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OFF-DIAGONAL ESTIMATES FOR CUBE SKELETON MAXIMAL OPERATORS

BY

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Abstract. We provide off-diagonal estimates for maximal operators arising from a geometric problem of estimating the size of a certain geometric configuration of k-skeletons in $\mathbb{R}^n$. This is achieved by interpolating a weak-type endpoint estimate with the known diagonal bounds. The endpoint estimate is proved by combining a geometric result about k-skeletons and adapting an argument used to prove off-diagonal estimates for the circular maximal function in the plane.

1. Introduction. In this work we present off-diagonal estimates for maximal operators associated to averaging over (neighborhoods of) squares in the plane and, more generally, over k-skeletons of cubes with arbitrary dimension. Roughly speaking, a k-skeleton is the k-dimensional boundary of an n-dimensional cube with axes-parallel sides in $\mathbb{R}^n$, for $0 \leq k < n$.

The interest in this type of operators emerges, on the one hand, from a geometric problem about the size of a set containing re-scaled and translated copies of the k-skeleton of the unit cube in $\mathbb{R}^n$ around every point of a set of a given size. On the other hand, they are natural variants of the celebrated spherical maximal operator of Bourgain–Stein.

The problem of finding minimal values of the size of sets in $\mathbb{R}^n$ containing k-skeletons centered at any point of a given set was introduced by Keleti, Nagy and Shmerkin [2] for $n = 2$, and by Thornton [5] for $n \geq 3$. Unlike the situation of spheres, where a set containing a sphere with center at every point of a set of positive Lebesgue measure must have positive Lebesgue measure, a set containing the $(n-1)$-skeleton of an n-dimensional cube with axes-parallel sides and center at every point of $\mathbb{R}^n$ can have zero Lebesgue measure. Even more, it can have Hausdorff dimension $n - 1$ (see [5]), the same as a single $(n-1)$-skeleton. This result indicates that Hausdorff dimension does not fit well with this problem, and for this reason in [2, 5] the authors obtained estimates for other notions of fractal dimension, as for

2020 *Mathematics Subject Classification*: Primary 42B25; Secondary 43A85.
Key words and phrases: averages over skeletons, maximal functions, off-diagonal estimates.
Received 9 November 2020; revised 17 January 2021.
Published online 27 May 2021.

DOI: 10.4064/cm8439-2-2021

example box dimension and packing dimension. The arguments from [2, 5] are direct and do not involve any maximal operators. Nevertheless, in [2] the authors introduced a maximal operator associated with the aforementioned geometric problem. More precisely, for each $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $0 < \delta < 1$ and $0 \leq k < n \in \mathbb{N}$, the *$k$-skeleton maximal operator* with width δ is defined as

$$M_\delta^k f(x) = \sup_{1 \leq r \leq 2} \min_{j=1}^{N} \fint_{S^j_{k,\delta}(x,r)} |f(y)|\, dy, \tag{1.1}$$

where $S_{k,\delta}(x, r)$ is a δ-neighborhood of the k-skeleton of a cube with center x and side length $2r$, and the index j enumerates its $N = 2^{n-k}\binom{n}{k}$ faces. Here, as usual, the symbol $\fint_E f\, d\mu := \frac{1}{\mu(E)} \int_E f\, d\mu$ denotes the average of the function f with respect to the measure μ over the set E.

As pointed out in [2], it turns out necessary to take the minimum over all the faces of the k-skeleton to avoid natural and trivial results, analogous to the ones for Hausdorff dimension. It is worth noticing that the maximal operator defined above cannot be bounded from L^p to L^q for any finite p, q, otherwise a set containing a k-skeleton centered at every point of $\mathbb{R}^n$ would have positive measure. Following this line, Shmerkin and the present author studied in [3] discretized versions of M_δ^k, and proved nearly sharp L^p bounds for the k-skeleton maximal operator. One can deduce easily that M_δ^k is bounded on L^p, if $p > 1$, just by comparison with the Hardy–Littlewood maximal operator (see Theorem 1.1 below). Nonetheless, an interesting problem is to determine the rate at which the norm of M_δ^k increases as δ goes to zero.

Theorem 1.1 ([3, Theorem 1.2]). *Given $0 \leq k < n$, $1 \leq p < \infty$ and $\varepsilon > 0$, there exist positive constants $C'(n, k, \varepsilon), C(n, k)$ such that*

$$C'(n, k, \varepsilon) \cdot \delta^{\frac{k-n}{2np}+\varepsilon} \leq \|M_\delta^k\|_{L^p \to L^p} \leq C(n, k) \cdot \delta^{\frac{k-n}{2np}}$$

for all $\delta \in (0, 1)$.

The lower bound relies on a specific construction due to [5], and to obtain the upper bound there is in fact a result for a larger (normwise) maximal-type operator which is localized on a given cube and is linear (see Section 2 for more details). From this L^p bound it is possible to recover the known values for the box counting dimension of sets containing skeletons centered at any point of a prescribed set of centers under the condition of having full box dimension (see [3, Corollary 3.5]).

Motivated by the previous result, the purpose of this article is to study the off-diagonal case. Since the k-skeleton maximal operator is bounded on L^p, $p \geq 1$, and is trivially bounded on L^∞ and from L^1 to L^∞, by the classical Marcinkiewicz interpolation theorem, applied to the larger (normwise) linear maximal operator (see Definition 2.3), we can obtain the boundedness from

L^p to L^q, $1 < p \leq q$. Nevertheless, as mentioned before, we are interested in the rate at which the norm of the k-skeleton maximal operator from L^p to L^q increases as the parameter δ tends to 0, where as usual

$$\|M_\delta^k\|_{L^p \to L^q} = \sup_{f \neq 0} \frac{\|M_\delta^k f\|_{L^q}}{\|f\|_{L^p}}.$$

In what follows, we write $f(\delta) \approx g(\delta)$ if there exist positive constants c, c', not depending on δ, such that $c' g(\delta) \leq f(\delta) \leq c g(\delta)$.

Our main result is the following:

Main Theorem 1.2. *Given $1 < p \leq q < \infty$ and $\varepsilon > 0$, there exist positive constants $C = C(p, q, k, n)$ and $C' = C'(p, q, k, n, \varepsilon)$ such that*

$$C' \delta^{\frac{k-n}{2np} + \varepsilon} \leq \|M_\delta^k\|_{L^p \to L^q} \leq C \delta^{\frac{k-n}{2np}} \qquad \text{if } q \leq q^* p,$$
$$\|M_\delta^k\|_{L^p \to L^q} \approx \delta^{\frac{n}{q} - \frac{n-k}{p}} \qquad \text{if } q > q^* p,$$

for all $\delta \in (0, 1)$ and $q^* = \frac{2n^2}{(n-k)(2n-1)}$.

The lower bound follows by applying M_δ^k to an appropriate function. For the upper bounds, we apply a combinatorial method used by Schlag [4] for the circular maximal operator in the plane, combined with classical interpolation theorems and an estimate from the combinatorial Lemma 2.1, that we will present in the next section.

For the remaining case, $q > p$, it is straightforward to conclude that the k-skeleton maximal operator is unbounded. In fact, given $N \in \mathbb{N}$, let f be the characteristic function of an n-dimensional cube with side length N and consider another cube, N^*, with the same center and side length $N - 6$. By a simple calculation we obtain $M_\delta^k f(x) \geq 1$ for all $x \in N^*$. Therefore,

$$\|M_\delta^k\|_{L^p \to L^q} \geq (N-6)^{n/q} N^{-n/p},$$

which grows with N.

2. Preliminaries and notation. Most of the following definitions and results were introduced in [3]; for completeness here we give a brief summary. In particular, we define the linear maximal operator that turns out to be key to obtaining upper bounds.

We denote the half-open unit cube by Q_0, i.e. $Q_0 = [0, 1)^n$. Let $0 < \delta < 1$ be such that $1/\delta$ is an integer and consider the grid $Q_0 \cap \delta\mathbb{Z}^n$. We define $Q_0^* := \{x_1, \ldots, x_u\}$, $u = \delta^{-n}$, the centers of the half-open n-cubes, $Q_0(1), \ldots, Q_0(u)$, with side length δ determined by $Q_0 \cap \delta\mathbb{Z}^n$.

We define the function $\psi : Q_0 \to Q_0^*$, $x \mapsto x^*$, where x^* denotes the center of the corresponding half-open n-cube with center in Q_0^* and side length δ containing x. Observe that ψ is constant over each $Q_0(i)$ and that the sets $\psi^{-1}(Q_0(i))$, $i = 1, \ldots, u$, form a Borel partition of Q_0.

We use the letter C to denote positive constants, indicating any parameters they may depend on by subscripts, and their values may change from line to line. For example, C_n denotes a positive function of n.

The next combinatorial lemma is crucial to our work. It states, roughly speaking, that given a finite family of k-skeletons, we can extract one face from each skeleton in such a way that the overlaps between these faces are controlled.

LEMMA 2.1 ([3, Lemma 3.2]). *There is a constant $C_{n,k} < \infty$, depending only on n, k, such that the following holds. Let $\{S_k(x_i, r_i)\}_{i=1}^m$ be a finite collection of k-skeletons in $\mathbb{R}^n$. Then it is possible to choose one k-face of each skeleton with the following property: If V is an affine k-plane which is a translate of a coordinate k-plane, then V contains at most*

$$C_{n,k} m^{1-\frac{(n-k)(2n-1)}{2n^2}}$$

of the chosen k-faces.

DEFINITION 2.2. Let Γ_0 denote the family of all functions $\rho : Q_0^* \to [1, 2] \cap \delta\mathbb{Z}$. Fix also $\rho \in \Gamma_0$. For simplicity, write $\mathrm{S}_{k,i} = S_k(x_i, r_i)$, where $r_i = \rho(x_i)$ and $1 \le i \le u$. For this family of k-skeletons, we define the function Φ_ρ:

$$\Phi_\rho(S_k(x_i, r_i)) = \ell_k^i, \tag{2.1}$$

where ℓ_k^i denotes the face of $S_{k,i}$ chosen as in Lemma 2.1.

The k-skeleton maximal operator defined in (1.1), unlike most other kinds of maximal operators, is not sublinear. To deal with this inconvenience we introduce a discretized and linearized version of the problem.

DEFINITION 2.3. Given a function $\rho \in \Gamma_0$ and $0 < \delta < 1$, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we define the *(ρ, k)-skeleton maximal function* $\widetilde{M}^k_{\rho,\delta} f : Q_0 \to \mathbb{R}$ with width δ by

$$\widetilde{M}^k_{\rho,\delta} f(x) = \frac{1}{|\ell_{x,\delta}|} \int_{\ell_{x,\delta}} |f(y)|\, dy, \tag{2.2}$$

where $\ell_{x,\delta}$ is a δ-neighborhood of $\ell_x := \Phi_\rho(S_k(x^*, \rho(x^*)))$.

REMARK 2.4. By definition, $\widetilde{M}^k_{\rho,\delta}$ is constant over each set $Q_0(i)$, $i = 1, \dots, u$, and completely determined by its values over $Q_0^* = \{x_1, \dots, x_u\}$.

Let CQ_0 denote the n-cube with the same center as Q_0 and with side length C. Since r is bounded by 2, in Definition 2.3 it is enough to consider functions f supported on $7Q_0$, since for each $x \in Q_0$ we have $\ell_{x,\delta} \subset 7Q_0$.

The following result establishes a norm relation between the k-skeleton maximal operator and its linearized version.

LEMMA 2.5 ([3, Lemma 2.6]). *There exists a constant $C_{k,n} > 0$ such that if $0 < \delta < 1$, then*

$$\|M_\delta^k\|_{L^p \to L^q(Q_0)} \leq C_{k,n} \sup_{\rho \in \Gamma_0} \|\widetilde{M}_{\rho,3\delta}^k\|_{L^p \to L^q(Q_0)}.$$

In consequence, by obtaining $L^p \to L^q$ estimates on the discrete maximal operator uniformly on ρ, we will also obtain $L^p \to L^q$ bounds for M_δ^k, at least at a local level.

3. The weak endpoint estimate. In the present section we shall prove a weak-type $(1, q^*)$ estimate for the (ρ, k)-skeleton maximal function, the linearized version of M_δ^k. To achieve this, we will follow some ideas used in [4] to treat the circular maximal operator.

Although we are interested in k-skeletons, first we establish the setting for more general sets.

3.1. General setting. Let $\Omega \subseteq \mathbb{R}^n$ be a Borel set and I an index set. For each $x \in \Omega$, consider a family $\mathcal{A}_x = \{A_{x,r}\}_{r \in I}$ of sets with positive and finite Lebesgue measure in $\mathbb{R}^n$ such that $\inf_{r \in I} |A_{x,r}| \geq \delta^\tau$ for some $\tau \in \mathbb{R}$ and $0 < \delta < 1$. Intuitively, we can think that $\mathcal{A}_x$ is a family of sets at δ-scale with measure equal to δ^τ, for all $x \in \Omega$.

In addition, suppose there exists a constant $C > 0$, not depending on x, such that

$$\operatorname{diam}\Big(\{x\} \cup \bigcup_{r \in I} A_{x,r}\Big) \leq C, \tag{3.1}$$

and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ consider the maximal operator $Tf : \Omega \to \mathbb{R}$ given by

$$Tf(x) = \sup_{r \in I} \frac{1}{|A_{x,r}|} \int_{A_{x,r}} |f(y)|\, dy.$$

Let $E \subset \mathbb{R}^n$ be a bounded set, $0 < \lambda \leq 1$, and $\{x_j\}_{j=1}^m$ be a maximal δ-separated sequence in

$$F = \{x \in \Omega : T\mathbf{1}_E(x) > \lambda\}. \tag{3.2}$$

Note that, by (3.1), F is a bounded set. Pick $r_j \in I$ such that

$$|A_{x_j,r_j} \cap E| > \lambda |A_{x_j,r_j}|, \qquad 1 \leq j \leq m.$$

To simplify the notation, in what follows we write A_j instead of A_{x_j,r_j} and A_j^* instead of $A_{x_j,r_j} \cap E$. Consider the multiplicity function

$$\Upsilon = \sum_{j=1}^m \mathbf{1}_{A_j^*}$$

and define μ to be the smallest integer for which there exist at least $m/2$ values of j such that

$$|\{x \in A_j^* : \Upsilon(x) \le \mu\}| \ge \frac{\lambda}{2}|A_j|.$$

Observe that

$$\mu|E| \ge \int\limits_{\{x\in E:\,\Upsilon(x)\le\mu\}} \Upsilon\,dx = \sum_{j=1}^{m} |\{x \in A_j^* : \Upsilon(x) \le \mu\}| \ge \frac{\lambda}{2} m\delta^{\tau}. \tag{3.3}$$

The following lemma characterizes the estimates on μ required to obtain restricted weak-type (p,q) estimates for the maximal operator T.

Recall that T is said to be of *weak-type* (p,q) with norm K, written

$$\|Tf\|_{q,\infty} \le K\|f\|_p,$$

if for all $f \in L^p(\mathbb{R}^n)$ and $t \in (0,1]$,

$$|\{x \in \Omega : Tf(x) > t\}|^{1/q} \le Kt^{-1}\|f\|_{L^p}.$$

When the above inequality holds for characteristic functions of arbitrary sets in $\mathbb{R}^n$ of finite measure, we say that T is of *restricted weak type* (p,q).

LEMMA 3.1. *Let $\alpha \ge 0$ and $\beta < 1$. There exists a positive constant $C = C(n,q)$ such that, if $\mu \le H\lambda^{-\alpha}m^{\beta}$ for every choice of a bounded set $E \subset \mathbb{R}^n$, $0 < \lambda \le 1$ and $\delta > 0$, then T is of restricted weak type (p,q) with constant $CH^{1/p}\delta^{-\gamma}$, where $p = \alpha + 1$, $q = p(1-\beta)^{-1}$ and $\gamma = \tau/p - n/q$.*

Proof. We need to prove that

$$|\{x \in \Omega : T\mathbf{1}_E(x) > \lambda\}|^{1/q} \le CH^{1/p}\delta^{-\gamma}\lambda^{-1}|E|^{1/p} \tag{3.4}$$

for every set $E \subset \mathbb{R}^n$ of finite measure. Let $E \subset \mathbb{R}^n$ be a bounded set and $\{x_j\}_{j=1}^m$ be a maximal δ-separated sequence in F, as in (3.2). Then

$$|\{x \in \Omega : T\mathbf{1}_E(x) > \lambda\}| \le c_n\delta^n m,$$

where c_n denotes the measure of the n-dimensional ball with radius 1. In view of (3.3), i.e. $|E| \ge \mu^{-1}\frac{\lambda}{2}m\delta^{\tau}$, and by the assumption on μ we conclude that

$$\begin{aligned} H^{1/p}\delta^{-\gamma}\lambda^{-1}|E|^{1/p} &\ge H^{1/p}\delta^{-\gamma}\lambda^{-1}(H^{-1}(\lambda/2)^{1+\alpha}m^{1-\beta}\delta^{\tau})^{1/p} \\ &= \tfrac{1}{2}\delta^{-\gamma+\tau/p}m^{(1-\beta)/p} = \tfrac{1}{2}(m\delta^n)^{1/q}. \end{aligned}$$

Therefore, (3.4) follows with $C = 2c_n^{1/q}$. If E is not bounded, one can write it as a union of bounded sets and the result follows similarly. ∎

3.2. The case of k-skeletons. First, we introduce some notation and definitions.

Given $n \ge 2$, $0 \le k < n \in \mathbb{N}$, and $\{e_1,\dots,e_n\}$ the canonical base in $\mathbb{R}^n$, we denote by $\pi_1,\dots,\pi_{\binom{n}{k}}$ the $\binom{n}{k}$ subspaces of dimension k generated by vectors in the canonical base. From now on, we refer to them as k-planes. For example, if $n = 2$ and $k = 1$, then π_1 and π_2 are the usual coordinate axes in the plane. If $k = 0$, the 0-plane will be the origin.

Let $\rho \in \Gamma_0$ and $\ell_k^1, \ldots, \ell_k^u$ be as in (2.1). We say that ℓ_k^i, $1 \le i \le u$, is *parallel* to π_ω if there exists a point $v \in \mathbb{R}^n$ such that ℓ_k^i is contained in the affine k-plane $V = \pi_\omega + v$.

For each $\omega = 1, \ldots, \binom{n}{k}$, we define the sets

$$E_{\pi_w} := \{x_i \in Q_0^* : \ell_i^k \text{ is parallel to } \pi_w\}.$$

Let $E \subset \mathbb{R}^n$ be a bounded set, $0 < \lambda \le 1$, and consider the set

$$F_\omega := \{x \in \psi^{-1}(E_{\pi_\omega}) : \widetilde{M}^k_{\rho,\delta}\mathbf{1}_E(x) > \lambda\},$$

where the index ω is fixed. By Remark 2.4, F_ω is the union of those half-open cubes $Q_0(i)$ with center $x_i \in F_\omega$. Note that it is enough to consider those sets E such that $E \cap 7Q_0 \neq \emptyset$, otherwise $F_\omega = \emptyset$.

The set $Q_0^* \cap F_\omega := \{x_j\}_{j=1}^{m_\omega}$ is a maximal δ-separated set in F_ω, and for each x_j we have

$$|E \cap \ell^j_{k,\delta}| > \lambda |\ell^j_{k,\delta}|.$$

For simplicity, we write $(\ell^j_\delta)^*$ instead of $E \cap \ell^j_{k,\delta}$ and ℓ^j_δ instead of $\ell^j_{k,\delta}$.

We define the multiplicity function associated to F_ω by

$$\Upsilon_\omega = \sum_{j=1}^{m_\omega} \mathbf{1}_{(\ell^j_\delta)^*}$$

and μ, as before, to be the smallest integer such that there exist at least $m_w/2$ values of j such that

$$|\{x \in (\ell^j_\delta)^* : \Upsilon_\omega(x) \le \mu\}| \ge \frac{\lambda}{2}|\ell^j_\delta|.$$

Fix $1 \le j \le m_\omega$ and consider $x \in \ell^j_\delta$. Since $(\ell^j_\delta)^* \subseteq \ell^j_\delta$, we see that

$$\Upsilon_\omega \le \sum_{j=1}^{m_\omega} \mathbf{1}_{\ell^j_\delta}.$$

The faces $\{\ell_k^1, \ldots, \ell_k^{m_\omega}\}$ were chosen from a family of k-skeletons using Lemma 2.1 and each one of them belongs to an affine k-plane parallel to π_ω. Therefore, by means of the lemma mentioned, we obtain an estimate for the number of faces containing x. More precisely, for all $x \in \ell^j_\delta$ we have

$$\Upsilon_\omega(x) \le C_{n,k} m_\omega^{1-\frac{(n-k)(2n-1)}{2n^2}}.$$

Since this holds for every $j = 1, \ldots, m_\omega$, by the definition of μ, we obtain

$$\mu \le C_{n,k} m_\omega^{1-\frac{(n-k)(2n-1)}{2n^2}}. \tag{3.5}$$

The following lemma provides us with the weak-type $(1, q^*)$ estimate mentioned at the beginning of the section.

LEMMA 3.2. *Given $1 < q < \infty$, $\rho \in \Gamma_0$, $0 < \delta < 1$ and $0 \le k < n$, there exists a positive constant $C = C(k, n, q)$ such that*

$$\|\widetilde{M}^k_{\rho,\delta} f\|_{q,\infty} \le \begin{cases} C\,\delta^{\frac{k-n}{2np}}\|f\|_1 & \text{if } 1 < q \le q^*, \\ C\,\delta^{\frac{n}{q}+k-n}\|f\|_1 & \text{if } q^* < q < \infty, \end{cases}$$

for every $f \in L^1(7Q_0)$ and $q^* = \frac{2n^2}{(n-k)(2n-1)}$.

Proof. Consider the restricted maximal operator $\widetilde{M}^k_{\rho,\delta} : \psi^{-1}(E_{\pi_\omega}) \to \mathbb{R}$. By (3.5) we have

$$\mu \le C_{n,k} m_\omega^{1-\frac{(n-k)(2n-1)}{2n^2}}.$$

Applying Lemma 3.1 with $\alpha = 0$, $\beta = 1 - \frac{(n-k)(2n-1)}{2n^2}$, $H = C_{n,k}$ and $\tau = n - k$, we obtain

$$|\{x \in \psi^{-1}(E_{\pi_w}) : \widetilde{M}^k_{\rho,\delta}\mathbf{1}_E(x) > \lambda\}|^{1/q^*} \le C_{k,n}\delta^{\frac{k-n}{2n}}\lambda^{-1}|E| \tag{3.6}$$

for every bounded set $E \subset \mathbb{R}^n$.

Since this holds for every $\omega = 1, \dots, \binom{n}{k}$, and the sets $\psi^{-1}(E_{\pi_\omega})$ form a Borel partition of Q_0, we have

$$\begin{aligned} |\{x \in Q_0 : \widetilde{M}^k_{\rho,\delta}\mathbf{1}_E(x) > \lambda\}| &\le \sum_{\omega=1}^{\binom{n}{k}} |\{x \in \psi^{-1}(E_{\pi_\omega}) : \widetilde{M}^k_{\rho,\delta}\mathbf{1}_E(x) > \lambda\}| \\ &\le \binom{n}{k}(C_{n,k}\delta^{\frac{k-n}{2n}}\lambda^{-1}|E|)^{q^*}, \end{aligned}$$

and therefore we can conclude that $\widetilde{M}^k_{\rho,\delta}$ is of restricted weak type $(1, q^*)$.

If $1 < q < q^*$, then

$$\begin{aligned} |\{x \in Q_0 : \widetilde{M}^k_{\rho,\delta}\mathbf{1}_E > \lambda\}|^{1/q} &\le |\{x \in Q_0 : \widetilde{M}^k_{\rho,\delta}\mathbf{1}_E > \lambda\}|^{1/q^*} \\ &\le C_{k,n}\lambda^{-1}\delta^{\frac{k-n}{2n}}|E|, \end{aligned}$$

and $\widetilde{M}^k_{\rho,\delta}$ is of restricted weak-type $(1, q)$.

For the remaining case $q > q^*$, take a constant $v > 0$ with $1/q = 1/q^* - v$. Trivially, by (3.5),

$$\mu \le C_{n,k} m_\omega^{1-\frac{(n-k)(2n-1)}{2n^2}+v}.$$

Invoking Lemma 3.1 with $\beta = 1 - \frac{(n-k)(2n-1)}{2n^2} + v$, $\alpha = 0$ and $K = C_{n,k}$, we obtain

$$|\{x \in \psi^{-1}(E_{\pi_w}) : \widetilde{M}^k_{\rho,\delta}\mathbf{1}_E(x) > \lambda\}|^{1/q} \le C_{k,n}\delta^{\frac{n}{q}-(n-k)}\lambda^{-1}|E|.$$

Therefore,

$$|\{x \in Q_0 : \widetilde{M}^k_{\rho,\delta}\mathbf{1}_E(x) > \lambda\}|^{1/q} \leq C_{k,n}\delta^{\frac{n}{q}+k-n}\lambda^{-1}|E|,$$

which implies that $\widetilde{M}^k_{\rho,\delta}$ is of restricted weak type $(1,q)$ with $q > q^*$.

Finally, by [1, Theorem 5.5.3], we can conclude that $\widetilde{M}^k_{\rho,\delta}$ is of weak-type $(1,q)$. ■

4. Proof of the main theorem. In order to prove the upper bounds in Theorem 1.2, we first establish the following result.

PROPOSITION 4.1. *For every $1 < p \leq q$, $0 < \delta < 1$ and $\rho \in \Gamma_0$, there exist positive constants C and C' depending on k, n, p, q such that*

$$\|\widetilde{M}^k_{\rho,\delta}\|_{L^p \to L^q(Q_0)} \leq \begin{cases} C\delta^{\frac{k-n}{2np}} & \text{if } q \leq q^*p, \\ C'\delta^{\frac{n}{q}+\frac{k-n}{p}} & \text{if } q > q^*p. \end{cases}$$

Proof. Given (p,q) with $p \leq q$, we deduce the assertion from Lemma 3.2, the trivial bound

$$\|\widetilde{M}^k_{\rho,\delta}f\|_{L^\infty(Q_0)} \leq \|f\|_{L^\infty(7Q_0)},$$

and the Marcinkiewicz interpolation theorem (see e.g. [1, Theorem 4.4.13]). ■

For each $z = (z_1, \ldots, z_n) \in \mathbb{Z}^n$ we denote

$$Q_z = [z_1, z_1 + 1) \times \cdots \times [z_n, z_n + 1).$$

By translation invariance, Proposition 4.1 continues to hold if we replace Q_0 by Q_z.

Proof of Theorem 1.2. In both cases, the upper bounds follow from Lemma 2.5, Proposition 4.1 and the facts that $\mathbb{R}^n = \bigcup_z Q_z$ and $\|\sum_{z\in\mathbb{Z}^n} \mathbf{1}_{7Q_z}\|_\infty$ is finite.

If $q > q^*p$, to obtain the lower bound, we consider $f = \mathbf{1}_{B_{6\delta}}$, where B is the k-skeleton of an n-cube with side length 1 and center at some point $x_0 \in \mathbb{R}^n$. It is easy to see that $M^k_\delta f(x) \geq 1$ for all x in a δ-neighborhood of x_0 and $\|M^k_\delta f\|_{L^q} \geq \delta^{n/q}$. Since

$$\|f\|_{L^p} = |B_{6\delta}|^{n-k/p} \approx \delta^{n-k/p},$$

we have

$$\|M^k_\delta\|_{L^p \to L^q} \geq c_{n,k}\delta^{\frac{n}{q}+\frac{k-n}{p}}.$$

For the remaining lower bound in the case $q \leq q^*p$, see [3, Proposition 2.2]. ■

Acknowledgments. I thank my supervisor Pablo Shmerkin for his guidance and for his extremely valuable comments and suggestions. The author is partially supported by grant PIP (CONICET) 11220110101018.

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