (M,g), $n \geq 3$, at which S is not proportional to g, i.e., $\mathcal{U}_S = \{x \in M \mid S - (\kappa/n)g \neq 0$ at $x\}$. A semi-Riemannian manifold (M,g), $n \geq 3$, is called *Ricci-pseudosymmetric* if the tensors $R \cdot S$ and $Q(g,S)$ are linearly dependent at every point of M (see, e.g., [24], [3, Section 8.5.3], [15, 55]). This is equivalent on $\mathcal{U}_S \subset M$ to

$$R \cdot S = L_S Q(g,S), \tag{2.6}$$

where L_S is some function on $\mathcal{U}_S$. Every warped product manifold $\overline{M} \times_F \widetilde{N}$ with a 1-dimensional manifold $(\overline{M}, \overline{g})$ and an $(n-1)$-dimensional Einstein semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, $n \geq 3$, and a warping function F, is a Ricci-pseudosymmetric manifold (see, e.g., [3, Section 8.5.3], [6, Section 1], [18, Example 4.1], [24]).

A semi-Riemannian manifold (M, g) is said to be *pseudosymmetric of constant type* [2], resp., *Ricci-pseudosymmetric of constant type* [37], if the function L_R is constant on $\mathcal{U}_R \subset M$, resp., if the function L_S is constant on $\mathcal{U}_S \subset M$. Let $\mathcal{U}_C$ be the set of all points of a semi-Riemannian manifold (M, g), $n \geq 4$, at which $C \neq 0$. We note that $\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R$ (see, e.g., [13]).

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to have *pseudosymmetric Weyl tensor* if the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent at every point of M (see, e.g., [13, 15, 18]). This is equivalent on $\mathcal{U}_C \subset M$ to

$$C \cdot C = L_C Q(g, C), \tag{2.7}$$

where L_C is some function on $\mathcal{U}_C$. Every warped product manifold $\overline{M} \times_F \widetilde{N}$ with $\dim \overline{M} = \dim \widetilde{N} = 2$ satisfies (2.7) (see, e.g., [13, 15, 18] and references therein). Thus in particular, the Schwarzschild spacetime, the Kottler spacetime and the Reissner–Nordström spacetime satisfy (2.7). Recently, manifolds with (2.7) were investigated in [13, 18, 28]. Warped product manifolds $\overline{M} \times_F \widetilde{N}$, of dimension ≥ 4, satisfying on $\mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$ the condition

$$R \cdot R - Q(S, R) = L Q(g, C), \tag{2.8}$$

where L is some function on $\mathcal{U}_C$, were studied in [10, 18]. For instance, in [10] necessary and sufficient conditions for $\overline{M} \times_F \widetilde{N}$ to satisfy (2.8) are given, and it proved that any 4-dimensional warped product manifold $\overline{M} \times_F \widetilde{N}$ with a 1-dimensional base $(\overline{M}, \overline{g})$ satisfies (2.8) [10, Theorem 4.1]. The warped product manifold $\overline{M} \times_F \widetilde{N}$ with a 2-dimensional base $(\overline{M}, \overline{g})$ and an $(n-2)$-dimensional space of constant curvature $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, satisfies both (2.7) and (2.8) [18, Theorem 7.1(i)].

We refer to [6, 13, 15, 16, 18, 22, 28] for details on semi-Riemannian manifolds with (2.5) and (2.6)–(2.8), as well as other properties of this kind, called *pseudosymmetry type curvature conditions* or *pseudosymmetry type conditions*. It seems that (2.5) is the most important one of those curvature conditions (see, e.g., [18]). We also point out that the Schwarzschild spacetime, the Kottler spacetime, the Reissner–Nordström spacetime, as well as the Friedmann–Lemaître–Robertson–Walker spacetimes are the "oldest" examples of pseudosymmetric warped product manifolds (see, e.g., [18, 22, 33]).

Investigations on semi-Riemannian manifolds (M, g), $n \geq 4$, satisfying (2.5) and (2.7) or (2.5) and (2.8) on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ lead to the following condition ([34, Theorem 3.2(ii)], [26, Lemma 4.1], see also [18, Section 1]):

$$R = \frac{\phi}{2} S \wedge S + \mu g \wedge S + \frac{\eta}{2} g \wedge g, \tag{2.9}$$

where ϕ, μ and η are some functions on $\mathcal{U}_S \cap \mathcal{U}_C$. Note that if (2.9) holds at a point of $\mathcal{U}_S \cap \mathcal{U}_C$, then at this point we have $\operatorname{rank}(S - \alpha g) > 1$ for any $\alpha \in \mathbb{R}$. A semi-Riemannian manifold (M, g), $n \geq 4$, satisfying (2.9) on

$\mathcal{U}_S \cap \mathcal{U}_C \subset M$ is called a *Roter space*, or a *Roter type space*, or a *Roter type manifold* [12, 19, 20].

Curvature properties of 2-recurrent semi-Riemannian manifolds ($\nabla^2 R = R \otimes \psi$) were investigated by Witold Roter [50]. In that paper it was shown that

$$R = \frac{1}{2\kappa} S \wedge S \tag{2.10}$$

on some 2-recurrent manifolds [50, Theorem 1]. It seems that [50] is the earliest paper on manifolds with (2.10). Evidently, (2.10) is a special case of (2.9) (for $\mu = \eta = 0$), since then

$$R = \frac{\phi}{2} S \wedge S. \tag{2.11}$$

We refer to [35, Example 3.1], [43, Section 4] and [48, Example 3.1] for results on manifolds satisfying (2.11).

Curvature properties of semi-Riemannian manifolds of dimension ≥ 4 with parallel Weyl conformal curvature tensor ($\nabla C = 0$) which are neither conformally-flat ($C \neq 0$) nor locally symmetric ($\nabla R \neq 0$) were investigated in [11]. Such manifolds are also called *essentially conformally symmetric manifolds*, or *e.c.s. manifolds* for short. In [11] it was shown that the Weyl tensor C of some e.c.s. manifolds is of the form $C = (\phi/2) S \wedge S$. Since the scalar curvature κ of every e.c.s. manifold vanishes, the last equation yields $R = (\phi/2) S \wedge S + (1/(n-2)) g \wedge S$. Thus we have (2.9) with $\mu = 1/(n-2)$ and $\eta = 0$.

Roter spaces and in particular Roter hypersurfaces (i.e., hypersurfaces satisfying (2.9)) in semi-Riemannian spaces of constant curvature were studied in [12, 13, 16, 19, 30, 31, 32, 36, 44]. Roter spaces satisfy several pseudosymmetry type curvature conditions. Specifically, we have the following theorem.

Theorem 2.1 ([15, 36]). *If (M, g), $n \geq 4$, is a semi-Riemannian Roter space satisfying* (2.9) *on $\mathcal{U}_S \cap \mathcal{U}_C \subset M$ then, on this set,*

$$S^2 = \alpha_1 S + \alpha_2 g, \tag{2.12}$$

$$\alpha_1 = \kappa + \frac{(n-2)\mu - 1}{\phi}, \quad \alpha_2 = \frac{\mu\kappa + (n-1)\eta}{\phi},$$

$$R \cdot R = L_R Q(g, R), \quad L_R = \frac{1}{\phi}((n-2)(\mu^2 - \phi\eta) - \mu), \tag{2.13}$$

$$R \cdot C = L_R Q(g, C), \quad R \cdot S = L_R Q(g, S),$$

$$R \cdot R = Q(S,R) + LQ(g,C), \qquad L = L_R + \frac{\mu}{\phi} = \frac{n-2}{\phi}(\mu^2 - \phi\eta), \tag{2.14}$$

$$\begin{aligned} C \cdot C &= L_C Q(g,C), \qquad L_C = L_R + \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \alpha_1\right), \\ C \cdot R &= L_C Q(g,R), \qquad C \cdot S = L_C Q(g,S), \\ R \cdot C - C \cdot R &= \left(\frac{1}{\phi}\left(\mu - \frac{1}{n-2}\right) + \frac{\kappa}{n-1}\right) Q(g,R) \\ &\quad + \left(\frac{\mu}{\phi}\left(\mu - \frac{1}{n-2}\right) - \eta\right) Q(S,G), \\ C \cdot R - R \cdot C &= Q(S,C) - \frac{\kappa}{n-1} Q(g,C). \end{aligned} \tag{2.15}$$

REMARK 2.1. (i) In the standard Schwarzschild coordinates $(t; r; \theta; \phi)$, and the physical units $(c = G = 1)$, the Reissner–Nordström–de Sitter $(\Lambda > 0)$ and the Reissner–Nordström–anti-de Sitter $(\Lambda < 0)$ metrics are given by the line element (see, e.g., [52])

$$\begin{aligned} ds^2 &= -h(r)\, dt^2 + h(r)^{-1}\, dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2), \\ h(r) &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}, \end{aligned} \tag{2.16}$$

where M, Q and Λ are non-zero constants.

(ii) [14, Section 6] The metric (2.16) satisfies (2.9) with

$$\begin{aligned} \phi &= \tfrac{3}{2}(Q^2 - Mr)r^4 Q^{-4}, \qquad \mu = \tfrac{1}{2}(Q^4 + 3Q^2\Lambda r^4 - 3\Lambda M r^5)Q^{-4}, \\ \eta &= \tfrac{1}{12}(3Q^6 + 4Q^4\Lambda r^4 - 3Q^4 Mr + 9Q^2\Lambda^2 r^8 - 9\Lambda^2 M r^9)r^{-4}Q^{-4}. \end{aligned}$$

If we set $\Lambda = 0$ in (2.16) then we obtain the line element of the Reissner–Nordström spacetime (see, e.g., [38, Section 9.2] and references therein). It seems that the Reissner–Nordström spacetime is the oldest known example of a Roter warped product space.

(iii) Some comments on pseudosymmetric manifolds (also called *Deszcz symmetric spaces*), as well as Roter spaces, are given in [7, Section 1]:

> From a geometric point of view, the Deszcz symmetric spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms.

and

> From an algebraic point of view, Roter spaces may well be considered to be the simplest Riemannian manifolds next to the real space forms.

For further comments we refer to [55].

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be a *quasi-Einstein manifold* if

$$\operatorname{rank}(S - \alpha g) = 1 \tag{2.17}$$

on $\mathcal{U}_S \subset M$, where α is some function on this set. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and the investigation on quasi-umbilical hypersurfaces of conformally-flat spaces (see, e.g., [15, 18] and references therein). Quasi-Einstein manifolds satisfying various pseudosymmetry type conditions were investigated in [6, 13, 16, 28]. Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied in [29, 37]; see also [3, Section 6.2], [15, 55] and references therein. We mention that there are different extensions of the class of quasi-Einstein manifolds. For instance we have the class of almost quasi-Einstein manifolds [5] as well as the class of 2-quasi-Einstein manifolds (see, e.g., [17, 18]).

3. Geodesic mappings. Let (M, g) and $(\overline{M}, \overline{g})$ be n-dimensional semi-Riemannian manifolds and let a diffeomorphism $h : M \to \overline{M}$ be a geodesic mapping. It is known that in a common coordinate system $\{x^1, \ldots, x^n\}$, the Christoffel symbols, the curvature tensors and the Ricci tensors of (M, g) and $(\overline{M}, \overline{g})$ are related by (see [51], [47, Chapter 8])

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \psi_j + \delta^h_j \psi_i, \qquad \overline{R}^h{}_{ijk} = R^h{}_{ijk} + \delta^h_j \psi_{ik} - \delta^h_k \psi_{ij}, \tag{3.1}$$

$$\overline{S}_{ij} = S_{ij} - (n-1)\psi_{ij}, \tag{3.2}$$

$$\psi_{ij} = \nabla_j \psi_i - \psi_i \psi_j, \qquad \psi_i = \frac{1}{2(n+1)} \frac{\partial}{\partial x^i} \left(\log \left| \frac{\det \overline{g}}{\det g} \right| \right). \tag{3.3}$$

We will denote by $h : (M, g) \xrightarrow{\psi} (\overline{M}, \overline{g})$ a geodesic mapping of (M, g) onto $(\overline{M}, \overline{g})$ and the manifolds (M, g) and $(\overline{M}, \overline{g})$ will be called *geodesically related.* Further, a geodesic mapping $h : (M, g) \xrightarrow{\psi} (\overline{M}, \overline{g})$ is called *non-trivial* on M if the covector field ψ with local components ψ_i is non-zero. It is also known that a manifold (M, g) can be geodesically mapped onto $(\overline{M}, \overline{g})$ if and only if there exists a covector field ψ on M which is a gradient with the property that

$$\nabla_k \overline{g}_{ij} = 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik}. \tag{3.4}$$

We have the following theorem.

THEOREM 3.1 ([8, 23]). *If (M, g) is a pseudosymmetric semi-Riemannian manifold admitting a non-trivial geodesic mapping h onto a manifold $(\overline{M}, \overline{g})$ then $(\overline{M}, \overline{g})$ is also pseudosymmetric. Moreover,*

$$\psi_{ij} = L_R g_{ij} - L_{\overline{R}} \overline{g}_{ij},$$

and $L_R = \mathrm{const}$ *if and only if* $L_{\overline{R}} = \mathrm{const}$.

We point out that the above statement was presented in the survey paper [46], but without proof.

The paper [9] dealt with manifolds satisfying $R\cdot R = LQ(S,R)$ or $R\cdot R = Q(S,R)$ and admitting non-trivial geodesic mappings.

Let M be a 2-dimensional manifold with the metric

$$ds^2 = \mathfrak{a}(x)\,dx^2 + \mathfrak{b}(x)\,dy^2.$$

It is known ([41, 42], see also [47, p. 356]) that M maps geodesically onto $\overline{M}$ with the metric

$$d\bar{s}^2 = \frac{p\mathfrak{a}(x)}{(1+q\mathfrak{b}(x))^2}\,dx^2 + \frac{p\mathfrak{b}(x)}{1+q\mathfrak{b}(x)}\,dy^2,$$

where $p \neq 0$ and q are real parameters, x and y are common coordinates. Evidently we assume that $\mathfrak{a}(x) \neq 0$, $\mathfrak{b}(x) \neq 0$ and $1+q\mathfrak{b}(x) \neq 0$. Taking into account that $g_{11} = \mathfrak{a}(x)$, $g_{22} = \mathfrak{b}(x)$, $g_{12} = 0$ and

$$\bar{g}_{11} = \frac{p\mathfrak{a}(x)}{(1+q\mathfrak{b}(x))^2}, \qquad \bar{g}_{22} = \frac{p\mathfrak{b}(x)}{1+q\mathfrak{b}(x)}, \qquad \bar{g}_{12} = 0$$

it is easy to see that the only non-zero components of Christoffel symbols are

$$\Gamma^1_{11} = \frac{\mathfrak{a}'}{2\mathfrak{a}}, \qquad \Gamma^2_{12} = \frac{\mathfrak{b}'}{2\mathfrak{b}}, \qquad \Gamma^1_{22} = -\frac{\mathfrak{b}'}{2\mathfrak{a}}, \tag{3.5}$$

where $\mathfrak{a}' = \frac{d\mathfrak{a}}{dx}$, $\mathfrak{b}' = \frac{d\mathfrak{b}}{dx}$. Moreover, equality (3.4) is satisfied with ψ given by

$$\psi_1 = -\frac{1}{2}\,\frac{q\mathfrak{b}'}{1+q\mathfrak{b}}, \qquad \psi_2 = 0. \tag{3.6}$$

Since we are interested in non-trivial geodesic mappings, throughout this paper we assume that $q \neq 0$ and $\mathfrak{b}'(x) \neq 0$.

The following lemma is useful.

LEMMA 3.1. *Let the metric g on $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$ be of the form $g_{11} = \mathfrak{a}(x)$, $g_{22} = \mathfrak{b}(x)$, $g_{12} = 0$. For the Gauss curvature κ_G of g we have $\kappa_G = K = \mathrm{const}$ if and only if*

$$\mathfrak{a} = \frac{(\mathfrak{b}')^2}{\mathfrak{b}(E - 4K\mathfrak{b})}, \qquad E \in \mathbb{R}. \tag{3.7}$$

Proof. We have (cf. (4.2)) $R_{1221} = g_{11}(\partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{21} + \Gamma^r_{22}\Gamma^1_{r1} - \Gamma^r_{21}\Gamma^1_{r2})$ and, in virtue of (3.5), we obtain

$$R_{1221} = \frac{1}{2}\left(-\mathfrak{b}'' + \frac{\mathfrak{a}'\mathfrak{b}'}{2\mathfrak{a}} + \frac{(\mathfrak{b}')^2}{2\mathfrak{b}}\right).$$

On the other hand, $R_{1221} = \frac{\kappa}{2}(g_{11}g_{22} - g_{12}^2)$, i.e., $R_{1221} = \frac{\kappa}{2}\mathfrak{a}\mathfrak{b} = \kappa_G\mathfrak{a}\mathfrak{b}$. Consequently,

$$\kappa_G = -\frac{2\mathfrak{a}\mathfrak{b}\mathfrak{b}'' - \mathfrak{b}\mathfrak{a}'\mathfrak{b}' - \mathfrak{a}(\mathfrak{b}')^2}{4(\mathfrak{a}\mathfrak{b})^2}$$

and, by our assumption, $2\mathfrak{a}\mathfrak{b}\mathfrak{b}'' - \mathfrak{b}\mathfrak{a}'\mathfrak{b}' - \mathfrak{a}(\mathfrak{b}')^2 = -4K\mathfrak{a}^2\mathfrak{b}^2$. We thus obtain the following Bernoulli equation with respect to the unknown function $\mathfrak{a}$:

$$\mathfrak{a}' + \left(\frac{\mathfrak{b}'}{\mathfrak{b}} - \frac{2\mathfrak{b}''}{\mathfrak{b}'}\right)\mathfrak{a} = \frac{4K\mathfrak{b}}{\mathfrak{b}'}\mathfrak{a}^2.$$

A standard calculation now leads to the solution of the form (3.7). ∎

4. Warped product manifolds. Let $(\widehat{M}, \widehat{g})$ and $(\widetilde{N}, \widetilde{g})$, $\dim \widehat{M} = p$, $\dim \widetilde{N} = n - p$, $1 \le p < n$, be semi-Riemannian manifolds and F a positive smooth function on $\widehat{M}$. The warped product $\widehat{M} \times_F \widetilde{N}$ of $(\widehat{M}, \widehat{g})$ and $(\widetilde{N}, \widetilde{g})$ is the product manifold $\widehat{M} \times \widetilde{N}$ with the metric tensor g defined by $g = \widehat{g} \times_F \widetilde{g} = \pi_1^* \widehat{g} + (F \circ \pi_1)\pi_2^* \widetilde{g}$, where $\pi_1 : \widehat{M} \times \widetilde{N} \to \widehat{M}$ and $\pi_2 : \widehat{M} \times \widetilde{N} \to \widetilde{N}$ are the natural projections on $\widehat{M}$ and $\widetilde{N}$, respectively (see, e.g., [49] and references therein).

Let $(\widehat{M}, \widehat{g})$ and $(\widetilde{N}, \widetilde{g})$ be covered by systems of charts $\{U; x^a\}$ and $\{V; y^\alpha\}$, respectively, and let $\{U \times V; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\widehat{M} \times \widetilde{N}$. The local components g_{ij} of the metric $g = \widehat{g} \times_F \widetilde{g}$ with respect to this chart are $g_{ij} = \widehat{g}_{ab}$ if $i = a$ and $j = b$, $g_{ij} = F\widetilde{g}_{\alpha\beta}$ if $i = \alpha$ and $j = \beta$, and $g_{ij} = 0$ otherwise, where $a, b, c, d, f \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \delta \in \{p + 1, \ldots, n\}$ and $h, i, j, k, l, m, r, s \in \{1, \ldots, n\}$. We will mark with hats (resp., with tildes) tensors formed from $\widehat{g}$ (resp., $\widetilde{g}$).

The local components $\varGamma^h_{ij} = \frac{1}{2} g^{hs}(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij})$, $\partial_j = \frac{\partial}{\partial x^j}$, of the Levi-Civita connection ∇ of $\widehat{M} \times_F \widetilde{N}$ are (see, e.g., [18])

$$\begin{aligned} &\varGamma^a_{bc} = \widehat{\varGamma}^a_{bc}, \quad \varGamma^\alpha_{\beta\gamma} = \widetilde{\varGamma}^\alpha_{\beta\gamma}, \quad \varGamma^a_{\alpha\beta} = -\frac{1}{2}\widehat{g}^{ab} F_b \widetilde{g}_{\alpha\beta}, \\ &\varGamma^\alpha_{a\beta} = \frac{1}{2F} F_a \delta^\alpha_\beta, \quad \varGamma^a_{\alpha b} = \varGamma^\alpha_{ab} = 0, \quad F_a = \frac{\partial F}{\partial x^a}. \end{aligned} \tag{4.1}$$

The local components

$$R_{hijk} = g_{hs} R^s_{ijk} = g_{hs}(\partial_k \varGamma^s_{ij} - \partial_j \varGamma^s_{ik} + \varGamma^r_{ij}\varGamma^s_{rk} - \varGamma^r_{ik}\varGamma^s_{rj}) \tag{4.2}$$

of the Riemann–Christoffel curvature tensor R and the local components S_{ij} of the Ricci tensor S of the warped product $\widehat{M} \times_F \widetilde{N}$ which may not vanish identically are

$$R_{abcd} = \widehat{R}_{abcd}, \quad R_{\alpha ab\delta} = -\frac{1}{2} T_{ab} \widetilde{g}_{\alpha\delta}, \quad R_{\alpha\beta\gamma\delta} = F\widetilde{R}_{\alpha\beta\gamma\beta} - \frac{\varDelta_1 F}{4} \widetilde{G}_{\alpha\beta\gamma\delta},$$

$$S_{ab} = \widehat{S}_{ab} - \frac{n-p}{2F} T_{ab}, \quad S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \frac{1}{2}\left(\operatorname{tr} T + \frac{n-p-1}{2F} \varDelta_1 F\right)\widetilde{g}_{\alpha\beta}, \tag{4.3}$$

$$T_{ab} = \widehat{\nabla}_a F_b - \frac{1}{2F} F_a F_b, \quad \operatorname{tr} T = \widehat{g}^{ab} T_{ab}, \quad \varDelta_1 F = \varDelta_{1\widehat{g}} F = \widehat{g}^{ab} F_a F_b, \tag{4.4}$$

where T is the $(0, 2)$-tensor with local components T_{ab}. The scalar curvature κ of $\widehat{M} \times_F \widetilde{N}$ is given by $\kappa = \widehat{\kappa} + \frac{1}{F}\widetilde{\kappa} - \frac{n-p}{F}\big(\operatorname{tr} T + \frac{n-p-1}{4F} \varDelta_1 F\big)$.

Let $(\widehat{M},\widehat{g})$ be a 2-dimensional manifold with metric given by

$$\widehat{g}_{11}=\mathfrak{a}(x^1),\qquad \widehat{g}_{22}=\mathfrak{b}(x^1),\qquad \widehat{g}_{12}=0,$$

and $(\widetilde{N},\widetilde{g})$ be an $(n-2)$-dimensional, $n\geq 4$, semi-Riemannian space, assumed to be of constant curvature when $n\geq 5$. Next, let $\widehat{M}\times_F\widetilde{N}$ be the warped product with $F=F(x^1,x^2)$. Finally, let $(\overline{\widehat{M}},\overline{\widehat{g}})$ be a manifold geodesically related to $(\widehat{M},\widehat{g})$ with

$$\overline{\widehat{g}}_{11}=\frac{p\mathfrak{a}(x^1)}{(1+q\mathfrak{b}(x^1))^2},\qquad \overline{\widehat{g}}_{22}=\frac{p\mathfrak{b}(x^1)}{1+q\mathfrak{b}(x^1)},\qquad \overline{\widehat{g}}_{12}=0$$

and a covector field ψ as in (3.6).

We will find necessary and sufficient conditions in order that $\widehat{M}\times_F\widetilde{N}$ can be geodesically mapped onto $\overline{\widehat{M}}\times_{\overline{F}}\widetilde{N}$ with $\overline{F}=\overline{F}(x^1,x^2)$. Under our assumptions we have

$$\begin{aligned}
&g_{11}=\mathfrak{a}(x^1),\qquad g_{22}=\mathfrak{b}(x^1),\qquad g_{\alpha\beta}=F\widetilde{g}_{\alpha\beta},\\
&\overline{g}_{11}=\frac{p\mathfrak{a}(x^1)}{(1+q\mathfrak{b}(x^1))^2},\qquad \overline{g}_{22}=\frac{p\mathfrak{b}(x^1)}{1+q\mathfrak{b}(x^1)},\qquad \overline{g}_{\alpha\beta}=\overline{F}\widetilde{g}_{\alpha\beta},
\end{aligned}\tag{4.5}$$

and the remaining components of g and $\overline{g}$ vanish.

It is obvious that equality (3.4) is satisfied for $i=a$, $j=b$, $k=c$.

Considering the case $i=a$, $j=\alpha$, $k=\beta$ we have, in virtue of (4.1),

$$\begin{aligned}
&\frac{\partial\overline{g}_{a\alpha}}{\partial x^\beta}-\mathit{\Gamma}^s_{\beta a}\overline{g}_{s\alpha}-\mathit{\Gamma}^s_{\beta\alpha}\overline{g}_{sa}=2\psi_\beta\overline{g}_{a\alpha}+\psi_a\overline{g}_{\alpha\beta}+\psi_\alpha\overline{g}_{a\beta},\\
&-\mathit{\Gamma}^\epsilon_{\beta a}\overline{g}_{\epsilon\alpha}-\mathit{\Gamma}^\epsilon_{\beta\alpha}\overline{g}_{\epsilon a}-\mathit{\Gamma}^c_{\beta\alpha}\overline{g}_{ca}=\psi_a\overline{g}_{\alpha\beta},\\
&-\frac{1}{2F}F_a\overline{g}_{\alpha\beta}+\frac{1}{2}\frac{F^c}{F}\overline{g}_{ca}F\widetilde{g}_{\alpha\beta}=\psi_a\overline{g}_{\alpha\beta},
\end{aligned}$$

and finally

$$-\frac{\overline{F}}{2F}F_a+\frac{1}{2}F^c\overline{g}_{ca}=\overline{F}\psi_a.\tag{4.6}$$

Now, let $i=\alpha$, $j=\beta$, $k=a$. We have, successively,

$$\begin{aligned}
&\frac{\partial\overline{g}_{\alpha\beta}}{\partial x^a}-\mathit{\Gamma}^s_{a\alpha}\overline{g}_{s\beta}-\mathit{\Gamma}^s_{a\beta}\overline{g}_{s\alpha}=2\psi_a\overline{g}_{\alpha\beta}+\psi_\alpha\overline{g}_{a\beta}+\psi_\beta\overline{g}_{a\alpha},\\
&\frac{\partial\overline{F}}{\partial x^a}\widetilde{g}_{\alpha\beta}-\mathit{\Gamma}^\epsilon_{a\alpha}\overline{g}_{\epsilon\beta}-\mathit{\Gamma}^\epsilon_{a\beta}\overline{g}_{\epsilon\alpha}=2\psi_a\overline{g}_{\alpha\beta},\\
&\frac{\partial\overline{F}}{\partial x^a}-\frac{1}{2F}F_a\overline{F}-\frac{1}{2F}F_a\overline{F}=2\psi_a\overline{F},\\
&\frac{1}{\overline{F}}\frac{\partial\overline{F}}{\partial x^a}=\frac{F_a}{F}+2\psi_a,\qquad \frac{\partial\log\overline{F}}{\partial x^a}-\frac{\partial\log F}{\partial x^a}=2\psi_a,
\end{aligned}$$

and finally

$$\frac{\partial}{\partial x^a}\left(\log\frac{\overline{F}}{F}\right) = 2\psi_a. \tag{4.7}$$

It is easy to check that in the remaining cases (3.4) is also satisfied. Thus we have proved

Proposition 4.1. *Let $(\widehat{M}, \widehat{g})$ be a 2-dimensional manifold with metric given by $\widehat{g}_{11} = \mathfrak{a}(x^1)$, $\widehat{g}_{22} = \mathfrak{b}(x^1)$, $\widehat{g}_{12} = 0$, and $(\widetilde{N}, \widetilde{g})$ be an $(n-2)$-dimensional, $n \geq 4$, semi-Riemannian space, assumed to be of constant curvature when $n \geq 5$. Next, let $\widehat{M} \times_F \widetilde{N}$ be the warped product manifold with warping function $F = F(x^1, x^2)$ and let $(\overline{\widehat{M}}, \overline{\widehat{g}})$ be a manifold geodesically related to $(\widehat{M}, \widehat{g})$ with metric given by*

$$\overline{\widehat{g}}_{11} = \frac{p\mathfrak{a}(x^1)}{(1+q\mathfrak{b}(x^1))^2}, \qquad \overline{\widehat{g}}_{22} = \frac{p\mathfrak{b}(x^1)}{1+q\mathfrak{b}(x^1)}, \qquad \overline{\widehat{g}}_{12} = 0$$

and a covector field ψ as in (3.6). *Then $\widehat{M} \times_F \widetilde{N}$ can be geodesically mapped onto $\overline{\widehat{M}} \times_{\overline{F}} \widetilde{N}$ with $\overline{F} = \overline{F}(x^1, x^2)$ if and only if equalities* (4.6) *and* (4.7) *are satisfied.*

According to [18, Theorem 5.3] the warped product manifold $\widehat{M} \times_F \widetilde{N}$ with a 2-dimensional manifold $(\widehat{M}, \widehat{g})$ and an $(n-2)$-dimensional semi-Riemannian space $(\widetilde{N}, \widetilde{g})$, assumed to be of constant curvature when $n \geq 5$, is pseudosymmetric on $U_S \cap U_C$ if and only if T_{ab} is proportional to $\widehat{g}_{ab}$ on this set. Therefore let $\widehat{M} \times_F \widetilde{N}$ be as in Proposition 4.1 with

$$F = F(x^1, x^2) = f^2(x^1, x^2) \tag{4.8}$$

and consider now the condition $T = \lambda\widehat{g}$. In view of (4.5) this condition is equivalent to

$$\text{(i) } T_{12} = 0, \qquad \text{(ii) } \mathfrak{b}T_{11} = \mathfrak{a}T_{22}. \tag{4.9}$$

Further, by (4.4),

$$T_{12} = \widehat{\nabla}_1 F_2 - \frac{1}{2F}F_1F_2 = \partial_1 F_2 - F_s\Gamma^s_{12} - \frac{1}{2F}F_1F_2 = \partial_1 F_2 - F_2\Gamma^2_{12} - \frac{1}{2F}F_1F_2,$$

so using (4.9)(i), (3.5) and (4.8) we get

$$2(f_1f_2 + ff_{12}) - 2ff_2\frac{\mathfrak{b}'}{2\mathfrak{b}} - \frac{1}{2f^2}2ff_1 2ff_2 = 0,$$

$$f_{12} = f_2\frac{\mathfrak{b}'}{2\mathfrak{b}}. \tag{4.10}$$

Similarly,

$$
\begin{aligned}
T_{11} &= \partial_1 F_1 - \frac{\mathfrak{a}'}{2\mathfrak{a}} F_1 - \frac{1}{2F} F_1^2 = 2(f_1^2 + f f_{11}) - \frac{\mathfrak{a}'}{2\mathfrak{a}} 2 f f_1 - \frac{1}{2f^2} 4 f^2 f_1^2 \\
&= 2 f f_{11} - 2 f f_1 \frac{\mathfrak{a}'}{2\mathfrak{a}}, \\
T_{22} &= \partial_2 F_2 + \frac{\mathfrak{b}'}{2\mathfrak{a}} F_1 - \frac{1}{2F} F_2^2 = 2(f_2^2 + f f_{22}) + \frac{\mathfrak{b}'}{2\mathfrak{a}} 2 f f_1 - 2 f_2^2 \\
&= 2 f f_{22} + 2 f f_1 \frac{\mathfrak{b}'}{2\mathfrak{a}}.
\end{aligned}
$$

Thus (4.9)(ii) leads to

$$
\mathfrak{b} f_{11} - \mathfrak{a} f_{22} = f_1 \left(\frac{\mathfrak{b}\mathfrak{a}'}{2\mathfrak{a}} + \frac{\mathfrak{b}'\mathfrak{a}}{2\mathfrak{a}} \right) = \frac{f_1}{2\mathfrak{a}} (\mathfrak{a}\mathfrak{b})', \qquad f_{11} - \frac{\mathfrak{a}}{\mathfrak{b}} f_{22} = \frac{f_1}{2\mathfrak{a}\mathfrak{b}} (\mathfrak{a}\mathfrak{b})'. \tag{4.11}
$$

We will now find some conditions equivalent to (4.6) and (4.7).

Using (4.6) for $a = 2$, in virtue of (3.6), we have

$$
\frac{\overline{F}}{F} F_2 = F^c \overline{g}_{c2} = F_s \widehat{g}^{2s} \overline{g}_{22} = F_2 \frac{1}{\mathfrak{b}} \frac{p\mathfrak{b}}{1+q\mathfrak{b}} = \frac{pF_2}{1+q\mathfrak{b}}, \qquad \frac{\overline{F}}{F} = \frac{p}{1+q\mathfrak{b}}. \tag{4.12}
$$

This implies $\frac{\partial}{\partial x^2}\big(\log \frac{\overline{F}}{F}\big) = 0$. On the other hand $\frac{\partial}{\partial x^1}\big(\log \frac{\overline{F}}{F}\big) = -\frac{q\mathfrak{b}'}{1+q\mathfrak{b}} = 2\psi_1$. Thus, equality (4.7) is satisfied.

Condition (4.6) for $a = 1$ takes the form $-\frac{\overline{F}}{F} F_1 + F^c \overline{g}_{c1} = 2\overline{F}\psi_1$. Since

$$
F^c \overline{g}_{c1} = F_s \widehat{g}^{s1} \overline{g}_{11} = F_1 \frac{1}{\mathfrak{a}} \frac{p\mathfrak{a}}{(1+q\mathfrak{b})^2} = F_1 \frac{p}{(1+q\mathfrak{b})^2},
$$

using (4.12) we get $\mathfrak{b}' F = \mathfrak{b} F_1$, which in terms of f takes the form $2\mathfrak{b} f_1 = \mathfrak{b}' f$. Applying this equality to (4.10) we have $f f_{12} = f_1 f_2$. It is easy to see that the solution of this differential equation is $f(x^1, x^2) = A(x^1) B(x^2)$. Thus $f_1/f = A_1/A$. But $f_1/f = \mathfrak{b}'/(2\mathfrak{b})$ and we obtain $2A_1/A = \mathfrak{b}'/\mathfrak{b}$, which in particular gives $A^2 = \mathfrak{b}$ and, without loss of generality, $f(x^1, x^2) = \sqrt{\mathfrak{b}(x^1)}\, B(x^2)$. This leads to

$$
f_1 = \frac{\mathfrak{b}'}{2\sqrt{\mathfrak{b}}} B, \qquad f_2 = \sqrt{\mathfrak{b}}\, B_2, \qquad f_{22} = \sqrt{\mathfrak{b}}\, B_{22}, \qquad f_{11} = \frac{B}{2\sqrt{\mathfrak{b}}} \left(\mathfrak{b}'' - \frac{(\mathfrak{b}')^2}{2\mathfrak{b}} \right).
$$

Substituting these equalities into (4.11) we obtain

$$
\begin{aligned}
&\frac{B}{2\sqrt{\mathfrak{b}}} \left(\mathfrak{b}'' - \frac{(\mathfrak{b}')^2}{2\mathfrak{b}} \right) - \frac{\mathfrak{a}}{\mathfrak{b}} \sqrt{\mathfrak{b}}\, B_{22} = \frac{(\mathfrak{a}\mathfrak{b})'}{2\mathfrak{a}\mathfrak{b}} \frac{\mathfrak{b}' B}{2\sqrt{\mathfrak{b}}}, \\
&\mathfrak{b}'' - \frac{(\mathfrak{b}')^2}{2\mathfrak{b}} = 2\mathfrak{a} \frac{B_{22}}{B} + \frac{(\mathfrak{a}\mathfrak{b})'}{2\mathfrak{a}\mathfrak{b}} \mathfrak{b}'.
\end{aligned}
$$

The last equality implies

$$
\frac{B_{22}}{B} = C = \text{const.} \tag{4.13}
$$

Therefore

$$\mathfrak{b}\mathfrak{b}'' - (\mathfrak{b}')^2 = 2\mathfrak{a}\mathfrak{b}C + \frac{\mathfrak{a}'\mathfrak{b}'\mathfrak{b}}{2\mathfrak{a}}. \tag{4.14}$$

Rewriting this equation in the form $-\mathfrak{a}' + 2\mathfrak{a}\big(\frac{\mathfrak{b}''}{\mathfrak{b}'} - \frac{\mathfrak{b}'}{\mathfrak{b}}\big) = \frac{4C\mathfrak{a}^2}{\mathfrak{b}'}$ we have the Bernoulli equation with respect to the unknown function $\mathfrak{a}$, which leads to

$$\mathfrak{a} = \frac{(\mathfrak{b}')^2}{\mathfrak{b}(D\mathfrak{b} - 4C)}, \qquad D \in \mathbb{R}. \tag{4.15}$$

Comparing this equality with Lemma 3.1 we obtain the following.

Corollary 4.1. *Let (M, g) be a 2-dimensional manifold with metric*

$$g_{11} = \mathfrak{a}(x^1), \qquad g_{22} = \mathfrak{b}(x^1), \qquad g_{12} = 0$$

and let functions $\mathfrak{a}$ and $\mathfrak{b}$ satisfy (4.14). *Then $\mathfrak{a}$ satisfies relation* (4.15) *and the Gauss curvature of M is constant, namely*

$$\kappa_G = -D/4. \tag{4.16}$$

Taking into account the equality $f(x^1, x^2) = \sqrt{\mathfrak{b}(x^1)}\, B(x^2)$ and (4.8), we see that

$$F(x^1, x^2) = \mathfrak{b}(x^1) B^2(x^2). \tag{4.17}$$

Computing once more T_{22} and using (4.17), (4.13) and (4.15) we obtain $T_{22} = (D/2)\mathfrak{b}^2 B^2$. Thus, in view of (4.9)(ii) we have $\operatorname{tr} T = \widehat{g}^{11} T_{11} + \widehat{g}^{22} T_{22} = (2/\mathfrak{b}) T_{22} = D\mathfrak{b}B^2$, i.e.,

$$\operatorname{tr} T = DF. \tag{4.18}$$

For a function B satisfying (4.13) we have the following.

Remark 4.1. Let a function $B : \mathbb{R} \to \mathbb{R}$ satisfy $B''(t)/B(t) = C = \text{const}$. Then:

(i) $(B')^2 - CB^2 = \text{const}$;
(ii) (a) if $C > 0$, then $B(t) = C_1 e^{\sqrt{C}t} + C_2 e^{-\sqrt{C}t}$, $C_1, C_2 \in \mathbb{R}$,
(b) if $C < 0$, then $B(t) = C_1 \cos\sqrt{-C}\, t + C_2 \sin\sqrt{-C}\, t$, $C_1, C_2 \in \mathbb{R}$,
(c) if $C = 0$, then $B(t) = C_1 t + C_2$, $C_1, C_2 \in \mathbb{R}$.

Concerning conformal flatness of warped product manifolds we have the following.

Remark 4.2. Let $\widehat{M} \times_F \widetilde{N}$ be a warped product manifold with a 2-dimensional manifold $(\widehat{M}, \widehat{g})$ and an $(n-2)$-dimensional fiber $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, and let $(\widetilde{N}, \widetilde{g})$ be a semi-Riemannian space, assumed to be of constant curvature when $n \geq 5$. The local components C_{hijk} of the Weyl conformal curvature tensor C of $\widehat{M} \times_F \widetilde{N}$ are (see [18])

$$C_{abcd} = \frac{(n-3)\rho_0}{n-1} G_{abcd}, \qquad C_{\alpha bc\delta} = -\frac{(n-3)\rho_0}{(n-2)(n-1)} G_{\alpha bc\delta},$$

$$C_{\alpha\beta\gamma\delta} = \frac{2\rho_0}{(n-2)(n-1)} G_{\alpha\beta\gamma\delta}, \qquad C_{abc\delta} = C_{ab\gamma\delta} = C_{a\beta\gamma\delta} = 0,$$

$$\rho_0 = \frac{\widehat{\kappa}}{2} + \frac{\widetilde{\kappa}}{(n-3)(n-2)F} + \frac{\operatorname{tr} T}{2F} - \frac{\Delta_1 F}{4F^2}. \tag{4.19}$$

Hence, $\widehat{M} \times_F \widetilde{N}$ is conformally-flat if and only if $\rho_0 = 0$.

Taking into account (4.15) we find

$$\Delta_1 F = \frac{1}{\mathfrak{a}} F_1^2 + \frac{1}{\mathfrak{b}} F_2^2 = \mathfrak{b} B^2 (B^2 (D\mathfrak{b} - 4C) + 4B_2^2). \tag{4.20}$$

Thus, by using (4.19), (4.16) and (4.18), we obtain

$$\begin{aligned}\rho_0 &= -\frac{D}{4} + \frac{\widetilde{\kappa}}{(n-3)(n-2)} \frac{1}{F} + \frac{DF}{2F} - \frac{1}{F}\frac{1}{4}(B^2(D\mathfrak{b} - 4C) + 4B_2^2) \\ &= \frac{D}{4} + \frac{\widetilde{\kappa}}{(n-3)(n-2)F} - \frac{1}{F}\left(B^2\left(\frac{D}{4}\mathfrak{b} - C\right) + B_2^2\right) \\ &= \frac{1}{F}\left(\frac{\widetilde{\kappa}}{(n-3)(n-2)} - B_2^2 + CB^2\right).\end{aligned}$$

Therefore, the equality $\rho_0 = 0$ is equivalent to

$$\frac{\widetilde{\kappa}}{(n-3)(n-2)} = B_2^2 - CB^2. \tag{4.21}$$

Now we consider the following problem: when is the manifold $\widehat{M} \times_F \widetilde{N}$ quasi-Einstein or Einstein? In virtue of (4.3) we have

$$S_{ab} = \widehat{S}_{ab} - \frac{n-2}{F} T_{ab} = \left(\frac{\widehat{\kappa}}{2} - \frac{n-2}{2F}\frac{\operatorname{tr} T}{2}\right) g_{ab}$$

and in view of the relations $\widehat{\kappa}_G = \widehat{\kappa}/2$, (4.16) and (4.18) we get

$$S_{ab} = -\frac{n-1}{4} D g_{ab}. \tag{4.22}$$

Similarly,

$$\begin{aligned}S_{\alpha\beta} &= \widetilde{S}_{\alpha\beta} - \frac{1}{2}\left(\operatorname{tr} T + \frac{n-3}{2}\frac{\Delta_1 F}{F}\right)\widetilde{g}_{\alpha\beta} \\ &= \left(\frac{\widetilde{\kappa}}{n-2} - \frac{\operatorname{tr} T}{2} - \frac{n-3}{4F}\Delta_1 F\right)\frac{1}{F} g_{\alpha\beta}.\end{aligned}$$

Taking into account (4.20) we find that

$$S_{\alpha\beta} = \frac{1}{\mathfrak{b}B^2}\left(\frac{\widetilde{\kappa}}{n-2} - \frac{n-1}{4} D\mathfrak{b}B^2 + (n-3)(B^2 C - B_2^2)\right) g_{\alpha\beta}. \tag{4.23}$$

Equalities (2.17), (4.22) and (4.23) imply that $\widehat{M} \times_F \widetilde{N}$ cannot be quasi-Einstein and will be Einstein if and only if

$$\frac{\widetilde{\kappa}}{n-2} - \frac{n-1}{4} D\mathfrak{b}B^2 + (n-3)(B^2C - B_2^2) = -\frac{n-1}{4} D\mathfrak{b}B^2,$$

which reduces to (4.21).

Remark 4.3. Let $\widehat{M} \times_F \widetilde{N}$ be a warped product manifold with a 2-dimensional manifold $(\widehat{M}, \widehat{g})$ and an $(n-2)$-dimensional fiber $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, and let $(\widetilde{N}, \widetilde{g})$ be a semi-Riemannian space, assumed to be of constant curvature when $n \geq 5$. If $T = \frac{\operatorname{tr} T}{2} \widehat{g}$ on $U = U_S \cap U_C \subset \widehat{M} \times \widetilde{N}$ then we have (see [30, p. 12])

$$R_{abcd} = \rho_1 G_{abcd}, \quad \rho_1 = \frac{\widehat{\kappa}}{2}, \quad R_{\alpha bc\beta} = \rho_2 G_{\alpha bc\beta}, \quad \rho_2 = -\frac{\operatorname{tr} T}{4F},$$

$$R_{\alpha\beta\gamma\delta} = \rho_3 G_{\alpha\beta\gamma\delta}, \quad \rho_3 = \frac{1}{F}\left(\frac{\widetilde{\kappa}}{(n-3)(n-2)} - \frac{\Delta_1 F}{4F}\right),$$

$$S_{ab} = \mu_1 g_{ab}, \quad \mu_1 = \frac{1}{4F}(2F\widehat{\kappa} - (n-2)\operatorname{tr} T), \tag{4.24}$$

$$S_{\alpha\beta} = \mu_2 g_{\alpha\beta}, \quad \mu_2 = \frac{1}{F}\left(\frac{\widetilde{\kappa}}{n-2} - \frac{\operatorname{tr} T}{2} - (n-3)\frac{\Delta_1 F}{4F}\right). \tag{4.25}$$

In the case considered here, if the equality (4.21) does not hold then $U_S \cap U_C = \widehat{M} \times \widetilde{N}$. Computing now μ_1, μ_2 and ρ_1, ρ_2, ρ_3 from Remark 4.3, in view of (4.18) and (4.20), we have

$$\begin{aligned} \mu_1 &= -\frac{n-1}{4} D, \\ \mu_2 &= \frac{1}{F}\left(\frac{\widetilde{\kappa}}{n-2} - \frac{n-1}{4} FD - (n-3)(B_2^2 - CB^2)\right). \end{aligned} \tag{4.26}$$

Thus

$$\mu_2 - \mu_1 = \frac{1}{F}\left(\frac{\widetilde{\kappa}}{n-2} - (n-3)(B_2^2 - CB^2)\right), \tag{4.27}$$

$$\rho_1 = \frac{\widehat{\kappa}}{2} = -\frac{D}{4} = \rho_2,$$

$$\begin{aligned} \rho_3 &= \frac{1}{F(n-3)}\left(\frac{\widetilde{\kappa}}{n-2} - (n-3)(B_2^2 - CB^2) - (n-3)\frac{FD}{4}\right) \\ &= \frac{1}{n-3}(\mu_2 - \mu_1) - \frac{D}{4}. \end{aligned}$$

According to [30, Theorem 4.1], $\widehat{M} \times_F \widetilde{N}$ is a Roter space, i.e., (2.9) is satisfied, with $\phi = \nu(\rho_1 - 2\rho_2 + \rho_3)$, $\mu = \nu((\rho_2 - \rho_3)\mu_1 + (\rho_2 - \rho_1)\mu_2)$, $\eta = \nu(\rho_1 \mu_2^2 - 2\rho_2 \mu_1 \mu_2 + \rho_3 \mu_1^2)$, where $\nu = (\mu_2 - \mu_1)^{-2}$. Thus, applying these results, we obtain

$$
\begin{aligned}
&\phi = \frac{1}{(n-3)(\mu_2-\mu_1)}, \qquad \mu = -\frac{\mu_1}{(n-3)(\mu_2-\mu_1)},\\
&\eta = \rho_1 + \frac{\mu_1^2}{(n-3)(\mu_2-\mu_1)}.
\end{aligned} \tag{4.28}
$$

Substituting these equalities into (2.13) we get

$$
L_R = -\frac{D}{4} = \frac{\widehat{\kappa}}{2} = \widehat{\kappa}_G. \tag{4.29}
$$

We see that $\widehat{M} \times_F \widetilde{N}$ is a Roter space, in particular, a pseudosymmetric manifold of constant type, and admits a non-trivial geodesic mapping onto $\overline{\widehat{M}} \times_{\overline{F}} \widetilde{N}$, so that $\overline{\widehat{M}} \times_{\overline{F}} \widetilde{N}$ is a pseudosymmetric manifold of constant type (see Theorem 3.1). We would like to show that it is also a Roter space. First we compute the components of the tensor $\overline{T}$.

We observe, in view of (3.1), (3.5), (4.1) and (3.6), that

$$
\begin{aligned}
&\overline{\Gamma}^1_{11} = \Gamma^1_{11} + 2\psi_1 = \frac{\mathfrak{a}'}{2\mathfrak{a}} - \frac{q\mathfrak{b}'}{1+q\mathfrak{b}},\\
&\overline{\Gamma}^2_{12} = \Gamma^2_{12} + \psi_1 = \frac{\mathfrak{b}'}{2\mathfrak{b}(1+q\mathfrak{b})}, \qquad \overline{\Gamma}^1_{22} = \Gamma^1_{22} = -\frac{\mathfrak{b}'}{2\mathfrak{a}}.
\end{aligned} \tag{4.30}
$$

Taking into account (4.12) and (4.17) we have $\overline{F}(x^1,x^2) = \frac{p\mathfrak{b}(x^1)B^2(x^2)}{1+q\mathfrak{b}(x^1)}$, so that $\overline{F}_1 = \partial_1\overline{F} = \frac{\mathfrak{b}'}{(1+q\mathfrak{b})^2}pB^2$, $\overline{F}_2 = \partial_2\overline{F} = \frac{p\mathfrak{b}}{1+q\mathfrak{b}}2BB_2$ and

$$
\partial_2\overline{F}_2 = \frac{p\mathfrak{b}}{1+q\mathfrak{b}}2(B_2^2 + BB_{22}) = \frac{p\mathfrak{b}}{1+q\mathfrak{b}}2(B_2^2 + CB^2),
$$

in virtue of (4.13). Next, using (4.30) and (4.15) we obtain

$$
\begin{aligned}
\overline{T}_{12} &= \overline{\nabla}_1\overline{F}_2 - \frac{1}{2\overline{F}}\overline{F}_1\overline{F}_2 = \partial_1\overline{F}_2 - \overline{F}_2\overline{\Gamma}^2_{12} - \frac{1}{2\overline{F}}\overline{F}_1\overline{F}_2 = 0,\\
\overline{T}_{22} &= \overline{\nabla}_2\overline{F}_2 - \frac{1}{2\overline{F}}(\overline{F}_2)^2 = \partial_2\overline{F}_2 - \overline{F}_1\overline{\Gamma}^1_{22} - \frac{1}{2\overline{F}}(\overline{F}_2)^2\\
&= \frac{D+4qC}{2p}\,\frac{p\mathfrak{b}}{1+q\mathfrak{b}}\overline{F} = \frac{D+4qC}{2p}\overline{F}\overline{g}_{22},\\
\overline{T}_{11} &= \overline{\nabla}_1\overline{F}_1 - \frac{1}{2\overline{F}}(\overline{F}_{12})^2 = \partial_1\overline{F}_1 - \overline{F}_1\overline{\Gamma}^1_{11} - \frac{1}{2\overline{F}}(\overline{F}_1)^2\\
&= \frac{pB^2}{(1+q\mathfrak{b})^2}\left(\mathfrak{b}'' - \frac{1+2q\mathfrak{b}}{2\mathfrak{b}(1+q\mathfrak{b})}(\mathfrak{b}')^2 - \frac{\mathfrak{a}'\mathfrak{b}'}{2\mathfrak{a}}\right).
\end{aligned}
$$

But, in virtue of (4.14), $\mathfrak{b}'' - \frac{\mathfrak{a}'\mathfrak{b}'}{2\mathfrak{a}} = \frac{(\mathfrak{b}')^2}{\mathfrak{b}} + 2\mathfrak{a}C$, and

$$\begin{aligned}\overline{T}_{11} &= \frac{pB^2}{(1+q\mathfrak{b})^2}\left(\frac{(\mathfrak{b}')^2}{2\mathfrak{b}(1+q\mathfrak{b})} + 2\mathfrak{a}C\right)\\ &= \frac{p(\mathfrak{b}')^2}{\mathfrak{b}(D\mathfrak{b}-4C)(1+q\mathfrak{b})^2}\,\frac{p\mathfrak{b}B^2}{1+q\mathfrak{b}}\,\frac{D+4qC}{2p} = \overline{g}_{11}\overline{F}\frac{D+4qC}{2p}.\end{aligned}$$

Thus,

$$\overline{T}_{ab} = \frac{D+4qC}{2p}\overline{F}\overline{g}_{ab}. \tag{4.31}$$

From the first equation of (3.3), using (4.1) and (3.5), we find that $\psi_{11} = \partial_1\psi_1 - \psi_1\Gamma^1_{11} - (\psi_1)^2$ and, in virtue of (3.6), (4.14) and (4.15) we get

$$\psi_{11} = \frac{q(\mathfrak{b}')^2(4C - qD\mathfrak{b}^2 - 2D\mathfrak{b})}{4\mathfrak{b}(1+q\mathfrak{b})^2(D\mathfrak{b}-4C)}. \tag{4.32}$$

Similarly, $\psi_{22} = \partial_2\psi_2 - \psi_1\Gamma^1_{12} - (\psi_2)^2 = -\psi_1\Gamma^1_{12}$ and

$$\psi_{22} = -\frac{q\mathfrak{b}(D\mathfrak{b}-4C)}{4(1+q\mathfrak{b})}. \tag{4.33}$$

Using (3.5) we see that $\Gamma^1_{\alpha\beta} = -\frac{1}{2}\widehat{g}^{11}F_1\widetilde{g}_{\alpha\beta} = -\frac{1}{2\mathfrak{a}}F_1\widetilde{g}_{\alpha\beta}$. Substituting this equality into $\psi_{\alpha\beta} = \partial_\beta\psi_\alpha - \psi_1\Gamma^1_{\alpha\beta}$ we obtain

$$\psi_{\alpha\beta} = -\frac{q\mathfrak{b}B^2(D\mathfrak{b}-4C)}{4(1+q\mathfrak{b})}\widetilde{g}_{\alpha\beta}. \tag{4.34}$$

In the same manner we easily get

$$\psi_{12} = 0, \qquad \psi_{a\alpha} = 0. \tag{4.35}$$

Starting with (3.2) we calculate the local components of $\overline{S}$. As $\overline{S}_{11} = S_{11} - (n-1)\psi_{11}$, using (4.22), (4.32) and (4.15) we have $\overline{S}_{11} = -\frac{n-1}{4p}(D + 4qC)\overline{g}_{11}$. Similarly, using (4.22) and (4.33) we obtain $\overline{S}_{22} = -\frac{n-1}{4p}(D+4qC)\overline{g}_{22}$. Now the last two equations and $\overline{S}_{12} = 0$ yield

$$\overline{S}_{ab} = -\frac{n-1}{4p}(D+4qC)\overline{g}_{ab}. \tag{4.36}$$

Taking into account (4.23) and (4.34) we get

$$\overline{S}_{\alpha\beta} = \left(\frac{\widetilde{\kappa}}{n-2} + (n-3)(CB^2 - B_2^2) - \frac{n-1}{4}\mathfrak{b}B^2\frac{D+4qC}{1+q\mathfrak{b}}\right)\frac{1}{\overline{\overline{F}}}\overline{g}_{\alpha\beta}. \tag{4.37}$$

Finally, in view of (4.35) we have

$$\overline{S}_{a\alpha} = 0. \tag{4.38}$$

On the other hand, (4.3) leads to $\overline{S}_{ab} = \overline{\widehat{S}}_{ab} - \frac{n-2}{2\overline{F}}\overline{T}_{ab}$. Thus, substituting into this equality (4.36) and (4.31) we obtain $\overline{\widehat{S}}_{ab} = -\frac{D+4qC}{4p}\overline{g}_{ab}$, which yields

$$\overline{\widehat{\kappa}} = -\frac{D+4qC}{2p}. \tag{4.39}$$

Equalities (4.36), (4.37) and (4.38) imply that $\overline{\widehat{M}} \times_{\overline{F}} \widetilde{N}$ cannot be quasi-Einstein and will be Einstein if and only if

$$-\frac{n-1}{4p}(D+4qC)\,\frac{p\mathfrak{b}B^2}{1+q\mathfrak{b}} = \frac{\widetilde{\kappa}}{n-2}+(n-3)(CB^2-B_2^2)-\frac{n-1}{4p}\mathfrak{b}B^2\frac{D+4qC}{1+q\mathfrak{b}},$$

which reduces to equality (4.21).

According to Remark 4.2, $\overline{\widehat{M}}\times_{\overline{F}}\widetilde{N}$ is conformally-flat if and only if $\overline{\rho}_0 = 0$, that is,

$$\frac{\overline{\widehat{\kappa}}}{2}\overline{F} + \frac{\widetilde{\kappa}}{(n-3)(n-2)} + \frac{\operatorname{tr}\overline{T}}{2} = \frac{\overline{\Delta_1 F}}{4\overline{F}}. \tag{4.40}$$

Substituting (4.39) into (4.31) we have $\overline{T} = -\overline{\widehat{\kappa}}\overline{F}\overline{g}$, which gives $\operatorname{tr}\overline{T} = -2\overline{\widehat{\kappa}}\overline{F}$ and

$$\frac{\overline{\widehat{\kappa}}}{2}\overline{F} + \frac{\operatorname{tr}\overline{T}}{2} = -\frac{\overline{\widehat{\kappa}}}{2}\overline{F} = \frac{\mathfrak{b}B^2(D+4qC)}{4(1+q\mathfrak{b})}.$$

Next, since $\overline{\Delta_1 F} = \frac{1}{\overline{g}_{11}}(\overline{F}_1)^2 + \frac{1}{\overline{g}_{22}}(\overline{F}_2)^2$, using (4.15) we easily conclude that

$$\frac{\overline{\Delta_1 F}}{4\overline{F}} = B_2^2 + \frac{D\mathfrak{b}-4C}{1+q\mathfrak{b}}B^2.$$

Thus, equality (4.40) takes the form

$$\frac{\widetilde{\kappa}}{(n-3)(n-2)} = -\frac{\mathfrak{b}B^2(D+4qC)}{4(1+q\mathfrak{b})} + B_2^2 + \frac{(D\mathfrak{b}-4C)B^2}{1+q\mathfrak{b}} = B_2^2 - CB^2,$$

i.e., (4.21). Hence, if equality (4.21) does not hold then $U_{\overline{S}} \cap U_{\overline{C}} = \overline{\widehat{M}} \times \widetilde{N}$.

Applying Remark 4.3 to the warped product $\overline{\widehat{M}} \times_{\overline{F}} \widetilde{N}$ and using earlier results we get $\overline{\rho}_1 = \overline{\widehat{\kappa}}/2 = \overline{\rho}_2$, and

$$\overline{\mu}_1 = \frac{n-1}{2}\overline{\widehat{\kappa}}, \quad \overline{\mu}_2 = \frac{1}{\overline{\overline{F}}}\frac{\widetilde{\kappa}}{n-2} + \frac{1}{\overline{\overline{F}}}(n-3)(CB^2-B_2^2) + \frac{n-1}{2}\overline{\widehat{\kappa}},$$

so that

$$\overline{\mu}_2 - \overline{\mu}_1 = \frac{1}{\overline{\overline{F}}}\left(\frac{\widetilde{\kappa}}{n-2} + (n-3)(CB^2-B_2^2)\right),$$

$$\overline{\rho}_3 = \frac{1}{\overline{\overline{F}}}\left(\frac{\widetilde{\kappa}}{(n-2)(n-3)} + \overline{F}\frac{\overline{\widehat{\kappa}}}{2} + CB^2 - B_2^2\right) = \frac{1}{n-3}(\overline{\mu}_2 - \overline{\mu}_1) + \frac{\overline{\widehat{\kappa}}}{2}.$$

According to [30, Theorem 4.1], $\overline{\widehat{M}} \times_{\overline{F}} \widetilde{N}$ is a Roter space, i.e.,

$$\overline{R} = \frac{\overline{\phi}}{2}\overline{S}\wedge\overline{S} + \overline{\mu g}\wedge\overline{S} + \frac{\overline{\eta}}{2}\overline{g}\wedge\overline{g},$$

with $\overline{\phi}=\overline{\nu}(\overline{\rho}_1-2\overline{\rho}_2+\overline{\rho}_3)$, $\overline{\mu}=\overline{\nu}((\overline{\rho}_2-\overline{\rho}_3)\overline{\mu}_1+(\overline{\rho}_2-\overline{\rho}_1)\overline{\mu}_2)$, $\overline{\eta}=\overline{\nu}(\overline{\rho}_1\overline{\mu}_2^2-2\overline{\rho}_2\overline{\mu}_1\overline{\mu}_2+\overline{\rho}_3\overline{\mu}_1^2)$, where $\overline{\nu}=(\overline{\mu}_2-\overline{\mu}_1)^{-2}$. Using the above expressions for $\overline{\rho}_1, \overline{\rho}_2, \overline{\rho}_3$ we obtain

$$\overline{\phi}=\frac{1}{(n-3)(\overline{\mu}_2-\overline{\mu}_1)},\quad \overline{\mu}=-\frac{\overline{\mu}_1}{(n-3)(\overline{\mu}_2-\overline{\mu}_1)},\quad \overline{\eta}=\overline{\rho}_1+\frac{\overline{\mu}_1^2}{(n-3)(\overline{\mu}_2-\overline{\mu}_1)}.$$

Substituting these equalities into (2.13) we get

$$L_{\overline{R}}=-\frac{D+4qC}{4p}=\frac{\overline{\widehat{\kappa}}}{2}=\overline{\widehat{\kappa}}_G. \tag{4.41}$$

Thus we have proved the following.

THEOREM 4.1. *Let $(\widehat{M},\widehat{g})$ be a 2-dimensional manifold with metric $\widehat{g}$ given by $\widehat{g}_{11}=\mathfrak{a}(x^1)$, $\widehat{g}_{22}=\mathfrak{b}(x^1)$, $\widehat{g}_{12}=0$, where $\mathfrak{a}=\frac{(\mathfrak{b}')^2}{\mathfrak{b}(D\mathfrak{b}-4C)}$, $C,D\in\mathbb{R}$. Next let $(\widetilde{N},\widetilde{g})$ be an $(n-2)$-dimensional, $n\geq 4$, semi-Riemannian space, assumed to be of constant curvature when $n\geq 5$, and let $\widehat{M}\times_F\widetilde{N}$ be the warped product manifold with $F=F(x^1,x^2)=\mathfrak{b}(x^1)B^2(x^2)$, where B is a function described in Remark* 4.1 *such that equality* (4.21) *does not hold. Then $\widehat{M}\times_F\widetilde{N}$ is a Roter space which admits a non-trivial geodesic mapping onto $\overline{\widehat{M}}\times_{\overline{F}}\widetilde{N}$, where $(\overline{\widehat{M}},\overline{\widehat{g}})$ is a manifold geodesically related to $(\widehat{M},\widehat{g})$ with metric given by $\overline{\widehat{g}}_{11}=\frac{p\mathfrak{a}}{(1+q\mathfrak{b})^2}$, $\overline{\widehat{g}}_{22}=\frac{p\mathfrak{b}}{1+q\mathfrak{b}}$, $\overline{\widehat{g}}_{12}=0$, $p,q\in\mathbb{R}$, and with $\overline{F}=\frac{p}{1+q\mathfrak{b}}F$. Moreover, $\overline{\widehat{M}}\times_{\overline{F}}\widetilde{N}$ is also a Roter space.*

PROPOSITION 4.2. *Under the above assumptions we have $L_R=-D/4$ and $L_{\overline{R}}=-\frac{D+4qC}{4p}$, so that both manifolds (M,g) and $(\overline{M},\overline{g})$ are pseudosymmetric of constant type. Furthermore,*

$$L_R-\frac{\kappa}{n(n-1)}=\frac{p}{1+q\mathfrak{b}}\left(L_{\overline{R}}-\frac{\overline{\kappa}}{n(n-1)}\right). \tag{4.42}$$

Proof. Equalities (4.29) and (4.41) give the first part of the assertion. Using (4.24), (4.25) and (4.26) we obtain

$$\begin{aligned}\kappa&=g^{ab}S_{ab}+g^{\alpha\beta}S_{\alpha\beta}=g^{ab}g_{ab}\mu_1+g^{\alpha\beta}g_{\alpha\beta}\mu_2=2\mu_1+(n-2)\mu_2\\&=-\frac{2(n-1)D}{4}+\frac{n-2}{F}\left(\frac{\widetilde{\kappa}}{n-2}-\frac{n-1}{4}DF-(n-3)(B_2^2-CB^2)\right)\\&=-\frac{n(n-1)}{4}D+\frac{\widetilde{\kappa}}{F}-\frac{(n-3)(n-2)}{F}(B_2^2-CB^2)\end{aligned}$$

and in virtue of (4.29) we have

$$\kappa=n(n-1)L_R+\frac{\widetilde{\kappa}}{F}-\frac{(n-3)(n-2)}{F}(B_2^2-CB^2).$$

Similarly, using analogous equations for $(\overline{M},\overline{g})$ we get

$$\overline{\kappa}=n(n-1)L_{\overline{R}}+\frac{\widetilde{\kappa}}{\overline{F}}-\frac{(n-3)(n-2)}{\overline{F}}(B_2^2-CB^2).$$

Comparing the last two relations we see that

$$L_R - \frac{\kappa}{n(n-1)} = \frac{\overline{F}}{F}\Big(L_{\overline{R}} - \frac{\overline{\kappa}}{n(n-1)}\Big),$$

and taking into account (4.12) we obtain (4.42). ∎

Roter spaces satisfy various pseudosymmetry type curvature conditions. We will now derive formulas for the functions L_C and L in (2.15) and (2.14). Taking into account (2.12) and (2.15) we get

$$L_C = L_R - \frac{\kappa}{n-1} + \frac{1-(n-2)\mu}{(n-2)\phi}.$$

Next, using (4.26)–(4.28), after standard calculations we obtain

$$\frac{(n-2)^2}{n} L_C = L_R - \frac{\kappa}{n(n-1)}. \tag{4.43}$$

Similarly (for $\overline{g}$) we have $\frac{(n-2)^2}{n} L_{\overline{C}} = L_{\overline{R}} - \frac{\overline{\kappa}}{n(n-1)}$. Thus, in virtue of (4.42) we get $L_C = \frac{\overline{F}}{F} L_{\overline{C}} = \frac{p}{1+q\mathfrak{b}} L_{\overline{C}}$. Taking into account (2.14), (4.28) and (4.26) we easily obtain $L = -(n-2)L_R$, and similarly $\overline{L} = -(n-2)L_{\overline{R}}$. We have thus established the following.

Corollary 4.2. *For manifolds (M, g) and $(\overline{M}, \overline{g})$ satisfying the assumptions of Theorem* 4.1 *we have:*

(i) $C \cdot C = L_C Q(g, C)$, $\overline{C} \cdot \overline{C} = L_{\overline{C}} Q(\overline{g}, \overline{C})$, *where L_C is given by* (4.43) *and $L_C = (p/(1+q\mathfrak{b}))L_{\overline{C}}$.*
(ii) $R \cdot R = Q(S, R) - (n-2)L_R Q(g, C)$ *and* $\overline{R} \cdot \overline{R} = Q(\overline{S}, \overline{R}) - (n-2)L_{\overline{R}} Q(\overline{g}, \overline{C})$.

As mentioned at the end of Section 1, we continue investigations of geodesic mappings in Roter spaces and, for example, we obtained

Remark 4.4 ([27]). Let (M, g) be a pseudosymmetric non-semisymmetric semi-Riemannian manifold admitting a non-trivial geodesic mapping onto a Roter space $(\overline{M}, \overline{g})$. Then

$$(\overline{\kappa}\overline{\phi} + n\overline{\mu})B - ((\operatorname{tr} B)\overline{\phi} + (\operatorname{tr}\psi)\overline{\mu})\overline{S} + (\overline{\kappa}\overline{\mu} - n(L_{\overline{R}} - \overline{\eta}))\psi + ((\operatorname{tr}\psi)(L_{\overline{R}} - \overline{\eta}) - (\operatorname{tr} B)\overline{\mu})\overline{g} = 0,$$

where ψ and B are $(0,2)$-tensors with components given by (3.3) and $B_{mk} = \psi_{mr}\overline{S}^{r}_{k}$, respectively. Moreover, we found some conditions sufficient for the manifold (M, g) to be also a Roter space.

Acknowledgements. The authors gratefully thank the referee for a careful reading of the manuscript and many valuable comments and suggestions of improvements. The first named author is supported by a grant from the Wrocław University of Environmental and Life Sciences, Poland.

REFERENCES

[1] A. L. Besse, *Einstein Manifolds*, Ergeb. Math. Grenzgeb. (3) 10, Springer, Berlin, 1987.

[2] E. Boeckx, O. Kowalski and L. Vanhecke, *Riemannian Manifolds of Conullity Two*, World Sci., River Edge, NJ, 1996.

[3] T. E. Cecil and P. J. Ryan, *Geometry of Hypersurfaces*, Springer Monogr. Math., Springer, New York, 2015.

[4] B.-Y. Chen, *Differential Geometry of Warped Product Manifolds and Submanifolds*, World Sci., Hackensack, NJ, 2017.

[5] B.-Y. Chen, *Classification of torqued vector fields and its applications to Ricci solitons*, Kragujevac J. Math. 41 (2017), 239–250.

[6] J. Chojnacka-Dulas, R. Deszcz, M. Głogowska and M. Prvanović, *On warped product manifolds satisfying some curvature conditions*, J. Geom. Phys. 74 (2013), 328–341.

[7] S. Decu, M. Petrović-Torgašev, A. Šebeković and L. Verstraelen, *On the Roter type of Wintgen ideal submanifolds*, Rev. Roumaine Math. Pures Appl. 57 (2012), 75–90.

[8] F. Defever and R. Deszcz, *A note on geodesic mappings of pseudosymmetric Riemannian manifolds*, Colloq. Math. 62 (1991), 313–319.

[9] F. Defever and R. Deszcz, *Some results on geodesic mappings of Riemannian manifolds satisfying the condition* $R \cdot R = Q(S, R)$, Period. Math. Hungar. 29 (1994), 267–276.

[10] F. Defever, R. Deszcz and M. Prvanović, *On warped product manifolds satisfying some curvature condition of pseudosymmetric type*, Bull. Greek Math. Soc. 36 (1994), 43–62.

[11] A. Derdziński and W. Roter, *Some properties of conformally symmetric manifolds which are not Ricci-recurrent*, Tensor (N.S.) 34 (1980), 11–20.

[12] R. Deszcz, *On some Akivis–Goldberg type metrics*, Publ. Inst. Math. (Beograd) (N.S.) 74 (88) (2003), 71–83.

[13] R. Deszcz, M. Głogowska, H. Hashiguchi, M. Hotloś and M. Yawata, *On semi-Riemannian manifolds satisfying some conformally invariant curvature condition*, Colloq. Math. 131 (2013), 149–170.

[14] R. Deszcz, M. Głogowska, M. Hotloś, J. Jełowicki and G. Zafindratafa, *Curvature properties of some warped product manifolds*, poster, Conf. "Differential Geometry", Banach Conf. Center at Będlewo, June 19 to 24, 2017.

[15] R. Deszcz, M. Głogowska, M. Hotloś and K. Sawicz, *A survey on generalized Einstein metric conditions*, in: Advances in Lorentzian Geometry, M. Plaue et al. (eds.), AMS/IP Stud. Adv. Math. 49, Amer. Math. Soc., Providence, RI, 2011, 27–45.

[16] R. Deszcz, M. Głogowska, M. Hotloś and G. Zafindratafa, *On some curvature conditions of pseudosymmetry type*, Period. Math. Hungar. 70 (2015), 153–170.

[17] R. Deszcz, M. Głogowska, M. Hotloś and G. Zafindratafa, *Hypersurfaces in space forms satisfying some curvature conditions*, J. Geom. Phys. 99 (2016), 218–231.

[18] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, *Curvature properties of some class of warped product manifolds*, Int. J. Geom. Methods Modern Phys. 13 (2016), no. 1, art. 1550135, 36 pp.

[19] R. Deszcz, M. Głogowska, M. Petrović-Torgašev and L. Verstraelen, *On the Roter type of Chen ideal submanifolds*, Results Math. 59 (2011), 401–413.

[20] R. Deszcz, M. Głogowska, M. Petrović-Torgašev and L. Verstraelen, *Curvature properties of some class of minimal hypersurfaces in Euclidean spaces*, Filomat 29 (2015), 479–492.

[21] R. Deszcz, M. Głogowska, M. Plaue, K. Sawicz and M. Scherfner, *On hypersurfaces in space forms satisfying particular curvature conditions of Tachibana type*, Kragujevac J. Math. 35 (2011), 223–247.
[22] R. Deszcz, S. Haesen and L. Verstraelen, *On natural symmetries*, Ch. 6 of: Topics in Differential Geometry, A. Mihai et al. (eds.), Editura Academiei Române, Bucureşti, 2008, 249–308.
[23] R. Deszcz and M. Hotloś, *On geodesic mappings in pseudo-symmetric manifolds*, Bull. Inst. Math. Acad. Sinica 16 (1988), 251–262.
[24] R. Deszcz and M. Hotloś, *Remarks on Riemannian manifolds satisfying certain curvature condition imposed on the Ricci tensor*, Prace Nauk. Politechn. Szczec. Inst. Mat. 11 (1988), 23–34.
[25] R. Deszcz and M. Hotloś, *Notes on pseudo-symmetric manifolds admitting special geodesic mappings*, Soochow J. Math. 15 (1989), 19–27.
[26] R. Deszcz and M. Hotloś, *On a certain subclass of pseudosymmetric manifolds*, Publ. Math. Debrecen 53 (1998), 29–48.
[27] R. Deszcz and M. Hotloś, *On geodesically related Roter spaces*, to appear.
[28] R. Deszcz, M. Hotloś, J. Jełowicki, H. Kundu and A. A. Shaikh, *Curvature properties of Gödel metric*, Int. J. Geom. Methods Modern Phys. 11 (2014), no. 3, art. 1450025, 20 pp.
[29] R. Deszcz, M. Hotloś and Z. Şentürk, *On curvature properties of certain quasi-Einstein hypersurfaces*, Int. J. Math. 23 (2012), no. 7, art. 1250073, 17 pp.
[30] R. Deszcz and D. Kowalczyk, *On some class of pseudosymmetric warped products*, Colloq. Math. 97 (2003), 7–22.
[31] R. Deszcz, M. Plaue and M. Scherfner, *On Roter type warped products with 1-dimensional fibres*, J. Geom. Phys. 69 (2013), 1–11.
[32] R. Deszcz and M. Scherfner, *On a particular class of warped products with fibres locally isometric to generalized Cartan hypersurfaces*, Colloq. Math. 109 (2007), 13–29.
[33] R. Deszcz, L. Verstraelen and L. Vrancken, *The symmetry of warped product space-times*, Gen. Relativity Gravitation 23 (1991), 671–681.
[34] R. Deszcz and Ş. Yaprak, *Curvature properties of certain pseudosymmetric manifolds*, Publ. Math. Debrecen 45 (1994), 333–345.
[35] M. Głogowska, *Curvature conditions on hypersurfaces with two distinct principal curvatures*, in: Banach Center Publ. 69, Inst. Math., Polish Acad. Sci., Warszawa, 2005, 133–143.
[36] M. Głogowska, *On Roter type manifolds*, in: Pure and Applied Differential Geometry—PADGE 2007, Shaker Verlag, Aachen, 2007, 114–122.
[37] M. Głogowska, *On quasi-Einstein Cartan type hypersurfaces*, J. Geom. Phys. 58 (2008), 599–614.
[38] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity*, Cambridge Univ. Press, Cambridge, 2009.
[39] S. Haesen and L. Verstraelen, *Properties of a scalar curvature invariant depending on two planes*, Manuscripta Math. 122 (2007), 59–72.
[40] S. Haesen and L. Verstraelen, *Natural intrinsic geometrical symmetries*, SIGMA 5 (2009), art. 086, 15 pp.
[41] I. Hinterleitner, *Selected special vector fields and mappings in Riemannian geometry*, PhD thesis, Brno, 2009.
[42] I. Hinterleitner, *Special mappings of equidistant spaces*, J. Appl. Math. Aplimat 1 (2008), no. 2, 31–37.
[43] D. Kowalczyk, *On some class of semisymmetric manifolds*, Soochow J. Math. 27 (2001), 445–461.

[44] D. Kowalczyk, *On the Reissner–Nordström–de Sitter type spacetimes*, Tsukuba J. Math. 30 (2006), 363–381.

[45] Ü. Lumiste, *Semiparallel Submanifolds in Space Forms*, Springer, New York, 2009.

[46] J. Mikeš, *Geodesic mappings of special Riemannian spaces*, in: Topics in Differential Geometry (Hajduszoboszló, 1984), Colloq. Math. Soc. János Bolyai 46, Vol. II, North-Holland, Amsterdam, 1988, 793–813.

[47] J. Mikeš, *Differential Geometry of Special Mappings*, Palacký Univ. Press, Olomouc, 2015.

[48] C. Murathan, K. Arslan, R. Deszcz, R. Ezentaş and C. Özgür, *On some class of hypersurfaces of semi-Euclidean spaces*, Publ. Math. Debrecen 58 (2001), 587–604.

[49] B. O'Neill, *Semi-Riemannian Geometry. With Applications to Relativity*, Academic Press, New York, 1983.

[50] W. Roter, *A note on second order recurrent spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 12 (1964), 621–626.

[51] N. S. Sinyukov, *Geodesic Mappings of Riemannian Spaces*, Nauka, Moscow, 1979 (in Russian).

[52] Z. Stuchlík and S. Hledík, *Properties of the Reissner–Nordström spacetimes with a nonzero cosmological constant*, Acta Phys. Slovaca 52 (2002), 363–407.

[53] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying $R(X,Y) \cdot R = 0$. I. The local version*, J. Differential Geom. 17 (1982), 531–582.

[54] P. Venzi, *On geodesic mappings in Riemannian or pseudo-Riemannian manifolds*, Tensor (N.S.) 32 (1978), 193–198.

[55] L. Verstraelen, *Foreword*, in: [4], vii–xxi.

Ryszard Deszcz
Department of Applied Mathematics
Wrocław University of
Environmental and Life Sciences
Grunwaldzka 53
50-357 Wrocław, Poland
E-mail: Ryszard.Deszcz@upwr.edu.pl

Marian Hotloś
Department of Applied Mathematics
Wrocław University of
Science and Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: Marian.Hotlos@pwr.edu.pl