

CHARACTERIZATIONS OF MANIFOLDS MODELED ON ABSORBING SETS IN NON-SEPARABLE HILBERT SPACES AND THE DISCRETE CELLS PROPERTY

BY

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Abstract. We characterize infinite-dimensional manifolds modeled on absorbing sets in non-separable Hilbert spaces by using the discrete cells property, which is a general position property. Moreover, we study the discrete (locally finite) approximation property, which is an extension of the discrete cells property.

1. Introduction. Throughout this paper, spaces are metrizable, maps are continuous, and κ is an infinite cardinal. Let $\ell_2(\kappa)$ be the Hilbert space of density κ , and $\ell_2^f(\kappa)$ be the linear subspace spanned by the canonical orthonormal basis of $\ell_2(\kappa)$, that is,

$$\ell_2^f(\kappa) = \{(x(\gamma))_{\gamma < \kappa} \in \ell_2(\kappa) \mid x(\gamma) = 0 \text{ except for finitely many } \gamma < \kappa\}.$$

We denote $\ell_2 = \ell_2(\aleph_0)$ and $\ell_2^f = \ell_2^f(\aleph_0)$ for simplicity. Given a space E , we call a space X an E -manifold if each point $x \in X$ has an open neighborhood homeomorphic to some open subset of E . For a class \mathfrak{C} of spaces and a cardinal λ , write

- $\mathfrak{C}_\sigma = \{\bigcup_{n \in \omega} A_n \mid A_n \in \mathfrak{C} \text{ and } A_n \text{ is closed}\},$
- $\bigoplus_\lambda \mathfrak{C} = \{\bigoplus_{\gamma < \lambda} A_\gamma \mid A_\gamma \in \mathfrak{C}\},$
- $\mathfrak{C}(\lambda) = \{A \in \mathfrak{C} \mid A \text{ is of density } \leq \lambda\}.$

A closed set A in a space X is called a Z -set in X if for each open cover \mathcal{U} of X , there exists a map $f : X \rightarrow X$ such that f is \mathcal{U} -close to the identity map of X and $f(X) \cap A = \emptyset$. When moreover the closure $\text{cl } f(X)$ misses A , we say that A a *strong* Z -set in X . Recall that for maps $f : X \rightarrow Y$ and $g : X \rightarrow Y$, and for an open cover \mathcal{U} of Y , f is \mathcal{U} -close to g if for any point $x \in X$, there is a member $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subset U$. A [strong] Z_σ -set is a countable union of [strong] Z -sets. A map $f : X \rightarrow Y$ is a Z -embedding if it is an embedding and its image is a Z -set in Y . Given a

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class \mathfrak{C} , we say that a space X is *strongly \mathfrak{C} -universal* or *strongly universal for \mathfrak{C}* if the following condition is satisfied:

- Let $f : A \rightarrow X$ be a map from a space in \mathfrak{C} . Suppose that B is a closed subset of A and the restriction $f|_B$ is a Z -embedding. Then for each open cover \mathcal{U} of X , there exists a Z -embedding $g : A \rightarrow X$ such that g is \mathcal{U} -close to f and $g|_B = f|_B$.

For spaces $X \subset M$, X is said to be a *\mathfrak{C} -absorbing set* or an *absorbing set for \mathfrak{C}* in M (a notion which plays an important role in infinite-dimensional topology) if the following conditions are satisfied:

- (1) X is homotopy dense in M and $X \in \mathfrak{C}_\sigma$.
- (2) X is strongly \mathfrak{C} -universal.
- (3) X is a strong Z_σ -set in itself.

Here X is *homotopy dense* in M if there exists a homotopy $h : M \times \mathbf{I} \rightarrow M$ such that $h(M \times (0, 1]) \subset X$ and $h(x, 0) = x$ for each $x \in M$. For a cardinal λ , a space X has the *λ -discrete [locally finite] approximation property for \mathfrak{C}* if the following condition holds:

- Let $f : \bigoplus_{\gamma < \lambda} A_\gamma \rightarrow X$ be a map from a topological sum of members of \mathfrak{C} . For every open cover \mathcal{U} of X , there is a map $g : \bigoplus_{\gamma < \lambda} A_\gamma \rightarrow X$ such that g is \mathcal{U} -close to f and the family $\{g(A_\gamma) \mid \gamma < \lambda\}$ is discrete [locally finite] in X .

In particular, when $\mathfrak{C} = \{\mathbf{I}^n\}$, $n \in \omega$, we say that X has the *λ -discrete [locally finite] n -cells property*. These properties are general position properties based on “dimension” in the finite-dimensional case or “density” in the infinite-dimensional case. They are very useful for recognizing manifolds. For example, H. Toruńczyk [14, 15] characterized $\ell_2(\kappa)$ -manifolds by using the κ -discrete n -cells property. We will study these properties in Section 4.

We say that a class \mathfrak{C} is *topological* if every space homeomorphic to some member of \mathfrak{C} also belongs to \mathfrak{C} , and \mathfrak{C} is *closed hereditary* if any closed subspace of some member of \mathfrak{C} also belongs to \mathfrak{C} . In Section 5, using the discrete cells property, we shall characterize infinite-dimensional manifolds modeled on absorbing sets in Hilbert spaces as follows, which generalizes and improves the previous results in [4, 13, 8]:

THEOREM 1.1. *Let \mathfrak{C} be a topological and closed hereditary class, and Ω be a \mathfrak{C} -absorbing set in $\ell_2(\kappa)$. For a connected space $X \in \mathfrak{C}_\sigma$ of density $\leq \kappa$, the following are equivalent:*

- (1) X is an Ω -manifold.
- (2) X can be embedded into some $\ell_2(\kappa)$ -manifold as a \mathfrak{C} -absorbing set.
- (3) X satisfies the following conditions:
 - (a) X is an ANR;

- (b) X is strongly \mathfrak{C} -universal;
- (c) X has the κ -discrete n -cells property for every $n \in \omega$;
- (d) X is a strong Z_σ -set in itself.

A class \mathfrak{C} is **I-stable** if $A \times \mathbf{I} \in \mathfrak{C}$ for each $A \in \mathfrak{C}$. As is easily observed, the class $\mathfrak{M}_0^{\text{fd}}$ of finite-dimensional compact spaces is topological, closed hereditary and **I-stable**. The space $\ell_2^{\text{f}}(\kappa)$ is a $\bigoplus_\kappa \mathfrak{M}_0^{\text{fd}}$ -absorbing set in $\ell_2(\kappa)$. In Section 6, we will prove the following:

THEOREM 1.2. *Let $\aleph_0 \leq \kappa' < \kappa$ and \mathfrak{C} be a topological, closed hereditary and **I-stable** class. If Ω is a $\mathfrak{C}(\kappa')$ -absorbing set in $\ell_2(\kappa')$, then $\ell_2^{\text{f}}(\kappa) \times \Omega$ is a $\bigoplus_\kappa \mathfrak{C}(\kappa')$ -absorbing set in $\ell_2(\kappa) \times \ell_2(\kappa')$.*

Making use of the discrete cells property, we shall establish in Section 7 the following characterization:

THEOREM 1.3. *Let $\aleph_0 \leq \kappa' < \kappa$. Suppose that \mathfrak{C} is a topological, closed hereditary and **I-stable** class, and that Ω is a $\mathfrak{C}(\kappa')$ -absorbing set in $\ell_2(\kappa')$. A connected space $X \in (\bigoplus_\kappa \mathfrak{C}(\kappa'))_\sigma$ is an $(\ell_2^{\text{f}}(\kappa) \times \Omega)$ -manifold if and only if the following conditions are satisfied:*

- (1) X is an ANR;
- (2) X is strongly $\mathfrak{C}(\kappa')$ -universal;
- (3) X has the κ -discrete n -cells property for any $n \in \omega$;
- (4) X is a strong Z_σ -set in itself.

2. Characterizations of infinite-dimensional manifolds. In this section, some characterizations of infinite-dimensional manifolds related to Theorem 1.3 and its applications are introduced. A space X is said to be σ -locally compact if it is a countable union of locally compact subspaces. Note that any σ -locally compact space can be expressed as a countable union of closed subspaces that are discrete unions of compact sets [6, Proposition 5.1 and Remark 3]. In [13] (cf. [9]), K. Sakai and M. Yaguchi characterized $\ell_2^{\text{f}}(\kappa)$ -manifolds as follows:

THEOREM 2.1. *A connected space X is an $\ell_2^{\text{f}}(\kappa)$ -manifold if and only if the following conditions hold:*

- (1) X is a strongly countable-dimensional, σ -locally compact ANR of density $\leq \kappa$;
- (2) X is strongly universal for the class of strongly countable-dimensional, locally compact spaces of density $\leq \kappa$;
- (3) X is a strong Z_σ -set in itself.

The theory of infinite-dimensional manifolds is applied to studies of function spaces and hyperspaces because they are often homeomorphic to typical infinite-dimensional spaces. For example, some function spaces consisting of

PL maps and some hyperspaces consisting of finite subsets are homeomorphic to $\ell_2^f(\kappa)$ [10, 5, 7]. When we detect $\ell_2^f(\kappa)$ -manifolds by using the above characterization, it is difficult to verify strong universality. As is easily observed, the strong universality for the class of strongly countable-dimensional, locally compact spaces of density $\leq \kappa$ implies the one for $\mathfrak{M}_0^{\text{fd}}$ and the κ -discrete n -cells property for every $n \in \omega$. By making use of the discrete cells property, K. Sakai and M. Yaguchi's result was improved in [6] as follows:

THEOREM 2.2. *A connected space X is an $\ell_2^f(\kappa)$ -manifold if and only if the following conditions are satisfied:*

- (1) *X is a strongly countable-dimensional, σ -locally compact ANR of density $\leq \kappa$;*
- (2) *X is strongly $\mathfrak{M}_0^{\text{fd}}$ -universal;*
- (3) *X has the κ -discrete n -cells property for each $n \in \omega$;*
- (4) *every finite-dimensional compact subset of X is a strong Z -set ⁽¹⁾.*

Let α be a countable ordinal. Given a space X , let $\mathfrak{A}_0(X)$ be the collection of open sets in X , and let $\mathfrak{M}_0(X)$ be the family of closed sets in X . By transfinite induction, for $\alpha \geq 1$, we define the collections $\mathfrak{A}_\alpha(X)$ and $\mathfrak{M}_\alpha(X)$ as follows:

$$\mathfrak{A}_\alpha(X) = \left\{ \bigcup_{n \in \omega} A_n \mid A_n \in \bigcup_{\gamma < \alpha} \mathfrak{M}_\gamma(X) \right\},$$

$$\mathfrak{M}_\alpha(X) = \left\{ \bigcap_{n \in \omega} A_n \mid A_n \in \bigcup_{\gamma < \alpha} \mathfrak{A}_\gamma(X) \right\}.$$

Define \mathfrak{A}_α as a class such that $X \in \mathfrak{A}_\alpha$ if $X \in \mathfrak{A}_\alpha(Y)$ for any space Y containing X as a subspace, and define \mathfrak{M}_α similarly. We call \mathfrak{A}_α and \mathfrak{M}_α the *absolute Borel classes*. The absolute Borel classes are topological, closed hereditary and \mathbf{I} -stable. Note that $\mathfrak{A}_0 = \{\emptyset\}$. Moreover, \mathfrak{M}_0 is the class of compact spaces, \mathfrak{M}_1 is the one of completely metrizable spaces, and \mathfrak{A}_1 is the class of σ -locally compact spaces. It is known that for $\alpha \geq 1$, there exist absorbing sets for the absolute Borel classes $\mathfrak{A}_\alpha(\kappa)$ and $\mathfrak{M}_\alpha(\kappa)$ in the Hilbert space $\ell_2(\kappa)$ [4, 13, 8]. The spaces $\ell_2(\kappa) \times \ell_2^f$ and $\ell_2^f(\kappa) \times \mathbf{I}^\omega$, where \mathbf{I}^ω is the Hilbert cube, can be regarded as absorbing sets for $\mathfrak{M}_1(\kappa)$ and $\mathfrak{A}_1(\kappa)$ in $\ell_2(\kappa)$ respectively.

A space X is *σ -completely metrizable* if it can be written as a countable union of completely metrizable, closed subspaces. We say that X is *locally of density $\leq \kappa$* if every $x \in X$ has a neighborhood of density $\leq \kappa$. A space is *σ -locally of density $\leq \kappa$* if it can be expressed as a countable union of closed subspaces that are locally of density $\leq \kappa$. In particular, when $\kappa = \aleph_0$, the

⁽¹⁾ Conditions (4) of Theorems 2.2 and 2.5 can each be replaced with X being a strong Z_σ -set in itself.

space is called σ -locally separable. Infinite-dimensional manifolds modeled on the product spaces $\ell_2(\kappa) \times \ell_2^f$, $\ell_2^f(\kappa) \times \ell_2$ and $\ell_2^f(\kappa) \times \mathbf{I}^\omega$ are characterized as follows [9, 13, 6, 12] ⁽²⁾:

THEOREM 2.3. *A connected space X is an $(\ell_2(\kappa) \times \ell_2^f)$ -manifold if and only if the following conditions are satisfied:*

- (1) X is a σ -completely metrizable ANR of density $\leq \kappa$;
- (2) X is strongly $\mathfrak{M}_1(\kappa)$ -universal;
- (3) X has the κ -discrete n -cells property for any $n \in \omega$;
- (4) X is a strong Z_σ -set in itself.

THEOREM 2.4. *A connected space X is an $(\ell_2^f(\kappa) \times \ell_2)$ -manifold if and only if the following conditions are satisfied:*

- (1) X is a σ -locally separable, σ -completely metrizable ANR of density $\leq \kappa$;
- (2) X is strongly $\mathfrak{M}_1(\aleph_0)$ -universal;
- (3) X has the κ -discrete n -cells property for any $n \in \omega$;
- (4) X is a strong Z_σ -set in itself.

THEOREM 2.5. *A connected space X is an $(\ell_2^f(\kappa) \times \mathbf{I}^\omega)$ -manifold if and only if the following conditions are satisfied:*

- (1) X is a σ -locally compact ANR of density $\leq \kappa$;
- (2) X is strongly \mathfrak{M}_0 -universal;
- (3) X has the κ -discrete n -cells property for any $n \in \omega$;
- (4) each compact subset of X is a strong Z -set ⁽¹⁾.

REMARK 2.6. For $\kappa > \aleph_0$, Theorems 2.4 and 2.5 can be obtained as corollaries of Theorem 1.3. Indeed, $\ell_2^f(\kappa) \times \ell_2$ is homeomorphic to $\ell_2^f(\kappa) \times \ell_2^f \times \ell_2$, and $\ell_2^f \times \ell_2$ is homeomorphic to an $\mathfrak{M}_1(\aleph_0)$ -absorbing set in ℓ_2 . Moreover, $\ell_2^f(\kappa) \times \mathbf{I}^\omega$ is homeomorphic to $\ell_2^f(\kappa) \times \ell_2^f \times \mathbf{I}^\omega$, and $\ell_2^f \times \mathbf{I}^\omega$ is homeomorphic to an \mathfrak{M}_0 -absorbing set in ℓ_2 .

For simplicity, let \mathfrak{B}_α denote one of the absolute Borel classes \mathfrak{A}_α or \mathfrak{M}_α , and let $\Omega_\alpha(\kappa)$ be a $\mathfrak{B}_\alpha(\kappa)$ -absorbing set in $\ell_2(\kappa)$. For $\alpha \geq 2$, M. Bestvina, J. Mogilski, and K. Mine [4, 8] gave characterizations for manifolds modeled on $\Omega_\alpha(\kappa)$:

THEOREM 2.7. *For a countable ordinal $\alpha \geq 2$, a connected space X is an $\Omega_\alpha(\kappa)$ -manifold if and only if the following conditions hold:*

- (1) X is an ANR and $X \in \mathfrak{B}_\alpha(\kappa)$;
- (2) X is strongly $\mathfrak{B}_\alpha(\kappa)$ -universal;
- (3) X is a strong Z_σ -set in itself.

As a corollary of Theorem 1.3, we have the following:

⁽²⁾ In [12], their characterizations by the discrete cells property are given.

COROLLARY 2.8. *Let $\aleph_0 \leq \kappa' < \kappa$ and $\alpha \geq 2$ be a countable ordinal. A connected space X is an $(\ell_2^f(\kappa) \times \Omega_\alpha(\kappa'))$ -manifold if and only if the following conditions are satisfied:*

- (1) X is an ANR σ -locally of density $\leq \kappa'$ and $X \in \mathfrak{B}_\alpha(\kappa)$;
- (2) X is strongly $\mathfrak{B}_\alpha(\kappa')$ -universal;
- (3) X has the κ -discrete n -cells property for every $n \in \omega$;
- (4) X is a strong Z_σ -set in itself.

REMARK 2.9. Consider the case $\kappa = \kappa' \geq \aleph_0$. Suppose that \mathfrak{C} is a topological, closed hereditary and \mathbf{I} -stable class, and Ω is a $\mathfrak{C}(\kappa)$ -absorbing set in $\ell_2(\kappa)$. If $\mathfrak{C}(\kappa) = \bigoplus_\kappa \mathfrak{C}(\kappa)$, then $\ell_2^f(\kappa) \times \Omega$ is homeomorphic to Ω . In particular, if $\mathfrak{C} = \mathfrak{B}_\alpha$, $\alpha \geq 2$, then $\ell_2^f(\kappa) \times \Omega_\alpha(\kappa)$ is homeomorphic to $\Omega_\alpha(\kappa)$.

3. Z -sets and strong universality. In this section, we list some results on Z -sets and strong universality. The following properties of Z -sets and strong Z -sets in ANRs are useful [2, Corollary 1.4.5 and Exercise 1.4.1].

PROPOSITION 3.1. *Let X be an ANR.*

- (1) For every [strong] Z -set A in X and every open subset U of X , $A \cap U$ is a [strong] Z -set in U .
- (2) A locally finite union of [strong] Z -sets in X is a [strong] Z -set.

Due to [13, Lemma 3.3 and Proposition 3.4] (cf. [4, Corollary 1.8]) and [13, Proposition 3.11], we have the following proposition:

PROPOSITION 3.2. *Suppose that X is an ANR with the κ -discrete n -cells property for all $n \in \omega$.*

- (1) If a closed subset $A \subset X$ is compact or of density $< \kappa$, then A is a Z -set.
- (2) If A is a Z -set and a strong Z_σ -set in X , then A is a strong Z -set.

Using [11, Theorem 2.6.5], we can show the following:

PROPOSITION 3.3. *For any ANR X , if each point $x \in X$ has an open neighborhood that is a strong Z_σ -set in itself, then X is also a strong Z_σ -set in itself.*

A map $f : X \rightarrow Y$ is *closed over* $A \subset Y$ if for each point $a \in A$ and each open neighborhood U of $f^{-1}(a)$ in X , there is an open neighborhood V of a in Y such that $f^{-1}(V) \subset U$. Z -sets and strong Z -sets are characterized by homotopy density [4, Corollary 1.2].

LEMMA 3.4. *Let X be an ANR and A be a closed subset of X . Then the following are equivalent:*

- (1) A is a [strong] Z -set in X .
- (2) There is a homotopy $h : X \times \mathbf{I} \rightarrow X$ such that $h(X \times (0, 1]) \subset X \setminus A$ and $h(x, 0) = x$ for each $x \in X$ [and h is closed over A].

On Z -sets and strong Z -sets in product spaces, we prove the following proposition:

PROPOSITION 3.5. *Let X and Y be ANRs. For every [strong] Z -set A in X , $A \times Y$ is a [strong] Z -set in $X \times Y$.*

Proof. We will handle the case of A being a strong Z -set in X . According to Lemma 3.4, there exists a homotopy $h : X \times \mathbf{I} \rightarrow X$ such that $h(X \times (0, 1]) \subset X \setminus A$, $h(x, 0) = x$ for any $x \in X$, and h is closed over A . We define the desired homotopy $\tilde{h} : X \times Y \times \mathbf{I} \rightarrow X \times Y$ by $\tilde{h}(x, y, t) = (h(x, t), y)$. It is clear that

$$\tilde{h}(X \times Y \times (0, 1]) \subset X \times Y \setminus A \times Y$$

and $\tilde{h}(x, y, 0) = (x, y)$ for all $(x, y) \in X \times Y$. It remains to prove that \tilde{h} is closed over $A \times Y$. Take any $(a, y) \in A \times Y$ and any open neighborhood U of $\tilde{h}^{-1}(a, y)$ in $X \times Y \times \mathbf{I}$. Note that $\tilde{h}^{-1}(a, y) = \{(a, y, 0)\}$ and $h^{-1}(a) = \{(a, 0)\}$. Hence we can choose an open neighborhood U_1 of $(a, 0)$ in $X \times \mathbf{I}$ and an open neighborhood U_2 of y in Y such that $U_1 \times U_2 \subset U$. Since h is closed over A , there is an open neighborhood V of a in X such that $h^{-1}(V) \subset U_1$. Then $V \times U_2$ is an open neighborhood of (a, y) in $X \times Y$ and

$$\tilde{h}^{-1}(V \times U_2) \subset h^{-1}(V) \times U_2 \subset U_1 \times U_2 \subset U.$$

The proof is complete. ■

Proposition 2.1 of [4] is valid without separability:

PROPOSITION 3.6. *Let \mathfrak{C} be a topological and closed hereditary class. Suppose that an ANR X is strongly \mathfrak{C} -universal. Then so is every open set in X .*

Applying [11, Theorem 2.6.5], we can prove the following proposition ([4, Proposition 2.7] and [13]):

PROPOSITION 3.7. *Let \mathfrak{C} be a topological and closed hereditary class of spaces and X be an ANR. If every point $x \in X$ has an open neighborhood that is strongly \mathfrak{C} -universal, then X is also strongly \mathfrak{C} -universal.*

Proposition 2.6 of [4] holds without separability:

PROPOSITION 3.8. *Let \mathfrak{C} be a topological and \mathbf{I} -stable class. Suppose that X and Y are ANRs such that every Z -set in $X \times Y$ is a strong Z -set. If X is strongly \mathfrak{C} -universal, then so is the product space $X \times Y$.*

According to [13, Proposition 3.5], we have the following ⁽³⁾:

⁽³⁾ A class \mathfrak{C} is *additive* if every space written as a union of two closed subsets belonging to \mathfrak{C} is in \mathfrak{C} . Proposition 3.5 of [13] is valid without additivity. Moreover, this proposition can be shown by using the κ -discrete n -cells property instead of the assumption that $\mathbf{I}^n \times \kappa \in \mathfrak{C}$.

PROPOSITION 3.9. *Let \mathfrak{C} be a topological and closed hereditary class of spaces and X be an ANR of density $\leq \kappa$. Suppose that X has the κ -discrete n -cells property for each $n \in \omega$ and is a strong Z_σ -set in itself. If X is strongly \mathfrak{C} -universal, then it is strongly \mathfrak{C}_σ -universal.*

4. The discrete approximation property. This section is devoted to investigating the discrete (locally finite) approximation property. For a homotopy $h : X \times \mathbf{I} \rightarrow Y$ and an open cover \mathcal{U} of Y , h is a \mathcal{U} -homotopy if for each $x \in X$, the homotopy track $h(\{x\} \times \mathbf{I})$ is contained in some member of \mathcal{U} . When Y is a metric space and $\epsilon > 0$, h is an ϵ -homotopy if the diameter of $h(\{x\} \times \mathbf{I})$ is less than ϵ . The locally finite approximation property is open and homotopy dense hereditary.

PROPOSITION 4.1. *Let \mathfrak{C} be a class. Suppose that X is an ANR and U is an open set in X . If X has the κ -locally finite approximation property for \mathfrak{C} , then so does U .*

Proof. Suppose that $f : A = \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow U$ is a map, where $A_\gamma \in \mathfrak{C}$ for all $\gamma < \kappa$, and \mathcal{U} is an open cover of U . We will construct a map $h : A \rightarrow U$ such that h is \mathcal{U} -close to f and the family $\{h(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in U .

Take an open cover \mathcal{U}' of U that is a star-refinement of \mathcal{U} . We can write $U = \bigcup_{i \in \omega} U_i$, where U_i is an open set in X and $\text{cl} U_i \subset U_{i+1}$ for every $i \in \omega$. Let $B_i = f^{-1}(\text{cl} U_i \setminus U_{i-1})$, $i \in \omega$, where $U_{-1} = \emptyset$. We define an open cover \mathcal{U}'' of U and open covers \mathcal{V}_i of X , $i \in \omega$, as follows:

$$\begin{aligned} \mathcal{U}'' &= \{U' \cap U_i \setminus \text{cl} U_{i-2} \mid U' \in \mathcal{U}', i > 0\}, \\ \mathcal{V}_i &= \{U'' \cap U_{2i+1} \mid U'' \in \mathcal{U}''\} \cup \{X \setminus \text{cl} U_{2i}\}. \end{aligned}$$

Since X has the κ -locally finite approximation property for \mathfrak{C} , there exist maps $g_i : A \rightarrow X$, $i \in \omega$, such that g_i is \mathcal{V}_i -homotopic to f and $\{g_i(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in X . Then $g_i|_{B_{2i}}$ is \mathcal{U}'' -homotopic to $f|_{B_{2i}}$ for all $i \in \omega$. By the Homotopy Extension Theorem, we can obtain a map $g : A \rightarrow U$ such that g is \mathcal{U}'' -homotopic to f and $g|_{B_{2i}} = g_i|_{B_{2i}}$ for each $i \in \omega$. It is easy to see that $\{g(A_\gamma \cap B_{2i}) \mid \gamma < \kappa\}$ is locally finite in $U_{2i+1} \setminus \text{cl} U_{2i-2}$, where $U_{-2} = \emptyset$. Therefore $\{g(A_\gamma \cap B_{2i}) \mid \gamma < \kappa, i \in \omega\}$ is locally finite in U .

Next, we can find an open cover \mathcal{W} of U that refines \mathcal{U}' and satisfies the following:

- For every map $h : A \rightarrow U$, if h is \mathcal{W} -close to g , then the collection $\{h(A_\gamma \cap B_{2i}) \mid \gamma < \kappa, i \in \omega\}$ is locally finite in U .

Note that $g(B_{2i+1}) \subset U_{2i+2} \setminus \text{cl} U_{2i-1}$ for each $i \in \omega$. By a similar construction to that for g , we can obtain a map $h : A \rightarrow U$ such that h is \mathcal{W} -close to g and $\{h(A_\gamma \cap B_{2i+1}) \mid \gamma < \kappa, i \in \omega\}$ is locally finite in U . It follows from the definition of \mathcal{W} that $\{h(A_\gamma \cap B_{2i}) \mid \gamma < \kappa, i \in \omega\}$ is locally finite in U .

Therefore $\{h(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in U . Moreover, h is \mathcal{U} -close to f . Thus the proof is complete. ■

PROPOSITION 4.2. *Let \mathfrak{C} be a class and X be an ANR with the κ -locally finite approximation property for \mathfrak{C} . Then any homotopy dense subset $Y \subset X$ also has the κ -locally finite approximation property for \mathfrak{C} .*

Proof. Suppose that $f : A = \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow Y$ is a map, where each A_γ is in \mathfrak{C} , and \mathcal{U} is an open cover of Y . For each $U \in \mathcal{U}$, there is an open set \tilde{U} in X such that $U = \tilde{U} \cap Y$. Let $\tilde{\mathcal{U}} = \{\tilde{U} \mid U \in \mathcal{U}\}$ and $V = \bigcup \tilde{\mathcal{U}}$. Take a star refinement \mathcal{V} of $\tilde{\mathcal{U}}$ that covers V . Since V is an open subset of X and X has the κ -locally finite approximation property for \mathfrak{C} , V has the same property by Proposition 4.1. Hence we can find a map $g : A \rightarrow V$ such that g is \mathcal{V} -close to f and $\{g(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in V . Take an open cover \mathcal{W} of V that is a refinement of \mathcal{V} and satisfies the following condition:

- For any map $h : A \rightarrow V$ that is \mathcal{W} -close to g , the collection $\{h(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in V .

Since Y is homotopy dense in X , it is so in V , which implies that there exists a map $\phi : V \rightarrow V$ such that ϕ is \mathcal{W} -close to the identity map of V and $\phi(V) \subset Y$. Let $h = \phi g$, so h is \mathcal{W} -close to g . Hence h is \mathcal{U} -close to f and $\{h(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in Y . The proof is complete. ■

We denote the cardinality of a set X by $\text{card } X$. The next proposition detects the locally finite approximation property for a class \mathfrak{C} of manifolds modeled on spaces with that property.

PROPOSITION 4.3. *Let \mathfrak{C} be a closed hereditary class. An ANR X has the κ -locally finite approximation property for \mathfrak{C} if and only if each point $x \in X$ has an open neighborhood with the κ -locally finite approximation property for \mathfrak{C} .*

Proof. The “only if” part is obvious. We shall prove the “if” part. By [11, Theorem 2.6.5], we need only show that the following conditions hold:

- (a) For any open subsets $V \subset U$ of X , if U has the κ -locally finite approximation property for \mathfrak{C} , then so does V .
- (b) For any open sets $U, V \subset X$, if both U and V have the κ -locally finite approximation property for \mathfrak{C} , then so does $U \cup V$.
- (c) For every discrete family $\{U_\gamma \mid \gamma < \lambda\}$ of open sets in X , if each U_γ has the κ -locally finite approximation property for \mathfrak{C} , then so does $\bigcup_{\gamma < \lambda} U_\gamma$.

As is easily observed, (c) holds. The condition (a) follows from Proposition 4.1.

We shall show (b). For simplicity, we write $W = U \cup V$. Let $f : A = \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow W$ be a map, where $A_\gamma \in \mathfrak{C}$ for all $\gamma < \kappa$, and let \mathcal{W} be an open cover of W . We need only construct a map $h : A \rightarrow W$ such that h is

\mathcal{W} -close to f and the family $\{h(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in W . Take open sets $W_i \subset W$, $i = 1, 2, 3, 4$, such that

$$\begin{aligned} U \setminus V \subset W_1 \subset \text{cl}_W W_1 \subset W_2 \subset \text{cl}_W W_2 \\ \subset W_3 \subset \text{cl}_W W_3 \subset W_4 \subset \text{cl}_W W_4 \subset U, \end{aligned}$$

where $\text{cl}_W W_i$ is the closure of W_i in W . Then we can find an open cover \mathcal{W}' of W such that \mathcal{W}' is a star-refinement of \mathcal{W} and refines

$$\{W_1, W_2 \cap V, W_3 \setminus \text{cl}_W W_1, W_4 \setminus \text{cl}_W W_2, W \setminus \text{cl}_W W_3\}.$$

Since U has the κ -locally finite approximation property for \mathfrak{C} and each $A_\gamma \cap f^{-1}(\text{cl}_W W_4)$ is in \mathfrak{C} , which is a closed hereditary class, there exists a map $g : f^{-1}(\text{cl}_W W_4) \rightarrow U$ such that g is $\mathcal{W}'|_U$ -homotopic to $f|_{f^{-1}(\text{cl}_W W_4)}$ and $\{g(A_\gamma \cap f^{-1}(\text{cl}_W W_4)) \mid \gamma < \kappa\}$ is locally finite in U . By the Homotopy Extension Theorem, we can obtain a map $\tilde{g} : A \rightarrow W$ such that \tilde{g} is \mathcal{W}' -close to f and $\tilde{g}|_{f^{-1}(\text{cl}_W W_4)} = g$. Then $\{g(A_\gamma \cap f^{-1}(\text{cl}_W W_3)) \mid \gamma < \kappa\}$ is locally finite in W . Indeed, fix any point $x \in W$. When $x \in U$, there exists an open neighborhood U_x of x in U such that

$$\text{card} \{\gamma < \kappa \mid \tilde{g}(A_\gamma \cap f^{-1}(\text{cl}_W W_3)) \cap U_x \neq \emptyset\} < \infty$$

because $\{g(A_\gamma \cap f^{-1}(\text{cl}_W W_3)) \mid \gamma < \kappa\}$ is locally finite in U . Note that U_x is open in W . When $x \in V \setminus U$, the subset $V \setminus \text{cl}_W W_4$ is an open neighborhood of x in W and

$$\tilde{g}(A_\gamma \cap f^{-1}(\text{cl}_W W_3)) \cap V \setminus \text{cl}_W W_4 = \emptyset$$

since $\tilde{g}(f^{-1}(\text{cl}_W W_3)) \subset W_4$.

Let \mathcal{W}'' be an open cover of W that refines \mathcal{W}' and

$$\{W_2, W_3 \setminus \text{cl}_W W_1, W_4 \setminus \text{cl}_W W_2, W \setminus \text{cl}_W W_3\},$$

and satisfies the following:

- For every map $h : f^{-1}(\text{cl}_W W_3) \rightarrow W$, if h is \mathcal{W}'' -close to $\tilde{g}|_{f^{-1}(\text{cl}_W W_3)}$, then $\{h(A_\gamma \cap f^{-1}(\text{cl}_W W_3)) \mid \gamma < \kappa\}$ is locally finite in W .

Note that each $A_\gamma \cap f^{-1}(V \setminus W_1)$ is in \mathfrak{C} and $\tilde{g}(f^{-1}(V \setminus W_1)) \subset V$. Using the κ -locally finite approximation property of V , we can find a map $h : A \rightarrow W$ such that h is \mathcal{W}'' -close to \tilde{g} and $\{h(A_\gamma \cap f^{-1}(V \setminus W_3)) \mid \gamma < \kappa\}$ is locally finite in W by the same argument as above. It is easy to see that h is \mathcal{W} -close to f . Due to the definition of \mathcal{W}'' , $\{h(A_\gamma \cap f^{-1}(\text{cl}_W W_3)) \mid \gamma < \kappa\}$ is locally finite in W . Therefore $\{h(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in W . The proof is complete. ■

Replacing $\mathfrak{M}_0(n)$ with \mathfrak{C} and $D_\gamma = \mathbf{I}^{2n+1}$ with $D_\gamma = C$ in the proof of [6, Proposition 4.3], we obtain the following lemma:

LEMMA 4.4. *Let \mathfrak{C} be a class and C be a space such that every space of \mathfrak{C} can be embedded into C as a closed subset. Suppose that W is an open subset of an ANR X which is contractible in X . If X has the κ -locally finite approximation property for $\{C\}$, then W has the κ -locally finite approximation property for \mathfrak{C} .*

The class of compact spaces of dimension $\leq n$ is denoted by \mathfrak{M}_0^n . As a corollary of Proposition 4.3, we can establish the following:

COROLLARY 4.5. *Let X be an ANR. Then the following are equivalent:*

- (1) *X has the κ -discrete n -cells property for any $n \in \omega$.*
- (2) *Each point $x \in X$ has an open neighborhood with the κ -discrete n -cells property for any $n \in \omega$.*
- (3) *Each point $x \in X$ has an open neighborhood with the κ -discrete approximation property for \mathfrak{M}_0^n for any $n \in \omega$.*
- (4) *X has the κ -discrete approximation property for \mathfrak{M}_0^n for any $n \in \omega$.*

Proof. The implications (4) \Rightarrow (1) \Rightarrow (2) are obvious. Combining Lemma 4.4 (cf. [6, Proposition 4.3]) with [6, Lemma 4.2], we have (2) \Rightarrow (3). The implication (3) \Rightarrow (4) follows from Proposition 4.3 and [6, Lemma 4.2]. ■

In [6, Proposition 4.4], it is shown that the κ -discrete approximation property for $\mathfrak{C} \subset \mathfrak{M}_0$ can be replaced with a stronger property. That proposition holds even if we omit some assumptions. Applying similar techniques, we can prove the following lemma. For completeness, we give the proof.

LEMMA 4.6. *Let \mathfrak{C} be a class of spaces and X be an ANR with the κ -discrete approximation property for \mathfrak{C} . The following holds:*

- *Let $f : A = \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow X$ be a map, where each A_γ is in \mathfrak{C} , let B be a closed subset of A , and let \mathcal{U} be an open cover of X . Suppose that each $\text{cl } f(A_\gamma \cap B)$ is a strong Z -set in X and the collection $\{f(A_\gamma \cap B) \mid \gamma < \kappa\}$ is discrete in X . Then there exists a map $g : A \rightarrow X$ such that g is \mathcal{U} -close to f , $g|_B = f|_B$ and $\{g(A_\gamma) \mid \gamma < \kappa\}$ is discrete in X .*

Proof. Choose open covers \mathcal{U}_1 and \mathcal{U}_2 of X so that \mathcal{U}_1 is a star-refinement of \mathcal{U} and \mathcal{U}_2 is a refinement of \mathcal{U}_1 . Let $B_\gamma = A_\gamma \cap B$ for each $\gamma < \kappa$. Since $\{f(B_\gamma) \mid \gamma < \kappa\}$ is discrete in X , we can obtain a discrete collection $\{U_\gamma \mid \gamma < \kappa\}$ of open sets in X such that $\text{cl } f(B_\gamma) \subset U_\gamma$ for each $\gamma < \kappa$. It follows from Proposition 3.1(2) that the discrete union $\text{cl } f(B) = \bigoplus_{\gamma < \kappa} \text{cl } f(B_\gamma)$ is a strong Z -set in X . Taking an open cover \mathcal{U}'_2 of X that refines \mathcal{U}_2 and

$$\{U_\gamma, X \setminus \text{cl } f(B) \mid \gamma < \kappa\},$$

we can find a \mathcal{U}'_2 -homotopy $h : X \times \mathbf{I} \rightarrow X$ and an open neighborhood W of $\text{cl } f(B)$ in X such that $h(x, 0) = x$ for any $x \in X$ and $h(X \times \{1\}) \subset X \setminus W$. Define $h' : A \times \mathbf{I} \rightarrow X$ by $h'(x, t) = h(f(x), t)$, so h' is a \mathcal{U}'_2 -homotopy and

$h'(x, 0) = f(x)$ for any $x \in A$. Note that $h'(B_\gamma \times \mathbf{I}) \subset U_\gamma$ for every $\gamma < \kappa$. By the compactness of \mathbf{I} , each B_γ has an open neighborhood V_γ in A_γ such that $h'(V_\gamma \times \mathbf{I}) \subset U_\gamma$. Take a map $k : A \rightarrow \mathbf{I}$ such that $k^{-1}(0) = B$ and $k^{-1}(1) = A \setminus \bigcup_{\gamma < \kappa} V_\gamma$, and define a map $f' : A \rightarrow X$ by $f'(x) = h'(x, k(x))$. Let $W_\gamma = W \cap U_\gamma$ for each $\gamma < \kappa$. Observe that f' is \mathcal{U}'_2 -close to f , $f'|_B = f|_B$ and

(*) $f'(A \setminus V_\gamma) \cap W_\gamma = \emptyset$ for any $\gamma < \kappa$.

Let W'_γ and W''_γ be open neighborhoods of $\text{cl } f(B_\gamma)$ for each $\gamma < \kappa$ such that

$$\text{cl } W''_\gamma \subset W'_\gamma \subset \text{cl } W'_\gamma \subset W_\gamma.$$

Note that both $\{\text{cl } W'_\gamma \mid \gamma < \kappa\}$ and $\{\text{cl } W''_\gamma \mid \gamma < \kappa\}$ are discrete families of closed sets in X . Choose an open cover \mathcal{U}'_1 of X such that \mathcal{U}'_1 refines \mathcal{U}_1 and

$$\left\{ W''_\gamma, W'_\gamma \setminus \text{cl } f(B_\gamma), W_\gamma \setminus \text{cl } W''_\gamma, X \setminus \bigcup_{\gamma' < \kappa} \text{cl } W'_{\gamma'} \mid \gamma < \kappa \right\}.$$

Using the κ -discrete approximation property for \mathfrak{C} , we can obtain a \mathcal{U}'_1 -homotopy $h'' : A \times \mathbf{I} \rightarrow X$ such that $h''(x, 0) = f'(x)$ for all $x \in A$ and

(**) $\{h''(A_\gamma \times \{1\}) \mid \gamma < \kappa\}$ is discrete in X .

Since h'' is a \mathcal{U}'_1 -homotopy and $h''(x, 0) = f'(x)$ for any $x \in B$, we see that $h''(B_\gamma \times \mathbf{I}) \subset W''_\gamma$ for each $\gamma < \kappa$. Then there exists an open neighborhood G_γ of B_γ in A_γ for every $\gamma < \kappa$ such that $h''(G_\gamma \times \mathbf{I}) \subset W''_\gamma$. Taking a map $k' : A \rightarrow \mathbf{I}$ such that $(k')^{-1}(0) = B$ and $(k')^{-1}(1) = A \setminus \bigcup_{\gamma < \kappa} G_\gamma$, we can define the desired map $g : A \rightarrow X$ by $g(x) = h''(x, k'(x))$. Then g is \mathcal{U}'_1 -close to f' , which means that g is \mathcal{U} -close to f , and $g|_B = f|_B$. It remains to show that $\{g(A_\gamma) \mid \gamma < \kappa\}$ is discrete in X .

Fix any $x \in X$. According to (**), the collection $\{g(A_\gamma \setminus G_\gamma) \mid \gamma < \kappa\}$ is discrete in X . Therefore the point x has an open neighborhood U_x such that

$$\text{card } \{\gamma < \kappa \mid g(A_\gamma \setminus G_\gamma) \cap U_x \neq \emptyset\} \leq 1.$$

If $x \in X \setminus \bigcup_{\gamma < \kappa} \text{cl } W''_\gamma$, the subset $U'_x = U_x \setminus \bigcup_{\gamma < \kappa} \text{cl } W''_\gamma$ is an open neighborhood of x in X . Since $g(G_\gamma) \subset W''_\gamma$ for each γ , we have $U'_x \cap g(G_\gamma) = \emptyset$, which implies that

$$\text{card } \{\gamma < \kappa \mid g(A_\gamma) \cap U'_x \neq \emptyset\} \leq 1.$$

If $x \in \bigcup_{\gamma < \kappa} \text{cl } W''_\gamma$, there exists a unique $\gamma_0 < \kappa$ such that $x \in \text{cl } W''_{\gamma_0}$. Then $U'_x = U_x \setminus \bigcup_{\gamma \neq \gamma_0} \text{cl } W''_\gamma$ is an open neighborhood of x such that $U'_x \cap g(G_\gamma) = \emptyset$ for all $\gamma \neq \gamma_0$. Note that $x \notin \text{cl } g(A_\gamma \setminus G_\gamma)$ for any $\gamma \neq \gamma_0$. Suppose not, so $x \in \text{cl } g(A_\gamma \setminus G_\gamma)$ for some $\gamma \neq \gamma_0$. Then there is a sequence $\{a_n\} \subset A_\gamma \setminus G_\gamma$ such that $g(a_n) \rightarrow x$. We may assume that $\{g(a_n)\} \subset W'_{\gamma_0}$. Then $\{f'(a_n)\} \subset W_{\gamma_0}$ because g is \mathcal{U}'_1 -close to f' . On the other hand, since

$A_\gamma \subset A \setminus V_{\gamma_0}$, it follows from (*) that $f'(A_\gamma) \cap W_{\gamma_0} = \emptyset$. This is a contradiction. Taking the open neighborhood

$$U_x'' = U_x' \setminus \bigcup_{\gamma \neq \gamma_0} \text{cl } g(A_\gamma \setminus G_\gamma)$$

of x , we have

$$\text{card} \{ \gamma < \kappa \mid g(A_\gamma) \cap U_x'' \neq \emptyset \} \leq 1.$$

Consequently, $\{g(A_\gamma) \mid \gamma < \kappa\}$ is discrete in X . ■

For a subclass $\mathfrak{C} \subset \mathfrak{M}_0$, the κ -locally finite approximation property for \mathfrak{C} is equivalent to the κ -discrete approximation property for \mathfrak{C} [6, Lemma 4.2] (cf. [1, Lemma 4.6]). We prove the following:

LEMMA 4.7. *Let \mathfrak{C} be a class and X be an ANR. Suppose that for every map $\phi : A \rightarrow X$, where $A \in \mathfrak{C}$, $\text{cl } \phi(A)$ is a Z -set in X . If X has the κ -locally finite approximation property for \mathfrak{C} , then X has the κ -discrete approximation property for \mathfrak{C} .*

Proof. We will show that for each map $f : \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow X$, where $A_\gamma \in \mathfrak{C}$, and each open cover \mathcal{U} of X , there is a map $h : \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow X$ such that h is \mathcal{U} -close to f and $\{h(A_\gamma) \mid \gamma < \kappa\}$ is discrete in X . Take an open cover \mathcal{V} of X that is a star-refinement of \mathcal{U} . Applying the κ -locally finite approximation property for \mathfrak{C} , we can obtain a map $g : \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow X$ such that g is \mathcal{V} -close to f and $\{g(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in X . Then the family $\{\text{cl } g(A_\gamma) \mid \gamma < \kappa\}$ is also locally finite. Choose an open cover \mathcal{W} of X such that \mathcal{W} refines \mathcal{V} and the following condition is satisfied:

- For any map $\phi : \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow X$ that is \mathcal{W} -close to g , the family $\{\phi(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in X .

By transfinite induction, for each $\gamma < \kappa$, we can find a map $h_\gamma : A_\gamma \rightarrow X$ such that h_γ is \mathcal{W} -close to $g|_{A_\gamma}$ and $\{\text{cl } h_\gamma(A_\gamma) \mid \gamma < \kappa\}$ is pairwise disjoint. Then the map $h : \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow X$ defined by $h|_{A_\gamma} = h_\gamma$ is the desired map. Suppose that $\gamma < \kappa$ and $h_{\gamma'}$ has been obtained for any $\gamma' < \gamma$. By assumption, $\text{cl } h_{\gamma'}(A_{\gamma'})$ is a Z -set in X . The family $\{\text{cl } h_{\gamma'}(A_{\gamma'}) \mid \gamma' < \gamma\}$ is locally finite in X by the definition of \mathcal{W} , and hence the locally finite union $\bigcup_{\gamma' < \gamma} \text{cl } h_{\gamma'}(A_{\gamma'})$ is a Z -set due to Proposition 3.1(2). Therefore, taking a map $\phi : X \rightarrow X$ such that ϕ is \mathcal{W} -close to the identity map on X and

$$\phi(X) \cap \bigcup_{\gamma' < \gamma} \text{cl } h_{\gamma'}(A_{\gamma'}) = \emptyset,$$

we can define $h_\gamma = \phi g|_{A_\gamma}$. Thus the proof is complete. ■

Concerning the locally finite approximation property of product spaces, the following proposition holds:

PROPOSITION 4.8. *Let \mathfrak{C} be a topological and closed hereditary class and X be an ANR. If X has the κ -locally finite approximation property for \mathfrak{C} , then for any space Y , the product space $X \times Y$ also has the κ -locally finite approximation property for \mathfrak{C} .*

Proof. Taking admissible metrics d_X and d_Y on X and Y respectively, we define an admissible metric d on $X \times Y$ by

$$d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$$

for $x, x' \in X$ and $y, y' \in Y$. We shall show that for every map $f : A = \bigoplus_{\gamma < \kappa} A_\gamma \rightarrow X \times Y$, where $A_\gamma \in \mathfrak{C}$ for each $\gamma < \kappa$, and for every map $\alpha : X \times Y \rightarrow (0, 1]$, there exists a map $h : A \rightarrow X \times Y$ such that for each $a \in A$ we have $d(h(a), f(a)) < \alpha(f(a))$ and the family $\{h(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in $X \times Y$.

Let $\text{pr}_X : X \times Y \rightarrow X$ and $\text{pr}_Y : X \times Y \rightarrow Y$ be the projections, and

$$B_n = \{a \in A \mid \alpha(f(a)) \geq 2^{-n}\}, \quad n \in \omega.$$

By induction, we will construct maps $f_n : A \rightarrow X$, $n \in \omega$, such that

- (1) $f_n|_{B_{n-2} \cup (A \setminus B_{n+1})} = f_{n-1}|_{B_{n-2} \cup (A \setminus B_{n+1})}$,
- (2) $d_X(f_n(a), f_{n-1}(a)) < 2^{-n-2}$ for every $a \in A$,
- (3) $\{f_n(A_\gamma \cap B_n) \mid \gamma < \kappa\}$ is locally finite in X ,

where $f_{-1} = \text{pr}_X f$ and $B_{-2} = B_{-1} = \emptyset$. Suppose that f_i , $i \leq n-1$, have been obtained. Due to the inductive assumption, the family $\{f_{n-1}(A_\gamma \cap B_{n-1}) \mid \gamma < \kappa\}$ is locally finite in X . Hence there exists an open cover \mathcal{U} of X with mesh $< 2^{-n-2}$ and such that the following holds:

- For every map $\phi : A \rightarrow X$, if ϕ is \mathcal{U} -close to f_{n-1} , then $\{\phi(A_\gamma \cap B_{n-1}) \mid \gamma < \kappa\}$ is locally finite in X .

Note that each $A_\gamma \cap B_n \setminus \text{int } B_{n-1}$ is in \mathfrak{C} , where $\text{int } B_{n-1}$ is the interior of B_{n-1} . Applying the κ -locally finite approximation property for \mathfrak{C} of X , we can take a map $\phi : B_n \setminus \text{int } B_{n-1} \rightarrow X$ such that ϕ is \mathcal{U} -homotopic to $f_{n-1}|_{B_n \setminus \text{int } B_{n-1}}$ and

$$\{\phi(A_\gamma \cap B_n \setminus \text{int } B_{n-1}) \mid \gamma < \kappa\}$$

is locally finite in X . By the Homotopy Extension Theorem, there is a map $f_n : A \rightarrow X$ \mathcal{U} -close to f_{n-1} , which implies that $d_X(f_n(a), f_{n-1}(a)) < 2^{-n-2}$ for any $a \in A$, such that

$$f_n|_{B_{n-2} \cup (A \setminus B_{n+1})} = f_{n-1}|_{B_{n-2} \cup (A \setminus B_{n+1})}$$

and $f_n|_{B_n \setminus \text{int } B_{n-1}} = \phi$. The definition of \mathcal{U} shows that $\{f_n(A_\gamma \cap B_{n-1}) \mid \gamma < \kappa\}$ is locally finite in X , and hence $\{f_n(A_\gamma \cap B_n) \mid \gamma < \kappa\}$ is locally finite in X .

After completing the inductive construction, we can define a map $g : A \rightarrow X$ by $g|_{B_n} = f_{n+1}|_{B_n}$. Then $d_X(g(a), \text{pr}_X(f(a))) < \alpha(f(a))$ for each $a \in A$. Indeed, fix any $a \in A$, so we can find $n \in \omega$ such that $a \in B_n \setminus B_{n-1}$, that is, $2^{-n} \leq \alpha(f(a)) < 2^{-n+1}$. Observe that

$$\begin{aligned} d_X(g(a), \text{pr}_X(f(a))) &= d_X(f_{n+1}(a), f_{n-2}(a)) \\ &\leq d_X(f_{n+1}(a), f_n(a)) + d_X(f_n(a), f_{n-1}(a)) \\ &\quad + d_X(f_{n-1}(a), f_{n-2}(a)) \\ &< 2^{-n-3} + 2^{-n-2} + 2^{-n-1} < 2^{-n} \leq \alpha(f(a)). \end{aligned}$$

Now we can define the desired map $h : A \rightarrow X \times Y$ by $h(a) = (g(a), \text{pr}_Y(f(a)))$. For each $a \in A$,

$$\begin{aligned} d(h(a), f(a)) &= d_X(g(a), \text{pr}_X(f(a))) + d_Y(\text{pr}_Y(f(a)), \text{pr}_Y(f(a))) \\ &< \alpha(f(a)). \end{aligned}$$

It remains to show that $\{h(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in $X \times Y$. Suppose otherwise, so we can choose $(x, y) \in X \times Y$ and $a_i \in A_{\gamma_i}$, where $\gamma_i \neq \gamma_j$ if $i \neq j$, such that $h(a_i)$ converges to (x, y) . When $\liminf \alpha(f(a_i)) = 0$, replacing $\{a_i\}$ with a subsequence, we have $\lim \alpha(f(a_i)) = 0$. Since $d(h(a_i), f(a_i)) < \alpha(f(a_i))$ and $h(a_i) \rightarrow (x, y)$, $f(a_i)$ converges to (x, y) . Then $\alpha(f(a_i)) \rightarrow \alpha(x, y) > 0$, a contradiction. When $\liminf \alpha(f(a_i)) > 0$, there is $n \in \omega$ such that $2^{-n} \leq \liminf \alpha(f(a_i)) < 2^{-n+1}$. Taking a subsequence, we may assume that $2^{-n-1} < \alpha(f(a_i)) < 2^{-n+1}$. Then any a_i is in B_{n+1} . Since $h(a_i) \rightarrow (x, y)$, we have

$$f_{n+2}(a_i) = g(a_i) = \text{pr}_X(h(a_i)) \rightarrow x,$$

which contradicts $\{f_{n+2}(A_\gamma \cap B_{n+1}) \mid \gamma < \kappa\}$ being locally finite in X . Therefore $\{h(A_\gamma) \mid \gamma < \kappa\}$ is locally finite in $X \times Y$. ■

5. Characterizing absorbing sets in Hilbert manifolds. In this section, we shall prove Theorem 1.1. Theorem 5.1 of [4] and Theorem 3.7 of [13] can be rewritten as follows ⁽⁴⁾:

PROPOSITION 5.1. *Let \mathfrak{C} be a topological and closed hereditary class, and Y be a \mathfrak{C} -absorbing set in an $\ell_2(\kappa)$ -manifold. Suppose that a space $X \in \mathfrak{C}_\sigma$ of density $\leq \kappa$ is an ANR, is strongly \mathfrak{C} -universal, is a strong Z_σ -set in itself, and has the κ -discrete n -cells property for every $n \in \omega$. Then every fine homotopy equivalence $f : Y \rightarrow X$ is a near-homeomorphism.*

⁽⁴⁾ Theorem 5.1 of [4] and Theorem 3.7 of [13] are valid without the assumption that classes are additive and contain $\mathbf{I}^n \times \kappa$. Indeed, we can modify the proofs of [4, Proposition 2.3 and Theorem 3.1] and [13, Proposition 3.5].

Corollary 5.6(i) of [4] and Theorem 3.9(1) of [13] hold without some conditions on classes. The following proposition is proven in [12] ⁽⁵⁾. For the sake of completeness, we give its proof.

PROPOSITION 5.2. *Suppose that \mathfrak{C} is a topological and closed hereditary class, and Ω is a \mathfrak{C} -absorbing set in $\ell_2(\kappa)$. Then any $\ell_2(\kappa)$ -manifold M contains a \mathfrak{C} -absorbing set.*

Proof. By the Open Embedding Theorem for Hilbert manifolds, we can regard M as an open subset of $\ell_2(\kappa)$. Let $Y = M \cap \Omega$, so it is a \mathfrak{C} -absorbing set in M . Indeed, as is easily observed, Y is homotopy dense in M because Ω is homotopy dense in $\ell_2(\kappa)$ and M is open in $\ell_2(\kappa)$. Since Y is open in Ω , Y is in \mathfrak{C}_σ , is strongly \mathfrak{C} -universal, and is a strong Z_σ -set in itself by Propositions 3.6 and 3.1(1). Hence Y is a \mathfrak{C} -absorbing set in M . ■

Now we prove Theorem 1.1. The implication (2) \Rightarrow (1) is shown in [12].

Proof of Theorem 1.1. First, we show the implication (1) \Rightarrow (3). Since X is an Ω -manifold, each point $x \in X$ has an open neighborhood U_x that is homeomorphic to some open set in Ω . Observe that U_x is an ANR, is strongly \mathfrak{C} -universal, and is a strong Z_σ -set in itself due to Propositions 3.6 and 3.1(1). Since Ω is homotopy dense in $\ell_2(\kappa)$, and has the κ -discrete n -cells property for any $n \in \omega$, Ω has the κ -locally finite n -cells property by Proposition 4.2. Moreover, it follows from Proposition 4.1 and [6, Lemma 4.2] (cf. [1, Lemma 4.6]) that each U_x also has the κ -discrete n -cells property for every $n \in \omega$. According to [11, 6.2.10(4)], Propositions 3.7 and 3.3, and Corollary 4.5, X satisfies the conditions (a)–(d) of (3).

Next, we prove (3) \Rightarrow (2). Combining Proposition 5.2 with [13, Theorem 3.6] (cf. [4, Theorem 4.2]), we can obtain a \mathfrak{C} -absorbing set Y in some $\ell_2(\kappa)$ -manifold and a fine homotopy equivalence $f : Y \rightarrow X$. By Proposition 5.1, f is a near-homeomorphism, so X is homeomorphic to Y .

Finally, we show (2) \Rightarrow (1). Regarding X as a \mathfrak{C} -absorbing set in some $\ell_2(\kappa)$ -manifold M , we can see that each point $x \in X$ has an open neighborhood U_x in M that is homeomorphic to $\ell_2(\kappa)$. Then $X \cap U_x$ is a \mathfrak{C} -absorbing set in U_x by the same argument as in the proof of Proposition 5.2. Note that Theorem 3.1 of [4] holds without additivity of classes and separability (cf. [13]), so each point $x \in X$ has the open neighborhood $X \cap U_x$ homeomorphic to Ω . Therefore X is an Ω -manifold. ■

⁽⁵⁾ In [4, 13], Proposition 5.2 is shown by applying the Triangulation Theorem for Hilbert manifolds under the assumption that \mathfrak{C} is **I**-stable. On the other hand, using the Open Embedding Theorem for Hilbert manifolds, K. Sakai proved it without **I**-stability in [12].

6. A $\bigoplus_{\kappa} \mathfrak{C}(\kappa')$ -absorbing set in $\ell_2(\kappa)$. In this section, we shall show Theorem 1.2. From now on, let $\aleph_0 \leq \kappa' < \kappa$, \mathfrak{C} be a topological, closed hereditary and **I**-stable class, and Ω be a $\mathfrak{C}(\kappa')$ -absorbing set in $\ell_2(\kappa')$. Since $\ell_2^f(\kappa)$ and Ω are homotopy dense in $\ell_2(\kappa)$ and $\ell_2(\kappa')$ respectively, we have the following proposition:

PROPOSITION 6.1. *The space $\ell_2^f(\kappa) \times \Omega$ is homotopy dense in $\ell_2(\kappa) \times \ell_2(\kappa')$.*

PROPOSITION 6.2. *The space $\ell_2^f(\kappa) \times \Omega$ is in $(\bigoplus_{\kappa} \mathfrak{C}(\kappa'))_{\sigma}$.*

Proof. Since $\ell_2^f(\kappa)$ is strongly countable-dimensional, σ -locally compact and of density κ , and $\Omega \in (\mathfrak{C}(\kappa'))_{\sigma}$, we can write $\ell_2^f(\kappa) = \bigcup_{n \in \omega} (\bigoplus_{\gamma < \kappa} A_{(n, \gamma)})$ and $\Omega = \bigcup_{m \in \omega} \Omega_m$, where $A_{(n, \gamma)} \in \mathfrak{M}_0^{\text{fd}}$ and $\Omega_m \in \mathfrak{C}(\kappa')$ is closed in Ω . Then

$$\ell_2^f(\kappa) \times \Omega = \bigcup_{n, m \in \omega} \left(\bigoplus_{\gamma < \kappa} A_{(n, \gamma)} \times \Omega_m \right).$$

Since \mathfrak{C} is topological, closed hereditary and **I**-stable, each closed subset $A_{(n, \gamma)} \times \Omega_m$ is in $\mathfrak{C}(\kappa')$. Hence $\ell_2^f(\kappa) \times \Omega \in (\bigoplus_{\kappa} \mathfrak{C}(\kappa'))_{\sigma}$. ■

Let $X = (X, d)$ be a metric space. For a class \mathfrak{M} and a cardinal λ , we say that X has the λ -discrete separation property for \mathfrak{M} if the following is satisfied:

- For each $\epsilon > 0$, there exists $\delta > 0$ such that for any map $f : \bigoplus_{\gamma < \lambda} A_{\gamma} \rightarrow X$, where each $A_{\gamma} \in \mathfrak{M}$, there is a map $g : \bigoplus_{\gamma < \lambda} A_{\gamma} \rightarrow X$ such that g is ϵ -homotopic to f and $d(g(A_{\gamma}), g(A_{\gamma'})) \geq \delta$ for any $\gamma < \gamma' < \lambda$.

In the case $\mathfrak{M} = \{\mathbf{I}^n\}$, $n \in \omega$, refer to [3]. By the same argument, Lemmas 5.2 and 6.1 of [3] are generalized as follows:

LEMMA 6.3. *Let \mathfrak{M} be a class. If a metric space X has the κ -discrete separation property for $\mathfrak{M}(\kappa')$, then X has the κ -discrete approximation property for $\mathfrak{M}(\kappa')$.*

LEMMA 6.4. *Let \mathfrak{M} be a class. Suppose that a metric space $X = (X, \mathbf{0})$ with base point $\mathbf{0}$ has a pseudo-translation ⁽⁶⁾ with respect to $\mathbf{0}$, and is locally path-connected at $\mathbf{0}$. Then X has the κ -discrete separation property for $\mathfrak{M}(\kappa')$ if and only if any neighborhood of $\mathbf{0}$ contains a separated subset ⁽⁶⁾ of cardinality κ .*

Using these lemmas, we can obtain the following:

PROPOSITION 6.5. *Let \mathfrak{M} be a class. The space $\ell_2^f(\kappa)$ has the κ -discrete approximation property for $\mathfrak{M}(\kappa')$.*

⁽⁶⁾ Refer to [3, Section 6].

Proof. Take an admissible linear metric d on $\ell_2^f(\kappa)$ defined as follows:

$$d(x, y) = \left(\sum_{\gamma < \kappa} |x(\gamma) - y(\gamma)|^2 \right)^{1/2}$$

for $x = (x(\gamma)), y = (y(\gamma)) \in \ell_2^f(\kappa)$. By Lemmas 6.3 and 6.4, we only need to show that any neighborhood of $\mathbf{0} \in \ell_2^f(\kappa)$ contains a separated subset of cardinality κ . Indeed, for each $\epsilon > 0$, the ϵ -ball centered at $\mathbf{0}$ contains an ϵ -discrete subset $\{(\epsilon/\sqrt{2})\mathbf{e}_\gamma \mid \gamma < \kappa\}$ of cardinality κ , where $\mathbf{e}_\gamma = (x(\gamma'))_{\gamma' < \kappa}$ is the unit element such that $x(\gamma') = 1$ if $\gamma' = \gamma$, and $x(\gamma') = 0$ if $\gamma' \neq \gamma$. ■

Note that $\mathfrak{M}_0^{\text{fd}} \subset \mathfrak{C}$ because \mathfrak{C} is topological, closed hereditary and \mathbf{I} -stable. Therefore any space with the κ -locally finite approximation property for \mathfrak{C} has the κ -locally finite n -cells property for every $n \in \omega$, and hence it has the κ -discrete n -cells property by [6, Lemma 4.2] (cf. [1, Lemma 4.6]). Due to the combination of Propositions 4.8, 6.5 and Lemma 4.7, we have the following:

PROPOSITION 6.6. *The space $\ell_2^f(\kappa) \times \Omega$ has the κ -discrete approximation property for $\mathfrak{C}(\kappa')$.*

Proof. According to Proposition 6.5, the space $\ell_2^f(\kappa)$ has the κ -discrete approximation property for $\mathfrak{C}(\kappa')$. The product $\ell_2^f(\kappa) \times \Omega$ has the κ -locally finite approximation property for $\mathfrak{C}(\kappa')$ by Proposition 4.8, and hence it has the κ -discrete n -cells property for every $n \in \omega$ due to [6, Lemma 4.2] (cf. [1, Lemma 4.6]). Observe that for any map $\phi : A \rightarrow \ell_2^f(\kappa) \times \Omega$, where $A \in \mathfrak{C}(\kappa')$, the closure $\text{cl } \phi(A)$ is of density $\leq \kappa' < \kappa$, so it is a Z -set in $\ell_2^f(\kappa) \times \Omega$ by Proposition 3.2(1). It follows from Lemma 4.7 that $\ell_2^f(\kappa) \times \Omega$ has the κ -discrete approximation property for $\mathfrak{C}(\kappa')$. ■

REMARK 6.7. Proposition 6.6 can be proven by using Proposition 4.2 instead of Proposition 4.8. Indeed, by the same argument as for Proposition 6.5, the product $\ell_2(\kappa) \times \ell_2(\kappa')$, which is homeomorphic to $\ell_2(\kappa)$, has the κ -discrete approximation property for $\mathfrak{C}(\kappa')$. Then the homotopy dense subset $\ell_2^f(\kappa) \times \Omega \subset \ell_2(\kappa) \times \ell_2(\kappa')$ has the κ -locally finite approximation property for $\mathfrak{C}(\kappa')$. It follows from Lemma 4.7 that $\ell_2^f(\kappa) \times \Omega$ has the κ -discrete approximation property for $\mathfrak{C}(\kappa')$.

The space $\ell_2^f(\kappa)$ (or Ω) is a strong Z_σ -set in itself, so we can obtain the following proposition due to Proposition 3.5:

PROPOSITION 6.8. *The space $\ell_2^f(\kappa) \times \Omega$ is a strong Z_σ -set in itself.*

By virtue of Proposition 3.8, we have the following:

PROPOSITION 6.9. *The space $\ell_2^f(\kappa) \times \Omega$ is strongly $\mathfrak{C}(\kappa')$ -universal.*

Proof. Since $\mathfrak{M}_0^{\text{fd}} \subset \mathfrak{C}$, the space $\ell_2^f(\kappa) \times \Omega$ has the κ -discrete n -cells property for any $n \in \omega$ by Proposition 6.6. Moreover, $\ell_2^f(\kappa) \times \Omega$ is a strong

Z_σ -set in itself due to Proposition 6.8. It follows from Proposition 3.2(2) that every Z -set in $\ell_2^f(\kappa) \times \Omega$ is a strong Z -set. Because Ω is strongly $\mathfrak{C}(\kappa')$ -universal, the product space $\ell_2^f(\kappa) \times \Omega$ is also strongly $\mathfrak{C}(\kappa')$ -universal by Proposition 3.8. ■

Proof of Theorem 1.2. According to Propositions 6.9, 6.6 and Lemma 4.6, the space $\ell_2^f(\kappa) \times \Omega$ is strongly $\bigoplus_\kappa \mathfrak{C}(\kappa')$ -universal. Combining this with Propositions 6.1, 6.2, and 6.8, we can conclude that $\ell_2^f(\kappa) \times \Omega$ is a $\bigoplus_\kappa \mathfrak{C}(\kappa')$ -absorbing set in $\ell_2(\kappa) \times \ell_2(\kappa')$. ■

7. Characterizing $(\ell_2^f(\kappa) \times \Omega)$ -manifolds. This section is devoted to proving Theorem 1.3. By Propositions 3.1(1), 3.3 and 6.8, we can obtain the following:

PROPOSITION 7.1. *Every $(\ell_2^f(\kappa) \times \Omega)$ -manifold is a strong Z_σ -set in itself.*

Combining the above proposition with the following theorem, we can easily obtain Theorem 1.3.

THEOREM 7.2. *For a connected ANR $X \in (\bigoplus_\kappa \mathfrak{C}(\kappa'))_\sigma$ which is a strong Z_σ -set in itself, the following conditions are equivalent:*

- (1) X is an $(\ell_2^f(\kappa) \times \Omega)$ -manifold.
- (2) X is strongly $\bigoplus_\kappa \mathfrak{C}(\kappa')$ -universal.
- (3) (a) X is strongly $\mathfrak{C}(\kappa')$ -universal.
(b) X has the κ -discrete approximation property for $\mathfrak{C}(\kappa')$.
- (4) (a) X is strongly $\mathfrak{C}(\kappa')$ -universal.
(b) X has the κ -discrete approximation property for $\{\Omega\}$.
- (5) (a) X is strongly $\mathfrak{C}(\kappa')$ -universal.
(b) X has the κ -discrete n -cells property for every $n \in \omega$.

Proof. Note that strong $\bigoplus_\kappa \mathfrak{C}(\kappa')$ -universality implies the κ -discrete n -cells property for any $n \in \omega$. Due to Theorems 1.1 and 1.2, the equivalence (1) \Leftrightarrow (2) holds. The implications (2) \Rightarrow (3) \Rightarrow (5) are obvious. According to Proposition 3.9, (2) \Rightarrow (4) holds because for each topological copy A_γ of $\Omega \in (\mathfrak{C}(\kappa'))_\sigma$, $\gamma < \kappa$, we have

$$\bigoplus_{\gamma < \kappa} A_\gamma \in \bigoplus_\kappa (\mathfrak{C}(\kappa'))_\sigma \subset \left(\bigoplus_\kappa \mathfrak{C}(\kappa') \right)_\sigma.$$

The implication (3) \Rightarrow (2) follows from Lemma 4.6.

We will prove (4) \Rightarrow (1). Since X is locally contractible, each point of X has an open neighborhood W which is contractible in X . It is enough to show that W is an $(\ell_2^f(\kappa) \times \Omega)$ -manifold, that is, W satisfies (3). As is easily observed, the open set W is an ANR and is in $(\bigoplus_\kappa \mathfrak{C}(\kappa'))_\sigma$. Since X is strongly $\mathfrak{C}(\kappa')$ -universal and is a strong Z_σ -set in itself, so is W by Propositions 3.6 and 3.1(1). It follows from Lemma 4.4 that W has the

κ -locally finite approximation property for $\mathfrak{C}(\kappa')$. Since $\kappa > \kappa'$, the subset W has the κ -discrete approximation property for $\mathfrak{C}(\kappa')$ by the combination of Proposition 3.2(1) and Lemma 4.7. Consequently, W satisfies (3).

It remains to show (5) \Rightarrow (4). Since X has the κ -discrete n -cells property for every $n \in \omega$, X has the κ -discrete approximation property for \mathfrak{M}_0^n by Corollary 4.5. Note that $\mathfrak{M}_0^n \subset \mathfrak{C}$. Hence X is strongly universal for the class $\bigcup_{n \in \omega} (\bigoplus_{\kappa} \mathfrak{M}_0^n)$ by Lemma 4.6, which means that it is strongly $(\bigcup_{n \in \omega} (\bigoplus_{\kappa} \mathfrak{M}_0^n))_{\sigma}$ -universal due to Proposition 3.9. By [11, Theorems 4.9.6 and 6.6.2], the ANR Ω is approximated by a metric polyhedron [11, Section 4.5]. Observe that the metric polyhedron is in the class $(\bigcup_{n \in \omega} (\bigoplus_{\kappa'} \mathfrak{M}_0^n))_{\sigma}$ by [6, Proposition 6.1]. Therefore $\bigoplus_{\gamma < \kappa} A_{\gamma}$, where each A_{γ} is homeomorphic to Ω , can be approximated by some space belonging in $(\bigcup_{n \in \omega} (\bigoplus_{\kappa} \mathfrak{M}_0^n))_{\sigma}$, and hence X has the κ -discrete approximation property for $\{\Omega\}$. Thus the proof is complete. ■

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