

## RECOLLEMENTS OF EXTRIANGULATED CATEGORIES

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**Abstract.** We give a simultaneous generalization of recollements of abelian categories and triangulated categories, which we call recollements of extriangulated categories. For a recollement  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  of extriangulated categories, we show that cotorsion pairs in  $\mathcal{A}$  and  $\mathcal{C}$  induce cotorsion pairs in  $\mathcal{B}$  under certain conditions. As an application, our main result recovers a result given by Chen for recollements of triangulated categories, and it also shows a new phenomenon when it is applied to abelian categories.

**1. Introduction.** Abelian categories and triangulated categories are two fundamental kinds of structures in algebra and geometry. Recollements of triangulated categories were introduced by Beilinson, Bernstein and Deligne [2] in connection with derived categories of sheaves on topological spaces with the idea that one triangulated category may be “glued together” from two others. Recollements of abelian categories first appeared in the construction of the category of perverse sheaves on a singular space in [2]. Recollements of abelian categories and triangulated categories play an important role in algebraic geometry and representation theory: see for instance [1, 6, 13]. Tilting theory and  $\tau$ -tilting theory are important tools in the representation theory of artin algebras. Ma, Xie and Zhao [10] study the gluing of support  $\tau$ -tilting modules in a recollement of module categories; see also [8, 7] for support  $\tau$ -tilting modules. In recollements of abelian categories, the relations with tilting modules and torsion pairs have been studied in [11] and [9], respectively. Chen [5] studied the relationship of cotorsion pairs among three triangulated categories in a recollement of triangulated categories.

Recently, Nakaoka and Palu [12] introduced an extriangulated category which is extracting properties on triangulated categories and exact categories. Recollements of abelian categories and triangulated categories are closely related, and they enjoy similar properties in many respects. This inspires us to give a simultaneous generalization of recollements of abelian categories and triangulated categories, which we call recollements of extri-

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angulated categories. Then we study the relationship of cotorsion pairs in a recollement of extriangulated categories. In order to achieve this goal, we need to consider the WIC Condition (cf. [12, Condition 5.8]), introduce compatible morphisms, and then define left exact sequences, right exact sequences, left exact functors, and right exact functors in extriangulated categories.

The paper is organized as follows: We summarize some basic definitions and properties of extriangulated categories and exact functors in Section 2. In Section 3, we introduce the recollement of extriangulated categories and collect some basic properties. Section 4 is devoted to giving conditions such that the glued pair with respect to cotorsion pairs in  $\mathcal{A}$  and  $\mathcal{C}$  is a cotorsion pair in  $\mathcal{B}$  for a recollement  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  of extriangulated categories. Moreover, we show that the converse also holds for some special cotorsion pairs in  $\mathcal{B}$ . As an application, we recover the corresponding results in the triangulated category case.

**1.1. Conventions and notation.** For an additive category  $\mathcal{C}$ , its subcategories are assumed to be full and closed under isomorphisms. A subcategory  $\mathcal{D}$  of  $\mathcal{C}$  is said to be *contravariantly finite* in  $\mathcal{C}$  if for each object  $M \in \mathcal{C}$ , there exists a morphism  $f : X \rightarrow M$  with  $X \in \mathcal{D}$  such that  $\mathcal{C}(\mathcal{D}, f)$  is an epimorphism. Dually, one defines *covariantly finite* subcategories in  $\mathcal{C}$ . Given an object  $M \in \mathcal{C}$ , we denote by  $\text{add}(M)$  the *additive closure* of  $M$ , that is, the full subcategory of  $\mathcal{C}$  whose objects are the finite direct sums of direct summands of  $M$ . Let  $Q$  be a finite acyclic quiver. We denote by  $S_i$  the one-dimensional simple (left)  $kQ$ -module associated to the vertex  $i$  of  $Q$ , and denote by  $P_i$  and  $I_i$  the projective cover and the injective envelope of  $S_i$ , respectively.

## 2. Preliminaries

**2.1. Extriangulated categories.** Let us recall from [12] some notions concerning extriangulated categories.

Let  $\mathcal{C}$  be an additive category and let  $\mathbb{E} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}$  be a biadditive functor. For any pair of objects  $A, C \in \mathcal{C}$ , an element  $\delta \in \mathbb{E}(C, A)$  is called an  $\mathbb{E}$ -*extension*. The zero element  $0 \in \mathbb{E}(C, A)$  is called the *split  $\mathbb{E}$ -extension*. For any morphism  $a \in \mathcal{C}(A, A')$  and  $c \in \mathcal{C}(C', C)$ , we have  $\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A')$  and  $\mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A)$ . We simply denote these elements by  $a_*\delta$  and  $c^*\delta$ , respectively. A morphism  $(a, c) : \delta \rightarrow \delta'$  of  $\mathbb{E}$ -extensions is a pair of morphisms  $a \in \mathcal{C}(A, A')$  and  $c \in \mathcal{C}(C, C')$  satisfying  $a_*\delta = c^*\delta'$ .

By Yoneda's lemma, any  $\mathbb{E}$ -extension  $\delta \in \mathbb{E}(C, A)$  induces natural transformations

$$\delta_{\#} : \mathcal{C}(-, C) \rightarrow \mathbb{E}(-, A) \quad \text{and} \quad \delta^{\#} : \mathcal{C}(A, -) \rightarrow \mathbb{E}(C, -).$$

For any  $X \in \mathcal{C}$ , these  $(\delta_{\#})_X$  and  $(\delta^{\#})_X$  are defined by  $(\delta_{\#})_X : \mathcal{C}(X, C) \rightarrow \mathbb{E}(X, A)$ ,  $f \mapsto f^* \delta$  and  $(\delta^{\#})_X : \mathcal{C}(A, X) \rightarrow \mathbb{E}(C, X)$ ,  $g \mapsto g_* \delta$ .

Two sequences of morphisms  $A \xrightarrow{x} B \xrightarrow{y} C$  and  $A \xrightarrow{x'} B' \xrightarrow{y'} C$  in  $\mathcal{C}$  are said to be *equivalent* if there exists an isomorphism  $b \in \mathcal{C}(B, B')$  such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \parallel & & \downarrow b \simeq & & \parallel \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \end{array}$$

is commutative. We denote the equivalence class of  $A \xrightarrow{x} B \xrightarrow{y} C$  by  $[A \xrightarrow{x} B \xrightarrow{y} C]$ . In addition, for any  $A, C \in \mathcal{C}$ , we denote

$$0 = [A \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} A \oplus C \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} C].$$

For any two equivalence classes  $[A \xrightarrow{x} B \xrightarrow{y} C]$  and  $[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$ , we denote

$$[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'].$$

DEFINITION 2.1. Let  $\mathfrak{s}$  be a correspondence which associates an equivalence class  $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$  to any  $\mathbb{E}$ -extension  $\delta \in \mathbb{E}(C, A)$ . Then  $\mathfrak{s}$  is called a *realization* of  $\mathbb{E}$  if for any morphism  $(a, c) : \delta \rightarrow \delta'$  with  $\mathfrak{s}(\delta) = [\Delta_1]$  and  $\mathfrak{s}(\delta') = [\Delta_2]$ , there is a commutative diagram as follows:

$$\begin{array}{ccccc} \Delta_1 & & A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow & & \downarrow a & & \downarrow b & & \downarrow c \\ \Delta_2 & & A & \xrightarrow{x'} & B & \xrightarrow{y'} & C \end{array}$$

A realization  $\mathfrak{s}$  of  $\mathbb{E}$  is said to be *additive* if it satisfies the following conditions:

- (a) For any  $A, C \in \mathcal{C}$ , the split  $\mathbb{E}$ -extension  $0 \in \mathbb{E}(C, A)$  satisfies  $\mathfrak{s}(0) = 0$ .
- (b)  $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$  for any pair of  $\mathbb{E}$ -extensions  $\delta$  and  $\delta'$ .

Let  $\mathfrak{s}$  be an additive realization of  $\mathbb{E}$ . If  $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ , then the sequence  $A \xrightarrow{x} B \xrightarrow{y} C$  is called a *conflation*,  $x$  is called an *inflation* and  $y$  is called a *deflation*. In this case, we say  $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}$  is an  $\mathbb{E}$ -*triangle*. We will write  $A = \text{cocone}(y)$  and  $C = \text{cone}(x)$  if necessary. We say an  $\mathbb{E}$ -triangle is *splitting* if it realizes 0.

DEFINITION 2.2 ([12, Definition 2.12]). We call a triplet  $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$  an *extriangulated category* if it satisfies the following conditions:

- (ET1)  $\mathbb{E} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}$  is a biadditive functor.
- (ET2)  $\mathfrak{s}$  is an additive realization of  $\mathbb{E}$ .

(ET3) Let  $\delta \in \mathbb{E}(C, A)$  and  $\delta' \in \mathbb{E}(C', A')$  be any pair of  $\mathbb{E}$ -extensions, realized as  $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ ,  $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$ . For any commutative square

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ a \downarrow & & \downarrow b & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

in  $\mathcal{C}$ , there exists a morphism  $(a, c): \delta \rightarrow \delta'$  which is realized by  $(a, b, c)$ .

(ET3)<sup>op</sup> Dual of (ET3).

(ET4) Let  $\delta \in \mathbb{E}(D, A)$  and  $\delta' \in \mathbb{E}(F, B)$  be a pair of  $\mathbb{E}$ -extensions realized by  $A \xrightarrow{f} B \xrightarrow{f'} D$  and  $B \xrightarrow{g} C \xrightarrow{g'} F$ , respectively. Then there exist an object  $E \in \mathcal{C}$ , a commutative diagram

$$(2.1) \quad \begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \parallel & & \downarrow g & & \downarrow d \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\ & & \downarrow g' & & \downarrow e \\ & & F & = & F \end{array}$$

in  $\mathcal{C}$ , and an  $\mathbb{E}$ -extension  $\delta'' \in \mathbb{E}(E, A)$  realized by  $A \xrightarrow{h} C \xrightarrow{h'} E$ , which satisfy the following compatibilities:

- (i)  $D \xrightarrow{d} E \xrightarrow{e} F$  realizes  $\mathbb{E}(F, f')(\delta')$ ,
- (ii)  $\mathbb{E}(d, A)(\delta'') = \delta$ ,
- (iii)  $\mathbb{E}(E, f)(\delta'') = \mathbb{E}(e, B)(\delta')$ .

(ET4)<sup>op</sup> Dual of (ET4).

Let  $\mathcal{C}$  be an extriangulated category, and  $\mathcal{D}, \mathcal{D}' \subseteq \mathcal{C}$ . We write  $\mathcal{D} * \mathcal{D}'$  for the full subcategory of objects  $X$  admitting an  $\mathbb{E}$ -triangle  $D \rightarrow X \rightarrow D' \dashrightarrow$  with  $D \in \mathcal{D}$  and  $D' \in \mathcal{D}'$ . A subcategory  $\mathcal{D}$  of  $\mathcal{C}$  is *extension-closed* if  $\mathcal{D} * \mathcal{D} = \mathcal{D}$ . An object  $P$  in  $\mathcal{C}$  is called *projective* if for any conflation  $A \xrightarrow{x} B \xrightarrow{y} C$  and any morphism  $c$  in  $\mathcal{C}(P, C)$ , there exists  $b$  in  $\mathcal{C}(P, B)$  such that  $yb = c$ . We denote the full subcategory of projective objects in  $\mathcal{C}$  by  $\mathcal{P}(\mathcal{C})$ . Dually, the *injective* objects are defined, and the full subcategory of injective objects in  $\mathcal{C}$  is denoted by  $\mathcal{I}(\mathcal{C})$ . We say that  $\mathcal{C}$  *has enough projectives* if for any object  $M \in \mathcal{C}$ , there exists an  $\mathbb{E}$ -triangle  $A \rightarrow P \rightarrow M \dashrightarrow$  satisfying  $P \in \mathcal{P}(\mathcal{C})$ . Dually, we define  $\mathcal{C}$  *having enough injectives*. In particular, if  $\mathcal{C}$  is a triangulated category, then  $\mathcal{C}$  has enough projectives and injectives with  $\mathcal{P}(\mathcal{C})$  and  $\mathcal{I}(\mathcal{C})$  consisting of zero objects.

EXAMPLE 2.3. (a) Exact categories, triangulated categories and extension-closed subcategories of triangulated categories are extriangulated categories (cf. [12]).

(b) Let  $\mathcal{C}$  be an extriangulated category. Then  $\mathcal{C}/(\mathcal{P}(\mathcal{C}) \cap \mathcal{I}(\mathcal{C}))$  is an extriangulated category which is neither exact nor triangulated in general (cf. [12, Proposition 3.30]).

PROPOSITION 2.4 ([12, Proposition 3.3]). *Let  $\mathcal{C}$  be an extriangulated category. For any  $\mathbb{E}$ -triangle  $A \rightarrow B \rightarrow C \xrightarrow{\delta}$ , the following sequences of natural transformations are exact:*

$$\begin{aligned} \mathcal{C}(C, -) \rightarrow \mathcal{C}(B, -) \rightarrow \mathcal{C}(A, -) &\xrightarrow{\delta^{\sharp}} \mathbb{E}(C, -) \rightarrow \mathbb{E}(B, -), \\ \mathcal{C}(-, A) \rightarrow \mathcal{C}(-, B) \rightarrow \mathcal{C}(-, C) &\xrightarrow{\delta_{\sharp}} \mathbb{E}(-, A) \rightarrow \mathbb{E}(-, B). \end{aligned}$$

Let  $\mathcal{C}$  be an extriangulated category. A finite sequence

$$X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0$$

in  $\mathcal{C}$  is said to be an  $\mathbb{E}$ -triangle sequence if there exist  $\mathbb{E}$ -triangles  $X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{f_{n-1}} K_{n-1} \dashrightarrow$ ,  $K_{i+1} \xrightarrow{g_{i+1}} X_i \xrightarrow{f_i} K_i \dashrightarrow$ ,  $1 < i < n-1$ , and  $K_2 \xrightarrow{g_2} X_1 \xrightarrow{d_1} X_0 \dashrightarrow$  such that  $d_i = g_i f_i$  for any  $1 < i < n$ .

Two morphism sequences  $\eta_1 : A \xrightarrow{f} B \xrightarrow{g} C$  and  $\eta_2 : A' \xrightarrow{f'} B' \xrightarrow{g'} C'$  in  $\mathcal{C}$  are said to be *isomorphic*, denoted  $\eta_1 \simeq \eta_2$ , if there are isomorphisms  $x : A \rightarrow A'$ ,  $y : B \rightarrow B'$  and  $z : C \rightarrow C'$  in  $\mathcal{C}$  such that  $yf = f'x$  and  $zg = g'y$ .

LEMMA 2.5. *Let  $\mathcal{C}$  be an extriangulated category.*

- (1) *Let  $\eta_1$  and  $\eta_2$  be morphism sequences in  $\mathcal{C}$  such that  $\eta_1 \simeq \eta_2$ . Then  $\eta_1$  is a conflation if and only if  $\eta_2$  is a conflation.*
- (2) *Let  $A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow$  be an  $\mathbb{E}$ -triangle in  $\mathcal{C}$ . Then  $f$  is an isomorphism if and only if  $C \cong 0$ . Similarly,  $g$  is an isomorphism if and only if  $A \cong 0$ .*

*Proof.* This is easily proved by [12, Corollary 3.6 and Proposition 3.7]. ■

**2.2. Exact functors.** In what follows, we will always assume that  $\mathcal{C}$  is an extriangulated category. In addition, we assume the following conditions for the rest of this paper (see [12, Condition 5.8]):

CONDITION 2.6 (WIC).

- (1) *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be any composable pair of morphisms in  $\mathcal{C}$ . If  $gf$  is an inflation, then  $f$  is an inflation.*
- (2) *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be any composable pair of morphisms in  $\mathcal{C}$ . If  $gf$  is a deflation, then  $g$  is a deflation.*

REMARK 2.7. If  $\mathcal{C}$  is a triangulated category or a weakly idempotent complete exact category [4, Proposition 7.6], then Condition 2.6 is satisfied.

DEFINITION 2.8. A morphism  $f$  in  $\mathcal{C}$  is called *compatible* if “ $f$  is both an inflation and a deflation” implies “ $f$  is an isomorphism”. That is, the compatible morphisms are the morphisms in the following class:

$$\{f \mid f \text{ is not an inflation, or } f \text{ is not a deflation, or } f \text{ is an isomorphism}\}.$$

It is clear that the morphisms in an exact category are all compatible. On the other hand, the compatible morphisms in a triangulated category  $\mathcal{C}$  are just the isomorphisms in  $\mathcal{C}$ .

DEFINITION 2.9. A sequence  $A \xrightarrow{f} B \xrightarrow{g} C$  in  $\mathcal{C}$  is said to be *right exact* if there exists an  $\mathbb{E}$ -triangle  $K \xrightarrow{h_2} B \xrightarrow{g} C \dashrightarrow$  and a deflation  $h_1 : A \rightarrow K$  which is compatible, such that  $f = h_2 h_1$ . Dually one can also define the *left exact* sequences.

A 4-term  $\mathbb{E}$ -triangle sequence  $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$  is called *right exact* (resp. *left exact*) if there exist  $\mathbb{E}$ -triangles  $A \xrightarrow{f} B \xrightarrow{g_1} K \dashrightarrow$  and  $K \xrightarrow{g_2} C \xrightarrow{h} D \dashrightarrow$  such that  $g = g_2 g_1$  and  $g_1$  (resp.  $g_2$ ) is compatible.

For the convenience of statements, given a morphism  $f : A \rightarrow B$  in  $\mathcal{C}$ , we denote by  $\Phi_f$  the set consisting of all pairs  $(h_1, h_2)$  such that  $h_1 : A \rightarrow K$  is a deflation,  $h_2 : K \rightarrow B$  is an inflation and  $f = h_2 h_1$ .

LEMMA 2.10. *Let  $\eta : A \xrightarrow{f} B \xrightarrow{g} C$  be a right exact sequence in  $\mathcal{C}$ .*

- (1) *If  $f$  is an inflation, then  $\eta$  is a conflation.*
- (2) *If  $A = 0$ , then  $g$  is an isomorphism.*

*Proof.* (1) As  $\eta$  is right exact, there is an  $\mathbb{E}$ -triangle  $K \xrightarrow{h_2} B \xrightarrow{g} C \dashrightarrow$  and a compatible morphism  $h_1$  such that  $(h_1, h_2) \in \Phi_f$ . Since  $f$  is an inflation, by Condition 2.6(1), we deduce that  $h_1$  is an inflation. Since  $h_1$  is also a deflation, we find that  $h_1$  is an isomorphism. Then by Lemma 2.5(1), we conclude that  $\eta$  is a conflation.

(2) Noting that  $0 \rightarrow B$  is an inflation, we deduce that  $0 \rightarrow B \xrightarrow{g} C$  is an  $\mathbb{E}$ -triangle. Then by Lemma 2.5(2), we conclude that  $g$  is an isomorphism. ■

We omit the dual statement to Lemma 2.10.

REMARK 2.11. (1) A sequence  $\eta : A \xrightarrow{f} B \xrightarrow{g} C$  is both left exact and right exact if and only if  $\eta$  is a conflation.

(2) If  $\mathcal{C}$  is an abelian category, then  $A \xrightarrow{f} B \xrightarrow{g} C$  is right exact if and only if  $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  is exact. Similarly,  $A \xrightarrow{f} B \xrightarrow{g} C$  is left exact if and only if  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C$  is exact. If  $\mathcal{C}$  is a triangulated category with

the suspension functor [1], then  $\eta : A \xrightarrow{f} B \xrightarrow{g} C$  is right exact if and only if  $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow A[1]$  is a triangle if and only if  $\eta$  is left exact.

Let us give the notion of right (left) exact functors in extriangulated categories.

DEFINITION 2.12. Let  $(\mathcal{A}, \mathbb{E}_{\mathcal{A}}, \mathfrak{s}_{\mathcal{A}})$  and  $(\mathcal{B}, \mathbb{E}_{\mathcal{B}}, \mathfrak{s}_{\mathcal{B}})$  be extriangulated categories. An additive covariant functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is called a *right exact functor* if it satisfies the following conditions:

- (1) If  $f$  is a compatible morphism in  $\mathcal{A}$ , then  $Ff$  is compatible in  $\mathcal{B}$ .
- (2) If  $A \xrightarrow{a} B \xrightarrow{b} C$  is right exact in  $\mathcal{A}$ , then  $FA \xrightarrow{Fa} FB \xrightarrow{Fb} FC$  is right exact in  $\mathcal{B}$ . (Then for any  $\mathbb{E}_{\mathcal{A}}$ -triangle  $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta}$ , there exists an  $\mathbb{E}_{\mathcal{B}}$ -triangle  $A' \xrightarrow{x} FB \xrightarrow{Fg} FC \dashrightarrow$  such that  $Ff = xy$  and  $y : FA \rightarrow A'$  is a deflation and is compatible. Moreover,  $A'$  is uniquely determined up to isomorphism.)
- (3) There exists a natural transformation

$$\eta = \{\eta_{(C,A)} : \mathbb{E}_{\mathcal{A}}(C, A) \rightarrow \mathbb{E}_{\mathcal{B}}(F^{\text{op}}C, A')\}_{(C,A) \in \mathcal{A}^{\text{op}} \times \mathcal{A}}$$

$$\text{such that } \mathfrak{s}_{\mathcal{B}}(\eta_{(C,A)}(\delta)) = [A' \xrightarrow{x} FB \xrightarrow{Fg} FC].$$

Dually, we define the *left exact functor* between two extriangulated categories.

The *extriangulated functor* between two extriangulated categories has been defined in [3]. For our requirements, we modify the definition as follows:

DEFINITION 2.13. Let  $(\mathcal{A}, \mathbb{E}_{\mathcal{A}}, \mathfrak{s}_{\mathcal{A}})$  and  $(\mathcal{B}, \mathbb{E}_{\mathcal{B}}, \mathfrak{s}_{\mathcal{B}})$  be extriangulated categories. We say an additive covariant functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is an *exact functor* if the following conditions hold:

- (1) If  $f$  is a compatible morphism in  $\mathcal{A}$ , then  $Ff$  is compatible in  $\mathcal{B}$ .
- (2) There exists a natural transformation

$$\eta = \{\eta_{(C,A)}\}_{(C,A) \in \mathcal{A}^{\text{op}} \times \mathcal{A}} : \mathbb{E}_{\mathcal{A}}(-, -) \Rightarrow \mathbb{E}_{\mathcal{B}}(F^{\text{op}}-, F-).$$

- (3) If  $\mathfrak{s}_{\mathcal{A}}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ , then  $\mathfrak{s}_{\mathcal{B}}(\eta_{(C,A)}(\delta)) = [F(A) \xrightarrow{F(x)} F(B) \xrightarrow{F(y)} F(C)]$ .

PROPOSITION 2.14. *Let  $(\mathcal{A}, \mathbb{E}_{\mathcal{A}}, \mathfrak{s}_{\mathcal{A}})$  and  $(\mathcal{B}, \mathbb{E}_{\mathcal{B}}, \mathfrak{s}_{\mathcal{B}})$  be extriangulated categories. An additive covariant functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is exact if and only if  $F$  is both left exact and right exact.*

*Proof.* We only need to prove sufficiency. Let  $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta}$  be an  $\mathbb{E}_{\mathcal{A}}$ -triangle. Since  $F$  is both left exact and right exact, the sequence  $FA \xrightarrow{Ff}$

$FB \xrightarrow{Fg} FC$  is both left exact and right exact. By Remark 2.11(1),  $FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$  is a conflation. ■

REMARK 2.15. If the categories  $\mathcal{A}$  and  $\mathcal{B}$  are abelian, Definition 2.12 coincides with the usual right exact functor in abelian categories, and Definition 2.13 coincides with the usual exact functor. If the categories  $\mathcal{A}$  and  $\mathcal{B}$  are triangulated, by Remark 2.11 and Proposition 2.14, we know that  $F$  is a left exact functor if and only if  $F$  is a triangle functor if and only if  $F$  is a right exact functor.

LEMMA 2.16. *Let  $(\mathcal{A}, \mathbb{E}_{\mathcal{A}}, \mathfrak{s}_{\mathcal{A}})$  and  $(\mathcal{B}, \mathbb{E}_{\mathcal{B}}, \mathfrak{s}_{\mathcal{B}})$  be extriangulated categories and  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a functor which admits a right adjoint functor  $G$ .*

- (1) *If  $\mathcal{A}$  has enough projectives and  $F$  is an exact functor which preserves projectives, then  $\mathbb{E}_{\mathcal{B}}(FX, Y) \cong \mathbb{E}_{\mathcal{A}}(X, GY)$  for any  $X \in \mathcal{A}$  and  $Y \in \mathcal{B}$ .*
- (2) *If  $\mathcal{B}$  has enough injectives and  $G$  is an exact functor which preserves injectives, then  $\mathbb{E}_{\mathcal{B}}(FX, Y) \cong \mathbb{E}_{\mathcal{A}}(X, GY)$  for any  $X \in \mathcal{A}$  and  $Y \in \mathcal{B}$ .*

*Proof.* (1) For any  $X \in \mathcal{A}$ , there exists an  $\mathbb{E}_{\mathcal{A}}$ -triangle

$$(2.2) \quad M \rightarrow P \rightarrow X \dashrightarrow$$

with  $P \in \mathcal{P}(\mathcal{A})$ . Since  $F$  is an exact functor, and  $F$  preserves projectives, we obtain the  $\mathbb{E}_{\mathcal{B}}$ -triangle

$$(2.3) \quad FM \rightarrow FP \rightarrow FX \dashrightarrow$$

with  $FP \in \mathcal{P}(\mathcal{B})$ . Applying the functors  $\mathrm{Hom}_{\mathcal{A}}(-, GY)$  and  $\mathrm{Hom}_{\mathcal{B}}(-, Y)$  to (2.2) and (2.3), respectively, we get the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathcal{A}}(P, GY) & \rightarrow & \mathrm{Hom}_{\mathcal{A}}(M, GY) & \rightarrow & \mathbb{E}_{\mathcal{A}}(X, GY) & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Hom}_{\mathcal{B}}(FP, Y) & \rightarrow & \mathrm{Hom}_{\mathcal{B}}(FM, Y) & \rightarrow & \mathbb{E}_{\mathcal{B}}(FX, Y) & \rightarrow & 0 \end{array}$$

By the Five-Lemma, we obtain  $\mathbb{E}_{\mathcal{B}}(FX, Y) \cong \mathbb{E}_{\mathcal{A}}(X, GY)$ .

(2) This is similar to (1). ■

The following lemma is well-known. For the convenience of the reader we give a short proof.

LEMMA 2.17. *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two categories and  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a functor which admits a right adjoint functor  $G$ . Let  $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow GF$  be the unit and  $\epsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{B}}$  be the counit.*

- (1)  $\mathrm{Id}_{FX} = \epsilon_{FX} F(\eta_X)$  for any  $X \in \mathcal{A}$ .
- (2)  $\mathrm{Id}_{GY} = G(\epsilon_Y) \eta_{GY}$  for any  $Y \in \mathcal{B}$ .



*Proof.* (1) Let  $\theta : \text{Hom}_{\mathcal{B}}(F-, -) \Rightarrow \text{Hom}_{\mathcal{A}}(-, G-)$  be the adjoint isomorphism. For any  $X \in \mathcal{A}$ , consider the following commutative diagram:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{B}}(FGFX, FX) & \xrightarrow{\theta_{GFX, FX}} & \text{Hom}_{\mathcal{A}}(GFX, GFX) \\ (F\eta_X)^* \downarrow & & \downarrow \eta_X^* \\ \text{Hom}_{\mathcal{B}}(FX, FX) & \xrightarrow{\theta_{X, FX}} & \text{Hom}_{\mathcal{A}}(X, GFX) \end{array}$$

Then  $\theta_{X, FX}(\epsilon_{FX}F(\eta_X)) = (\theta_{GFX, FX}(\epsilon_{FX}))\eta_X = \eta_X = \theta_{X, FX}(\text{Id}_{FX})$ . Hence, we have  $\text{Id}_{FX} = \epsilon_{FX}F(\eta_X)$ . The proof of (2) is similar. ■

**3. Recollements.** Let us introduce the concept of recollements of extriangulated categories.

DEFINITION 3.1. Let  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  be three extriangulated categories. A *recollement* of  $\mathcal{B}$  relative to  $\mathcal{A}$  and  $\mathcal{C}$ , denoted by  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ , is a diagram

$$(3.1) \quad \begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ & \xleftarrow{i_*} & \mathcal{B} & \xleftarrow{j^*} & \mathcal{C} \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

given by two exact functors  $i_*, j^*$ , two right exact functors  $i^*, j_!$  and two left exact functors  $i^!, j_*$ , which satisfies the following conditions:

- (R1)  $(i^*, i_*, i^!)$  and  $(j_!, j^*, j_*)$  are adjoint triples.
- (R2)  $\text{Im } i_* = \text{Ker } j^*$ .
- (R3)  $i_*, j_!$  and  $j_*$  are fully faithful.
- (R4) For each  $X \in \mathcal{B}$ , there exists a left exact  $\mathbb{E}_{\mathcal{B}}$ -triangle sequence

$$(3.2) \quad i_*i^!X \xrightarrow{\theta_X} X \xrightarrow{\vartheta_X} j_*j^*X \longrightarrow i_*A$$

with  $A \in \mathcal{A}$ , where  $\theta_X$  and  $\vartheta_X$  are given by the adjunction morphisms.

- (R5) For each  $X \in \mathcal{B}$ , there exists a right exact  $\mathbb{E}_{\mathcal{B}}$ -triangle sequence

$$(3.3) \quad i_*A' \longrightarrow j_!j^*X \xrightarrow{\nu_X} X \xrightarrow{\nu_X} i_*i^*X$$

with  $A' \in \mathcal{A}$ , where  $\nu_X$  and  $\nu_X$  are given by the adjunction morphisms.

REMARK 3.2. (1) If the categories  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are abelian, then Definition 3.1 coincides with the definition of recollements of abelian categories (cf. [6], [13], [9]).

(2) If the categories  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are triangulated, then Definition 3.1 coincides with the definition of recollements of triangulated categories (cf. [2]).

Now, we collect some properties of recollements of extriangulated categories, which will be used in what follows.

LEMMA 3.3. *Let  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  be a recollement of extriangulated categories as in (3.1).*

(1) All the natural transformations

$$i^*i_* \Rightarrow \text{Id}_{\mathcal{A}}, \quad \text{Id}_{\mathcal{A}} \Rightarrow i^!i_*, \quad \text{Id}_{\mathcal{C}} \Rightarrow j^*j_!, \quad j^*j_* \Rightarrow \text{Id}_{\mathcal{C}}$$

are natural isomorphisms.

- (2)  $i^*j_! = 0$  and  $i^!j_* = 0$ .  
(3)  $i^*$  preserves projective objects and  $i^!$  preserves injective objects.  
(3')  $j_!$  preserves projective objects and  $j_*$  preserves injective objects.  
(4) If  $i^!$  (resp.  $j_*$ ) is exact, then  $i_*$  (resp.  $j^*$ ) preserves projective objects.  
(4') If  $i^*$  (resp.  $j_!$ ) is exact, then  $i_*$  (resp.  $j^*$ ) preserves injective objects.  
(5) If  $\mathcal{B}$  has enough projectives, then  $\mathcal{A}$  has enough projectives and  $\mathcal{P}(\mathcal{A}) = \text{add}(i^*(\mathcal{P}(\mathcal{B})))$ ; if  $\mathcal{B}$  has enough injectives, then  $\mathcal{A}$  has enough injectives and  $\mathcal{I}(\mathcal{A}) = \text{add}(i^!(\mathcal{I}(\mathcal{B})))$ .  
(6) If  $\mathcal{B}$  has enough projectives and  $j_*$  is exact, then  $\mathcal{C}$  has enough projectives and  $\mathcal{P}(\mathcal{C}) = \text{add}(j^*(\mathcal{P}(\mathcal{B})))$ ; if  $\mathcal{B}$  has enough injectives and  $j_!$  is exact, then  $\mathcal{C}$  has enough injectives and  $\mathcal{I}(\mathcal{C}) = \text{add}(j^*(\mathcal{I}(\mathcal{B})))$ .  
(7) If  $\mathcal{B}$  has enough projectives and  $i^!$  is exact, then  $\mathbb{E}_{\mathcal{B}}(i_*X, Y) \cong \mathbb{E}_{\mathcal{A}}(X, i^!Y)$  for any  $X \in \mathcal{A}$  and  $Y \in \mathcal{B}$ .  
(7') If  $\mathcal{C}$  has enough projectives and  $j_!$  is exact, then  $\mathbb{E}_{\mathcal{B}}(j_!Z, Y) \cong \mathbb{E}_{\mathcal{C}}(Z, j^*Y)$  for any  $Y \in \mathcal{B}$  and  $Z \in \mathcal{C}$ .  
(8) If  $i^*$  is exact, then  $j_!$  is exact.  
(8') If  $i^!$  is exact, then  $j_*$  is exact.

*Proof.* (1) This follows from the fact that  $i_*$ ,  $j_!$  and  $j_*$  are fully faithful.

(2) For any  $X \in \mathcal{C}$ , since  $j^*i_* = 0$ , we have

$$\text{Hom}_{\mathcal{A}}(i^*j_!X, i^*j_!X) \cong \text{Hom}_{\mathcal{B}}(j_!X, i_*i^*j_!X) \cong \text{Hom}_{\mathcal{C}}(X, j^*i_*i^*j_!X) = 0,$$

and we obtain  $i^*j_! = 0$ . The equality  $i^!j_* = 0$  follows in a similar way.

(3) Let  $P \in \mathcal{P}(\mathcal{B})$ . We need to show that  $i^*P \in \mathcal{P}(\mathcal{A})$ . Let

$$(3.4) \quad X \xrightarrow{f} Y \xrightarrow{g} Z \dashrightarrow$$

be an arbitrary  $\mathbb{E}_{\mathcal{A}}$ -triangle and  $h \in \text{Hom}_{\mathcal{A}}(i^*P, Z)$ . Applying  $i_*$  to (3.4), we have an  $\mathbb{E}_{\mathcal{B}}$ -triangle

$$(3.5) \quad i_*X \xrightarrow{i_*f} i_*Y \xrightarrow{i_*g} i_*Z \dashrightarrow.$$

Applying the functors  $\text{Hom}_{\mathcal{A}}(i^*P, -)$  and  $\text{Hom}_{\mathcal{B}}(P, -)$  to (3.4) and (3.5), respectively, we obtain the commutative diagram

$$\begin{array}{ccccc} \text{Hom}(i^*P, Y) & \longrightarrow & \text{Hom}(i^*P, Z) & \longrightarrow & \mathbb{E}_{\mathcal{A}}(i^*P, X) \\ \eta_{P,Y} \downarrow & & \eta_{P,Z} \downarrow & & \\ \text{Hom}(P, i_*Y) & \longrightarrow & \text{Hom}(P, i_*Z) & \longrightarrow & \mathbb{E}_{\mathcal{B}}(P, i_*X) \end{array}$$

Since  $P$  is projective in  $\mathcal{B}$  and then  $\mathbb{E}_{\mathcal{B}}(P, i_*X) = 0$ , there exists a morphism  $t : P \rightarrow i_*Y$  such that  $i_*g(t) = \eta_{P,Z}(h)$ . It follows that

$$g(\eta_{P,Y}^{-1}(t)) = \eta_{P,Z}^{-1}i_*g(t) = h.$$

Hence,  $i^*P \in \mathcal{P}(\mathcal{A})$ . It is proved dually that  $i^!$  preserves injective objects. The proofs of (3'), (4) and (4') are similar.

(5) For any  $X \in \mathcal{A}$ , there exists a deflation  $f : P \rightarrow i_*X$  with  $P \in \mathcal{P}(\mathcal{B})$ . Applying the functor  $i^*$ , we obtain a deflation  $i^*P \rightarrow X$  with  $i^*P \in \mathcal{P}(\mathcal{A})$ . Hence  $\mathcal{A}$  has enough projectives. Since  $i^*$  preserves projectives, we have  $\text{add}(i^*(\mathcal{P}(\mathcal{B}))) \subseteq \mathcal{P}(\mathcal{A})$ . Conversely, for  $Q \in \mathcal{P}(\mathcal{A})$ , as above, there exists a deflation  $i^*P \rightarrow Q$  with  $P \in \mathcal{P}(\mathcal{B})$ , which implies that  $Q$  is a direct summand of  $i^*P$ . That is,  $\mathcal{P}(\mathcal{A}) \subseteq \text{add}(i^*(\mathcal{P}(\mathcal{B})))$ . Similarly, the second statement in (5) can be proved.

(6) Since  $j_*$  (resp.  $j_!$ ) is exact, by (4) (resp. (4')), the functor  $j^*$  preserves projectives (resp. injectives). Then, using the similar proof of (5), we can prove (6).

(7) By (5) we deduce that  $\mathcal{A}$  has enough projectives. Since  $i^!$  is exact, we see that  $i_*$  preserves projectives. Then according to Lemma 2.16(1), the statement (7) follows.

(7') By using Lemma 2.16(1) we immediately obtain the assertion.

(8) Let  $X \xrightarrow{f} Y \xrightarrow{g} Z \dashrightarrow$  be an  $\mathbb{E}_{\mathcal{C}}$ -triangle. Since  $j_!$  is right exact, there is an  $\mathbb{E}_{\mathcal{B}}$ -triangle  $X' \xrightarrow{h_2} j_!Y \xrightarrow{j_!g} j_!Z \dashrightarrow$  and a compatible morphism  $h_1$  such that  $(h_1, h_2) \in \Phi_{j_!f}$ . Noting that  $j^*j_!X \xrightarrow{j^*j_!f} j^*j_!Y \xrightarrow{j^*j_!g} j^*j_!Z \dashrightarrow$  is an  $\mathbb{E}_{\mathcal{C}}$ -triangle since  $j^*j_! \cong \text{Id}_{\mathcal{C}}$ , and  $j^*j_!f = (j^*h_2)(j^*h_1)$ , we obtain that  $j^*h_1$  is an inflation and then  $j^*h_1$  is an isomorphism since  $j^*h_1$  is a deflation and is compatible. So  $j^*X' \cong j^*j_!X$ . Set  $M = \text{cocone}(h_1)$ . By Lemma 2.5(2), we have  $j^*M = 0$ . By (R2), there is an object  $N \in \mathcal{A}$  with  $i_*N = M$ . Since  $i^*$  is exact,  $i^*X' \xrightarrow{i^*h_2} i^*j_!Y \xrightarrow{i^*j_!g} i^*j_!Z \dashrightarrow$  is an  $\mathbb{E}_{\mathcal{A}}$ -triangle, which implies  $i^*X' = 0$  by  $i^*j_! = 0$ . Similarly, since  $i^*M \rightarrow i^*j_!X \xrightarrow{i^*h_1} i^*X' \dashrightarrow$  is an  $\mathbb{E}_{\mathcal{A}}$ -triangle, we see that  $i^*M = 0$ . Then  $M = i_*N \cong i_*(i^*i_*N) \cong i_*i^*M = 0$ . Hence,  $h_1$  is an isomorphism and  $j_!X \xrightarrow{j_!f} j_!Y \xrightarrow{j_!g} j_!Z \dashrightarrow$  is an  $\mathbb{E}_{\mathcal{B}}$ -triangle. For the condition (3) in the definition of an exact functor, it can be derived from the assumption that  $j_!$  is right exact. The proof of (8') is similar. ■

PROPOSITION 3.4. *Let  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  be a recollement of extriangulated categories as (3.1).*

(1) *If  $i^!$  is exact, then for each  $X \in \mathcal{B}$ , there is an  $\mathbb{E}_{\mathcal{B}}$ -triangle*

$$i_*i^!X \xrightarrow{\theta_X} X \xrightarrow{\vartheta_X} j_*j^*X \dashrightarrow$$

*where  $\theta_X$  and  $\vartheta_X$  are given by the adjunction morphisms.*

(2) If  $i^*$  is exact, then for each  $X \in \mathcal{B}$ , there is an  $\mathbb{E}_{\mathcal{B}}$ -triangle

$$j_*j^*X \xrightarrow{v_X} X \xrightarrow{\nu_X} i_*i^*X \dashrightarrow$$

where  $v_X$  and  $\nu_X$  are given by the adjunction morphisms.

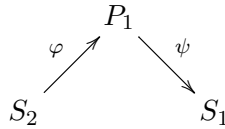
*Proof.* We only prove (1) since the proof of (2) is similar. By (R4), for each  $X \in \mathcal{B}$ , there is a left exact  $\mathbb{E}_{\mathcal{B}}$ -triangle sequence

$$i_*i^!X \xrightarrow{\theta_X} X \xrightarrow{\vartheta_X} j_*j^*X \xrightarrow{h} i_*A$$

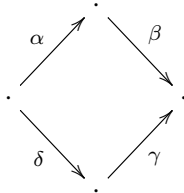
such that  $i_*i^!X \xrightarrow{\theta_X} X \xrightarrow{h_1} M \dashrightarrow$  and  $M \xrightarrow{h_2} j_*j^*X \xrightarrow{h} i_*A \dashrightarrow$  are  $\mathbb{E}_{\mathcal{B}}$ -triangles,  $h_2$  is compatible and  $\vartheta_X = h_2h_1$ . Since  $i^!$  is exact, we deduce that  $i^!i_*i^!X \xrightarrow{i^!\theta_X} i^!X \xrightarrow{i^!\vartheta_X} i^!j_*j^*X$  is left exact. By Lemma 3.3(2),  $i^!j_*j^*X = 0$ , so  $i^!h_2$  is an isomorphism. Thus,  $i^!M = 0$ . It follows that  $A \cong i^!i_*A = 0$ . By Lemma 2.5(2),  $h_2$  is an isomorphism and thus  $i_*i^!X \xrightarrow{\theta_X} X \xrightarrow{\vartheta_X} j_*j^*X \dashrightarrow$  is an  $\mathbb{E}_{\mathcal{B}}$ -triangle. ■

In what follows, we give an example of a recollement involving an extriangulated category which is neither abelian nor triangulated.

EXAMPLE 3.5. Let  $A$  be the path algebra of the quiver  $1 \xrightarrow{\alpha} 2$  over a field. The Auslander–Reiten quiver of  $\text{mod } A$  is as follows:



Then the triangular matrix algebra  $B = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$  is given by the quiver



with the relation  $\beta\alpha = \gamma\delta$ . It is well-known that  $\text{mod } B$  can be identified with the morphism category of  $A$ -modules. That is, each  $B$ -module can be written as a triple  $\begin{pmatrix} X \\ Y \end{pmatrix}_f$  such that  $f : Y \rightarrow X$  is a homomorphism of  $A$ -modules. In the following, we write  $\begin{pmatrix} X \\ Y \end{pmatrix}$  instead of  $\begin{pmatrix} X \\ Y \end{pmatrix}_0$ . The Auslander–Reiten quiver

of  $B$  is given by

$$\begin{array}{ccccccc}
 & & \begin{pmatrix} P_1 \\ 0 \end{pmatrix} & & \begin{pmatrix} 0 \\ S_2 \end{pmatrix} & & \begin{pmatrix} S_1 \\ S_1 \end{pmatrix}_1 \\
 & \nearrow & & \searrow & \nearrow & \searrow & \\
 \begin{pmatrix} S_2 \\ 0 \end{pmatrix} & & & & \begin{pmatrix} P_1 \\ S_2 \end{pmatrix}_\varphi & \longrightarrow & \begin{pmatrix} P_1 \\ P_1 \end{pmatrix}_1 & \longrightarrow & \begin{pmatrix} S_1 \\ P_1 \end{pmatrix}_\psi & & \begin{pmatrix} 0 \\ S_1 \end{pmatrix} \\
 & \searrow & & \nearrow & \searrow & \nearrow & \\
 & & \begin{pmatrix} S_2 \\ S_2 \end{pmatrix}_1 & & \begin{pmatrix} S_1 \\ 0 \end{pmatrix} & & \begin{pmatrix} 0 \\ P_1 \end{pmatrix}
 \end{array}$$

By [13, Example 2.12], we have a recollement of abelian categories

$$(3.6) \quad \text{mod } A \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \text{mod } B \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} \text{mod } A$$

such that  $i^*((\frac{X}{Y})_f) = \text{Coker } f$ ,  $i_*(X) = (\frac{X}{0})$ ,  $i^!((\frac{X}{Y})_f) = X$ ,  $j^!(Y) = (\frac{Y}{Y})_1$ ,  $j^*((\frac{X}{Y})_f) = Y$  and  $j_*(Y) = (\frac{0}{Y})$ .

Let  $\mathcal{X}_1 = \text{mod } A$  and  $\mathcal{X}_2 = \text{add}(S_1 \oplus P_1)$ . Observe that  $\mathcal{X}_2$  is an extriangulated category which is neither abelian nor triangulated. Indeed,  $\mathcal{X}_2$  is an extension-closed subcategory of  $\text{mod } B$ . On the one hand,  $P_1$  is a non-zero injective object. This implies  $\mathcal{X}_2$  is not triangulated. On the other hand,  $\text{Ker } \psi$  does not belong to  $\mathcal{X}_2$ . This implies  $\mathcal{X}_2$  is not abelian. In addition,

$$\mathcal{X} = \text{add}\left(\begin{pmatrix} S_2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P_1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} S_1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P_1 \\ P_1 \end{pmatrix}_1 \oplus \begin{pmatrix} S_1 \\ P_1 \end{pmatrix}_\psi \oplus \begin{pmatrix} S_1 \\ S_1 \end{pmatrix}_1 \oplus \begin{pmatrix} 0 \\ P_1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ S_1 \end{pmatrix}\right)$$

also is an extriangulated category.

We claim that

$$(3.7) \quad \mathcal{X}_1 \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \mathcal{X} \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} \mathcal{X}_2$$

is a recollement of an extriangulated category which is neither abelian nor triangulated. In fact, one can check that  $i_*, j^*$  are exact functors,  $i^*, j^!$  are right exact functors and  $i^!, j_*$  are left exact functors.

(R1) For any  $(\frac{X}{X'})_f \in \mathcal{X}$  and  $Y \in \mathcal{X}_2$ ,

$$\begin{aligned}
 \text{Hom}_{\mathcal{X}}\left(j^!Y, \begin{pmatrix} X \\ X' \end{pmatrix}_f\right) &\cong \text{Hom}_{\text{mod } A}\left(Y, j^*\begin{pmatrix} X \\ X' \end{pmatrix}_f\right) = \text{Hom}_{\mathcal{X}_2}\left(Y, j^*\begin{pmatrix} X \\ X' \end{pmatrix}_f\right), \\
 \text{Hom}_{\mathcal{X}_2}\left(j^*\begin{pmatrix} X \\ X' \end{pmatrix}_f, Y\right) &\cong \text{Hom}_{\text{mod } B}\left(\begin{pmatrix} X \\ X' \end{pmatrix}_f, j_*Y\right) = \text{Hom}_{\mathcal{X}}\left(\begin{pmatrix} X \\ X' \end{pmatrix}_f, j_*Y\right).
 \end{aligned}$$

It follows that  $(j^!, j^*, j_*)$  is an adjoint triple. Similarly, so is  $(i^*, i_*, i^!)$ .

(R2) Note that  $\text{Im } i_* = \text{Ker } j^* = \text{add}\left(\begin{pmatrix} P_1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} S_2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} S_1 \\ 0 \end{pmatrix}\right)$ .

(R3) Since the functors  $i_*$ ,  $j_!$  and  $j_*$  in (3.6) are fully faithful, it follows that  $i_*$ ,  $j_!$  and  $j_*$  in (3.7) are also fully faithful.

(R4) Note that  $i^!$  is exact. For any  $\begin{pmatrix} X \\ Y \end{pmatrix}_f \in \mathcal{X}$ , by [13, Proposition 2.6], there exists an exact sequence

$$0 \rightarrow i_* i^! \begin{pmatrix} X \\ Y \end{pmatrix}_f \rightarrow \begin{pmatrix} X \\ Y \end{pmatrix}_f \rightarrow j_* j^* \begin{pmatrix} X \\ Y \end{pmatrix}_f \rightarrow 0$$

in  $\text{mod } B$ , which also provides a left exact  $\mathbb{E}$ -triangle sequence

$$\begin{pmatrix} X \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} X \\ Y \end{pmatrix}_f \rightarrow \begin{pmatrix} 0 \\ Y \end{pmatrix} \rightarrow 0$$

in  $\mathcal{X}$ .

(R5) For  $\begin{pmatrix} X \\ Y \end{pmatrix}_f \in \mathcal{X}$ , by [13, Proposition 2.6], there exists an exact sequence

$$0 \rightarrow \begin{pmatrix} X' \\ 0 \end{pmatrix}_f \rightarrow j! j^* \begin{pmatrix} X \\ Y \end{pmatrix}_f \rightarrow \begin{pmatrix} X \\ Y \end{pmatrix}_f \rightarrow i_* i^* \begin{pmatrix} X \\ Y \end{pmatrix}_f \rightarrow 0$$

in  $\text{mod } B$  with  $X' \in \text{mod } A$ , which provides a right exact  $\mathbb{E}$ -triangle sequence

$$\begin{pmatrix} X' \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} Y \\ Y \end{pmatrix}_1 \rightarrow \begin{pmatrix} X \\ Y \end{pmatrix}_f \rightarrow \begin{pmatrix} \text{Coker } f \\ 0 \end{pmatrix}$$

in  $\mathcal{X}$ .

**4. Glued cotorsion pairs.** First of all, let us recall the definition of cotorsion pairs in an extriangulated category.

DEFINITION 4.1 ([12, Definition 4.1]). Let  $\mathcal{C}$  be an extriangulated category and  $\mathcal{T}, \mathcal{F} \subseteq \mathcal{C}$  be a pair of subcategories of  $\mathcal{C}$ . The pair  $(\mathcal{T}, \mathcal{F})$  is called a *cotorsion pair* in  $\mathcal{C}$  if it satisfies the following conditions:

- (a)  $\mathbb{E}(\mathcal{T}, \mathcal{F}) = 0$ .
- (b) For any  $C \in \mathcal{C}$ , there exists a conflation  $F \rightarrow T \rightarrow C$  such that  $F \in \mathcal{F}$ ,  $T \in \mathcal{T}$ .
- (c) For any  $C \in \mathcal{C}$ , there exists a conflation  $C \rightarrow F' \rightarrow T'$  such that  $F' \in \mathcal{F}$ ,  $T' \in \mathcal{T}$ .

REMARK 4.2. Let  $(\mathcal{U}, \mathcal{V})$  be a cotorsion pair in an extriangulated category  $\mathcal{C}$ . Then:

- $M \in \mathcal{U}$  if and only if  $\mathbb{E}(M, \mathcal{V}) = 0$ ;
- $N \in \mathcal{V}$  if and only if  $\mathbb{E}(\mathcal{U}, N) = 0$ ;
- $\mathcal{U}$  and  $\mathcal{V}$  are extension-closed;
- $\mathcal{U}$  is contravariantly finite and  $\mathcal{V}$  is covariantly finite in  $\mathcal{C}$ ;
- $\mathcal{P}(\mathcal{C}) \subseteq \mathcal{U}$  and  $\mathcal{I}(\mathcal{C}) \subseteq \mathcal{V}$ .

DEFINITION 4.3. Let  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  be a recollement of extriangulated categories as in (3.1). Given cotorsion pairs  $(\mathcal{T}_1, \mathcal{F}_1)$  and  $(\mathcal{T}_2, \mathcal{F}_2)$  in  $\mathcal{A}$  and  $\mathcal{C}$ , respectively, set

$$\begin{aligned}\mathcal{T} &= \{B \in \mathcal{B} \mid i^*B \in \mathcal{T}_1 \text{ and } j^*B \in \mathcal{T}_2\}, \\ \mathcal{F} &= \{B \in \mathcal{B} \mid i^!B \in \mathcal{F}_1 \text{ and } j^*B \in \mathcal{F}_2\}.\end{aligned}$$

Then we call  $(\mathcal{T}, \mathcal{F})$  the *glued pair* with respect to  $(\mathcal{T}_1, \mathcal{F}_1)$  and  $(\mathcal{T}_2, \mathcal{F}_2)$ .

For a subcategory  $\mathcal{X}$  of  $\mathcal{C}$ , we define the following full subcategory:

$$\mathcal{X}^{\perp 1} = \{M \in \mathcal{C} \mid \mathbb{E}(\mathcal{X}, M) = 0\}.$$

Recall from [12] that an extriangulated category  $\mathcal{C}$  is called *Frobenius* if  $\mathcal{C}$  has enough projectives and enough injectives and moreover the projectives coincide with the injectives. Note that each triangulated category is a Frobenius extriangulated category.

Now we are able to present one of the main results of this paper.

THEOREM 4.4. *Let  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  be a recollement of extriangulated categories as in (3.1), and  $(\mathcal{T}_1, \mathcal{F}_1)$  and  $(\mathcal{T}_2, \mathcal{F}_2)$  be cotorsion pairs in  $\mathcal{A}$  and  $\mathcal{C}$ , respectively. Let  $(\mathcal{T}, \mathcal{F})$  be the glued pair with respect to  $(\mathcal{T}_1, \mathcal{F}_1)$  and  $(\mathcal{T}_2, \mathcal{F}_2)$ . Assume that  $\mathcal{B}$  has enough projectives and  $i^!, j_!$  are exact.*

(1) *If one of the following conditions holds:*

- (i)  $\mathbb{E}_{\mathcal{B}}(\mathcal{T}, \mathcal{F}) = 0$ ,
- (ii)  $i^*$  is exact,
- (iii) for any morphism  $f : i_*A \rightarrow j_!T$  with  $A \in \mathcal{A}$  and  $T \in \mathcal{T}_2$ , the induced map  $f^* : \mathcal{B}(j_!T, F) \rightarrow \mathcal{B}(i_*A, F)$  is surjective for any  $F \in \mathcal{F}$ ,
- (iv)  $\mathcal{T} \subseteq j_!\mathcal{T}_2$  or  $i_*\mathcal{F}_1 \subseteq \mathcal{T}^{\perp 1}$ ,
- (v)  $\mathcal{A}$  and  $\mathcal{B}$  are Frobenius extriangulated categories,

*then  $(\mathcal{T}, \mathcal{F})$  is a cotorsion pair in  $\mathcal{B}$ .*

- (2) *If  $(\mathcal{U}, \mathcal{V})$  is a cotorsion pair in  $\mathcal{B}$  such that  $i_*i^!\mathcal{U} \subseteq \mathcal{U}$  and  $i_*i^*\mathcal{U} \subseteq \mathcal{U}$ , then  $(i^*\mathcal{U}, i^!\mathcal{V})$  is a cotorsion pair in  $\mathcal{A}$ .*
- (3) *If  $(\mathcal{U}, \mathcal{V})$  is a cotorsion pair in  $\mathcal{B}$  such that  $j_*j^*\mathcal{V} \subseteq \mathcal{V}$  or  $j_!j^*\mathcal{U} \subseteq \mathcal{U}$ , then  $(j^*\mathcal{U}, j^*\mathcal{V})$  is a cotorsion pair in  $\mathcal{C}$ .*

In what follows, we say that a commutative diagram is *exact* if every subdiagram of the form  $X \rightarrow Y \rightarrow Z$  is a conflation. Before proving Theorem 4.4, we give the following

LEMMA 4.5. *Keep the notation as in Definition 4.3.*

- (1) *If  $i^!$  is exact, then  $(\mathcal{T}, \mathcal{F})$  satisfies (b) in Definition 4.1.*
- (2) *If  $i^!$  and  $j_!$  are exact, then  $(\mathcal{T}, \mathcal{F})$  satisfies (c) in Definition 4.1.*

*Proof.* (1) For any  $M \in \mathcal{B}$ , there exists an  $\mathbb{E}_{\mathcal{C}}$ -triangle  $F_2 \rightarrow T_2 \rightarrow j^*M \dashrightarrow$  with  $F_2 \in \mathcal{F}_2$  and  $T_2 \in \mathcal{T}_2$ , since  $(\mathcal{T}_2, \mathcal{F}_2)$  is a cotorsion pair in  $\mathcal{C}$ .

Since  $i^!$  is exact, by Lemma 3.3(8'),  $j_*$  is exact. Applying  $j_*$  to the above  $\mathbb{E}_{\mathcal{C}}$ -triangle, we obtain an  $\mathbb{E}_{\mathcal{B}}$ -triangle  $j_*F_2 \rightarrow j_*T_2 \rightarrow j_*j^*M \dashrightarrow$ . Consider the following commutative diagram:

$$(4.1) \quad \begin{array}{ccccc} j_*F_2 & \longrightarrow & H & \longrightarrow & M \xrightarrow{\eta_M^* \delta} \triangleright \\ \parallel & & \downarrow & & \eta_M \downarrow \\ j_*F_2 & \longrightarrow & j_*T_2 & \longrightarrow & j_*j^*M \xrightarrow{\delta} \triangleright \end{array}$$

where  $\eta_M$  is the unit of the adjoint pair  $(j^*, j_*)$ . Applying  $j^*$  to (4.1) and using Lemma 2.17(1), we obtain  $j^*H \cong T_2$ . We also have an  $\mathbb{E}_{\mathcal{A}}$ -triangle  $F_1 \rightarrow T_1 \rightarrow i^*H \dashrightarrow$  with  $F_1 \in \mathcal{F}_1$  and  $T_1 \in \mathcal{T}_1$  since  $(\mathcal{T}_1, \mathcal{F}_1)$  is a cotorsion pair in  $\mathcal{A}$ . Then  $i_*F_1 \rightarrow i_*T_1 \rightarrow i_*i^*H \dashrightarrow$  is an  $\mathbb{E}_{\mathcal{B}}$ -triangle, since  $i_*$  is exact. For  $H \in \mathcal{B}$ , by (R5), there exists a commutative diagram

$$\begin{array}{ccccc} i_*A' & \longrightarrow & j!j^*H & \xrightarrow{v_H} & H \xrightarrow{h} i_*i^*H \\ & & \searrow & & \nearrow \\ & & & K & \\ & & h_2 \swarrow & & \searrow h_1 \end{array}$$

in  $\mathcal{B}$  such that  $i_*A' \rightarrow j!j^*H \xrightarrow{h_2} K \dashrightarrow$  and  $K \xrightarrow{h_1} H \xrightarrow{h} i_*i^*H \dashrightarrow$  are  $\mathbb{E}_{\mathcal{B}}$ -triangles and  $h_2$  is compatible, and moreover,  $j!j^*H \xrightarrow{v_H} H \xrightarrow{h} i_*i^*H$  is right exact. By [12, Proposition 3.15], we have the exact commutative diagram

$$(4.2) \quad \begin{array}{ccccc} & & K & \xlongequal{\quad} & K \\ & & \downarrow t & & \downarrow h_1 \\ i_*F_1 & \longrightarrow & T & \longrightarrow & H \\ \parallel & & \downarrow l & & \downarrow h \\ i_*F_1 & \longrightarrow & i_*T_1 & \longrightarrow & i_*i^*H \end{array}$$

Consider now the following commutative diagram:

$$(4.3) \quad \begin{array}{ccccc} j!j^*H & \longrightarrow & T & \xrightarrow{l} & i_*T_1 \\ & \searrow & & & \nearrow \\ & & & K & \\ & & h_2 \swarrow & & \searrow t \end{array}$$

where  $h_2$  is a deflation and compatible, and  $t$  is an inflation. Thus, the first row of (4.3) is right exact. Applying the right exact functor  $i^*$  to (4.3) and using Lemma 2.10(2), we obtain  $i^*T \cong i^*i_*T_1 \cong T_1$ . Applying  $j^*$  to the  $\mathbb{E}_{\mathcal{B}}$ -triangle in the second row of (4.2), we find that  $j^*T \cong j^*H \cong T_2$ . It



follows that  $T \in \mathcal{T}$ . Applying (ET4)<sup>op</sup> yields an exact commutative diagram

$$(4.4) \quad \begin{array}{ccccc} i_*F_1 & \longrightarrow & F & \longrightarrow & j_*F_2 \\ \parallel & & \downarrow & & \downarrow \\ i_*F_1 & \longrightarrow & T & \longrightarrow & H \\ & & \downarrow & & \downarrow \\ & & M & \longlongequal{\quad} & M \end{array}$$

Applying  $j^*$  to the  $\mathbb{E}_{\mathcal{B}}$ -triangle in the first row of (4.4), we deduce that  $j^*F \cong j^*j_*F_2 \cong F_2$ . Similarly, applying  $i^!$  to the  $\mathbb{E}_{\mathcal{B}}$ -triangle in the first row of (4.4), by the dual of Lemma 2.10, we have  $i^!F \cong i^!i_*F_1 \cong F_1$ . That is,  $F \in \mathcal{F}$ . So the second column in (4.4) gives a desired  $\mathbb{E}_{\mathcal{B}}$ -triangle. Therefore,  $(\mathcal{T}, \mathcal{F})$  satisfies (b) in Definition 4.1.

(2) For any  $M \in \mathcal{B}$ , there exists an  $\mathbb{E}_{\mathcal{C}}$ -triangle  $j^*M \rightarrow F_2 \rightarrow T_2 \dashrightarrow$  with  $F_2 \in \mathcal{F}_2$  and  $T_2 \in \mathcal{T}_2$ , since  $(\mathcal{T}_2, \mathcal{F}_2)$  is a cotorsion pair in  $\mathcal{C}$ . Since  $j_!$  is exact, we obtain an  $\mathbb{E}_{\mathcal{B}}$ -triangle

$$j_!j^*M \rightarrow j_!F_2 \rightarrow j_!T_2 \dashrightarrow.$$

Consider the following exact commutative diagram:

$$(4.5) \quad \begin{array}{ccccc} j_!j^*M & \longrightarrow & j_!F_2 & \longrightarrow & j_!T_2 - \overset{\delta}{\dashrightarrow} \\ \epsilon_M \downarrow & & \downarrow & & \parallel \\ M & \longrightarrow & H & \longrightarrow & j_!T_2 \overset{(\epsilon_M)_*\delta}{\dashrightarrow} \end{array}$$

where  $\epsilon_M$  is the counit of the adjoint pair  $(j_!, j^*)$ . Applying  $j^*$  to (4.5) and using Lemma 2.17(2), we obtain  $j^*H \cong F_2$ . We also have an  $\mathbb{E}_{\mathcal{A}}$ -triangle  $i^!H \rightarrow F_1 \rightarrow T_1 \dashrightarrow$  with  $F_1 \in \mathcal{F}_1$  and  $T_1 \in \mathcal{T}_1$ , since  $(\mathcal{T}_1, \mathcal{F}_1)$  is a cotorsion pair in  $\mathcal{A}$ . Then

$$i_*i^!H \rightarrow i_*F_1 \rightarrow i_*T_1 \dashrightarrow$$

is an  $\mathbb{E}_{\mathcal{B}}$ -triangle since  $i_*$  is exact.

Consider the following exact commutative diagram:

$$(4.6) \quad \begin{array}{ccccc} i_*i^!H & \longrightarrow & i_*F_1 & \longrightarrow & i_*T_1 - \overset{\delta}{\dashrightarrow} \\ \epsilon_H \downarrow & & \downarrow & & \parallel \\ H & \longrightarrow & F & \longrightarrow & i_*T_1 \overset{(\epsilon_H)_*\delta}{\dashrightarrow} \end{array}$$

where  $\epsilon_H$  is the counit of the adjoint pair  $(i_*, i^!)$ . Applying  $i^!$  to (4.6) and using Lemma 2.17(2), we get  $i^!F \cong F_1$ . Applying  $j^*$  to the  $\mathbb{E}_{\mathcal{B}}$ -triangle in the second row of (4.6) and using  $j^*i_* = 0$ , we obtain  $j^*F \cong j^*H \cong F_2$ . It

follows that  $F \in \mathcal{F}$ . Applying (ET4) yields an exact commutative diagram

$$(4.7) \quad \begin{array}{ccccc} M & \longrightarrow & H & \longrightarrow & j_!T_2 \\ \parallel & & \downarrow & & \downarrow \\ M & \longrightarrow & F & \longrightarrow & T \\ & & \downarrow & & \downarrow \\ & & i_*T_1 & \xlongequal{\quad} & i_*T_1 \end{array}$$

Applying  $j^*$  to the  $\mathbb{E}_{\mathcal{B}}$ -triangle in the third column of (4.7), we deduce that  $j^*T \cong j^*j_!T_2 \cong T_2$ . Similarly, applying  $i^*$  to the  $\mathbb{E}_{\mathcal{B}}$ -triangle in the third column of (4.7), by Lemma 2.10(2), we have  $i^*T \cong i^*i_*T_1 \cong T_1$ . That is,  $T \in \mathcal{T}$ . So the second row in (4.7) gives a desired  $\mathbb{E}_{\mathcal{B}}$ -triangle. Therefore,  $(\mathcal{T}, \mathcal{F})$  satisfies (c) in Definition 4.1. ■

*Proof of Theorem 4.4.* By Lemma 3.3(8'), (5) and (6), we see that  $j_*$  is exact and  $\mathcal{A}, \mathcal{C}$  have enough projectives.

(1) Take any objects  $T \in \mathcal{T}$  and  $F \in \mathcal{F}$ . By Lemma 4.5, we only need to prove  $\mathbb{E}_{\mathcal{B}}(T, F) = 0$ .

(i) Immediate.

(ii) Since  $i^*$  is exact, by Proposition 3.4(2), there is an  $\mathbb{E}_{\mathcal{B}}$ -triangle  $j_!j^*T \rightarrow T \rightarrow i_*i^*T \dashrightarrow$ . Applying  $\text{Hom}_{\mathcal{B}}(-, F)$  to this  $\mathbb{E}_{\mathcal{B}}$ -triangle, we get an exact sequence

$$\mathbb{E}_{\mathcal{B}}(i_*i^*T, F) \rightarrow \mathbb{E}_{\mathcal{B}}(T, F) \rightarrow \mathbb{E}_{\mathcal{B}}(j_!j^*T, F).$$

By Lemma 3.3(7), we deduce that  $\mathbb{E}_{\mathcal{B}}(i_*i^*T, F) \cong \mathbb{E}_{\mathcal{A}}(i^*T, i^!F) = 0$ , since  $i^*T \in \mathcal{T}_1$  and  $i^!F \in \mathcal{F}_1$ . Similarly,  $\mathbb{E}_{\mathcal{B}}(j_!j^*T, F) \cong \mathbb{E}_{\mathcal{C}}(j^*T, j^*F) = 0$ . It follows that  $\mathbb{E}_{\mathcal{B}}(T, F) = 0$ .

(iii) By (R5), there exists a commutative diagram

$$\begin{array}{ccccccc} i_*A' & \longrightarrow & j_!j^*T & \longrightarrow & T & \longrightarrow & i_*i^*T \longrightarrow \\ & & \searrow & & \nearrow & & \\ & & & & K & & \end{array}$$

in  $\mathcal{B}$  such that  $i_*A' \rightarrow j_!j^*T \rightarrow K \dashrightarrow$  and  $K \rightarrow T \rightarrow i_*i^*T \dashrightarrow$  are  $\mathbb{E}_{\mathcal{B}}$ -triangles. Applying  $\text{Hom}(-, F)$  to the two  $\mathbb{E}_{\mathcal{B}}$ -triangles above, we have two exact sequences

$$\mathbb{E}_{\mathcal{B}}(i_*i^*T, F) \rightarrow \mathbb{E}_{\mathcal{B}}(T, F) \rightarrow \mathbb{E}_{\mathcal{B}}(K, F)$$

and

$$\text{Hom}_{\mathcal{B}}(j_!j^*T, F) \xrightarrow{f^*} \text{Hom}_{\mathcal{B}}(i_*A', F) \rightarrow \mathbb{E}_{\mathcal{B}}(K, F) \rightarrow \mathbb{E}_{\mathcal{B}}(j_!j^*T, F).$$

By Lemma 3.3(7'), we obtain  $\mathbb{E}_{\mathcal{B}}(j_!j^*T, F) \cong \mathbb{E}_{\mathcal{C}}(j^*T, j^*F) = 0$ . Similarly,  $\mathbb{E}_{\mathcal{B}}(i_*i^*T, F) \cong \mathbb{E}_{\mathcal{A}}(i^*T, i^!F) = 0$ . By hypothesis,  $f^*$  is surjective, so we find that  $\mathbb{E}_{\mathcal{B}}(K, F) = 0$ . Consequently,  $\mathbb{E}_{\mathcal{B}}(T, F) = 0$ .

(iv) Since  $i^!$  is exact, by Proposition 3.4(1), there is an  $\mathbb{E}_{\mathcal{B}}$ -triangle

$$i_*i^!F \rightarrow F \rightarrow j_*j^*F \dashrightarrow.$$

Applying  $\text{Hom}(T, -)$  to the  $\mathbb{E}_{\mathcal{B}}$ -triangle above, we have an exact sequence

$$\mathbb{E}_{\mathcal{B}}(T, i_*i^!F) \rightarrow \mathbb{E}_{\mathcal{B}}(T, F) \rightarrow \mathbb{E}_{\mathcal{B}}(T, j_*j^*F).$$

Since  $i^!$  is exact, we deduce that  $j_*$  is exact and then  $j^*$  preserves projectives by Lemma 3.3(4). Thus, by Lemma 2.16(1), we see that  $\mathbb{E}_{\mathcal{B}}(T, j_*j^*F) \cong \mathbb{E}_{\mathcal{C}}(j^*T, j^*F) = 0$ . If  $\mathcal{T} \subseteq j_1\mathcal{T}_2$ , i.e., there exists  $T_2 \in \mathcal{T}_2$  such that  $T \cong j_1T_2$ , then  $\mathbb{E}_{\mathcal{B}}(T, i_*i^!F) \cong \mathbb{E}_{\mathcal{B}}(j_1T_2, i_*i^!F) \cong \mathbb{E}_{\mathcal{C}}(T_2, j^*i_*i^!F) = 0$ . If  $i_*\mathcal{F}_1 \subseteq \mathcal{T}^{\perp_1}$ , then  $\mathbb{E}_{\mathcal{B}}(T, i_*i^!F) = 0$  since  $i^!F \in \mathcal{F}_1$ . Hence,  $\mathbb{E}_{\mathcal{B}}(T, F) = 0$ .

(v) Since  $\mathcal{A}$  and  $\mathcal{B}$  are Frobenius extriangulated categories, projectives coincide with injectives, so by Lemma 3.3(4),  $i_*$  preserves injectives. Then by Lemma 2.16(2), we have the isomorphism  $\mathbb{E}_{\mathcal{A}}(i^*X, Y) \cong \mathbb{E}_{\mathcal{B}}(X, i_*Y)$  for any  $X \in \mathcal{B}$  and  $Y \in \mathcal{A}$ . Using the similar proof of (iv), we prove (v).

(2) For any  $X \in \mathcal{A}$ , there exists an  $\mathbb{E}_{\mathcal{B}}$ -triangle  $V \rightarrow U \rightarrow i_*X \dashrightarrow$  with  $V \in \mathcal{V}$  and  $U \in \mathcal{U}$ . Then we have an  $\mathbb{E}_{\mathcal{A}}$ -triangle  $i^!V \rightarrow i^!U \rightarrow X \dashrightarrow$ , since  $i^!$  is exact. Since  $i_*i^!U \in i_*i^!\mathcal{U} \subseteq \mathcal{U}$ , we find that  $i^!U \in i^*\mathcal{U}$ . That is,  $(i^*\mathcal{U}, i^!\mathcal{V})$  satisfies (b) Definition 4.1. Dually, we can prove that  $(i^*\mathcal{U}, i^!\mathcal{V})$  also satisfies (c) Definition 4.1. For any  $U \in \mathcal{U}$  and  $V \in \mathcal{V}$ , by Lemma 3.3(7), we have  $\mathbb{E}_{\mathcal{A}}(i^*U, i^!V) \cong \mathbb{E}_{\mathcal{B}}(i_*i^*U, V) \cong \mathbb{E}_{\mathcal{B}}(U', V) = 0$  for some  $U' \in \mathcal{U}$ . Hence,  $(i^*\mathcal{U}, i^!\mathcal{V})$  is a cotorsion pair in  $\mathcal{A}$ .

(3) The proof is analogous to that for (2). ■

By applying Theorem 4.4 to recollements of triangulated categories, we obtain the following

**COROLLARY 4.6** ([5, Theorem 3.1]). *Let  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  be a recollement of triangulated categories. Let  $(\mathcal{T}, \mathcal{F})$  be a glued pair with respect to cotorsion pairs  $(\mathcal{T}_1, \mathcal{F}_1)$  and  $(\mathcal{T}_2, \mathcal{F}_2)$  in  $\mathcal{A}$  and  $\mathcal{C}$ , respectively. Then  $(\mathcal{T}, \mathcal{F})$  is a cotorsion pair in  $\mathcal{B}$ .*

We finish this section with a straightforward example illustrating Theorem 4.4.

**EXAMPLE 4.7.** Keep the notation used in Example 3.5.

Observe that  $\mathcal{P}(\text{mod } A) = \text{add}(P_1 \oplus S_2)$  and  $\mathcal{I}(\text{mod } A) = \text{add}(P_1 \oplus S_1)$ . By Remark 4.2, we know that  $\text{mod } A$  has only two cotorsion pairs

$$\mathcal{H}_1 = (\mathcal{P}(\text{mod } A), \text{mod } A) \quad \text{and} \quad \mathcal{H}_2 = (\text{mod } A, \mathcal{I}(\text{mod } A)).$$

(1) Let  $(\mathcal{T}, \mathcal{F})$  be the glued pair with respect to  $\mathcal{H}_1$  and  $\mathcal{H}_1$ . Then

$$\mathcal{T} = \text{add} \left( \begin{pmatrix} S_2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P_1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} S_2 \\ S_2 \end{pmatrix}_1 \oplus \begin{pmatrix} 0 \\ S_2 \end{pmatrix} \oplus \begin{pmatrix} P_1 \\ P_1 \end{pmatrix}_1 \oplus \begin{pmatrix} S_1 \\ P_1 \end{pmatrix}_\psi \oplus \begin{pmatrix} 0 \\ P_1 \end{pmatrix} \right)$$

and  $\mathcal{F} = \text{mod } B$ .

(2) Let  $(\mathcal{T}, \mathcal{F})$  be the glued pair with respect to  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . Then

$$\begin{aligned}\mathcal{T} &= \text{mod } B \setminus \text{add} \left( \begin{pmatrix} P_1 \\ S_2 \end{pmatrix}_\varphi \oplus \begin{pmatrix} S_1 \\ 0 \end{pmatrix} \right), \\ \mathcal{F} &= \text{mod } B \setminus \text{add} \left( \begin{pmatrix} S_2 \\ S_2 \end{pmatrix}_1 \oplus \begin{pmatrix} P_1 \\ S_2 \end{pmatrix}_\varphi \oplus \begin{pmatrix} 0 \\ S_2 \end{pmatrix} \right).\end{aligned}$$

(3) Let  $(\mathcal{T}, \mathcal{F})$  be the glued pair with respect to  $\mathcal{H}_2$  and  $\mathcal{H}_2$ . Then  $\mathcal{T} = \text{mod } B$  and

$$\mathcal{F} = \text{mod } B \setminus \text{add} \left( \begin{pmatrix} S_2 \\ S_2 \end{pmatrix}_1 \oplus \begin{pmatrix} P_1 \\ S_2 \end{pmatrix}_\varphi \oplus \begin{pmatrix} 0 \\ S_2 \end{pmatrix} \right).$$

(4) Let  $(\mathcal{T}, \mathcal{F})$  be the glued pair with respect to  $\mathcal{H}_2$  and  $\mathcal{H}_1$ . Then

$$\begin{aligned}\mathcal{T} &= \text{mod } B \setminus \text{add} \left( \begin{pmatrix} S_1 \\ S_1 \end{pmatrix}_1 \oplus \begin{pmatrix} 0 \\ S_1 \end{pmatrix} \right), \\ \mathcal{F} &= \text{mod } B \setminus \text{add} \left( \begin{pmatrix} S_2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} S_2 \\ S_2 \end{pmatrix}_1 \right).\end{aligned}$$

However, no glued pair above is a cotorsion pair in  $\text{mod } B$  because  $\text{Ext}^1(\mathcal{T}, \mathcal{F}) \neq 0$ .

Observe that

$$\begin{aligned}\mathcal{P}(\text{mod } B) &= \text{add} \left( \begin{pmatrix} S_2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P_1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} S_2 \\ S_2 \end{pmatrix}_1 \oplus \begin{pmatrix} P_1 \\ P_1 \end{pmatrix}_1 \right), \\ \mathcal{I}(\text{mod } B) &= \text{add} \left( \begin{pmatrix} P_1 \\ P_1 \end{pmatrix}_1 \oplus \begin{pmatrix} S_1 \\ S_1 \end{pmatrix}_1 \oplus \begin{pmatrix} 0 \\ P_1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ S_1 \end{pmatrix} \right).\end{aligned}$$

(5) Take  $\mathcal{T} = \mathcal{P}(\text{mod } B)$  and  $\mathcal{F} = \text{mod } B$ . Then  $(\mathcal{T}, \mathcal{F})$  is a cotorsion pair in  $\text{mod } B$ . One can check that

$$i_* i^* \mathcal{T} = i_* i^! \mathcal{T} = \text{add} \left( \begin{pmatrix} S_2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P_1 \\ 0 \end{pmatrix} \right) \subseteq \mathcal{T}$$

and  $j_* j^* \mathcal{F} \subseteq \mathcal{F}$ . Therefore, by Theorem 4.4(2) and (3), we know that  $(i^* \mathcal{T}, i^! \mathcal{F})$  and  $(j^* \mathcal{T}, j^* \mathcal{F})$  are cotorsion pairs in  $\text{mod } A$ .

(5') Let

$$\begin{aligned}\mathcal{T} &= \text{add} \left( \begin{pmatrix} S_2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P_1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} S_2 \\ S_2 \end{pmatrix}_1 \oplus \begin{pmatrix} P_1 \\ P_1 \end{pmatrix}_1 \oplus \begin{pmatrix} 0 \\ S_1 \end{pmatrix} \right), \\ \mathcal{F} &= \text{add} \left( \begin{pmatrix} P_1 \\ P_1 \end{pmatrix}_1 \oplus \begin{pmatrix} S_1 \\ S_1 \end{pmatrix}_1 \oplus \begin{pmatrix} 0 \\ P_1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ S_1 \end{pmatrix} \oplus \begin{pmatrix} P_1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} S_2 \\ S_2 \end{pmatrix}_1 \oplus \begin{pmatrix} S_2 \\ 0 \end{pmatrix} \right).\end{aligned}$$

Then  $(\mathcal{T}, \mathcal{F})$  is also a cotorsion pair in  $\text{mod } B$ . We have

$$i_* i^* \mathcal{T} = i_* i^! \mathcal{T} = \text{add} \left( \begin{pmatrix} S_2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P_1 \\ 0 \end{pmatrix} \right) \subseteq \mathcal{T}.$$

Thus, by Theorem 4.4(2), we conclude that  $(i^*\mathcal{T}, i^!\mathcal{F})$  is a cotorsion pairs in  $\text{mod } A$ . But  $(j^*\mathcal{T}, j^*\mathcal{F})$  is not a cotorsion pair in  $\text{mod } A$  because  $\text{Ext}^1(j^*\mathcal{T}, j^*\mathcal{F}) \neq 0$ .

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