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COUNTEREXAMPLE TO THE OFF-TESTING CONDITION IN TWO DIMENSIONS

BY

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Abstract. In proving the local T_b theorem for two weights in one dimension, Sawyer, Shen and Uriarte-Tuero used a basic theorem of Hytönen to deal with estimates for measures living in adjacent intervals. Hytönen's theorem states that the off-testing constant for the Hilbert transform is controlled by Muckenhoupt's $\mathcal{A}_2$ and $\mathcal{A}_2^*$ constants. So in attempting to extend the two-weight T_b theorem to higher dimensions, it is natural to ask if a higher-dimensional analogue of Hytönen's theorem holds that permits analogous control of terms involving measures that live on adjacent cubes. In this paper, we show that this is not the case even in the presence of the energy conditions used in one dimension. Thus, in order to obtain a local T_b theorem in higher dimensions, it will be necessary to find some substantially new arguments to control the notoriously difficult "nearby form". More precisely, we show that Hytönen's off-testing constant for the two-weight fractional integral and the Riesz transform inequalities is not controlled by Muckenhoupt's $\mathcal{A}_2^\alpha$ and $\mathcal{A}_2^{\alpha,*}$ constants and energy constants.

1. Introduction. Characterizing two-weight norm inequalities for singular integrals is an important, long-standing open problem, only recently solved in one dimension by Lacey, Sawyer, Shen and Uriarte-Tuero in the two-part paper [LSSU]–[La]. Hytönen [Hy] later removed the technical hypothesis of no common point masses, and for his proof an important piece was to bound the bilinear form when two functions are supported on disjoint half-lines in terms only of (his variant of) the Muckenhoupt $\mathcal{A}_2$ constants. Sawyer, Shen and Uriarte-Tuero [SSU] used Hytönen's theorem to estimate the difficult "nearby form" in the one-dimensional local T_b theorem. It seems natural to ask whether a higher-dimensional analogue of Hytönen's theorem is true in order to estimate the "nearby form" in the higher-dimensional local T_b theorem. Our paper answers this question negatively, even if we assume the energy and dual energy conditions, as in the case of the one-dimensional two-weight local T_b theorem.

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The key idea is the construction of two measures on the plane placed close to each other (Figure 1) so that the off-testing condition fails but the $\mathcal{A}_2$ and energy conditions along with their duals hold using some one-dimensional results from [LSU]. Following closely the aforementioned work, we first construct two measures in $\mathbb{R}$ with the novelty being the use of a "wrong" homogeneity of the one-dimensional Riesz, Poisson and fractional integrals that accommodates all $0 < \alpha < 2$.

Let $0 \leq \alpha < n$. For any locally finite Borel measure σ, we define the fractional integral on $\mathbb{R}^n$ by

$$I^a(f\sigma)(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, d\sigma(y), \qquad x \notin \operatorname{supp}(f\sigma),$$

for any $f \in L^2(\sigma)$. The Riesz transforms are given by

$$R_m^\alpha(f\sigma)(x) = \int\limits_{\mathbb{R}^n} \frac{(t_m - x_m)f(t)}{|x-t|^{n+1-\alpha}}\, d\sigma(t), \qquad x \notin \operatorname{supp}(f\sigma),\ 1 \leq m \leq n,$$

where $x = (x_1, \dots, x_n)$, $t = (t_1, \dots, t_n)$. If ω is another locally finite Borel measure, we say that the pair of weights (σ, ω) satisfies the *fractional Muckenhoupt* $\mathcal{A}_2^\alpha$ and *dual* $\mathcal{A}_2^\alpha$ *conditions* in $\mathbb{R}^n$ if the the following $\mathcal{A}_2^\alpha$ and $\mathcal{A}_2^{\alpha,*}$ *constants* are finite:

$$\mathcal{A}_2^\alpha \equiv \sup_{Q\in\mathcal{I}} \mathcal{P}^\alpha(Q, \mathbf{1}_{Q^c}\sigma)\frac{\omega(Q)}{|Q|^{1-\alpha/n}}, \qquad \mathcal{A}_2^{\alpha,*} \equiv \sup_{Q\in\mathcal{I}} \mathcal{P}^\alpha(Q, \mathbf{1}_{Q^c}\omega)\frac{\sigma(Q)}{|Q|^{1-\alpha/n}},$$

where $\mathcal{I}$ denotes the collection of all cubes Q in $\mathbb{R}^n$ whose sides are parallel to the axes and

$$\mathcal{P}^\alpha(Q, \mu) = \int\limits_{\mathbb{R}^n} \left(\frac{|Q|^{1/n}}{(|Q|^{1/n} + |x - x_Q|)^2}\right)^{n-\alpha} d\mu(x),$$

with x_Q being the center of the cube, is the reproducing Poisson integral. We also say that the pair (σ, ω) satisfies the *energy* (resp. *dual energy*) *condition* if the following $\mathcal{E}_2^\alpha$ and $\mathcal{E}_2^{\alpha,*}$ constants are finite:

$$(\mathcal{E}_2^\alpha)^2 \equiv \sup_{Q=\dot{\bigcup} Q_r} \frac{1}{\sigma(Q)} \sum_{r=1}^\infty \left(\frac{\mathrm{P}^\alpha(Q_r, \mathbf{1}_Q\sigma)}{|Q_r|^{1/n}}\right)^2 \|x - m_{Q_r}^\omega\|^2_{L^2(\mathbf{1}_{Q_r}\omega)},$$

$$(\mathcal{E}_2^{\alpha,*})^2 \equiv \sup_{Q=\dot{\bigcup} Q_r} \frac{1}{\omega(Q)} \sum_{r=1}^\infty \left(\frac{\mathrm{P}^\alpha(Q_r, \mathbf{1}_Q\omega)}{|Q_r|^{1/n}}\right)^2 \|x - m_{Q_r}^\sigma\|^2_{L^2(\mathbf{1}_{Q_r}\sigma)},$$

with the supremum taken over arbitrary decompositions of a cube Q using a pairwise disjoint union of subcubes Q_r, where

$$\mathrm{P}^\alpha(Q, \mu) = \int\limits_{\mathbb{R}^n} \frac{|Q|^{1/n}}{(|Q|^{1/n} + |x - x_Q|)^{n+1-\alpha}}\, d\mu(x)$$

is the standard Poisson integral and

$$m_I^\mu \equiv \frac{1}{\mu(I)} \int x\, d\mu(x) = \left\langle \frac{1}{|I|_\mu} \int x_1\, d\mu(x), \ldots, \frac{1}{|I|_\mu} \int x_n\, d\mu(x) \right\rangle.$$

In the one-dimensional setting, Hytönen [Hy] has characterized the restricted bilinear inequality,

$$\left| \int_{\mathbb{R}\setminus I} \left(\int_I \frac{f(y)}{|x-y|}\, d\sigma(y) \right) g(x)\, d\omega(x) \right| \lesssim \mathcal{D} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} \tag{1.1}$$

for all intervals I, in terms of the Muckenhoupt conditions, namely

$$\mathcal{D} \approx \sqrt{\mathcal{A}_2^0} + \sqrt{\mathcal{A}_2^{0,*}},$$

where $\mathcal{D}$ is the best constant in (1.1). In [Hy] this inequality was proved for complementary half-lines, where it was noted that the passage to an interval and its complement is then routine. In [SSU], Hytönen's characterization was extended to fractional integrals on the line with the same proof. Namely,

$$\left| \int_{\mathbb{R}\setminus I} \left(\int_I \frac{f(y)}{|x-y|^{1-\alpha}}\, d\sigma(y) \right) g(x)\, d\omega(x) \right| \le \mathcal{D}^\alpha \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}$$

and $\sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,*}} \approx \mathcal{D}^\alpha$, where $\mathcal{D}^\alpha$ is the best constant in the inequality above.

We say that the *off-testing conditions* $\mathcal{T}_{\mathrm{off},\alpha}$ and $\mathcal{R}_{m,\mathrm{off},\alpha}$ hold in $\mathbb{R}^2$ if the following *off-testing constants* $\mathcal{T}_{\mathrm{off},\alpha}$ and $\mathcal{R}_{m,\mathrm{off},\alpha}$ are finite:

$$\mathcal{T}_{\mathrm{off},\alpha}^2 = \sup_Q \frac{1}{\omega(Q)} \int_{\mathbb{R}^2\setminus Q} \left(\int_Q \frac{1}{|x-y|^{2-\alpha}}\, d\omega(y) \right)^2 d\sigma(x),$$

$$\mathcal{R}_{m,\mathrm{off},\alpha}^2 = \sup_Q \frac{1}{\omega(Q)} \int_{\mathbb{R}^2\setminus Q} \left(\int_Q \frac{t_m - x_m}{|x-t|^{3-\alpha}}\, d\omega(t) \right)^2 d\sigma(x), \qquad 1 \le m \le 2,$$

for cubes $Q \subset \mathbb{R}^2$ whose sides are parallel to the axes.

Throughout the paper, the phrases "the $\mathcal{A}_2^\alpha$ condition holds" and "the $\mathcal{A}_2^\alpha$ constant is finite" are equivalent. The same holds for the other conditions/constants.

2. Main results. We show that in two dimensions, we can find a pair of measures such that $\mathcal{A}_2^\alpha$, $\mathcal{E}_2^\alpha$ and their dual conditions hold, but the off-testing condition fails. Thus, one cannot extend Hytonen's theorem in [Hy] to higher dimensions. Indeed, Theorem 2.2 provides a counterexample to the analogue of Hytönen's theorem in $\mathbb{R}^2$ as the Riesz transforms for $\alpha = 0$ are the extensions of the Hilbert transform in higher dimensions.

THEOREM 2.1. *For $0 \leq \alpha < 2$, there exists a pair of locally finite Borel measures σ, ω in $\mathbb{R}^2$ such that the fractional Muckenhoupt $\mathcal{A}_2^\alpha, \mathcal{A}_2^{\alpha,*}$ and the energy $\mathcal{E}_2^\alpha$, $\mathcal{E}_2^{\alpha,*}$ constants are finite, but the off-testing constant $\mathcal{T}_{\text{off},\alpha}$ is not.*

THEOREM 2.2. *For $0 \leq \alpha < 2$, there exists a pair of locally finite Borel measures σ, ω in $\mathbb{R}^2$ such that the fractional Muckenhoupt $\mathcal{A}_2^\alpha, \mathcal{A}_2^{\alpha,*}$ and the energy $\mathcal{E}_2^\alpha$, $\mathcal{E}_2^{\alpha,*}$ constants are finite, but the off-testing constants $\mathcal{R}_{m,\text{off},\alpha}$ are not.*

3. Proofs of the theorems. We begin with the proof of Theorem 2.1. The proof of Theorem 2.2 will be very similar and we will only have to deal with the cancellation occurring in the kernel with Lemma 3.1 being useful.

Proof of Theorem 2.1. First, we build two measures in $\mathbb{R}$, generalizing the work done in [LSU]; they will be used later for our two-dimensional construction.

The one-dimensional construction. Given $0 \leq \alpha < 2$, choose $\frac{1}{3} \leq b < 1$ such that $\frac{1}{9} \leq \left(\frac{1-b}{2}\right)^{2-\alpha} \leq \frac{1}{3}$. Let $s_0^{-1} = \left(\frac{1-b}{2}\right)^{2-\alpha}$. Recall the middle-$b$ Cantor set E_b and the Cantor measure $\ddot{\omega}$ on the closed interval $I_1^0 = [0,1]$. At the kth generation in the construction, there is a collection $\{I_j^k\}_{j=1}^{2^k}$ of 2^k pairwise disjoint closed intervals of length $|I_j^k| = \left(\frac{1-b}{2}\right)^k$. The Cantor set is defined by $\mathrm{E}_b = \bigcap_{k=1}^\infty \bigcup_{j=1}^{2^k} I_j^k$ and the Cantor measure $\ddot{\omega}$ is the unique probability measure supported in E_b that is equidistributed among the intervals $\{I_j^k\}_{j=1}^{2^k}$ at each scale k, i.e

$$\ddot{\omega}(I_j^k) = 2^{-k}, \qquad k \geq 0,\, 1 \leq j \leq 2^k.$$

We denote the removed open middle bth of I_j^k by G_j^k and by $\ddot{z}_j^k$ its center. Following closely [LSU], we define

$$\ddot{\sigma} = \sum_{k,j} s_j^k \delta_{\ddot{z}_j^k},$$

where the positive numbers s_j^k are chosen to satisfy $\frac{s_j^k \ddot{\omega}(I_j^k)}{|I_j^k|^{4-2\alpha}} = 1$, that is,

$$s_j^k = \left(\frac{2}{s_0^2}\right)^k, \qquad k \geq 0,\, 1 \leq j \leq 2^k.$$

The testing constant is infinite. Consider the operator

$$\ddot{T} f(x) = \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{2-\alpha}}\, dy.$$

Note that

$$\ddot{T}\ddot{\omega}(\ddot{z}_1^k) = \int_{I_1^0} \frac{d\ddot{\omega}(y)}{|\ddot{z}_1^k - y|^{2-\alpha}} \geq \int_{I_1^k} \frac{d\ddot{\omega}(y)}{|\ddot{z}_1^k - y|^{2-\alpha}} \geq \frac{\ddot{\omega}(I_1^k)}{\left(\frac{1}{2}\left(\frac{1-b}{2}\right)^k\right)^{2-\alpha}} \approx \left(\frac{s_0}{2}\right)^k,$$

since $|\ddot{z}_1^k - y| \le |\ddot{z}_1^k|$ for $y \in I_1^k$ and $\ddot{z}_1^k = \frac{1}{2}\left(\frac{1-b}{2}\right)^k$. Similar inequalities hold for the rest of $\ddot{z}_j^k$. This implies that the following testing constant is not finite:

$$\int\limits_{I_1^0} (\ddot{T}(\mathbf{1}_{I_1^0}\ddot{\omega})(x))^2 \, d\ddot{\sigma}(y) \gtrsim \sum_{k=1}^{\infty}\sum_{j=1}^{2^k} s_j^k \cdot \left(\frac{s_0}{2}\right)^{2k} = \sum_{k=1}^{\infty}\sum_{j=1}^{2^k} \frac{1}{2^k} = \infty. \tag{3.1}$$

The $\ddot{\mathcal{A}}_2$ constant. Let us now define

$$\ddot{\mathcal{P}}(I,\mu) = \int\limits_{\mathbb{R}} \left(\frac{|I|}{(|I|+|x-x_I|)^2}\right)^{2-\alpha} d\mu(x),$$

and the following variant of the $\mathcal{A}_2^\alpha$ constant:

$$\ddot{\mathcal{A}}_2^\alpha(\ddot{\sigma},\ddot{\omega}) = \sup_I \ddot{\mathcal{P}}(I,\ddot{\sigma}) \cdot \ddot{\mathcal{P}}(I,\ddot{\omega}),$$

where the supremum is taken over all intervals in $\mathbb{R}$. We verify that $\ddot{\mathcal{A}}_2^\alpha$ is finite for the pair $(\ddot{\sigma},\ddot{\omega})$. The starting point is the estimate

$$\ddot{\sigma}(I_r^\ell) = \sum_{(k,j):\, \ddot{z}_j^k \in I_r^\ell} s_j^k = \sum_{k=l}^{\infty} 2^{k-\ell} s_j^k = 2^{-\ell}\sum_{k=l}^{\infty}\left(\frac{4}{s_0^2}\right)^k \approx \left(\frac{2}{s_0^2}\right)^\ell = s_r^\ell$$

and from this it immediately follows that

$$\frac{\ddot{\sigma}(I_j^\ell)\ddot{\omega}(I_j^\ell)}{|I_j^\ell|^{4-2\alpha}} \approx \frac{s_j^\ell \ddot{\omega}(I_j^\ell)}{|I_j^\ell|^{4-2\alpha}} = 1 \quad \text{for } \ell \ge 0,\ 1 \le j \le 2^\ell. \tag{3.2}$$

Now from the definition of $\ddot{\sigma}$ we get

$$\begin{aligned} \ddot{\mathcal{P}}(I_r^\ell,\ddot{\sigma}) &\le \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \int\limits_{I_1^0\setminus I_r^\ell}\left(\frac{|I_r^\ell|}{\left(|I_r^\ell|+|x-x_{I_r^\ell}|\right)^2}\right)^{2-\alpha} d\ddot{\sigma}(x) \\ &\le \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \sum_{m=0}^{\ell}\sum_{k=m}^{\infty} \frac{2^{k-m}s_j^k |I_r^\ell|^{2-\alpha}}{\left(|I_r^\ell| + b\left(\frac{1-b}{2}\right)^m\right)^{4-2\alpha}} \\ &\lesssim \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \sum_{m=0}^{\ell} \frac{2^{-m}|I_r^\ell|^{2-\alpha}\left(\frac{4}{s_0^2}\right)^m}{\left(b\left(\frac{1-b}{2}\right)^{m-\ell}|I_r^\ell|\right)^{4-2\alpha}} \\ &= \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \frac{b^{2\alpha-4}}{|I_r^\ell|^{2-\alpha}}\left(\frac{1}{s_0^2}\right)^\ell \sum_{m=0}^{\ell} 2^m \\ &\lesssim \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \frac{s_r^\ell}{|I_r^\ell|^{2-\alpha}} \approx \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}}, \end{aligned} \tag{3.3}$$

and using the uniformity of $\ddot{\omega}$,

$$
\begin{aligned}
\ddot{\mathcal{P}}(I_r^\ell,\ddot{\omega}) &\le \frac{\ddot{\omega}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \int_{I_1^0\setminus I_r^\ell}\left(\frac{|I_r^\ell|}{\left(|I_r^\ell|+|x-x_{I_r^\ell}|\right)^2}\right)^{2-\alpha} d\ddot{\omega}(x) \\
&\le \frac{\ddot{\omega}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \sum_{k=1}^{\ell}\frac{|I_r^\ell|^{2-\alpha}\ddot{\omega}(I_{j_k}^k)}{\left(|I_r^\ell|+b\left(\frac{1-b}{2}\right)^{k-1}\right)^{4-2\alpha}} \\
&\le \frac{\ddot{\omega}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \sum_{k=1}^{\ell}\frac{|I_r^\ell|^{2-\alpha}\ddot{\omega}(I_{j_k}^k)}{\left(b\left(\frac{1-b}{2}\right)^{k-1-\ell}|I_r^\ell|\right)^{4-2\alpha}} \\
&\lesssim \frac{\ddot{\omega}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \frac{2^{-\ell}}{|I_r^\ell|^{2-\alpha}} = 2\frac{\ddot{\omega}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}},
\end{aligned} \tag{3.4}
$$

where $I_{j_k}^k \subset I_t^{k-1}$, $I_r^\ell \subset I_t^{k-1}$ and $I_{j_k}^k \cap I_r^\ell = \emptyset$, and where all the implied constants in the above calculations depend only on α. From (3.3), (3.4) and (3.2), we see that

$$\ddot{\mathcal{P}}(I_r^\ell,\ddot{\sigma})\ddot{\mathcal{P}}(I_r^\ell,\ddot{\omega}) \lesssim 1.$$

Let us now consider an interval $I \subset I_1^0$ and let $A > 1$ be fixed. Then, let k be the smallest integer such that $\ddot{z}_j^k \in AI$; if there is no such k, then $AI \subsetneq G_j^\ell$ for some ℓ. We have the following cases:

Case 1. Assume that $I \subset AI \subsetneq G_j^k \subset I_j^k$. If $|x_I - \ddot{z}_j^k| \le \operatorname{dist}(x_I, \partial G_j^k)$ then

$$
\begin{aligned}
&\ddot{\mathcal{P}}(I,\ddot{\sigma})\ddot{\mathcal{P}}(I,\ddot{\omega}) \\
&= |I|^{4-2\alpha}\int_{I_1^0}\frac{d\ddot{\sigma}(x)}{(|I|+|x-x_I|)^{4-2\alpha}}\int_{I_1^0}\frac{d\ddot{\omega}(x)}{(|I|+|x-x_I|)^{4-2\alpha}} \\
&\lesssim |I|^{4-2\alpha}\left(\frac{s_j^k}{|I|^{4-2\alpha}} + \frac{1}{|I_j^k|^{2-\alpha}}\int_{I_1^0\setminus G_j^k}\frac{|I_j^k|^{2-\alpha}\, d\ddot{\sigma}(x)}{(|I_j^k|+|x-x_{I_j^k}|)^{4-2\alpha}}\right)\frac{\ddot{\mathcal{P}}(I_j^k,\ddot{\omega})}{|I_j^k|^{2-\alpha}} \\
&\lesssim \frac{|I|^{4-2\alpha}}{|I_j^k|^{2-\alpha}}\left(\frac{s_j^k}{|I|^{4-2\alpha}} + \frac{\ddot{\sigma}(I_j^k)}{|I_j^k|^{4-2\alpha}}\right)\frac{\ddot{\omega}(I_j^k)}{|I_j^k|^{2-\alpha}} \lesssim \frac{\ddot{\sigma}(I_j^k)\ddot{\omega}(I_j^k)}{|I_j^k|^{4-2\alpha}} \approx 1,
\end{aligned} \tag{3.5}
$$

where in the first inequality we use the fact that $|x - x_I| \approx |x - \ddot{z}_j^k| \gtrsim |I_j^k|$ when $x \notin G_j^k$, since x_I is "close" to the center of G_j^k, and for the second inequality we use (3.3) and (3.4).

If $|x_I - \ddot{z}_j^k| > \operatorname{dist}(x_I, \partial G_j^k)$, we can assume $b\left(\frac{1-b}{2}\right)^{m-1} \le |I| \le b\left(\frac{1-b}{2}\right)^m$ for some $m > k$, since for $m = k$ we have $|I| \approx |I_j^k|$, $|x - x_I| \gtrsim |x - x_{I_j^k}|$ for $x \notin G_j^k$ and we can repeat the proof of (3.5). Now let I_t^m be the mth generation interval that is closer to I and that touches the boundary of G_j^k. We see, using $|x_{I_t^m} - \ddot{z}_j^\ell| \lesssim |x_I - \ddot{z}_j^\ell|$ for all $\ell \geq 1$, $1 \leq j \leq 2^\ell$, that $\ddot{\mathcal{P}}(I,\ddot{\sigma}) \lesssim \ddot{\mathcal{P}}(I_t^m,\ddot{\sigma})$ and $\ddot{\mathcal{P}}(I,\ddot{\omega}) \lesssim \ddot{\mathcal{P}}(I_t^m,\ddot{\omega})$, which imply

$$\ddot{\mathcal{P}}(I,\ddot{\sigma})\ddot{\mathcal{P}}(I,\ddot{\omega}) \lesssim 1.$$

Case 2. Now assume $G_j^k \subset AI$. If $I_j^k \cap I = \emptyset$, then, using the minimality of k, we get $I \subset G_t^m$ for some $m < k$ and we can repeat the proof of (3.5). If $I_j^k \cap I \neq \emptyset$ then $|I| \lesssim |I_j^k|$ since otherwise AI would contain $\ddot{z}_t^{k-1}$, contradicting the minimality of k if we fix A large enough depending only on α. Hence

$$|G_j^k| + |x - \ddot{z}_j^k| \leq |G_j^k| + |x_I - \ddot{z}_j^k| + |x - x_I| \leq \left(A + \frac{A}{2}\right)|I| + |x - x_I| \ ,$$

which implies that

$$\begin{aligned}\ddot{\mathcal{P}}(I,\ddot{\sigma}) &\lesssim \int_{I_1^0} \frac{|I|^{2-\alpha}}{\left(|G_j^k| + |x - \ddot{z}_j^k|\right)^{4-2\alpha}}\, d\ddot{\sigma}(x)\\ &\lesssim \frac{|I|^{2-\alpha}}{|I_j^k|^{2-\alpha}} \int_{I_1^0} \frac{|I_j^k|^{2-\alpha}}{\left(|I_j^k| + |x - \ddot{z}_j^k|\right)^{4-2\alpha}}\, d\ddot{\sigma}(x),\end{aligned}$$

and similarly

$$\ddot{\mathcal{P}}(I,\ddot{\omega}) \lesssim \frac{|I|^{2-\alpha}}{|I_t^k|^{2-\alpha}}\ddot{\mathcal{P}}(I_j^k,\ddot{\omega}) \leq \ddot{\mathcal{P}}(I_j^k,\ddot{\omega}),$$

which leads to

$$\ddot{\mathcal{P}}(I,\ddot{\sigma})\ddot{\mathcal{P}}(I,\ddot{\omega}) \lesssim 1.$$

Case 3. If neither $G_j^k \cap AI \neq G_j^k$ nor $G_j^k \cap AI \neq AI$, then we note that $G_j^k \subset 3AI$ and we repeat the proof of Case 2.

Thus, for any interval $I \subset I_1^0$, we have shown that $\ddot{\mathcal{P}}(I,\ddot{\sigma})\ddot{\mathcal{P}}(I,\ddot{\omega}) \lesssim 1$, which implies

$$\ddot{\mathcal{A}}_2^\alpha(\ddot{\sigma},\ddot{\omega}) < \infty. \tag{3.6}$$

The energy constants $\ddot{\mathcal{E}}$ and $\ddot{\mathcal{E}}^*$. Now define the following variant of the energy constants:

$$\begin{aligned}\ddot{\mathcal{E}} &= \sup_{I=\dot{\bigcup} I_r} \frac{1}{\ddot{\sigma}(I)} \sum_{r\geq 1} \ddot{\omega}(I_r) E(I_r,\ddot{\omega})^2 \ddot{\mathrm{P}}(I_r,\mathbf{1}_I\ddot{\sigma})^2,\\ \ddot{\mathcal{E}}^* &= \sup_{I=\dot{\bigcup} I_r} \frac{1}{\ddot{\omega}(I)} \sum_{r\geq 1} \ddot{\sigma}(I_r) E(I_r,\ddot{\sigma})^2 \ddot{\mathrm{P}}(I_r,\mathbf{1}_I\ddot{\omega})^2,\end{aligned}$$

where the suprema are taken over all intervals I and all decompositions

$I = \dot{\bigcup}_{r\geq 1} I_r$, and

$$\begin{aligned}\ddot{\mathrm{P}}(I,\mu) &= \int_{\mathbb{R}} \frac{|I|}{(|I|+|x-x_I|)^{3-\alpha}}\, d\mu(x),\\ E(I,\mu)^2 &= \frac{1}{2}\frac{1}{\mu(I)^2}\int_I\int_I \frac{(x-x')^2}{|I|^2}\, d\mu(x')\, d\mu(x)\\ &= \frac{1}{\mu(I)}\cdot \|x-m_I^\mu\|^2_{L^2(\mathbf{1}_I\mu)} \leq 1.\end{aligned}$$

We first show that $\ddot{\mathcal{E}}$ is finite. We have

$$\begin{aligned}\ddot{\mathrm{P}}(I,\ddot{\sigma}) = \int \frac{|I|}{(|I|+|x-x_I|)^{3-\alpha}}\, d\ddot{\sigma}(x) &\lesssim \sum_{n=0}^{\infty} \frac{\ddot{\sigma}\big((2^n+1)I\big)}{2^n|2^nI|^{2-\alpha}}\\ \leq \sum_{n=0}^{\infty} \inf_{x\in I} M^\alpha\ddot{\sigma}(x)2^{-n} &\lesssim \inf_{x\in I} M^\alpha\ddot{\sigma}(x)\ ,\end{aligned}$$

where

$$M^\alpha\mu(x) = \sup_{I\ni x}\frac{1}{|I|^{2-\alpha}}\int_I d\mu$$

and the implied constants depend only on α. Thus, given an interval $I = \dot{\bigcup}_{r\geq 1} I_r$, we have

$$\sum_{r\geq 1}\ddot{\omega}(I_r)\ddot{\mathrm{P}}(I_r,\mathbf{1}_I\ddot{\sigma})^2 \leq \sum_{r\geq 1}\ddot{\omega}(I_r)\inf_{x\in I}(M^\alpha\mathbf{1}_I\ddot{\sigma})^2(x) \leq \int_I (M^\alpha\mathbf{1}_I\ddot{\sigma})^2(x)\, d\ddot{\omega}(x)$$

and so we are left with estimating the right hand term of the above inequality. We will prove

$$\int_{I_r^\ell}(M^\alpha\mathbf{1}_{I_r^\ell}\ddot{\sigma})^2(x)\, d\ddot{\omega}(x) \leq C\ddot{\sigma}(I_r^\ell), \tag{3.7}$$

where the constant C depends only on α. This will be enough, since for an interval I containing a point mass $\ddot{z}_r^\ell$ but no masses $\ddot{z}_j^k$ for $k<\ell$, we have

$$\begin{aligned}\int_I (M^\alpha\ddot{\sigma})^2(x)\, d\ddot{\omega}(x) &= \int_{I\cap I_r^\ell}(M^\alpha\mathbf{1}_{I\cap I_r^\ell}\ddot{\sigma})^2(x)\, d\ddot{\omega}(x)\\ &\leq \int_{I_r^\ell}(M^\alpha\mathbf{1}_{I_r^\ell}\ddot{\sigma})^2(x)\, d\ddot{\omega}(x) \leq \ddot{\sigma}(I_r^\ell)\approx\ddot{\sigma}(I).\end{aligned}$$

Since the measure $\ddot{\omega}$ is supported in the Cantor set E_b, we can use the fact that for $x \in I_r^\ell \cap \mathrm{E}_b$,

$$M^\alpha(\mathbf{1}_{I_r^\ell}\ddot{\sigma})(x) \lesssim \sup_{(k,j)\colon x\in I_j^k} \frac{1}{|I_j^k|^{2-\alpha}} \int_{I_j^k\cap I_r^\ell} d\ddot{\sigma} \approx \sup_{(k,j)\colon x\in I_j^k} \frac{s_0^{-2(k\vee\ell)} 2^{k\vee\ell}}{s_0^{-k}}$$
$$\approx \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} \approx \left(\frac{2}{s_0}\right)^\ell.$$

Fix m, and let the approximations $\ddot{\omega}^{(m)}$ and $\ddot{\sigma}^{(m)}$ to the measures ω and $\ddot{\sigma}$ be given by

$$d\ddot{\omega}^{(m)}(x) = \sum_{i=1}^{2^m} 2^{-m} \frac{1}{|I_i^m|} \mathbf{1}_{I_i^m}(x)dx \quad \text{and} \quad \ddot{\sigma}^{(m)} = \sum_{k<m} \sum_{j=1}^{2^k} s_j^k \delta_{z_j^k}.$$

For these approximations we have in the same way the estimate for $x \in \bigcup_{i=1}^{2^m} I_i^m$:

$$M^\alpha(\mathbf{1}_{I_r^\ell}\ddot{\sigma}^{(m)})(x) \lesssim \sup_{(k,j)\colon x\in I_j^k} \frac{1}{|I_j^k|^{2-\alpha}} \int_{I_j^k\cap I_r^\ell} d\ddot{\sigma}$$
$$\approx \sup_{(k,j)\colon x\in I_j^k} \frac{\left(\frac{1}{s_0}\right)^{k\vee\ell}\left(\frac{2}{s_0}\right)^{k\vee\ell}}{\left(\frac{1}{s_0}\right)^k} \le C\left(\frac{2}{s_0}\right)^\ell.$$

Thus, for each $m \ge n \ge \ell$,

$$\int_{I_r^\ell} M^\alpha(\mathbf{1}_{I_r^\ell}\ddot{\sigma}^{(n)})^2\, d\ddot{\omega}^{(m)} \le C \sum_{i\colon I_i^m\subset I_r^\ell} \left(\frac{2}{s_0}\right)^{2\ell} 2^{-m} = C2^{m-\ell}\left(\frac{2}{s_0}\right)^{2\ell} 2^{-m}$$
$$= Cs_r^\ell \approx C\int_{I_r^\ell} d\ddot{\sigma}.$$

Now since $\ddot{\omega}^m$ converges weakly to $\ddot{\omega}$, and using the fact that M^α is lower semicontinuous, we obtain

$$\int_{I_r^\ell} M^\alpha(\mathbf{1}_{I_r^\ell}\ddot{\sigma}^{(n)})^2\, d\ddot{\omega} \le \liminf_{m\to\infty} \int_{I_r^\ell} M^\alpha(\mathbf{1}_{I_r^\ell}\ddot{\sigma}^{(n)})^2\, d\ddot{\omega}^{(m)} \le C\ddot{\sigma}(I_r^\ell).$$

Letting $n \to \infty$, by monotone convergence we get (3.7). This proves

$$\sum_{r\ge1} \ddot{\omega}(I_r)\ddot{\mathrm{P}}(I_r, \mathbf{1}_I\ddot{\sigma})^2 \le C\ddot{\sigma}(I), \tag{3.8}$$

which in turn implies $\ddot{\mathcal{E}} < \infty$ as $E(I_r, \ddot{\omega}) \le 1$.

Finally, we show that the dual energy constant $\ddot{\mathcal{E}}^*$ is finite. Let us show that for $I \subset I_1^0$,

$$\ddot{\sigma}(I)E(I,\ddot{\sigma})^2\ddot{\mathrm{P}}(I,\ddot{\omega})^2 \lesssim \ddot{\omega}(I), \tag{3.9}$$

because if we let $\{I_r : r \geq 1\}$ be any partition of I, then (3.9) gives

$$\sum_{r\geq 1} \ddot{\sigma}(I_r) E(I_r, \ddot{\sigma})^2 \ddot{\mathrm{P}}(I_r, \ddot{\omega})^2 \lesssim \sum_{r\geq 1} \ddot{\omega}(I_r) = \ddot{\omega}(I).$$

Now let us establish (3.9). We can assume that $E(I, \ddot{\sigma}) \neq 0$. Let k be the smallest integer for which there is an r with $\ddot{z}_r^k \in I$. And let n be the smallest integer such that for some s we have $\ddot{z}_s^{k+n} \in I$ and $\ddot{z}_s^{k+n} \neq \ddot{z}_r^k$. Then

$$\begin{aligned} E(I, \ddot{\sigma})^2 &= \frac{1}{2} \frac{1}{\ddot{\sigma}(I)^2} \int_I\int_I \frac{|x - x'|^2}{|I|^2}\, d\ddot{\sigma}(x)\, d\ddot{\sigma}(x') \\ &= \frac{1}{\ddot{\sigma}(I)^2}\left[\ddot{\sigma}(\ddot{z}_r^k)\int_I \frac{|x - \ddot{z}_r^k|^2}{|I|^2} d\ddot{\sigma}(x) + \int_I \int_{I\setminus\{\ddot{z}_r^k\}} \frac{|x - x'|^2}{|I|^2}\, d\ddot{\sigma}(x)\, d\ddot{\sigma}(x')\right] \\ &\lesssim \frac{\ddot{\sigma}(\ddot{z}_r^k)\ddot{\sigma}(I\setminus\{\ddot{z}_r^k\})}{\ddot{\sigma}(I)^2} + \frac{\ddot{\sigma}(I \setminus \{\ddot{z}_r^k\})}{\ddot{\sigma}(I)} \lesssim \left(\frac{2}{s_0^2}\right)^n. \end{aligned}$$

Finally, $\ddot{\sigma}(I) \approx (2/s_0^2)^k$, $\ddot{\omega}(I) \approx 2^{-k-n}$, and $\ddot{\mathrm{P}}(I, \ddot{\omega}) \approx (s_0/2)^k$, which proves (3.9).

The two-dimensional construction. It is time now to define the two-dimensional measures that prove the statement of Theorem 2.1. For any set $E \subset \mathbb{R}^2$ let

$$\omega(E) = \sum_{n=0}^{\infty} \ddot{\omega}_n(E),$$

where $\ddot{\omega}_0(E) = \ddot{\omega}(E_x \cap I_1^0)$, E_x is the projection of E on the x-axis, and $\ddot{\omega}_n$ are copies of $\ddot{\omega}_0$ on the intervals $[a_n, a_n + 1] \times \{0\}$ with $k_n = a_{n+1} - (a_n + 1)$ to be determined later. In the same way, let

$$\sigma(E) = \sum_{n=0}^{\infty} \ddot{\sigma}_n(E),$$

where $\ddot{\sigma}_0(E) = \ddot{\sigma}([E \cap (I_1^0 \times \{\gamma_0\})]_x)$, and $\ddot{\sigma}_n$ are copies of $\ddot{\sigma}_0$ on the intervals $[a_n, a_n + 1] \times \{\gamma_n\}$, where the height γ_n will be determined later.

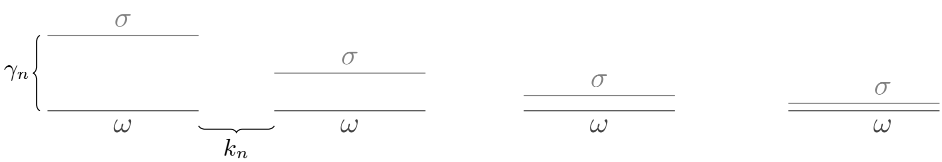

Fig. 1

The $\mathcal{A}_2$ constants. We will now prove that both $\mathcal{A}_2^{\alpha}$ and $\mathcal{A}_2^{\alpha,*}$ constants are finite. Let Q be a cube in $\mathbb{R}^2$, $J_0^n = [a_n, a_n+1] \times \{0\}$ and $J_{\gamma_n}^n = [a_n, a_n+1] \times \{\gamma_n\}$. We consider several cases for Q. If Q intersects only one of the intervals J_0^n, say J_0^0 for convenience, and $(Q \cap J_0^0)_x =: J_0$, we have

$$\mathcal{P}^\alpha(Q, \mathbf{1}_{Q^c}\sigma)\frac{\omega(Q)}{|Q|^{1-\alpha/2}} \lesssim \ddot{\mathcal{P}}(J_0, \ddot{\sigma})\frac{\ddot{\omega}(J_0)}{|J_0|^{2-\alpha}} + \mathcal{P}^\alpha(Q, \mathbf{1}_{(J^1_{\gamma_1})^c}\sigma)\frac{\ddot{\omega}(I_1^0)}{|Q|^{1-\alpha/2}}$$
$$\leq \ddot{\mathcal{A}}_2^\alpha(\ddot{\sigma}, \ddot{\omega}) + C < \infty,$$

using (3.6) and taking k_n large enough so that the second summand is bounded independently of the interval (the value $k_n = 4^{2n\cdot\max\{(2-\alpha)^{-1},1\}}$ would do here). If Q intersects more than one of the intervals J_0^n, it is easy to see, taking into account that Q is very big (since it intersects more than one of the intervals) and that k_n is also large, that

$$\mathcal{P}^\alpha(Q, \mathbf{1}_{Q^c}\sigma)\frac{\omega(Q)}{|Q|^{1-\alpha/2}} \lesssim 1,$$

which of course shows that $\mathcal{A}_2^\alpha$ is finite. Essentially using the same calculations we see that $\mathcal{A}_2^{\alpha,*}$ is finite as well.

Off-testing constant. Let us now check that the off-testing constant is infinite. Choose the cube $Q_n = [a_n, a_n + 1] \times [0, -1]$. Then

$$\frac{1}{\omega(Q_n)} \int_{Q_n^c} \left[\int_{Q_n} \frac{d\omega(y)}{|x-y|^{2-\alpha}} \right]^2 d\sigma(x)$$
$$\geq \frac{1}{\ddot{\omega}(I_1^0)} \int_{I_1^0} \left[\int_{I_1^0} \frac{d\ddot{\omega}(y_1)}{\sqrt{(x_1-y_1)^2+\gamma_n^2}^{\,2-\alpha}} \right]^2 d\ddot{\sigma}(x_1)$$

for $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Taking γ_n such that the last expression in the display above equals n (note that this is feasible, since for $\gamma_n = 0$, formula (3.1) gives infinity in the last expression) we have

$$\mathcal{T}_{\mathrm{off},\alpha}^2 \geq \frac{1}{\omega(Q_n)} \int_{Q_n^c} \left[\int_{Q_n} \frac{d\omega(y)}{|x-y|^{2-\alpha}} \right]^2 d\sigma(x) \geq n,$$

and by letting $n \to \infty$ we conclude that the off-testing constant is infinite.

The energy constants. For the energy constant $\mathcal{E}_2^\alpha$ first, let Q be a cube and $Q = \dot{\bigcup} Q_r$, where $\{Q_r\}_{r=1}^\infty$ is a decomposition of Q. Then

$$\frac{1}{\sigma(Q)} \sum_{r=1}^\infty \left(\frac{\mathrm{P}^\alpha(Q_r, \mathbf{1}_Q\sigma)}{|Q_r|^{1/2}} \right)^2 \|x - m_{Q_r}^\omega\|^2_{L^2(\mathbf{1}_{Q_r}\omega)}$$
$$\leq \frac{2}{\sigma(Q)} \sum_{r=1}^\infty \omega(Q_r)(\mathrm{P}^\alpha(Q_r, \mathbf{1}_Q\sigma))^2.$$

Assume that Q intersects m intervals of the form J_0^n. Then we have $m-2 \lesssim \sigma(Q) \lesssim m$. The case $m = 1$ is exactly the same as the one-dimensional analog for $\ddot{\mathcal{E}}$. Assume $m = 2$. Now we need to consider several cases for Q_r:

- Let Q^1 be the set of cubes Q_r that intersect only one of the intervals J_0^n. Then, following the proof of (3.8), we have

$$\sum_{Q_r\in Q^1} \omega(Q_r)(\mathrm{P}^\alpha(Q_r,\mathbf{1}_Q\sigma))^2 \leq C\sigma(Q).$$

- If Q_r intersects both of the intervals J_0^n then this Q_r is unique since the family $\{Q_r\}_{r\in\mathbb{N}}$ forms a decomposition of Q. Therefore we have

$$\omega(Q_r)(\mathrm{P}^\alpha(Q_r,\mathbf{1}_Q\sigma))^2 \lesssim \frac{\omega(Q_r)\sigma(Q)}{|Q_r|^{2-\alpha}}\sigma(Q) \lesssim \sigma(Q)$$

  using the fact that $|Q_r| \gtrsim 4^2$ since Q_r intersects both of the intervals J_0^n, and $\omega(Q_r) \lesssim 2$, $\sigma(Q) \lesssim 2$.

For $m \geq 3$, again we consider several cases for Q_r:

- If Q_r intersects only one J_0^n, we again have, following the proof of (3.8),

$$\sum_{Q_r\in Q^1} \omega(Q_r)(\mathrm{P}^\alpha(Q_r,\mathbf{1}_Q\sigma))^2 \leq C\sigma(Q).$$

- If Q_r intersects more than one of the intervals J_0^n, the last one being $J_0^{n_0}$, we have

$$\omega(Q_r)(\mathrm{P}^\alpha(Q_r,\mathbf{1}_Q\sigma))^2 \lesssim \frac{\omega(Q_r)\sigma(Q_r^-)^2}{|Q_r|^{2-\alpha}} + \omega(Q_r)\sum_{k=1}^{m}\frac{1}{4^{2k}|Q_r|^{2-\alpha}} \lesssim 2,$$

  where Q_r^- contains all the intervals J_0^n such that $n \leq n_0$. Again in the last inequality we use the fact that Q_r is very big since it intersects at least two intervals J_0^n. Now since the Q_r form a decomposition of Q, we can have at most $m-1$ of these.

Combining the above cases, we obtain

$$\sum_{r=1}^{\infty}\omega(Q_r)(\mathrm{P}^\alpha(Q_r,\mathbf{1}_Q\sigma))^2 \leq C\sigma(Q) + 2m - 2 \leq 2C\sigma(Q),$$

and that proves the energy constant is finite.

The dual energy $\mathcal{E}_2^{\alpha,*}$ can be proved to be finite with the same calculations as for the energy constant, following the proof of (3.9) instead of (3.8) as in the first case above.

This completes the proof of Theorem 2.1. ∎

To obtain the same result for the Riesz transforms, we need to deal with the fact that the kernel is not positive. This prevents us from placing the masses for $\ddot{\sigma}$ at the center of the intervals G_j^k, as we did in the proof of Theorem 2.1: the point mass $\ddot{\sigma}$ located at the center of G_j^k would result in the cancellation of much of the mass, not letting us deduce that the off-testing constant for the Riesz transform is infinite. The following lemma, whose proof follows closely the work in [LSU] but with a two-dimensional twist, helps us overcome this problem, showing that, while not being able to place the point masses in the middle of G_j^k, we can place them far from the boundary. This enables us to show that the $\ddot{\mathcal{A}}_2$ constant is finite, as in the proof of Theorem 2.1. First we need to define the operator

$$\ddot{R}f(x) = \int_{\mathbb{R}} \frac{(x-y)f(y)}{|x-y|^{3-\alpha}}\,dy.$$

LEMMA 3.1. *For $k \geq 1$, $1 \leq j \leq 2^k$, write $G_j^k = (a_j^k, b_j^k)$. Then there exists $0 < c < 1$ that depends only on α such that*

$$\ddot{R}\ddot{\omega}\left(a_j^k + c\left(\frac{1-b}{2}\right)^k b\right) \approx \left(\frac{s_0}{2}\right)^k,$$

where $\ddot{\omega}$ is the measure defined above.

Proof. Fix k. We have

$$\ddot{R}\ddot{\omega}\left(a_1^k + c\left(\frac{1-b}{2}\right)^k b\right) \leq \ddot{R}\ddot{\omega}\left(a_j^k + c\left(\frac{1-b}{2}\right)^k b\right) \leq \ddot{R}\ddot{\omega}\left(a_{2^k}^k + c\left(\frac{1-b}{2}\right)^k b\right)$$

from monotonicity. So it is enough to prove that

$$\left(\frac{s_0}{2}\right)^k \lesssim \ddot{R}\ddot{\omega}\left(a_1^k + c\left(\frac{1-b}{2}\right)^k b\right) \leq \ddot{R}\ddot{\omega}\left(a_{2^k}^k + c\left(\frac{1-b}{2}\right)^k b\right) \lesssim \left(\frac{s_0}{2}\right)^k.$$

We start with the rightmost inequality. From the definitions of $\ddot{R}, \ddot{\omega}$ we get

$$\begin{aligned}
\ddot{R}\ddot{\omega}\left(a_{2^k}^k + c\left(\frac{1-b}{2}\right)^k b\right) &\leq \int_{[0,a_{2^k}^k]} \frac{d\ddot{\omega}(y)}{\left(a_{2^k}^k + c\left(\frac{1-b}{2}\right)^k b - y\right)^{2-\alpha}} \\
&\leq \sum_{\ell=1}^{k} \frac{2^{-\ell}}{\left(a_{2^k}^k + c\left(\frac{1-b}{2}\right)^k b - \left[1 - \left(\frac{1-b}{2}\right)^{\ell-1}\frac{1+b}{2}\right]\right)^{2-\alpha}} \\
&\approx \frac{2^{-k}}{c^{2-\alpha}s_0^{-k}} + \sum_{\ell=1}^{k-1} \frac{2^{-\ell}}{s_0^{-\ell}\left[\frac{1+b}{2} + \left(\frac{1-b}{2}\right)^{k-\ell+1}\left[cb - \frac{1+b}{2}\right]\right]^{2-\alpha}} \\
&\leq \frac{2^{-k}}{c^{2-\alpha}s_0^{-k}} + \sum_{\ell=1}^{k-1} \frac{2^{-\ell}}{s_0^{-\ell}\left[\frac{1+b}{2} - \frac{1+b}{2}\left(\frac{1-b}{2}\right)^{k-\ell+1}\right]^{2-\alpha}}
\end{aligned}$$

since $a_{2^k}^k = 1 - \frac{1+b}{2}\left(\frac{1-b}{2}\right)^k$. The square bracket inside the last fraction is minimized for $\ell = k-1$ and we get the inequality

$$\ddot{R}\ddot{\omega}\left(a_{2^k}^k + c\left(\frac{1-b}{2}\right)^k b\right) \lesssim \frac{2^{-k}}{c^{2-\alpha}s_0^{-k}} + \sum_{\ell=1}^{k-1}\left(\frac{s_0}{2}\right)^\ell \lesssim \frac{1}{c^{2-\alpha}}\left(\frac{s_0}{2}\right)^k,$$

where the implied constants depend again only on α. We should note here that the summand with $\ell = k$ is the dominant one in the above inequality.

Now we consider the left hand inequality. We see that $\ddot{R}\,\ddot{\omega}\big(a_1^k + c\big(\frac{1-b}{2}\big)^k b\big)$ equals

$$\ddot{R}\ddot{\omega}\mathbf{1}_{I_1^{k+1}}\left(a_1^k + c\left(\frac{1-b}{2}\right)^k b\right) + \sum_{\ell=1}^{k+1} \ddot{R}\ddot{\omega}\mathbf{1}_{I_2^\ell}\left(a_1^k + c\left(\frac{1-b}{2}\right)^k b\right) \tag{3.10}$$

and following the argument for the previous inequality we see that

$$\left|\sum_{\ell=1}^{k+1} \ddot{R}\ddot{\omega}\mathbf{1}_{I_2^\ell}\left(a_1^k + c\left(\frac{1-b}{2}\right)^k b\right)\right| \leq A\left(\frac{s_0}{2}\right)^k,$$

where A depends only on α but not on c. The first summand of (3.10) gives

$$\begin{aligned}\int_{I_1^{k+1}} \frac{d\ddot{\omega}(y)}{\big(a_1^k + c\big(\frac{1-b}{2}\big)^k b - y\big)^{2-\alpha}} &\geq \sum_{\ell=k+1}^{\infty} \frac{2^{-\ell-1}}{\big(\big(\frac{1-b}{2}\big)^\ell + c\big(\frac{1-b}{2}\big)^k b\big)^{2-\alpha}} \\ &\approx \frac{s_0^k}{2^k} \sum_{\ell=k+1}^{\infty} \frac{2^{-\ell+k-1}}{\big(\big(\frac{1-b}{2}\big)^{\ell-k} + cb\big)^{2-\alpha}} \\ &= \frac{s_0^k}{2^k} \sum_{\ell=1}^{\infty} \frac{2^{-\ell-1}}{\big(\big(\frac{1-b}{2}\big)^{\ell} + cb\big)^{2-\alpha}}.\end{aligned}$$

Choosing c small enough not depending on k (since the last sum does not depend on k), we obtain

$$\int_{I_1^{k+1}} \frac{d\ddot{\omega}(y)}{\big(a_1^k + c\big(\frac{1-b}{2}\big)^k b - y\big)^{2-\alpha}} \geq C_1\left(\frac{s_0}{2}\right)^k$$

with $C_1 > 2A$, and we conclude the proof. ■

Proof of Theorem 2.2. Set $\dot{z}_j^k = a_j^k + cb\big(\frac{1-b}{2}\big)^k$ and define the measure $\dot{\sigma} = \sum_{k,j} s_j^k \delta_{\dot{z}_j^k}$, where $s_j^k = (2/s_0^2)^k$, as before. Following verbatim the calculations of Theorem 2.1, one can show that $\ddot{A}_2(\dot{\sigma}, \ddot{\omega}) < \infty$. Now define the measures ω and σ, as before, for any measurable set $E \subset \mathbb{R}^2$ by

$$\omega(E) = \sum_{n=0}^{\infty} \ddot{\omega}_n(E) \quad \text{and} \quad \sigma(E) = \sum_{n=0}^{\infty} \dot{\sigma}_n(E),$$

where $\dot{\sigma}_0(E) = \dot{\sigma}([E \cap (I_1^0 \times \{\gamma_0\})]_x)$, and $\dot{\sigma}_n$ are copies of $\dot{\sigma}_0$ on the intervals $[a_n, a_n + 1] \times \{\gamma_n\}$, and where the height γ_n will be determined later. Again, as before, it is easy to see that both $\mathcal{A}_2^\alpha$ and $\mathcal{A}_2^{\alpha,*}$ and both $\mathcal{E}_2^\alpha$ and $\mathcal{E}_2^{\alpha,*}$ are finite.

Let us now finish the proof by showing that the off-testing constants for the Riesz transforms are infinite. From Lemma 3.1 we have $\ddot{R}\ddot{\omega}(\dot{z}_j^k) \gtrsim (s_0/2)^k$, which implies

$$\int_{I_1^0} (\ddot{R}(\mathbf{1}_{I_1^0}\ddot{\omega})(x))^2 \, d\dot{\sigma}(y) \gtrsim \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} s_j^k \cdot \left(\frac{s_0}{2}\right)^{2k} = \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} \frac{1}{2^k} = \infty. \tag{3.11}$$

Now choose the cube $Q_n = [a_n, a_n + 1] \times [0, -1]$. Then

$$\begin{aligned} \mathcal{R}_{1,\text{off},\alpha}^2 &\geq \frac{1}{\omega(Q_n)} \int_{Q_n^c} \left[\int_{Q_n} \frac{(x_1 - y_1)\, d\omega(y)}{|x-y|^{3-\alpha}} \right]^2 d\sigma(x) \\ &\geq \frac{1}{\omega(Q_n)} \int_{I_1^0} \left[\int_{I_1^0} \frac{(x_1 - y_1)\, d\ddot{\omega}(y_1)}{\sqrt{(x_1-y_1)^2 + \gamma_n^2}^{3-\alpha}} \right]^2 d\dot{\sigma}(x_1) = \frac{n}{\omega(Q_n)} \end{aligned}$$

by choosing the height γ_n so that

$$\int_{I_1^0} \left[\int_{I_1^0} \frac{(x_1 - y_1)\, d\ddot{\omega}(y_1)}{\sqrt{(x_1-y_1)^2 + \gamma_n^2}^{3-\alpha}} \right]^2 d\dot{\sigma}(x_1) = n$$

by (3.11). Letting $n \to \infty$, we see that the off-testing constant is infinite. ∎

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