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ONE-RELATOR SASAKIAN GROUPS

BY

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Abstract. We prove that any one-relator group G is the fundamental group of a compact Sasakian manifold if and only if G is either finite cyclic or isomorphic to the fundamental group of a compact Riemann surface of genus $g > 0$ with at most one orbifold point of order $n \geq 1$. We also classify all groups of deficiency at least 2 that are also the fundamental group of some compact Sasakian manifold.

1. Introduction. Fundamental groups of compact Sasakian manifolds are called *Sasakian groups*. While the Kähler groups have been studied for long (see [ABCKT] and references therein), the study of their odd-dimensional relatives, viz. Sasakian groups, has been started relatively recently [Ch], [BM2]. In this paper, we extend the results of [BM1], [Ko] to Sasakian groups by analyzing one-relator Sasakian groups and Sasakian groups of deficiency greater than one.

We prove the following generalization of the corresponding statement for Kähler groups [Ko] (see Theorem 3.3 below):

THEOREM 1.1. *Let G be a Sasakian group with* $\mathrm{def}(G) > 1$. *Then G must be an orbifold surface group with genus greater than* 1.

The following theorem classifies one-relator Sasakian groups (see Theorem 4.10 below):

THEOREM 1.2. *Let G be an infinite one-relator group. Then G is Sasakian if and only if it is isomorphic to*

$$\Big\langle a_1, b_1, \ldots, a_i, b_i, \ldots, a_g, b_g \;\Big|\; \Big(\prod_{i=1}^{g} [a_i, b_i]\Big)^n \Big\rangle,$$

where g and n are some positive integers.

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We observe that each of the groups

$$\Big\langle a_1, b_1, \ldots, a_g, b_g \;\Big|\; \Big(\prod_{i=1}^{g}[a_i, b_i]\Big)^n \Big\rangle, \qquad g, n > 0,$$

can in fact be realized as the fundamental group of a closed Sasakian manifold as follows. It was shown in [BM1] that these groups are fundamental groups of smooth projective varieties. Since the fundamental group of a smooth projective variety is also a Sasakian group [Ch, Proposition 1.2], the above groups are also Sasakian groups.

It was shown in [Ch] that all finite groups are Sasakian. Since the only finite one-relator groups are finite cyclic groups, it follows that finite one-relator Sasakian groups are precisely the finite cyclic groups.

A useful tool we establish along the way is the following generalization of the corresponding fact for Kähler groups in [ABCKT, Ch. 4] (see Theorem 3.1 below):

THEOREM 1.3. *If G is a Sasakian group that satisfies $\beta^2_1(G) > 0$, then G is virtually a surface group.*

In Theorem 1.3, $\beta^2_1(G)$ denotes the first l^2 Betti number of G.

2. Sasakian groups of positive deficiency. We refer to [BoGa] for the definition and basic properties of Sasakian manifolds. The Sasakian structure on any Sasakian manifold M can be perturbed in order to make the Sasakian structure quasi-regular [Ru], [OV, Theorem 1.2]. In particular, every Sasakian group is the fundamental group of some closed quasi-regular Sasakian manifold. In view of this, henceforth all Sasakian manifolds considered will be assumed to be quasi-regular.

Let M be a quasi-regular closed Sasakian manifold. The group $\mathrm{U}(1)$ acts on M; let

$$\alpha : M \times \mathrm{U}(1) \to M \tag{2.1}$$

be the action map. The Kähler orbifold base $M/\mathrm{U}(1)$ of M will be denoted by B [Ru], [BoGa, Theorem 7.1.3]. Let

$$f : M \to M/\mathrm{U}(1) = B \tag{2.2}$$

be the quotient map. Let

$$G = \pi_1(M)$$

be the fundamental group. For notational convenience we will omit mentioning the base point for any fundamental group.

Take a regular point $x_0 \in B$. This means that the action of $\mathrm{U}(1)$ on the fiber $f^{-1}(x_0)$ is free. Let $i : f^{-1}(x_0) \to M$ be the inclusion map of the regular fiber. The map $\mathrm{U}(1) \to f^{-1}(x_0)$ defined by $\lambda \to x_1\lambda$, where $x_1 \in f^{-1}(x_0)$

is any given point, is a diffeomorphism. The image $i_*\pi_1(f^{-1}(x_0)) = i_*\mathbb{Z}$ is a (finite or infinite) cyclic group, which we shall denote by C; to clarify, C may be the trivial group. Let

$$Q := \pi_1^{\mathrm{orb}}(B)$$

be the orbifold fundamental group of B regarded as the orbifold quotient of M by the action of $\mathrm{U}(1)$. Then there is a short exact sequence

$$1 \to C \to G = \pi_1(M) \to Q \to 1. \tag{2.3}$$

LEMMA 2.1 ([BlGo], [Fu], [Ta], [Ch]). *For the group G in* (2.3)*, the first Betti number $b_1(G)$ is even.*

Note that from Lemma 2.1 it follows that a non-trivial free group with finitely many generators cannot be a Sasakian group.

The *deficiency* of a finitely presented group Γ is the maximum of $n - r$ taken over all possible finite presentations of Γ, where n and r are the numbers of generators and relations respectively.

COROLLARY 2.2. *If G in* (2.3) *is of positive deficiency, then $b_1(G) \geq 2$. If G in* (2.3) *is an infinite one-relator group, then $b_1(G) \geq 2$.*

Proof. Since G has positive deficiency, we get $b_1(G) \geq 1$. From Lemma 2.1 it follows that $b_1(G) \geq 2$.

Since infinite one-relator groups have positive deficiency, the second statement follows from the first. ∎

We shall denote the first l^2 Betti number of a group H by $\beta_1^2(H)$. As Kotschick did in [Ko], we shall make essential use of the following theorem due to Gaboriau and Lück.

THEOREM 2.3 ([Lü], [Ga]). *If $1 \to N \to H \to Q_1 \to 1$ is a short exact sequence of infinite finitely generated groups with H finitely presented, then $\beta_1^2(H) = 0$.*

COROLLARY 2.4. *Assume that G in* (2.3) *is of positive deficiency* (*for example, it is an infinite one-relator group*)*. If C in* (2.3) *is infinite, then $\beta_1^2(G) = 0$.*

Proof. In view of Theorem 2.3 it suffices to show that Q in (2.3) is infinite. Towards a contradiction, suppose Q is finite. Then G has a subgroup $C = \mathbb{Z}$ of finite index in it. Let N be the finite covering of M such that $C = \pi_1(N) \subset \pi_1(M) = G$. So N is a quasi-regular closed Sasakian manifold with fundamental group $\mathbb{Z}$. This contradicts Lemma 2.1. ∎

We shall also need the following fundamental inequality:

LEMMA 2.5 ([Gr]). *Let $\mathrm{def}(H)$ denote the deficiency of H. Then*

$$\mathrm{def}(H) - 1 \leq \beta_1^2(H).$$

Combining Corollary 2.4 and Lemma 2.5 leads to:

LEMMA 2.6. *Assume that G in* (2.3) *is of deficiency at least* 2. *Then the cyclic group C in* (2.3) *is finite.*

Proof. Since G is of deficiency at least 2, Lemma 2.5 says that $\beta_1^2(G) \geq 1$. On the other hand, if C is infinite, then $\beta_1^2(G) = 0$ by Corollary 2.4. Hence the cyclic group C in (2.3) is finite. ∎

The following lemma is proved in [Ko].

LEMMA 2.7 ([Ko, Lemma 3]). *If C in* (2.3) *is finite, then Q in* (2.3) *satisfies the condition*

$$\beta_1^2(Q) = p \cdot \beta_1^2(G),$$

where p is the order of C.

LEMMA 2.8. *Assume that G in* (2.3) *is of deficiency at least* 2. *Then Q in* (2.3) *is the fundamental group of a smooth projective orbifold with $\beta_1^2(Q) > 0$.*

Proof. Since M in (2.2) is a closed quasi-regular Sasakian manifold, the quotient B in (2.2) is a projective orbifold.

Since G is of deficiency at least 2, Lemma 2.6 implies that C in (2.3) is finite. in view of Lemma 2.5, the deficiency assumption also implies that $\beta_1^2(G) > 0$. Hence $\beta_1^2(Q) > 0$ by Lemma 2.7. ∎

3. Sasakian groups with deficiency greater than 1

3.1. From Sasakian groups to virtual surface groups. A group H is said to be *virtually a surface group* if some finite index subgroup of H is the fundamental group of a closed surface of positive first Betti number. The following theorem was proved in [Gr], [ABR], [ABCKT, p. 47, Theorem 4.1] for Kähler groups.

THEOREM 3.1. *If G in* (2.3) *satisfies the condition $\beta_1^2(G) > 0$, then G is virtually a surface group.*

Proof. Let C, G, Q be as in (2.3). As we saw in the proof of Corollary 2.4, if Q is finite then C is finite. On the other hand, if Q is infinite, then Theorem 2.3 implies that C is finite. Therefore, we conclude that C is finite.

Hence, by Lemma 2.7 we have $\beta_1^2(Q) > 0$. Note that Q equals the orbifold fundamental group of the orbifold B in (2.2).

Consider the Sasakian metric g_M on the quasi-regular Sasakian manifold M in (2.2). In the proof of [ABCKT, p. 47, Theorem 4.1] substitute (M, g_M) in place of the Kähler metric. Then it is straightforward to check that all results in [ABCKT, Sections 4.2 and 4.3] remain valid.

Let

$$\psi : \widetilde{M} \to M$$

be the universal covering of the quasi-regular Sasakian manifold M in (2.2). Consider the action of the fundamental group $\pi_1(M)$ on $\widetilde{M}$. Set

$$\widehat{M} := \widetilde{M}/C,$$

where $C \subset \pi_1(M) = G$ is the subgroup in (2.3). Let

$$p_0 : \widehat{M} \to M \tag{3.1}$$

be the natural projection.

The action of $\mathrm{U}(1)$ on M in (2.1) canonically lifts to an action of $\mathrm{U}(1)$ on $\widehat{M}$. To prove this, let

$$\chi = \alpha \circ (p_0 \times \mathrm{Id}_{\mathrm{U}(1)}) : \widehat{M} \times \mathrm{U}(1) \to M$$

be the composition, where α is the map in (2.1). The image of the homomorphism

$$\chi_* : \pi_1(\widehat{M} \times \mathrm{U}(1)) \to \pi_1(M)$$

is clearly the subgroup $C \subset G = \pi_1(M)$. From this it follows that the action of $\mathrm{U}(1)$ on M canonically lifts to an action of $\mathrm{U}(1)$ on $\widehat{M}$. Let

$$\varphi : \widehat{M} \to \widehat{B} := \widehat{M}/\mathrm{U}(1) \tag{3.2}$$

be the orbifold quotient for this action of $\mathrm{U}(1)$ on $\widehat{M}$.

So we have a commutative diagram

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{p_0} & M \\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle f} \\ \widehat{B} & \xrightarrow{q} & B \end{array} \tag{3.3}$$

where p_0, φ and f are the maps in (3.1), (3.2) and (2.2) respectively. The map q in (3.3) is an étale Galois covering, in the orbifold category, with Galois group $Q = G/C$.

Following the proof of Theorem 4.14 in [ABCKT, Section 4.4, p. 53] we see that there are a proper holomorphic map to the unit disk

$$h : \widehat{B} \to \mathbb{D}^2 := \{z \in \mathbb{C} \mid |z| < 1\} \tag{3.4}$$

and a homomorphism $\rho : Q \to \mathrm{Aut}(\mathbb{D}^2) = \mathrm{PSL}(2, \mathbb{R})$ such that

- the fibers of h are connected,
- h is Q-equivariant for the action of Q on $\mathbb{D}^2$ given by ρ and the Galois action of Q on $\widehat{B}$, and
- the homomorphism $h^* : \mathcal{H}^1_{(2)}(\mathbb{D}^2) \to \mathcal{H}^1_{(2)}(\widehat{B})$ corresponding to h is an isomorphism.

Consider the complex one-dimensional orbifold $\mathcal{O} = \mathbb{D}^2/\rho(Q)$ with orbifold fundamental group $Q_1 = \rho(Q)$. Since h in (3.4) is Q-equivariant, we conclude that

- h induces a holomorphic (orbifold) map $h_1 : B \to \mathcal{O}$ inducing a surjective homomorphism $h_{1*} : Q \to Q_1$ of orbifold fundamental groups, and
- the fibers of h_1 are compact and connected, because h is proper with connected fibers.

Consequently, there exists a short exact sequence

$$1 \to K \to Q \to Q_1 \to 1 \tag{3.5}$$

given by $\rho = h_{1*}$. It was observed at the beginning of the proof that $\beta_1^2(Q) > 0$. Also, note that Q_1 is an infinite group, because B, being compact, does note have any non-constant map to a finite quotient of $\mathbb{D}^2$. In view of these facts, by Theorem 2.3, the group K in (3.5) must be finite.

Since $\mathcal{O}$ is a compact complex one-dimensional orbifold (it is compact because B has a non-constant map to it), we conclude that Q_1 is virtually a surface group. Passing to a further finite index subgroup Q_2 of Q_1 if necessary, we may assume that Q_2 is a surface group acting trivially on the finite normal subgroup K in (3.5). Hence Q is virtually a surface group.

Finally, since C is finite, as pointed out at the beginning of the proof, the same argument applied to G now shows that G is virtually a surface group. ∎

By Theorem 3.1 and Lemma 2.5, we immediately have:

PROPOSITION 3.2. *If G in* (2.3) *satisfies the condition* $\text{def}(G) > 1$*, then G is virtually a surface group.*

3.2. From virtual surface groups to orbifold groups. In this subsection, we classify virtual surface groups of positive deficiency. Let $\mathcal{G}$ be a virtual surface group; equivalently, there exists an exact sequence

$$1 \to \pi_1(S) \to \mathcal{G} \to F \to 1, \tag{3.6}$$

where S is a compact surface of positive genus, and F is a finite group acting by automorphisms on $\pi_1(S)$.

THEOREM 3.3. *Assume that G in* (2.3) *satisfies the condition* $\text{def}(G) > 1$*. Then G must be an orbifold surface group with genus greater than* 1*.*

Proof. By Proposition 3.2, G is a virtual surface group.

Euclidean case. Suppose that the genus of the surface S in (3.6) for $\mathcal{G} = G$ is one. Then there exists a two-dimensional crystallographic group H such that

$$G = H \times F_1,$$

where F_1 is the subgroup of F in (3.6) acting trivially on $\pi_1(S)$. The deficiency of any crystallographic group other than $\mathbb{Z} \times \mathbb{Z}$, and the Klein bottle group, is non-negative. The deficiency of any finite group is also non-negative. Since $\mathbb{Z} \times \mathbb{Z}$ and the Klein bottle group have deficiency exactly 1, and need at least two generators, it follows that if F_1 is non-trivial, then $\mathrm{def}(G) \geq 0$. Thus, F_1 is trivial, and we have one of the two possibilities for G:

- $\mathbb{Z} \times \mathbb{Z}$,
- the Klein bottle group.

Since the Klein bottle group has first Betti number 1, it cannot be Sasakian (see Lemma 2.1). Finally, $\mathbb{Z} \times \mathbb{Z}$ has deficiency 1 and is ruled out by the hypothesis.

Hyperbolic case. Next, suppose that the genus of S in (3.6) is greater than 1. By the Nielsen realization theorem [Ke], there exists a hyperbolic orbifold $\mathcal{O}$ with $\pi_1(\mathcal{O}) = H$ such that

$$G = H \times B_1$$

with B_1 finite. As in the genus one case, it follows that B_1 is trivial. If $\mathcal{O}$ is non-orientable, then $b_1(\mathcal{O})$ is odd, forcing $\mathcal{O}$ to be an orientable hyperbolic orbifold (see Lemma 2.1 again). Further, from the assumption that the deficiency is greater than 1 it follows that the genus of $\mathcal{O}$ is greater than 1. ∎

4. One-relator groups

4.1. One-relator Sasakian groups are projective orbifold groups. Murasugi has described in detail the centers of one-relator groups:

THEOREM 4.1 ([Mu, Theorems 1, 2]). *Let $\mathcal{G} = \langle x_1, \ldots, x_k \mid w^n \rangle$ be a one-relator group with $n \geq 1$. If $k \geq 3$, then the center $Z(\mathcal{G})$ is trivial. If $\mathcal{G}$ is not abelian with $k = 2$, and $Z(\mathcal{G})$ is not trivial, then $Z(\mathcal{G})$ is infinite cyclic.*

THEOREM 4.2 ([KS, p. 219]). *Let $\mathcal{G}$ be a one-relator group having a* (*non-trivial*) *finitely presented normal subgroup H of infinite index. Then $\mathcal{G}$ is torsion-free and has two generators. Further, $\mathcal{G}$ is an infinite cyclic or infinite dihedral extension of a finitely generated free group N* (*meaning $N \subset \mathcal{G}$*) *satisfying the following:*

- *$H \subset N$ if H is not cyclic, and*
- *$H \cap N$ is trivial if H is cyclic.*

PROPOSITION 4.3. *Assume that G in* (2.3) *is an infinite one-relator group. Then*

- *C in* (2.3) *is trivial, and*
- *$G = Q$ is a projective orbifold group.*

Proof. By Theorem 4.1, C is trivial if $\operatorname{def}(G) > 1$. In that case, $G = Q$ is a projective orbifold group.

On the other hand, if G is abelian, then $G = \mathbb{Z} \times \mathbb{Z}$ and G is a projective orbifold group, as it is the fundamental group of an elliptic curve.

Therefore, assume that $\operatorname{def}(G) = 1$ and G is non-abelian.

Since $\operatorname{def}(G) = 1$, the minimum number of generators of the one-relator group G, which we shall denote by k, is 2.

Since C is contained in the center $Z(G)$ of G, if $Z(G)$ is trivial, then the proposition follows. So we assume that $Z(G)$ is non-trivial.

By Theorem 4.1, $C = \mathbb{Z}$. Hence, by Theorem 4.2, $G = N \times \mathbb{Z}$, where $N = F_r$ is free of rank r greater than one; we note that if N is free of rank 1, then $G = \mathbb{Z} \times \mathbb{Z}$ and $C \neq \mathbb{Z}$.

Next, since G is one-relator, we have $b_2(G) \leq 1$. On the other hand,

$$b_2(N \times \mathbb{Z}) = r > 1,$$

so we get a contradiction. Thus, if $k = 2$, then $Z(G)$ is trivial, forcing C to be trivial. Consequently, $G = Q$ is a projective orbifold group. ∎

We shall need the following result.

PROPOSITION 4.4 ([FKS, Theorem 1]). *Let $\mathcal{G} = \langle x_1, \ldots, x_k \mid w^n \rangle$ be a one-relator group with $n \geq 1$. If $k = 1$, then G is torsion-free. Else, every torsion element in $\mathcal{G}$ is conjugate to a power of w, and the subgroup generated by torsion elements in $\mathcal{G}$ is the free product of the conjugates of w.*

4.2. From orbifold projective groups to projective groups. In this subsection we assume that G in (2.3) is a one-relator Sasakian group

$$G = \langle x_1, \ldots, x_k \mid w^n \rangle.$$

If $\operatorname{def}(G) > 1$, Theorem 3.3 shows that $G = \pi_1(\mathcal{O})$, where $\mathcal{O}$ is a hyperbolic orbifold of genus greater than 1.

Therefore, we assume that $\operatorname{def}(G) = 1$, so $k = 2$.

Since $k = 2$, Proposition 4.3 further shows that

$$G = \pi_1(B), \tag{4.1}$$

where B is the projective orbifold in (2.2).

It can be shown that $\dim_{\mathbb{C}} B > 1$. Indeed, if $\dim_{\mathbb{C}} B = 1$, then M in (2.2) is a 3-dimensional Seifert-fibered manifold. Since $G = \pi_1(M)$ is infinite, it follows that the center of G is infinite cyclic [He, Ch. 12]. But this contradicts Proposition 4.3. So we conclude that $\dim_{\mathbb{C}} B > 1$.

Structure of singularities. We analyze the locus of the points of the orbifold B in (2.2) with non-trivial inertia group. Such examples exist even when, at the global level, the Sasakian manifold is simply connected [CMST,

MST, MT]. Let $\mathcal{S}$ be a connected component of this locus of B. Let $U_{\mathcal{S}}$ denote a regular neighborhood of $\mathcal{S}$ in B.

Lemma 4.5. *Let $2m$ denote the real dimension of B. There exists a lens space L of dimension $2l-1$, with $l \geq 2$, such that the topological space for the orbifold $U_{\mathcal{S}}$ is homeomorphic to $cL \times \mathbb{D}_{2m-2l}$, where cL denotes the cone on L and $\mathbb{D}_{2m-2l}$ denotes a ball of real dimension $2m-2l$.*

Proof. Recall $B = M/\mathrm{U}(1)$ with M being a smooth manifold. All the isotropy subgroups for the action of $\mathrm{U}(1)$ on M are finite cyclic groups. The local structure of B follows from this (see [BoGa, Theorem 4.7.7]). ∎

Let $i_{\mathcal{S}} : U_{\mathcal{S}} \to B$ be the inclusion map between orbifolds, and let $i_{\mathcal{S},*}$ denote the induced homomorphism between orbifold fundamental groups.

Lemma 4.6. *Suppose $i_{\mathcal{S},*}(\pi_1^{\mathrm{orb}}(U_{\mathcal{S}}))$ is trivial for all components $\mathcal{S}$. Then the group G is projective. In particular, if G is torsion-free, it is projective.*

Proof. We may blow up the orbifold B and obtain another orbifold

$$\varpi : B' \to B$$

such that the underlying topological space for B' is a smooth projective variety. We note that the underlying topological space for an orbifold B'' is a smooth projective variety if the locus in B'' of points with non-trivial inertia is a normal crossing divisor. In view of the given condition, we may choose B' such that $\pi_1(B)$ coincides with the fundamental group of the underlying topological space for B'. But the underlying topological space for B' is a smooth projective variety, so $\pi_1(B)$ is a projective group.

If G is torsion-free, then $i_{\mathcal{S},*}(\pi_1^{\mathrm{orb}}(U_{\mathcal{S}}))$ must be trivial for all components $\mathcal{S}$. The conclusion of the lemma follows. ∎

Lemma 4.7. *Suppose $i_{\mathcal{S},*}(\pi_1^{\mathrm{orb}}(U_{\mathcal{S}}))$ is non-trivial for some $\mathcal{S}$. Then any torsion element of G has a non-trivial power that is conjugate to an element of $i_{\mathcal{S},*}(\pi_1^{\mathrm{orb}}(U_{\mathcal{S}}))$.*

Proof. This is an immediate consequence of Proposition 4.4. ∎

We next analyze the case where the underlying topological space for B is smooth.

Lemma 4.8. *Suppose that the underlying topological space for the projective orbifold B is smooth. Then G is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V_1 with at most one cone-point. Furthermore, V_1 is orientable.*

Proof. Let

$$G = \langle x_1, \ldots, x_k \mid w^n \rangle, \qquad n > 1.$$

Then, replacing the orbifold B by its underlying topological space $\mathcal{B}$, we note by Proposition 4.4 and Lemma 4.7 that $\pi_1(\mathcal{B})$ is the quotient of $\pi_1(B)$ by some powers of w. Hence $G_1 = \pi_1(\mathcal{B})$ is of the form

$$G_1 = \langle x_1, \ldots, x_k \mid w^r \rangle, \qquad r \geq 1.$$

Since $\mathcal{B}$ is a smooth projective variety, G_1 is a one-relator Kähler group. Hence, by the classification of one-relator Kähler groups [BM1] (see also [Ko]), G_1 is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V with at most one cone-point y_0. If $r = 1$, then G_1 is torsion-free. Else the loop that goes around y_0 represents the conjugacy class of w. Replacing r by n, we note that G is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V_1, where V_1 differs from V only in the order of the cone-point y_0. Thus, G is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V with at most one cone-point. ∎

Lemma 4.5 furnishes the local structure of singularities of B. We now describe the blow-up. We first note that a lens space cL on a lens space L of dimension $2k-1$ with fundamental group $\mathbb{Z}/q\mathbb{Z}$ can be realized as the topological boundary of the total space E_q of a twisted line bundle (more precisely, the qth tensor power of the tautological line bundle) over $\mathbb{C}P^{k-1}$. Hence, the blow-up of cL may be obtained by removing an open neighborhood of the cone point $y_0 \in cL$ and attaching a copy of E_q along the resulting boundary L. Let $\mathrm{BU}(L)$ denote the blow-up of cL, and let $\mathrm{ED}(L) \subset \mathrm{BU}(L)$ be the exceptional divisor, which is homeomorphic to $\mathbb{C}P^{k-1}$. Then we note the following two properties of $\mathrm{BU}(L)$:

Remark 4.9.

(1) If cL has an orbifold cone-point of order q then $\mathrm{BU}(L)$ is also an orbifold so that $\mathrm{ED}(L)$ is an orbifold locus of ramification order q.
(2) The underlying topological space of $\mathrm{BU}(L)$ is a smooth manifold.

Next, we describe the local structure of a singularity of B as obtained in Lemma 4.5 along with the $\mathrm{U}(1)$ bundle over it. Let $E_{\mathcal{S}}$ denote the circle bundle over $U_{\mathcal{S}}$ induced by the inclusion of $U_{\mathcal{S}}$ into B. Thus, $E_{\mathcal{S}} \subset M$. Then we have the local structure of the Sasakian manifold M given by the following commutative diagram:

$$\begin{array}{ccc} D^{2k} \times U(1) & \longrightarrow & E_{\mathcal{S}} \\ \downarrow & & \downarrow \\ D^{2k} & \longrightarrow & U_{\mathcal{S}} = cL \end{array} \tag{4.2}$$

Theorem 4.10. *Let*

$$G = \langle x_1, \ldots, x_n \mid w^k \rangle, \qquad k > 1,$$

be a one-relator Sasakian group. Then G is isomorphic to the fundamental *group of a (real) two-dimensional compact orbifold V with at most one cone-point. Further, the underlying manifold of V is orientable.*

Proof. As discussed at the beginning of Section 4.2, it remains to deal with the case $n = 2$. If $k = 1$, then G is torsion-free by Proposition 4.4, and the hypothesis of Lemma 4.6 is satisfied, forcing G to be the fundamental group of a smooth projective variety. The main theorem of [BM1] now furnishes the result.

Else, the local structure of singularities is given by Lemma 4.5. Blowing up and using Remark 4.9, we obtain an orbifold B_1 satisfying the following:

- The underlying topological space $\mathcal{B}_1$ of B_1 is a smooth projective variety.
- The orbifold locus of B_1 is a divisor.
- Any loop that goes around a component of the divisor represents an element of $\pi_1^{\mathrm{orb}}(B) = G$ that is conjugate to w^r for some r that divides k.

Hence, by Lemma 4.8, $G_1 = \pi_1(\mathcal{B}_1)$ is isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V with at most one cone-point. The rest of the argument is a replica of the last part of the proof of Lemma 4.8 forcing G to be isomorphic to the fundamental group of a (real) two-dimensional compact orbifold V_1, where V_1 differs from V only in the order of the cone-point. ∎

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