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INDIVIDUAL ERGODIC THEOREMS FOR INFINITE MEASURE

BY

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Abstract. Given a σ-finite infinite measure space (Ω, μ), it is shown that any Dunford–Schwartz operator $T : \mathcal{L}^1(\Omega) \to \mathcal{L}^1(\Omega)$ can be uniquely extended to the space $\mathcal{L}^1(\Omega) + \mathcal{L}^\infty(\Omega)$. This allows one to find the largest subspace $\mathcal{R}_\mu$ of $\mathcal{L}^1(\Omega) + \mathcal{L}^\infty(\Omega)$ such that the ergodic averages $n^{-1}\sum_{k=0}^{n-1} T^k(f)$ converge almost uniformly (in Egorov's sense) for every $f \in \mathcal{R}_\mu$ and every Dunford–Schwartz operator T. Utilizing this result, almost uniform convergence of the averages $n^{-1}\sum_{k=0}^{n-1} \beta_k T^k(f)$ for every $f \in \mathcal{R}_\mu$, any Dunford–Schwartz operator T and any bounded Besicovitch sequence $\{\beta_k\}$ is established. Further, given a measure preserving transformation $\tau : \Omega \to \Omega$, Assani's extension of Bourgain's Return Times theorem to σ-finite measures is employed to show that for each $f \in \mathcal{R}_\mu$ there exists a set $\Omega_f \subset \Omega$ such that $\mu(\Omega \setminus \Omega_f) = 0$ and the averages $n^{-1}\sum_{k=0}^{n-1} \beta_k f(\tau^k\omega)$ converge for all $\omega \in \Omega_f$ and any bounded Besicovitch sequence $\{\beta_k\}$. Applications to fully symmetric subspaces $E \subset \mathcal{R}_\mu$ are outlined.

1. Introduction. The celebrated Dunford–Schwartz and Wiener–Wintner-type ergodic theorems are two of the major themes of ergodic theory. Due to their fundamental roles, these theorems have been revisited ever since their first appearance. For instance, Garsia [G70] gave an elegant self-contained proof of the Dunford–Schwartz theorem, and Assani [A99, A03] extended Bourgain's Return Times theorem to the σ-finite setting.

If the measure in question is finite, then the ultimate goal in the study of individual convergence of ergodic averages is to show that these averages converge almost everywhere—equivalently, almost uniformly—for each function in the space $\mathcal{L}^1$. In the case of infinite measure, one should begin by asking

(A) whether the Dunford–Schwartz pointwise ergodic theorem is valid for some functions within the space $\mathcal{L}^1 + \mathcal{L}^\infty$ but outside the union of spaces $\mathcal{L}^p$, $1 \le p < \infty$;

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(B) whether almost everywhere convergence in the Dunford–Schwartz theorem can be replaced by the generally stronger almost uniform (in Egorov's sense) convergence.

To answer (A), one needs to extend a Dunford–Schwartz operator $T : \mathcal{L}^1 \to \mathcal{L}^1$ to the space $\mathcal{L}^1 + \mathcal{L}^\infty$. Thus, we begin by showing, in Theorem 3.3, that such an extension $\widetilde{T}$ exists and is unique if $\widetilde{T}|_{\mathcal{L}^\infty}$ is $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$-continuous.

This fact allows us to assume without loss of generality that any Dunford–Schwartz operator is defined on the entire space $\mathcal{L}^1 + \mathcal{L}^\infty$ and contracts $\mathcal{L}^\infty$.

Note that if a Dunford–Schwartz operator T is defined on $\mathcal{L}^1 + \mathcal{L}^\infty$ and contracts $\mathcal{L}^\infty$, positive solutions to (A) and (B) can be found in [CL19, Theorem 3.1]. In fact, the largest subspace $\mathcal{R}_\mu$ of $\mathcal{L}^1 + \mathcal{L}^\infty$ in which the ergodic averages converge almost uniformly was found (see [CL18], [K-K19], and [CL19, Theorem 3.4]). Since Theorem 3.3 was not yet known, in [CL19], a Dunford–Schwartz operator T was assumed to satisfy these properties.

In Section 4, this extension is employed to show almost uniform convergence of Besicovitch weighted ergodic averages generated by a Dunford–Schwartz operator in $\mathcal{R}_\mu$ (see Theorem 4.10).

In Section 5, we utilize Assani's extension of the Return Times theorem to σ-finite measure to generalize the Wiener–Wintner ergodic theorem. Namely, we show that it holds for each function in $\mathcal{R}_\mu$ and with the original weights $\{\lambda^k\}$, $\lambda \in \mathbb{C}_1$, expanded to the set all bounded Besicovitch sequences $\{\beta_k\}$ (see Theorem 5.8).

Section 6 is devoted to applications of the above results to fully symmetric spaces $E \subset \mathcal{L}^1 + \mathcal{L}^\infty$ such that $\mathbf{1} \notin E$. It is demonstrated that the class of fully symmetric spaces E with $\mathbf{1} \notin E$ is significantly wider than the class of $\mathcal{L}^p$-spaces, $1 \le p < \infty$, and includes such well-known Banach spaces as Orlicz, Lorentz and Marcinkiewicz spaces of measurable functions.

2. Preliminaries. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and let $\mathcal{L}^0 = \mathcal{L}^0(\Omega)$ be the $*$-algebra of equivalence classes of almost everywhere (a.e.) finite complex-valued measurable functions on Ω. Given $1 \le p \le \infty$, let $\mathcal{L}^p \subset \mathcal{L}^0$ be the $\mathcal{L}^p$-space on Ω equipped with the standard Banach norm $\|\cdot\|_p$.

A net $\{f_\alpha\} \subset \mathcal{L}^0$ is said to converge *almost uniformly* (*a.u.*) to $f \in \mathcal{L}^0$ (in Egorov's sense) if for every $\varepsilon > 0$ there exists a set $G \subset \Omega$ such that $\mu(\Omega \setminus G) \le \varepsilon$ and $\|(f - f_\alpha)\chi_G\|_\infty \to 0$, where χ_G is the characteristic function of G. It is clear that every a.u. convergent net converges almost everywhere (a.e.) and that the converse is not true in general.

Define

$$\mathcal{R}_\mu = \{f \in \mathcal{L}^1 + \mathcal{L}^\infty : \mu\{|f| > \lambda\} < \infty \text{ for all } \lambda > 0\}.$$

It is clear that $\bigcup_{1 \le p < \infty} \mathcal{L}^p \subsetneq \mathcal{R}_\mu$.

The following characterization of $\mathcal{R}_\mu$ is crucial.

PROPOSITION 2.1. *Let $f \in \mathcal{L}^1 + \mathcal{L}^\infty$. Then $f \in \mathcal{R}_\mu$ if and only if for each $\varepsilon > 0$ there exist $g_\varepsilon \in \mathcal{L}^1$ and $h_\varepsilon \in \mathcal{L}^\infty$ such that*

$$f = g_\varepsilon + h_\varepsilon \quad \textit{and} \quad \|h_\varepsilon\|_\infty \le \varepsilon.$$

Proof. Pick $f \in \mathcal{R}_\mu$ and let

$$\Omega_\varepsilon = \{|f| > \varepsilon\}, \qquad g_\varepsilon = f\chi_{\Omega_\varepsilon}, \qquad h_\varepsilon = f\chi_{\Omega\setminus\Omega_\varepsilon}.$$

Then $\|h_\varepsilon\|_\infty \le \varepsilon$; moreover, as $f \in \mathcal{L}^1 + \mathcal{L}^\infty$, we have

$$f = g_\varepsilon + h_\varepsilon = g + h$$

for some $g \in \mathcal{L}^1$, $h \in \mathcal{L}^\infty$. Therefore, since $f \in \mathcal{R}_\mu$, we have $\mu(\Omega_\varepsilon) < \infty$, which implies that

$$g_\varepsilon = g\,\chi_{\Omega_\varepsilon} + (h - h_\varepsilon)\,\chi_{\Omega_\varepsilon} \in \mathcal{L}^1.$$

Conversely, let $f \in \mathcal{L}^1 + \mathcal{L}^\infty$, $\lambda > 0$, and denote $E = \{|f| > \lambda\}$. Let $g_{\lambda/2} \in \mathcal{L}^1$ and $h_{\lambda/2} \in \mathcal{L}^\infty$ be such that

$$f = g_{\lambda/2} + h_{\lambda/2} \quad \text{and} \quad \|h_{\lambda/2}\|_\infty \le \lambda/2.$$

Then

$$|f|\chi_E \le |g_{\lambda/2}|\chi_E + |h_{\lambda/2}|\chi_E,$$

implying that

$$\begin{aligned}\mu\{|f|\chi_E > \lambda\} &\le \mu\{|g_{\lambda/2}|\chi_E > \lambda/2\} + \mu\{|h_{\lambda/2}|\chi_E > \lambda/2\}\\ &= \mu\{|g_{\lambda/2}|\chi_E > \lambda/2\} < \infty. \ \blacksquare\end{aligned}$$

PROPOSITION 2.2. *$\mathcal{R}_\mu$ is closed with respect to a.u. convergence.*

Proof. Let $\mathcal{R}_\mu \ni f_\alpha \to f$ a.u. Fix $\lambda > 0$ and denote $F = \{|f| > \lambda\}$. Let $\varepsilon > 0$. Then there is $E \subset \Omega$ such that

$$\mu(\Omega \setminus E) < \varepsilon \quad \text{and} \quad \|(f - f_\alpha)\chi_E\|_\infty \to 0.$$

Since $\|(f - f_\alpha)\chi_{F\cap E}\|_\infty \to 0$ and

$$\|(f - f_\alpha)\chi_{F\cap E}\|_\infty \ge |f\chi_{F\cap E} - f_\alpha\chi_{F\cap E}| \ge \big||f|\chi_{F\cap E} - |f_\alpha|\chi_{F\cap E}\big|,$$

it follows from $|f|\chi_{F\cap E} > \lambda$ that there exists α_0 such that $|f_{\alpha_0}|\chi_{F\cap E} > \lambda$. Therefore, as $f_{\alpha_0} \in \mathcal{R}_\mu$, we have $\mu(F \cap E) < \infty$, implying $\mu(F) < \infty$. ■

3. Extension of a Dunford–Schwartz operator to $\mathcal{L}^1 + \mathcal{L}^\infty$. A linear operator $T : \mathcal{L}^1 \to \mathcal{L}^1$ is called a *Dunford–Schwartz operator* (see [DS88, Ch. VIII, §6], [G70], [K85, §§4.1, 4.2]), and we write $T \in \mathrm{DS}$, if

$$\|T(f)\|_1 \le \|f\|_1 \quad \forall f \in \mathcal{L}^1 \quad \text{and} \quad \|T(f)\|_\infty \le \|f\|_\infty \quad \forall f \in \mathcal{L}^\infty \cap \mathcal{L}^1.$$

Given $\mathcal{L} \subset \mathcal{L}^0$, set $\mathcal{L}_+ = \{f \in \mathcal{L} : f \ge 0\}$. If $T \in \mathrm{DS}$ is such that $T(\mathcal{L}^1_+) \subset \mathcal{L}^1_+$, then we say that T is *positive* and write $T \in \mathrm{DS}^+$.

We will need the following well-known properties of bounded linear operators $T : \mathcal{L}^1 \to \mathcal{L}^1$ (resp. $T : \mathcal{L}^\infty \to \mathcal{L}^\infty$) (see, for example, [K85, §4.1, Theorem 1.1, Proposition 1.2(d), Theorem 1.3]).

PROPOSITION 3.1. *For any bounded linear operator $T : \mathcal{L}^1 \to \mathcal{L}^1$ (respectively, $T : \mathcal{L}^\infty \to \mathcal{L}^\infty$) there exists a unique positive bounded linear operator $|T| : \mathcal{L}^1 \to \mathcal{L}^1$ (respectively, $|T| : \mathcal{L}^\infty \to \mathcal{L}^\infty$) such that*

(1) $\| |T| \| = \|T\|$,
(2) $|T^k(f)| \leq |T|^k(|f|)$, $k = 1, 2, \ldots$, $\forall f \in \mathcal{L}^1$ *(respectively,* $\forall f \in \mathcal{L}^\infty$*),*
(3) $|T^*| = |T|^*$, *where* $T^* : \mathcal{L}^\infty \to \mathcal{L}^\infty$ *is the adjoint operator of an operator* $T : \mathcal{L}^1 \to \mathcal{L}^1$.

The operator $|T|$ is called the *linear modulus* of T.

We will also utilize another fundamental fact, which can be found, for example, in [P02, Corollary 2.9].

THEOREM 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, and let $\mathbf{1}$ be the unit of $\mathcal{A}$. If $T : \mathcal{A} \to \mathcal{B}$ is a positive linear map, then $\|T\| = \|T(\mathbf{1})\|$.*

In what follows, let $\mathbf{1} = \chi_\Omega$.

Let $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$ stand for the *weak topology* induced on $\mathcal{L}^\infty$ by the pairing $(\mathcal{L}^\infty, \mathcal{L}^1, \phi)$, where $\phi(f, g) = \int_\Omega fg \, d\mu$, $f \in \mathcal{L}^\infty$, $g \in \mathcal{L}^1$.

THEOREM 3.3. *For any Dunford–Schwartz operator $T : \mathcal{L}^1 \to \mathcal{L}^1$ there exists a unique linear operator $\widetilde{T} : \mathcal{L}^1 + \mathcal{L}^\infty \to \mathcal{L}^1 + \mathcal{L}^\infty$ such that*

$$\widetilde{T}(f) = T(f) \quad \forall f \in \mathcal{L}^1, \qquad \|\widetilde{T}(f)\|_\infty \leq \|f\|_\infty \quad \forall f \in \mathcal{L}^\infty,$$

and $\widetilde{T}|_{\mathcal{L}^\infty}$ is $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$-continuous.

Proof. Assume first that $T \in \mathrm{DS}^+$. Since $(\mathcal{L}^1)^* = \mathcal{L}^\infty$, the adjoint operator T^* acts in $\mathcal{L}^\infty$ and is $\sigma(\mathcal{L}^\infty, \mathcal{L}^1)$-continuous. Moreover, since

$$\int_\Omega T^*(f) g \, d\mu = \int_\Omega f T(g) \, d\mu \quad \forall f \in \mathcal{L}^\infty, \, g \in \mathcal{L}^1,$$

it follows that the linear operator T^* is positive.

Choose $F_n \subset \Omega$, $n = 1, 2, \ldots$, satisfying

$$F_n \subset F_{n+1}, \quad \mu(F_n) < \infty \quad \text{for all } n \quad \text{and} \quad \bigcup_{n=1}^\infty F_n = \Omega.$$

As $0 \leq T(\chi_{F_n}) \leq \mathbf{1}$ for each n, given $f \in \mathcal{L}^1 \cap \mathcal{L}^\infty_+$, it follows that

$$\begin{aligned} \|T^*(f)\|_1 = \int_\Omega T^*(f) \, d\mu &= \lim_{n\to\infty} \int_\Omega T^*(f) \chi_{F_n} \, d\mu \\ &= \lim_{n\to\infty} \int_\Omega f T(\chi_{F_n}) \, d\mu \leq \int_\Omega f \, d\mu = \|f\|_1. \end{aligned}$$

Therefore, T^* is $\|\cdot\|_1$-continuous on $\mathcal{L}^1\cap\mathcal{L}^\infty_+$, hence on $\mathcal{L}^1\cap\mathcal{L}^\infty$. Since $\mathcal{L}^1\cap\mathcal{L}^\infty$ is dense in $\mathcal{L}^1$, T^* uniquely extends to a positive linear $\|\cdot\|_1$-continuous operator $\widehat{T^*}:\mathcal{L}^1\to\mathcal{L}^1$.

Next, replacing in the above argument T by $\widehat{T^*}$, we uniquely extend the operator $(\widehat{T^*})^*|_{\mathcal{L}^1\cap\mathcal{L}^\infty}:\mathcal{L}^1\cap\mathcal{L}^\infty\to\mathcal{L}^1\cap\mathcal{L}^\infty$ to a positive $\|\cdot\|_1$-continuous linear operator $\widetilde{T}:\mathcal{L}^1\to\mathcal{L}^1$. Since

$$\int_\Omega f(\widehat{T^*})^*(g)\,d\mu=\int_\Omega \widehat{T^*}(f)g\,d\mu=\int_\Omega T^*(f)g\,d\mu=\int_\Omega fT(g)\,d\mu\quad\forall f,g\in\mathcal{L}^1\cap\mathcal{L}^\infty,$$

it follows that $\widetilde{T}(f)=(\widehat{T^*})^*(f)=T(f)$ for all $f\in\mathcal{L}^1\cap\mathcal{L}^\infty$. Consequently, $\widetilde{T}$ coincides with T on $\mathcal{L}^1$.

Furthermore, as $\widetilde{T}|_{\mathcal{L}^\infty\cap\mathcal{L}^1}=(\widehat{T^*})^*|_{\mathcal{L}^\infty\cap\mathcal{L}^1}$ is $\sigma(\mathcal{L}^\infty,\mathcal{L}^1)$-continuous and $\mathcal{L}^1\cap\mathcal{L}^\infty$ is $\sigma(\mathcal{L}^\infty,\mathcal{L}^1)$-dense in $\mathcal{L}^\infty$, it follows that $\widetilde{T}|_{\mathcal{L}^1\cap\mathcal{L}^\infty}$ uniquely extends to an operator on $\mathcal{L}^\infty$, which coincides with $(\widehat{T^*})^*:\mathcal{L}^\infty\to\mathcal{L}^\infty$.

Let us now show that $\|\widetilde{T}\|_{\mathcal{L}^\infty\to\mathcal{L}^\infty}\le 1$. Indeed, given $f\in\mathcal{L}^1\cap\mathcal{L}^\infty_+$, we have

$$\int_\Omega f\widetilde{T}(\mathbf{1})\,d\mu=\int_\Omega f(\widehat{T^*})^*(\mathbf{1})\,d\mu=\int_\Omega \widehat{T^*}(f)\,d\mu=\int_\Omega T^*(f)\,d\mu\le\int_\Omega f\,d\mu,$$

and we conclude that $\widetilde{T}(\mathbf{1})\le\mathbf{1}$, hence $\|\widetilde{T}(\mathbf{1})\|_\infty\le 1$. Therefore, in view of Theorem 3.2 with $\mathcal{A}=\mathcal{B}=\mathcal{L}^\infty$, we have

$$\|\widetilde{T}\|_{\mathcal{L}^\infty\to\mathcal{L}^\infty}=\|\widetilde{T}(\mathbf{1})\|_\infty\le 1.$$

This completes the proof of the theorem in the case $T\in\mathrm{DS}^+$, since the operator $\widetilde{T}:\mathcal{L}^1+\mathcal{L}^\infty\to\mathcal{L}^1+\mathcal{L}^\infty$ defined by

$$\widetilde{T}(f)=T(f)\quad\forall f\in\mathcal{L}^1,\qquad \widetilde{T}(g)=(\widehat{T^*})^*(g)\quad\forall g\in\mathcal{L}^\infty,$$

satisfies the required conditions.

Let now $T\in\mathrm{DS}$. Since $|T|\in\mathrm{DS}^+$, it follows as above that $|T|^*:\mathcal{L}^\infty\to\mathcal{L}^\infty$ uniquely extends to a positive continuous linear operator $\widehat{|T|^*}:\mathcal{L}^1\to\mathcal{L}^1$ and, since by Proposition 3.1,

$$\|T^*f\|_1\le\big\||T^*|(f)\big\|_1=\big\||T|^*(f)\big\|_1=\big\|\widehat{|T|^*}(f)\big\|_1\quad\forall f\in\mathcal{L}^1\cap\mathcal{L}^\infty_+,$$

T^* is $\|\cdot\|_1$-continuous on $\mathcal{L}^1\cap\mathcal{L}^\infty$. Therefore, T^* admits a unique $\|\cdot\|_1$-continuous extension $\widehat{T^*}$ to $\mathcal{L}^1$, implying as above that $\widetilde{T}=(\widehat{T^*})^*$ is the unique extension of T to $\mathcal{L}^\infty$.

Next, $\widehat{T^*}(f)=T^*(f)$ for all $f\in\mathcal{L}^1\cap\mathcal{L}^\infty$ implies that

$$|\widehat{T^*}|(f)=|T^*|(f)=|T|^*(f)=\widehat{|T|^*}(f),\qquad f\in\mathcal{L}^1\cap\mathcal{L}^\infty,$$

hence $|\widehat{T^*}|(g)=\widehat{|T|^*}(g)$ for all $g\in\mathcal{L}^\infty$, since $|T|^*$ is $\sigma(\mathcal{L}^\infty,\mathcal{L}^1)$-continuous on $\mathcal{L}^1\cap\mathcal{L}^\infty$. Since, as above, we have

$$\|(\widehat{|T|^*})^*\|_{\mathcal{L}^\infty\to\mathcal{L}^\infty}\le 1,$$

it now follows by Proposition 3.1 that

$$\begin{aligned}\|\widetilde{T}\|_{\mathcal{L}^\infty\to\mathcal{L}^\infty} &= \|(\widehat{T^*})^*\|_{\mathcal{L}^\infty\to\mathcal{L}^\infty} = \big\||(\widehat{T^*})^*|\big\|_{\mathcal{L}^\infty\to\mathcal{L}^\infty}\\ &= \big\||\widehat{T^*}|^*\big\|_{\mathcal{L}^\infty\to\mathcal{L}^\infty} = \|(\widehat{|T|^*})^*\|_{\mathcal{L}^\infty\to\mathcal{L}^\infty} \le 1. \ \blacksquare\end{aligned}$$

REMARK 3.4. Theorem 3.3 implies that one can (and we will in what follows) assume without loss of generality that any $T \in \mathrm{DS}$ is defined on the entire space $\mathcal{L}^1 + \mathcal{L}^\infty$ and satisfies the conditions

$$\|T(f)\|_1 \le \|f\|_1 \quad \forall f \in \mathcal{L}^1 \quad \text{and} \quad \|T(f)\|_\infty \le \|f\|_\infty \quad \forall f \in \mathcal{L}^\infty. \tag{3.1}$$

4. Almost uniform convergence of Besicovitch weighted averages. In this section we will show that pointwise convergence of Besicovitch weighted ergodic averages (see, for example, [CLO98], and also [LOT99]) can be extended to the context of a.u. convergence and a Dunford–Schwartz operator acting in $\mathcal{R}_\mu$ (Theorem 4.10 below).

Let $\mathbb{C}_1$ be the unit circle in the field $\mathbb{C}$ of complex numbers, and let $\mathbb{Z}$ be the set of integers. A function $P : \mathbb{Z} \to \mathbb{C}$ is said to be a *trigonometric polynomial* if $P(k) = \sum_{j=1}^s z_j \lambda_j^k$, $k \in \mathbb{Z}$, for some $s \in \mathbb{N}$, $\{z_j\}_{j=1}^s \subset \mathbb{C}$ and $\{\lambda_j\}_{j=1}^s \subset \mathbb{C}_1$, where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers. A sequence $\{\beta_k\}_{k=0}^\infty \subset \mathbb{C}$ is called a *bounded Besicovitch sequence* if

(i) $|\beta_k| \le C < \infty$ for all $k = 0, 1, \ldots$ and some $C > 0$;
(ii) for every $\varepsilon > 0$ there exists a trigonometric polynomial P such that

$$\limsup_n \frac{1}{n} \sum_{k=0}^{n-1} |\beta_k - P(k)| < \varepsilon. \tag{4.1}$$

Let E be a Banach space, and let $A_n : E \to \mathcal{L}^0$ be a sequence of linear maps. Given $f \in E$, the function

$$A^*(f) = \sup_n |A_n(f)|$$

is called the *maximal function* of f. If $A^*(f) \in \mathcal{L}^0$ for every $f \in E$, then

$$A^* : E \to \mathcal{L}^0, \qquad f \in E,$$

is called the *maximal operator* of the sequence $\{A_n\}$.

Here is the well-known maximal ergodic inequality for $\{A_n(T)\}$, where

$$A_n(T) = \frac{1}{n} \sum_{k=0}^{n-1} T^k, \qquad T \in \mathrm{DS}$$

(see, for example, [CL19, Theorem 3.3]):

THEOREM 4.1. *Let* $T \in \mathrm{DS}$. *If*

$$A(T)^*(f) = \sup_n |A_n(T)(f)|, \qquad f \in \mathcal{L}^1,$$

the maximal operator of the sequence $\{A_n(T)\}$ *on* $E = \mathcal{L}^1$, *then*

$$\mu\{A(T)^*(|f|) > \lambda\} \le \frac{\|f\|_1}{\lambda} \quad \text{for all } f \in \mathcal{L}^1,\ \lambda > 0.$$

Given $T \in \mathrm{DS}$, $\{\beta_k\}_{k=0}^{\infty} \subset \mathbb{C}$, and $f \in \mathcal{L}^1 + \mathcal{L}^\infty$, denote

$$B_n(f) = B_n(T)(f) = \frac{1}{n}\sum_{k=0}^{n-1} \beta_k T^k(f). \tag{4.2}$$

Corollary 4.2. *Let* $\{\beta_k\}_{k=0}^{\infty} \subset \mathbb{C}$ *be such that* $|\beta_k| \le C < \infty$ *for every* k. *If* $T \in \mathrm{DS}$, *then*

$$\mu\{B_n(T)^*(|f|) > \lambda\} \le 6C\frac{\|f\|_1}{\lambda} \quad \forall f \in \mathcal{L}^1,\ \lambda > 0.$$

Proof. We have

$$B_n(T) = \frac{1}{n}\sum_{k=0}^{n-1}(\operatorname{Re}\beta_k + C)T^k + \frac{i}{n}\sum_{k=0}^{n-1}(\operatorname{Im}\beta_k + C)T^k - C(1+i)A_n(T).$$

Therefore, as $0 \le \operatorname{Re}\beta_k + C \le 2C$ and $0 \le \operatorname{Im}\beta_k + C \le 2C$ for every k, it follows that

$$|B_n(T)(f)| \le 6CA_n(|T|)(|f|) \quad \text{for every } f \in \mathcal{L}^1 + \mathcal{L}^\infty \text{ and } n,$$

and Theorem 4.1 implies that

$$\begin{aligned}\mu\{B(T)^*(|f|) > \lambda\} &= \mu\Big\{\sup_n |B_n(T)(|f|)| > \lambda\Big\} \\ &\le \mu\Big\{6C\sup_n |A_n(|T|)(|f|)| > \lambda\Big\} \\ &= \mu\Big\{A(|T|)^*(|f|) > \frac{\lambda}{6C}\Big\} \le 6C\frac{\|f\|_1}{\lambda}. \quad \blacksquare\end{aligned}$$

Let us denote

$$\mathcal{L}^0_\mu = \{f \in \mathcal{L}^0 : \mu\{|f| > \lambda\} < \infty \text{ for some } \lambda > 0\}.$$

Proposition 4.3 (see [CL19, Proposition 3.1]). *The* $*$-*subalgebra* $\mathcal{L}^0_\mu$ *of* $\mathcal{L}^0$ *is complete with respect to a.u. convergence.*

In what follows, t_μ will stand for the *measure topology* in $\mathcal{L}^0$, that is, the topology given by the following system of neighborhoods of zero:

$$\mathcal{N}(\varepsilon,\delta) = \{f \in \mathcal{L}^0 : \mu\{|f| > \delta\} \le \varepsilon\}, \quad \varepsilon, \delta > 0.$$

It is well-known that $(\mathcal{L}^0, t_\mu)$ is a complete metrizable topological vector space; see, for example, [KA82, Ch. I, §6, Theorem 15]. Since $\mathcal{L}^0_\mu$ is a closed linear subspace of $(\mathcal{L}^0, t_\mu)$, it follows that $(\mathcal{L}^0_\mu, t_\mu)$ is also a complete metrizable topological vector space.

A proof of the next fact is given in [CL19, Lemma 3.1].

Lemma 4.4. *Let $(E, \|\cdot\|)$ be a Banach space. If the maximal operator $A^* : E \to \mathcal{L}^0$ of a sequence of linear maps $A_n : (E, \|\cdot\|) \to (\mathcal{L}^0_\mu, t_\mu)$ is continuous at zero, then the set*

$$E_c = \{f \in E : \{A_n(f)\} \text{ converges a.u.}\}$$

is closed in E.

Since Corollary 4.2 entails that the sequence $B_n(T) : (\mathcal{L}^1, \|\cdot\|_1) \to (\mathcal{L}^0_\mu, t_\mu)$ is continuous at zero for every $T \in \mathrm{DS}$, we arrive at the following.

Corollary 4.5. *If $T \in \mathrm{DS}$ and $\{\beta_k\} \subset \mathbb{C}$ is such that $|\beta_k| \le C < \infty$ for all k, then the set*

$$\mathcal{L}^1_c = \{f \in \mathcal{L}^1 : \{B_n(T)(f)\} \text{ converges a.u.}\}$$

is closed in $\mathcal{L}^1$.

Note that Proposition 2.1 implies that $T(\mathcal{R}_\mu) \subset \mathcal{R}_\mu$ for any $T \in \mathrm{DS}$. The following theorem was established in [CL19, Theorems 3.1, 3.4] (see also [K-K19]) under the initial assumption that the operator T satisfied conditions (3.1). Also, although it was proved for real-valued functions, the argument remains valid in the general case.

Theorem 4.6. *If $T \in \mathrm{DS}$, then for every $f \in \mathcal{R}_\mu$ the averages $A_n(T)(f)$ converge a.u. to some $\widehat{f} \in \mathcal{R}_\mu$. Conversely, if $f \in (\mathcal{L}^1 + \mathcal{L}^\infty) \setminus \mathcal{R}_\mu$, then there exists $T \in \mathrm{DS}$ such that the sequence $\{A_n(T)(f)\}$ does not converge a.e., hence a.u.*

In particular, Theorem 4.6 entails that the Dunford–Schwartz pointwise ergodic theorem holds for $f \in \mathcal{L}^1 + \mathcal{L}^\infty$ and for any $T \in \mathrm{DS}$ if and only if $f \in \mathcal{R}_\mu$.

Lemma 4.7. *Let (X, ν) and (Y, μ) be σ-finite measure spaces, and let $\{g_n\} \subset \mathcal{L}^0(X \otimes Y, \nu \otimes \mu)$ be such that $g_n \to g$ a.u. on $X \otimes Y$. Then $g_n(x, \cdot) \to g(x, \cdot)$ a.u. on Y for almost all $x \in X$.*

(For the definition of the product measure space $(X \otimes Y, \nu \otimes \mu)$ and measurability of sections see, for example, [R87, Ch. 8, Theorem 8.5].)

Proof of Lemma 4.7. Fix $\varepsilon > 0$. Given $k \in \mathbb{N}$, there exists $G_k \subset X \otimes Y$ such that

$$(\nu \otimes \mu)((X \otimes Y) \setminus G_k) < \varepsilon^2/k \quad \text{and} \quad \|(g - g_n)\chi_{G_k}\|_\infty \to 0 \quad \text{as } n \to \infty.$$

If $x \in X$ and

$$G_k(x) = \{y \in Y : (x, y) \in G_k\},$$

then, with

$$X_k = \{x \in X : \mu(Y \setminus G_k(x)) < \varepsilon\},$$

we have

$$\varepsilon^2/k > (\nu\otimes\mu)((X\otimes Y)\setminus G_k) = \int_X \mu(Y\setminus G_k(x))\,d\nu(x) \geq \int_{X\setminus X_k} \mu(Y\setminus G_k(x))\,d\nu(x).$$

Hence $\nu(X\setminus X_k) < \varepsilon/k$, implying that if $X' = \bigcup_k X_k$, then $\nu(X\setminus X') = 0$.

Now, if $x \in X'$, then $x \in X_{k_0}$ for some k_0, so, if $Y_x = G_{k_0}(x)$, then $\mu(Y\setminus Y_x) < \varepsilon$ and

$$\|(g(x,\cdot)-g_n(x,\cdot))\chi_{Y_x}\|_{\mathcal{L}^\infty(Y)} \leq \|(g(x,\cdot)-g_n(x,\cdot))\chi_{G_{k_0}}\|_{\mathcal{L}^\infty(X\otimes Y)} \to 0,$$

that is, $g_n(x,\cdot)\to g(x,\cdot)$ a.u. on Y. ■

The following fact can be easily verified.

Lemma 4.8. *Let $\{g_n\}\subset\mathcal{L}^\infty$ be a sequence such that, given $\varepsilon>0$, there exists an a.u. convergent sequence $\{f_n\}\subset\mathcal{L}^\infty$ for which $\|g_n-f_n\|_\infty\leq\varepsilon$ for all large enough n. Then $\{g_n\}$ itself converges a.u.*

Theorem 4.9. *Let $T\in\mathrm{DS}$, and let $\{\beta_k\}$ be a bounded Besicovitch sequence. Then for every $f\in\mathcal{L}^1$ the averages* (4.2) *converge a.u.*

Proof. In view of Corollary 4.5, to prove that the averages $B_n(T)$ converge a.u. in $\mathcal{L}^1$ for every $T\in\mathrm{DS}$, it is sufficient to present a dense subset D of $\mathcal{L}^1$ such that $\{B_n(T)(f)\}$ converges a.u. for each $f\in D$.

Following the scheme in [R-N75], we begin by showing that, given a trigonometric polynomial P and $f\in\mathcal{L}^1$, the averages

$$A_n^{(P)}(T)(f) = \frac{1}{n}\sum_{k=0}^{n-1} P(k)T^k(f)$$

converge a.u. Consider the product space $(\mathbb{C}_1,\nu)\otimes(\Omega,\mu)$, where ν is Lebesgue measure in $\mathbb{C}_1$. Fix $\lambda\in\mathbb{C}_1$ and define an operator T_λ on $\mathcal{L}^1(\mathbb{C}_1\otimes\Omega)$ as follows: if $\widetilde{f}\in\mathcal{L}^1(\mathbb{C}_1\otimes\Omega)$, $z\in\mathbb{C}_1$, and $\omega\in\Omega$, put

$$T_\lambda(\widetilde{f})(z,\omega) = T(f_{\lambda z})(\omega), \quad \text{where} \quad f_z(\omega) = \widetilde{f}(z,\omega)$$

(note that $f_z\in\mathcal{L}^1$ for almost all $z\in\mathbb{C}_1$). It is easily verified that $T_\lambda\in\mathrm{DS}$ on $\mathcal{L}^1(\mathbb{C}_1\otimes\Omega)+\mathcal{L}^\infty(\mathbb{C}_1\otimes\Omega)$. For instance, given $\widetilde{f}\in\mathcal{L}^1(\mathbb{C}_1\otimes\Omega)$, we have

$$\int_{\mathbb{C}_1\otimes\Omega} |T_\lambda(\widetilde{f})(z,\omega)|\,d(\nu\otimes\mu) = \int_{\mathbb{C}_1}\int_{\Omega} |T(f_{\lambda z})(\omega)|\,d\mu\,d\nu \leq \int_{\mathbb{C}_1}\int_{\Omega} |f_{\lambda z}(\omega)|\,d\mu\,d\nu$$
$$= \int_{\Omega}\int_{\mathbb{C}_1} |f_{\lambda z}(\omega)|\,d\nu\,d\mu = \int_{\Omega}\int_{\mathbb{C}_1} |f_z(\omega)|\,d\nu\,d\mu = \int_{\mathbb{C}_1\otimes\Omega} |\widetilde{f}(z,\omega)|\,d(\nu\otimes\mu) = \|\widetilde{f}\|_1,$$

hence $T_\lambda(\widetilde{f})\in\mathcal{L}^1(\mathbb{C}_1\otimes\Omega)$ and $\|T_\lambda(\widetilde{f})\|_1\leq\|\widetilde{f}\|_1$.

It follows by induction that

$$(T_\lambda^k(\widetilde{f}))_z = T^k(f_{\lambda^k z}), \quad k = 1, 2, \dots.$$

Indeed, $(T_\lambda(\widetilde{f}))_z(\omega) = T_\lambda(\widetilde{f})(z,\omega) = T(f_{\lambda z})(\omega)$, so that $(T_\lambda(\widetilde{f}))_z = T(f_{\lambda z})$, and if $(T_\lambda^k(\widetilde{f}))_z = T^k(f_{\lambda^k z})$ for some $k \in \mathbb{N}$, then

$$\begin{aligned} T_\lambda^{k+1}(\widetilde{f})_z(\omega) &= T_\lambda(T_\lambda^k(\widetilde{f}))(z,\omega) = T(T_\lambda^k(\widetilde{f})_{\lambda z})(\omega) \\ &= T(T^k(f_{\lambda^{k+1} z}))(\omega) = T^{k+1}(f_{\lambda^{k+1} z})(\omega). \end{aligned}$$

Therefore,

$$T_\lambda^k(\widetilde{f})(z,\omega) = (T_\lambda^k(\widetilde{f}))_z(\omega) = T^k(f_{\lambda^k z})(\omega), \quad k = 1, 2, \dots.$$

Now, if $\widetilde{f} \in \mathcal{L}^1(\mathbb{C}_1 \otimes \Omega)$ is given by $\widetilde{f}(z,\omega) = zf(\omega)$, then $f_{\lambda^k z}(\omega) = \widetilde{f}(\lambda^k z, \omega) = \lambda^k z f(\omega)$, and we obtain

$$T_\lambda^k(\widetilde{f})(z,\omega) = T^k(f_{\lambda^k z})(\omega) = \lambda^k z T^k(f(\omega)), \quad k = 1, 2, \dots.$$

By Theorem 4.6, the averages

$$\frac{1}{n}\sum_{k=0}^{n-1} T_\lambda^k(\widetilde{f})(z,\omega) = z\,\frac{1}{n}\sum_{k=0}^{n-1} \lambda^k T^k(f(\omega))$$

converge a.u. on $(z,\omega) \in \mathbb{C}_1 \otimes \Omega$. Thus, by Lemma 4.7, they converge a.u. on Ω for some $z \in \mathbb{C}_1$, which implies that the averages

$$\frac{1}{n}\sum_{k=0}^{n-1} \lambda^k T^k(f)$$

converge a.u. Therefore, by linearity, $A_n^{(P)}(T)(f)$ converge a.u.

Now, assume that $f \in D = \mathcal{L}^1 \cap \mathcal{L}^\infty$. If we fix $\varepsilon > 0$ and take P to satisfy the inequality (4.1), then

$$\|A_n^{(P)}(T)(f) - B_n(T)(f)\|_\infty \le \|f\|_\infty \frac{1}{n}\sum_{k=0}^{n-1} |\beta_k - P(k)| < \varepsilon \|f\|_\infty$$

for all large enough n. Thus, Lemma 4.8 entails a.u. convergence of the sequence $\{B_n(T)(f)\}$, which completes the proof as the set D is dense in $\mathcal{L}^1$. ∎

Now we can present the main result of the section:

THEOREM 4.10. *Let $T \in \mathrm{DS}$, and let $\{\beta_k\}$ be a bounded Besicovitch sequence. Then, given $f \in \mathcal{R}_\mu$, the averages* (4.2) *converge a.u. to some $\widehat{f} \in \mathcal{R}_\mu$.*

Proof. Let $C \neq 0$ be such that $\sup\{|\beta_k|\} \le C$. Fix $\varepsilon > 0$ and $\delta > 0$. In view of Proposition 2.1, there exist $g \in \mathcal{L}^1$ and $h \in \mathcal{L}^\infty$ such that

$$f = g + h, \quad g \in \mathcal{L}^1,\ \|h\|_\infty \le \frac{\delta}{3C}.$$

Since $g \in \mathcal{L}^1$, Theorem 4.9 implies that there exist $E \subset \Omega$ and $N \in \mathbb{N}$ satisfying the conditions

$$\mu(\Omega \setminus E) \le \varepsilon \quad \text{and} \quad \|(B_m(g) - B_n(g))\chi_E\|_\infty \le \delta/3 \quad \forall m, n \ge N.$$

Then, given $m, n \ge N$, we have

$$\begin{aligned} &\|(B_m(f) - B_n(f))\chi_E\|_\infty \\ &\qquad \le \|(B_m(g) - B_n(g))\chi_E\|_\infty + \|(B_m(h) - B_n(h))\chi_E\|_\infty \\ &\qquad \le \delta/3 + \|B_m(h)\|_\infty + \|B_n(h)\|_\infty \le \delta/3 + 2C\|h\|_\infty \le \delta, \end{aligned}$$

which, by Propositions 4.3 and 2.2, implies that $\{B_n(f)\}$ converges a.u. to some $\widehat{f} \in \mathcal{R}_\mu$. ∎

5. A Wiener–Wintner-type ergodic theorem in $\mathcal{R}_\mu$. Recall that (Ω, μ) is a σ-finite measure space, and let $\tau : \Omega \to \Omega$ be a measure preserving transformation (m.p.t.). Assume that (X, ν) is a finite measure space and $\phi : X \to X$ is also a m.p.t. Given $f \in \mathcal{L}^0$ and $g \in \mathcal{L}^1(X)$, denote

$$A_n(f, g)(\omega, x) = \frac{1}{n} \sum_{k=0}^{n-1} g(\phi^k x) f(\tau^k \omega). \tag{5.1}$$

Here is an extension of Bourgain's Return Times theorem to infinite measures [A03, p. 101].

THEOREM 5.1. *Let $F \subset \Omega$ with $\mu(F) < \infty$. Then there exists $\Omega_F \subset \Omega$ such that $\mu(\Omega \setminus \Omega_F) = 0$ and for any (X, ν, ϕ) and $g \in \mathcal{L}^1(X)$ the averages*

$$A_n(\chi_F, g)(\omega, x) = \frac{1}{n} \sum_{k=0}^{n-1} g(\phi^k x) \chi_F(\tau^k \omega)$$

converge ν-a.e. for all $\omega \in \Omega_F$.

Note that, since μ is not necessarily finite, one can consider replacing in Theorem 5.1 the factor $1/n$ by $1/n(F, \omega)$, where

$$n(F, \omega) = \sum_{k=0}^{n-1} \chi_F(\tau^k \omega).$$

To this end, it was shown in [L97] that in this generality Theorem 5.1 is no longer valid. Namely, there exist a σ-finite measure space (Ω, μ), a m.p.t. $\tau : \Omega \to \Omega$, and $F \subset \Omega$ with $0 < \mu(F) < \infty$ such that for every finite measure space (X, ν) and every aperiodic m.p.t. $\phi : X \to X$ there exists $g \in \mathcal{L}^2(X, \nu)$ such that the averages

$$\frac{1}{n(F, \omega)} \sum_{k=0}^{n-1} g(\phi^k x) \chi_F(\tau^k \omega)$$

diverge a.e. on X for almost every $\omega \in F$.

The next result is a version of Theorem 5.1 where the functions χ_F and $g \in \mathcal{L}^1(X)$ are replaced by $f \in \mathcal{L}^1(\Omega)$ and $g \in \mathcal{L}^\infty(X)$, respectively.

THEOREM 5.2. *Given* $f \in \mathcal{L}^1(\Omega)$, *there exists a set* $\Omega_f \subset \Omega$ *with* $\mu(\Omega \setminus \Omega_f) = 0$ *such that for any* (X, ν, ϕ) *and* $g \in \mathcal{L}^\infty(X)$ *the averages* (5.1) *converge* ν*-a.e. for all* $\omega \in \Omega_f$.

Proof. Let $f \in \mathcal{L}^1(\Omega)$. Then there exist $\{\lambda_{m,i}\} \subset \mathbb{C}$ and $F_{m,i} \subset \Omega$ with $\mu(F_{m,i}) < \infty$, $m = 1, 2, \dots$, $1 \le i \le l_m$, such that

$$\|f - f_m\|_1 \to 0, \quad \text{where} \quad f_m = \sum_{i=1}^{l_m} \lambda_{m,i} \chi_{F_{m,i}}.$$

If

$$\Omega_{m,j} = \left\{ \omega \in \Omega : \sup_n \frac{1}{n} \sum_{k=0}^{n-1} |f - f_m|(\tau^k \omega) > \frac{1}{j} \right\},$$

then, due to the maximal ergodic inequality, we have

$$\mu(\Omega_{m,j}) \le j \|f - f_m\|_1,$$

which implies that $\mu(\bigcap_m \Omega_{m,j}) = 0$ for a fixed j. Therefore, denoting

$$\Omega_0 = \Omega \setminus \bigcup_j \bigcap_m \Omega_{m,j},$$

we obtain $\mu(\Omega \setminus \Omega_0) = 0$.

If $\omega \in \Omega_0$, then $\omega \notin \Omega_{m_j,j}$ for every j and some m_j, and therefore

$$\sup_n \frac{1}{n} \sum_{k=0}^{n-1} |f - f_{m_j}|(\tau^k \omega) \le \frac{1}{j} \quad \text{for all } j \text{ and } \omega \in \Omega_0. \tag{5.2}$$

Now, by Theorem 5.1, there exist $G_{j,i} \subset \Omega$ with $\mu(\Omega \setminus G_{j,i}) = 0$ such that for every (X, ν, ϕ) and $g \in \mathcal{L}^\infty(X)$ the averages

$$\frac{1}{n} \sum_{k=0}^{n-1} g(\phi^k x) \chi_{F_{m_j,i}}(\tau^k \omega)$$

converge ν-a.e. for all $\omega \in G_{j,i}$. Then, letting

$$\Omega_f = \Big(\bigcup_{j=1}^{\infty} \bigcap_{i=1}^{l_{m_j}} G_{j,i} \Big) \cap \Omega_0,$$

we obtain $\mu(\Omega \setminus \Omega_f) = 0$.

If we pick any (X, ν, ϕ) and $g \in \mathcal{L}^\infty(X)$, then the averages $A_n(f_{m_j}, g)(\omega, x)$ converge ν-a.e. for every j and all $\omega \in \Omega_f$, and it follows that there are $X_0 \subset X$ with $\nu(X \setminus X_0) = 0$ and $C > 0$ such that $|g(\phi^k x)| \le C$ for all k and

$x \in X_0$ and

$$\liminf_n \operatorname{Re} A_n(f_{m_j}, g)(\omega, x) = \limsup_n \operatorname{Re} A_n(f_{m_j}, g)(\omega, x),$$
$$\liminf_n \operatorname{Im} A_n(f_{m_j}, g)(\omega, x) = \limsup_n \operatorname{Im} A_n(f_{m_j}, g)(\omega, x)$$

for all $x \in X_0$, j, and $\omega \in \Omega_f$.

Let $\omega \in \Omega_f$ and $x \in X_0$. Given j, taking into account (5.2), we have

$$\begin{aligned}\Delta(\omega, x) &= \limsup_n \operatorname{Re} A_n(f, g)(\omega, x) - \liminf_n \operatorname{Re} A_n(f, g)(\omega, x) \\ &= \limsup_n \operatorname{Re} A_n(f - f_{m_j}, g)(\omega, x) - \liminf_n \operatorname{Re} A_n(f - f_{m_j}, g)(\omega, x) \\ &\le 2 \sup_n A_n(|f - f_{m_j}|, |g|)(\omega, x) \\ &\le 2C \sup_n \frac{1}{n} \sum_{k=0}^{n-1} |f - f_{m_j}|(\tau^k \omega) \le \frac{2C}{j}.\end{aligned}$$

Therefore, $\Delta(\omega, x) = 0$. Similarly,

$$\limsup_n \operatorname{Im} A_n(f, g)(\omega, x) = \liminf_n \operatorname{Im} A_n(f, g)(\omega, x),$$

and we conclude that the averages (5.1) converge ν-a.e. for all $\omega \in \Omega_f$. ∎

Now we extend Theorem 5.2 to $\mathcal{R}_\mu$.

THEOREM 5.3. *Given $f \in \mathcal{R}_\mu$, there exists a set $\Omega_f \subset \Omega$ with $\mu(\Omega \setminus \Omega_f) = 0$ such that for any finite measure space (X, ν), any m.p.t. $\phi : X \to X$, and any $g \in \mathcal{L}^\infty(X)$ the averages* (5.1) *converge ν-a.e. for all $\omega \in \Omega_f$.*

Proof. Due to Proposition 2.1, given a natural m, there exist $f_m \in \mathcal{L}^1(\Omega)$ and $h_m \in \mathcal{L}^\infty(\Omega)$ such that $f = f_m + h_m$ and $\|h_m\|_\infty \le 1/m$. Then there is $\Omega_0 \subset \Omega$ such that $\mu(\Omega \setminus \Omega_0) = 0$, $|h_m(\tau^k \omega)| \le 1/m$ for all $m \in \mathbb{N}$, $k = 0, 1, \ldots$, and $\omega \in \Omega_0$.

By Theorem 5.2, as $\{f_m\}_{m=1}^\infty \subset \mathcal{L}^1(\Omega)$, for every m there is a set $\Omega_m \subset \Omega$ with $\mu(\Omega \setminus \Omega_m) = 0$ such that for every (X, ν, ϕ) and $g \in \mathcal{L}^\infty(X)$ the averages

$$A_n(f_m, g)(\omega, x) = \frac{1}{n} \sum_{k=0}^{n-1} g(\phi^k x) f_m(\tau^k \omega) \tag{5.3}$$

converge ν-a.e. for all $\omega \in \Omega_m$. So, if $\Omega_f = \bigcap_{m=0}^\infty \Omega_m$, then $\mu(\Omega \setminus \Omega_f) = 0$, $|h_m(\tau^k \omega)| \le 1/m$ for all $m \in \mathbb{N}$, $k = 0, 1, \ldots$, and $\omega \in \Omega_f$, and for every (X, ν, ϕ) and $g \in \mathcal{L}^\infty(X)$, the averages (5.3) converge ν-a.e. for all m and $\omega \in \Omega_f$.

Fix $\omega \in \Omega_f$, (X, ν, ϕ), $g \in \mathcal{L}^\infty(X, \nu)$; we will show that the averages (5.1) converge ν-a.e. Indeed, as the averages (5.3) converge ν-a.e. for each m, there is a set $X_1 \subset X$ with $\nu(X \setminus X_1) = 0$ such that the sequence (5.3) converges for every m and $x \in X_1$. Also, since the averages

$$\frac{1}{n}\sum_{k=0}^{n-1} |g|(\phi^k x)$$

converge ν-a.e., there is a set $X_2 \subset X$ such that $\nu(X \setminus X_2) = 0$ and the sequence $n^{-1}\sum_{k=0}^{n-1} |g|(\phi^k x)$ converges for all $x \in X_2$. Then, letting $X_0 = X_1 \cap X_2$, we conclude that $\nu(X \setminus X_0) = 0$, $\sup_n n^{-1}\sum_{k=0}^{n-1} |g|(\phi^k x) < \infty$, and the sequence (5.3) converges for all m and $x \in X_0$. Now, if $x \in X_0$, we have

$$\liminf_n \operatorname{Re} A_n(f_m, g)(\omega, x) = \limsup_n \operatorname{Re} A_n(f_m, g)(\omega, x),$$
$$\liminf_n \operatorname{Im} A_n(f_m, g)(\omega, x) = \limsup_n \operatorname{Im} A_n(f_m, g)(\omega, x),$$

which implies that, for every m,

$$\begin{aligned}\Delta(\omega, x) &= \limsup_n \operatorname{Re} A_n(f, g)(\omega, x) - \liminf_n \operatorname{Re} A_n(f, g)(\omega, x)\\ &= \limsup_n \operatorname{Re} A_n(f - f_m, g)(\omega, x) - \liminf_n \operatorname{Re} A_n(f - f_m, g)(\omega, x)\\ &\le 2\sup_n \frac{1}{n}\sum_{k=0}^{n-1} |g(\phi^k x)| \cdot |h_m(\tau^k\omega)| \le \frac{2}{m}\sup_n \frac{1}{n}\sum_{k=0}^{n-1} |g|(\phi^k x).\end{aligned}$$

Therefore, $\Delta(\omega, x) = 0$. Similarly,

$$\limsup_n \operatorname{Im} A_n(f, g)(\omega, x) = \liminf_n \operatorname{Im} A_n(f, g)(\omega, x),$$

and we conclude that the averages (5.1) converge ν-a.e. ∎

If in Theorem 5.3 we let $X = \mathbb{C}_1 = \{x \in \mathbb{C} : |x| = 1\}$ with the Lebesgue measure ν, $\phi_\lambda(x) = \lambda x$, $x \in X$, for a given $\lambda \in X$, and $g(x) = x$ whenever $x \in X$, we obtain the Wiener–Wintner theorem for $\mathcal{R}_\mu$:

THEOREM 5.4. *If $f \in \mathcal{R}_\mu$, then there is a set $\Omega_f \subset \Omega$ with $\mu(\Omega \setminus \Omega_f) = 0$ such that the averages*

$$\frac{1}{n}\sum_{k=0}^{n-1} \lambda^k f(\tau^k\omega)$$

converge for all $\omega \in \Omega_f$ and $\lambda \in \mathbb{C}_1$.

Let $P(k) = \sum_{j=1}^{s} z_j\lambda_j^k$, $k = 0, 1, 2, \ldots$, be a trigonometric polynomial (see Section 4). Then, by linearity, Theorem 5.4 implies the following.

COROLLARY 5.5. *Given $f \in \mathcal{R}_\mu$, there exists a set $\Omega_f \subset \Omega$ with $\mu(\Omega \setminus \Omega_f) = 0$ such that the averages*

$$A_n(\{P(k)\}, f)(\omega) = \frac{1}{n}\sum_{k=0}^{n-1} P(k) f(\tau^k\omega)$$

converge for every $\omega \in \Omega_f$ and any trigonometric polynomial $P(k)$.

We will need the following.

PROPOSITION 5.6. *If* $f \in \mathcal{L}^1 \cap \mathcal{L}^\infty$, *then there exists* $\Omega_f \subset \Omega$ *with* $\mu(\Omega \setminus \Omega_f) = 0$ *such that the averages*

$$A_n(\overline{\beta}, f)(\omega) = \frac{1}{n} \sum_{k=0}^{n-1} \beta_k f(\tau^k \omega) \tag{5.4}$$

converge for every $\omega \in \Omega_f$ *and any bounded Besicovitch sequence* $\overline{\beta} = \{\beta_k\}$.

Proof. By Corollary 5.5, there exists a set $\Omega_{f,1} \subset \Omega$, $\mu(\Omega \setminus \Omega_{f,1}) = 0$, such that the sequence $n^{-1} \sum_{k=0}^{n-1} P(k) f(\tau^k \omega)$ converges for every $\omega \in \Omega_{f,1}$ and any trigonometric polynomial $P(k)$. Also, since $f \in \mathcal{L}^\infty$, there is a set $\Omega_{f,2} \subset \Omega$, $\mu(\Omega \setminus \Omega_{f,2}) = 0$ such that $|f(\tau^k \omega)| \le \|f\|_\infty$ for every k and $\omega \in \Omega_{f,2}$. If we set $\Omega_f = \Omega_{f,1} \cap \Omega_{f,2}$, then $\mu(\Omega \setminus \Omega_f) = 0$.

Now, let $\omega \in \Omega_f$, and let $\overline{\beta} = \{\beta_k\}$ be a Besicovitch sequence. Fix $\varepsilon > 0$, and choose a trigonometric polynomial $P(k)$ that satisfies (4.1). Then

$$\begin{aligned} \Delta(\omega) &= \limsup_n \operatorname{Re} A_n(\overline{\beta}, f)(\omega) - \liminf_n \operatorname{Re} A_n(\overline{\beta}, f)(\omega) \\ &= \limsup_n \operatorname{Re} A_n(\{\beta_k - P(k)\}, f)(\omega) \\ &\quad - \liminf_n \operatorname{Re} A_n(\{\beta_k - P(k)\}, f)(\omega) \\ &\le 2\|f\|_\infty \sup_n \frac{1}{n} \sum_{k=0}^{n-1} |\beta_k - P(k)| < 2\|f\|_\infty \varepsilon \end{aligned}$$

for all sufficiently large n. Therefore, $\Delta(\omega) = 0$, and we conclude that the sequence $\{\operatorname{Re} A_n(\overline{\beta}, f)(\omega)\}$ converges. Similarly, we obtain the convergence of $\{\operatorname{Im} A_n(\overline{\beta}, f)(\omega)\}$. ∎

THEOREM 5.7. *If* $f \in \mathcal{L}^1$, *then there exists a set* $\Omega_f \subset \Omega$ *with* $\mu(\Omega \setminus \Omega_f) = 0$ *such that the averages* (5.4) *converge for every* $\omega \in \Omega_f$ *and any bounded Besicovitch sequence* $\overline{\beta} = \{\beta_k\}$.

Proof. Let $\{f_m\} \subset \mathcal{L}^1 \cap \mathcal{L}^\infty$ be a sequence such that $\|f - f_m\|_1 \to 0$. As in the proof of Theorem 5.2, we construct a subsequence $\{f_{m_j}\}$ and a set $\Omega_0 \subset \Omega$ with $\mu(\Omega \setminus \Omega_0) = 0$ such that

$$\sup_n \frac{1}{n} \sum_{k=0}^{n-1} |f - f_{m_j}|(\tau^k \omega) \le \frac{1}{j} \quad \forall j \text{ and } \omega \in \Omega_0.$$

By Proposition 5.6, given j, there is $\Omega_j \subset \Omega$ with $\mu(\Omega \setminus \Omega_j) = 0$ such that the sequence $\{n^{-1} \sum_{k=0}^{n-1} \beta_k f_{m_j}(\tau^k \omega)\}$ converges for every $\omega \in \Omega_j$ and any Besicovitch sequence $\{\beta_k\}$.

If we set $\Omega_f = \bigcap_{j=1}^\infty \Omega_j \cap \Omega_0$, then $\mu(\Omega \setminus \Omega_f) = 0$, and for any $\omega \in \Omega_f$ and any bounded Besicovitch sequence $\overline{\beta} = \{\beta_k\}$ such that $\sup_k |\beta_k| \le C$ we have

$$\begin{aligned}\Delta(\omega) &= \limsup_n \operatorname{Re} A_n(\overline{\beta}, f)(\omega) - \liminf_n \operatorname{Re} A_n(\overline{\beta}, f)(\omega)\\ &= \limsup_n \operatorname{Re} A_n(\overline{\beta}, f - f_{m_j})(\omega) - \liminf_n \operatorname{Re} A_n(\overline{\beta}, f - f_{m_j})(\omega)\\ &\le 2 \sup_n \frac{1}{n} \sum_{k=0}^{n-1} |\beta_k|\, |f - f_{m_j}|(\tau^k \omega) \le \frac{2C}{j}.\end{aligned}$$

Therefore, $\Delta(\omega) = 0$, hence $\{\operatorname{Re} A_n(\overline{\beta}, f)(\omega)\}$ is convergent. Similarly, we derive the convergence of $\{\operatorname{Im} A_n(\overline{\beta}, f)(\omega)\}$. ∎

Taking into account that the sequence $\{\beta_k\}$ is bounded, we obtain, as in the proof of Theorem 5.3, the following extension of the Wiener–Wintner theorem.

Theorem 5.8. *Given $f \in \mathcal{R}_\mu$, there exists a set $\Omega_f \subset \Omega$ with $\mu(\Omega \setminus \Omega_f) = 0$ such that the averages* (5.4) *converge for every $\omega \in \Omega_f$ and every bounded Besicovitch sequence $\{\beta_k\}$.*

6. Applications to fully symmetric spaces. For any $f \in \mathcal{L}^0_\mu$ the *non-increasing rearrangement* of f is defined as

$$f^*(t) = \inf\{\lambda > 0 :\ \mu\{|f| > \lambda\} \le t\}, \qquad t > 0$$

(see [BS88, Ch. II, §2]).

Let ν be the Lebesgue measure on $(0, \infty)$. A non-zero linear subspace $E \subset \mathcal{L}^0_\nu$ with a Banach norm $\|\cdot\|_E$ is called *symmetric* (respectively, *fully symmetric*) on $((0, \infty), \nu)$ if

$$f \in E, \qquad g \in \mathcal{L}^0_\nu, \qquad g^*(t) \le f^*(t) \ \ \forall t > 0$$

(respectively,

$$f \in E, \qquad g \in \mathcal{L}^0_\nu, \qquad \int_0^s g^*(t)\, dt \le \int_0^s f^*(t)\, dt \quad \forall s > 0$$

(written $g \prec\prec f$)), implies that $g \in E$ and $\|g\|_E \le \|f\|_E$.

Let $(E, \|\cdot\|_E)$ be a symmetric (or fully symmetric) space on $((0, \infty), \nu)$. Define

$$E(\Omega) = E(\Omega, \mu) = \{f \in \mathcal{L}^0_\mu : f^*(t) \in E\}$$

and set

$$\|f\|_{E(\Omega)} = \|f^*(t)\|_E, \qquad f \in E(\Omega).$$

It is shown in [KS08] (see also [LSZ13, Sec. 3.5]) that $(E(\Omega), \|\cdot\|_{E(\Omega)})$ is a Banach space and the conditions $f \in E(\Omega)$, $g \in \mathcal{L}^0_\mu$, $g^*(t) \le f^*(t)$ for every $t > 0$ ($g \prec\prec f$) imply that $g \in E(\Omega)$ and $\|g\|_{E(\Omega)} \le \|f\|_{E(\Omega)}$. In such a case, we say that $(E(\Omega), \|\cdot\|_{E(\Omega)})$ is the symmetric (respectively, fully symmetric) space on (Ω, μ) *generated* by the symmetric (respectively, fully symmetric) space $(E, \|\cdot\|_E)$. Throughout, if it does not cause confusion, we will write $(E, \|\cdot\|_E)$ or simply E instead of $(E(\Omega), \|\cdot\|_{E(\Omega)})$.

Immediate examples of fully symmetric spaces are the spaces $\mathcal{L}^p(\Omega, \mu)$, $1 \le p \le \infty$, with the standard norms $\|\cdot\|_p$, the space $\mathcal{L}^1 \cap \mathcal{L}^\infty$ with the norm

$$\|f\|_{\mathcal{L}^1 \cap \mathcal{L}^\infty} = \max\{\|f\|_1, \|f\|_\infty\},$$

and the space $\mathcal{L}^1 + \mathcal{L}^\infty$ with the norm

$$\|f\|_{\mathcal{L}^1 + \mathcal{L}^\infty} = \inf\{\|g\|_1 + \|h\|_\infty : f = g + h,\ g \in \mathcal{L}^1,\ h \in \mathcal{L}^\infty\}.$$

Note that, alternatively,

$$\mathcal{R}_\mu = \{f \in \mathcal{L}^1 + \mathcal{L}^\infty : f^*(t) \to 0 \text{ as } t \to \infty\}$$

and $(\mathcal{R}_\mu, \|\cdot\|_{\mathcal{L}^1 + \mathcal{L}^\infty})$ is a symmetric space [KPS82, Ch. II, §4, Lemma 4.4]. In addition, $\mathcal{R}_\mu$ is the closure of $\mathcal{L}^1 \cap \mathcal{L}^\infty$ in $(\mathcal{L}^1 + \mathcal{L}^\infty, \|\cdot\|_{\mathcal{L}^1 + \mathcal{L}^\infty})$ (see [KPS82, Ch. II, §3, Sec. 1]). Furthermore, it follows from the definitions of $\mathcal{R}_\mu$ and $\|\cdot\|_{\mathcal{L}^1 + \mathcal{L}^\infty}$ that if

$$f \in \mathcal{R}_\mu, \quad g \in \mathcal{L}^1 + \mathcal{L}^\infty \quad \text{and} \quad g \prec\prec f,$$

then $g \in \mathcal{R}_\mu$ and $\|g\|_{\mathcal{L}^1 + \mathcal{L}^\infty} \le \|f\|_{\mathcal{L}^1 + \mathcal{L}^\infty}$. Therefore, $(\mathcal{R}_\mu, \|\cdot\|_{\mathcal{L}^1 + \mathcal{L}^\infty})$ is also a fully symmetric space. If $\mu(\Omega) < \infty$, then $\mathcal{R}_\mu = \mathcal{L}^1$.

Also, given $T \in \mathrm{DS}$, we have $T(E) \subset E$ and $\|T\|_{E \to E} \le 1$ for any fully symmetric space E (see [KPS82, Ch. II, §4, Theorem 4.1]). In addition,

$$\int_0^s T(f)^*(t)\, dt \le \int_0^s f^*(t)\, dt \qquad \forall s > 0,$$

that is, $T(f) \prec\prec f$ for every $f \in \mathcal{L}^1 + \mathcal{L}^\infty$ (see, for example, [KPS82, Ch. II, §3, Section 4]).

Proposition 6.1. *If $\mu(\Omega) = \infty$, then a symmetric space E is contained in $\mathcal{R}_\mu$ if and only if $\mathbf{1} \notin E$.*

Proof. As $\mu(\Omega) = \infty$, we have $\mathbf{1}^*(t) = 1$ for all $t > 0$, hence $\mathbf{1} \notin \mathcal{R}_\mu$. Therefore, E is not contained in $\mathcal{R}_\mu$ whenever $\mathbf{1} \in E$.

Let $\mathbf{1} \notin E$. If $f \in E$ and $\lim_{t \to \infty} f^*(t) = \alpha > 0$, then

$$\mathbf{1}^*(t) \equiv 1 \le \alpha^{-1} f^*(t),$$

implying $\mathbf{1} \in E$, a contradiction. Thus $\mathbf{1} \notin E$ entails $E \subset \mathcal{R}_\mu$. ∎

The following is a version of Theorems 4.10 for fully symmetric spaces.

Theorem 6.2. *Let E be a fully symmetric space such that $\mathbf{1} \notin E$. If $\{\beta_k\}$ is a bounded Besicovitch sequence, then for every $T \in \mathrm{DS}$ and $f \in E$ the averages* (4.2) *converge a.u. to some $\widehat{f} \in E$.*

Proof. Since, by Proposition 6.1, $E \subset \mathcal{R}_\mu$, it follows from Theorem 4.10 that the averages $B_n(T)(f)$ converge a.u., hence in measure topology, to some $\widehat{f} \in \mathcal{R}_\mu$. Therefore, $(B_n(T)(f))^* \to (\widehat{f})^*$ a.e. on $(0,\infty)$ [KPS82, Ch. II, §2, Property 11°].

With $M = \max\{1, \sup|\beta_k|\}$, we have $M^{-1}B_n(T) \in \mathrm{DS}$, hence

$$M^{-1}B_n(T)(f) \prec\prec f$$

for every n [KPS82, Ch. II, §3, Section 4]. Since

$$(M^{-1}B_n(T)(f))^* \to (M^{-1}\widehat{f})^* \quad \text{a.e.} \quad \text{on } (0,s),$$

Fatou's Lemma entails

$$\int_0^s (M^{-1}\widehat{f})^*(t)\,dt \le \liminf_n \int_0^s (M^{-1}B_n(T)(f))^*\,dt \le \int_0^s f^*\,dt$$

for all $s > 0$, that is, $(\widehat{f})^* \prec\prec Mf^*$. As E is a fully symmetric space and $f \in E$, it follows that $\widehat{f} \in E$. ∎

The next variant of Theorems 5.8 for fully symmetric spaces is straightforward.

THEOREM 6.3. *Let E be a fully symmetric space and let $\mathbf{1} \notin E$. Then for every $f \in E$ there exists a set $\Omega_f \subset \Omega$ with $\mu(\Omega \setminus \Omega_f) = 0$ such that the averages* (5.4) *converge for every $\omega \in \Omega_f$ and every bounded Besicovitch sequence $\{\beta_k\}$.*

A symmetric space $(E, \|\cdot\|_E)$ is said to have an *order continuous norm* if $\|f_n\|_E \downarrow 0$ whenever $f_n \in E$ and $f_n \downarrow 0$. It is known that a symmetric space E with order continuous norm is fully symmetric and $E \subset \mathcal{R}_\mu$ [KPS82, Ch. II, §4].

Since $E \subset \mathcal{R}_\mu$ for a symmetric space E with order continuous norm, it follows that Theorems 6.2 and 6.3 are valid for any symmetric space with order continuous norm.

Now we give applications of Theorems 6.2 and 6.3 to Orlicz, Lorentz, and Marcinkiewicz spaces.

1. Let Φ be an *Orlicz function*, that is, $\Phi : [0,\infty) \to [0,\infty)$ is convex, continuous at 0 and such that $\Phi(0) = 0$ and $\Phi(u) > 0$ if $u \ne 0$. Let

$$\mathcal{L}^\Phi = \Big\{ f \in \mathcal{L}^0_\mu : \int_\Omega \Phi(a^{-1}|f|)\,d\mu < \infty \text{ for some } a > 0 \Big\}$$

be the corresponding *Orlicz space*, and let

$$\|f\|_\Phi = \inf\Big\{ a > 0 : \int_\Omega \Phi(a^{-1}|f|)\,d\mu \le 1 \Big\}$$

be the *Luxemburg norm* in $\mathcal{L}^\Phi$. Then $(\mathcal{L}^\Phi, \|\cdot\|_\Phi)$ is a fully symmetric space (see, for example, [ES92, Ch. 2]). As $\mu(\Omega) = \infty$, we have $\int_\Omega \Phi(a^{-1})\,d\mu = \infty$

for all $a > 0$, hence $\mathbf{1} \notin \mathcal{L}^{\Phi}$. Therefore, Theorems 6.2 and 6.3 hold for any Orlicz space $\mathcal{L}^{\Phi}$.

2. Let $\varphi(t)$ be a concave increasing function on $[0, \infty)$ such that $\varphi(0) = 0$, $\varphi(t) > 0$ if $t > 0$, and let

$$\Lambda_\varphi = \Big\{ f \in \mathcal{L}_\mu^0 : \|f\|_{\Lambda_\varphi} = \int_0^\infty f^*(t)\, d\varphi(t) < \infty \Big\}$$

be the corresponding *Lorentz space*. Then $(\Lambda_\varphi, \|\cdot\|_{\Lambda_\varphi})$ is a fully symmetric space. In addition, if $\varphi(\infty) = \lim_{t\to\infty} \varphi(t) = \infty$, then $\mathbf{1} \notin \Lambda_\varphi$ [KPS82, Ch. II, §5]. Therefore, Theorems 6.2 and 6.3 are valid for any Lorentz space Λ_φ with $\varphi(\infty) = \infty$.

3. Let φ be as above, and let

$$M_\varphi = \Big\{ f \in \mathcal{L}_\mu^0 : \|f\|_{M_\varphi} = \sup_{0<s<\infty} \frac{1}{\varphi(s)} \int_0^s f^*(t)\, dt < \infty \Big\}$$

be the corresponding *Marcinkiewicz space*. It is known that $(M_\varphi, \|\cdot\|_{M_\varphi})$ is a fully symmetric space and that $\mathbf{1} \notin M_\varphi$ if and only if $\lim_{t\to\infty} \varphi(t)/t = 0$ [KPS82, Ch. II, §5]. Thus, Theorems 6.2 and 6.3 hold for any Marcinkiewicz space M_φ such that $\lim_{t\to\infty} \varphi(t)/t = 0$.

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