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EXPLICIT AVERAGES OF SQUARE-FREE SUPPORTED FUNCTIONS: TO THE EDGE OF THE CONVOLUTION METHOD

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Abstract. We give a general statement of the convolution method so that one can provide explicit asymptotic estimations for all averages of square-free supported arithmetic functions that have a sufficiently regular behavior on the prime numbers and observe how the nature of this method gives error estimations of order $X^{-\delta}$, where δ belongs to an open set I of positive reals. In order to have a better error estimation, a natural question is whether or not we can achieve an error term of critical order $X^{-\delta_0}$, where δ_0, the critical exponent, is the right endpoint of I. We answer this in the affirmative by presenting a new method that improves qualitatively almost all instances of the convolution method under some regularity conditions; now, the asymptotic estimation of averages of well-behaved square-free supported arithmetic functions can be given with its critical exponent and a reasonable explicit error constant. We illustrate this new method by analyzing a particular average related to the work of Ramaré–Akhilesh (2017), which leads to notable improvements when imposing non-trivial coprimality conditions.

1. Details and basic definitions. In the present work, we write $f(X) = O^*(h(X))$ as $X \to a$ to indicate that $|f(X)| \leq h(X)$ in a neighborhood of a, where, if not specified, a is ∞. We also consider the *Euler* φ_s *and Kappa* κ_s functions: let s be any complex number; we define $\varphi_s : \mathbb{Z}_{>0} \to \mathbb{C}$ as $q \mapsto q^s \prod_{p|q}\big(1 - \frac{1}{p^s}\big)$ and $\kappa_s : \mathbb{Z}_{>0} \to \mathbb{C}$ as $q \mapsto q^s \prod_{p|q}\big(1 + \frac{1}{p^s}\big)$.

Computational details. Every constant in this article has been estimated using interval arithmetic. Early numerical analysis was carried out using the ARB implementation, under the SageMath commands RBF and RIF, implemented in Python. We decided, however, to use Platt's implementation in C++, used for example in [5], as it provides results with double precision, when compared to ARB, and at higher performance and faster speed.

Throughout our calculations, we have set a precision order equal to $6 \cdot 10^9$ and run a .cpp script compiled with C++. We have also written a .ipynd script (compiled by SageMath) to verify some of our results.

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2. Introduction. The convolution method terminology was made popular by Ramaré in 1995, particularly in [6, Lemma 3.2], where it was given in a somewhat hidden version compared to this article. It is a technique, already present in [4] and [9], among many other places, that relies upon a convolution identity and helps to obtain explicit estimations of averages of arithmetic functions under some conditions. It is particularly useful when these arithmetic functions are supported on the square-free numbers, having a sufficiently regular behavior on all large prime numbers.

While the convolution method provides the main term of an asymptotic expansion for the average of an arithmetic function with ease, it is at the remainder term where it shows its true potential, as it succeeds in giving a good enough estimation, explicit, for the error term: if the average is performed for the range $(0, X]$, where $X > 0$, then the convolution method gives explicit error term estimations of order $X^{-\delta}$ when δ belongs to a maximal open interval I of positive reals.

However, the nature of the convolution method does not allow one to obtain an error estimation of order $X^{-\delta_0}$ where δ_0 is the right endpoint of I. Since it is usually of interest in number theory to improve error term orders, it is natural to ask whether one can provide, necessarily by a different method, an error term of critical order δ_0 so that the overall estimation is qualitatively improved, going thus to the edge of the method of convolution.

We first present in §3 a special form of the convolution method involving sufficiently regular square-free supported functions, in Theorem 3.3. This method is related to a typical complex analytic approach for estimating the asymptotic expansion for the average of an arithmetic function by means of residue theory.

Our main result, presented in §4, differs from complex analysis. In §4.2, we see how the use of some very particular estimations, given in §4.1, constitutes the main ingredient to obtain reasonable explicit estimations of the critical exponent in almost all cases where the convolution method may be applied. Indeed, since our technique also relies upon the convergence of infinite products, some extra conditions on the regularity of the arithmetic function that is being averaged are needed, as Theorem 4.6 shows; therefore there is a small range of functions that are not considered in our improvements, namely when the values of α and β defined in Theorem 4.6 differ in absolute value by $1/2$ or less. However, since the applications we mention throughout this article do not involve that missing case, we claim that they can all be improved up to their critical exponent.

Previous work towards obtaining error terms of critical exponent can be found, for some particular averages, in [1] and [9]. In [8] and [7], the critical exponent is reached by a completely different approach, using some results known as the *covering remainder lemma* and the *unbalanced Dirichlet hyperbola formula* as well as strong explicit bounds on some summatory functions involving the Möbius functions that oscillate, unlike in our case. Furthermore, it is important to point out that whereas a similar path to [8] or [7] could have been followed, the results there make use of specific properties of the functions being averaged and thus are not easy to generalize. This is the reason why [8, Thm. 1.2] improves on the classic convolution method result presented in Corollary 3.4(**a**) but still requires the convolution method to estimate related averages of more complicated arithmetic functions; for example, with the result of Theorem 4.6, one can now immediately derive stronger estimations for [8, Lemmas 5.2, 7.1, 7.2, 7.6–7.9] that may lead to further improvements on the cited article of Ramaré–Akhilesh. In that respect, our result might help as a reference for further improvements on many places where the convolution method is employed; it reads in a simplified manner as follows.

Theorem. *Let $X>0$ be a real number and q a positive integer. Consider a multiplicative function $f:\mathbb{Z}^+\to\mathbb{C}$ such that $f(p)\neq -1$ for any prime number p satisfying $(p,q)=1$ and for every sufficiently large prime number p coprime to q, we have $f(p)=1/p^\alpha+O\left(1/p^\beta\right)$, where α, β are real numbers satisfying $\beta>\alpha$, $\beta-\alpha>1/2$. Then one can determine a constant $\mathrm{W}_\alpha^q>0$ such that*

$$\sum_{\substack{\ell\leq X\\(\ell,q)=1}}\mu^2(\ell)f(\ell)=F_\alpha^q(X)+\begin{cases}O^*(\mathrm{W}_\alpha^q X^{1/2-\alpha}) & \text{if } \alpha\neq 1/2,\\ O^*(\mathrm{W}_\alpha^q\log(X)) & \text{if } \alpha=1/2,\end{cases}$$

where

$$F_\alpha^q(X)=\frac{M_\alpha^q\zeta(\alpha)\varphi_\alpha(q)}{q^\alpha}-\frac{N_\alpha^q\varphi(q)}{(\alpha-1)q}\frac{1}{X^{\alpha-1}}\qquad \text{if } \alpha>1/2,\ \alpha\neq 1,$$

$$F_1^q(X)=\frac{M_1^q\varphi(q)}{q}\left(\log(X)+T_f^q+\gamma+\sum_{p|q}\frac{\log(p)}{p-1}\right),$$

$$T_f^q=\sum_{p\nmid q}\frac{\log(p)(1-(p-2)f(p))}{(f(p)+1)(p-1)},$$

$$F_\alpha^q(X)=\frac{M_\alpha^q\varphi(q)}{(1-\alpha)q}X^{1-\alpha}\qquad \text{if } \alpha\leq 1/2,$$

$$M_\alpha^q=\begin{cases}\prod_{p\nmid q}\left(1-\frac{1-f(p)p^\alpha+f(p)}{p^\alpha}\right) & \text{if } \alpha>1/2,\\ N_\alpha^q & \text{if } \alpha\leq 1/2,\end{cases}$$

$$N_\alpha^q=\prod_{p\nmid q}\left(1-\frac{p^{1-\alpha}-f(p)p+f(p)}{p^{2-\alpha}}\right).$$

As an application of the above theorem, we deduce how the improvement on the convolution method produces better savings on the error term constant of $\sum_{\ell\leq X,\,(\ell,q)=1}\mu^2(\ell)/\varphi(\ell)$, $X>0$, $q\in\mathbb{Z}_{>0}$, than the one in [8, Thm. 1.1], when prime coprimality conditions are introduced. This situation is examined in §4.3, and we have for instance the improvement on the constant 4.956, given in [8, Thm. 1.1], to 2.169, according to the following result.

Lemma. *Let $X>0$. Then*

$$\sum_{\substack{\ell\leq X\\(\ell,2)=1}}\frac{\mu^2(\ell)}{\varphi(\ell)}=\frac{1}{2}(\log(X)+\mathfrak{a}_2)+O^*\left(\frac{2.169}{\sqrt{X}}\right),$$

where $\mathfrak{a}_2=1.679\ldots$.

3. A special version of the method of convolution. In the convolution method, it is crucial to preserve regularity conditions, that is, conditions that do not impose specific ranges other than the variable itself being a positive integer, under, perhaps, some coprimality restrictions.

To give an example, when one carries out a summation over a variable $e\in\mathbb{Z}_{>0}$ such that $e\leq X/d$ for a real number $X>0$ and a positive integer d, it is often implicitly assumed that $X/d\geq 1$, so that the set $\{e\in\mathbb{Z}_{>0}: e\leq X/d\}$ is not empty. If d is itself a variable, that means that we have the range condition $\{d\leq X\}$ on the variable d. Hence, if we are able to estimate asymptotically a sum over $e\in\mathbb{Z}_{>0}$ such that $e\leq X/d$, regardless of whether or not an empty summation is performed, then the range condition on the variable d will be absent.

3.1. Regularity conditions: estimating empty summations

Lemma 3.1. *Let $\alpha\in\mathbb{R}^+\setminus\{1\}$ and $X>0$. Then*

$$\sum_{n\leq X}\frac{1}{n^\alpha}=\zeta(\alpha)-\frac{1}{(\alpha-1)X^{\alpha-1}}+O^*\left(\frac{1}{X^\alpha}\right).$$

Proof. By definition of $\zeta(s)$ for $\Re(s)>1$, and by analytic continuation for all $s\neq 1$ with $\Re(s)>0$,

$$\zeta(s)-\frac{1}{(s-1)X^{s-1}}-\sum_{n\leq X}\frac{1}{n^s}=\sum_{n=1}^{\infty}\left(\int_{n-1}^{n}\frac{dx}{(X+x)^s}-\frac{1}{(\lfloor X\rfloor+n)^s}\right). \tag{3.1}$$

Set $s=\alpha$; clearly $(\lfloor X\rfloor+n)^{-\alpha}\geq(X+n)^{-\alpha}$ and by convexity of $t\mapsto 1/t^\alpha$,

$$\int_{n-1}^{n}\frac{dx}{(X+x)^\alpha}\leq\frac{1}{2}\left(\frac{1}{(X+n-1)^\alpha}+\frac{1}{(X+n)^\alpha}\right).$$

Hence, the right hand side of (3.1) is at most

$$\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{(X+n-1)^{\alpha}}-\frac{1}{(X+n)^{\alpha}}\right)\leq\frac{1}{2X^{\alpha}}.$$

On the other hand, by the mean value theorem, for any $n\in\mathbb{Z}_{>0}$ there exists $r\in[n-1,n]$ such that

$$\int_{n-1}^{n}\frac{dx}{(X+x)^{\alpha}}-\frac{1}{(\lfloor X\rfloor+n)^{\alpha}}=\frac{1}{(X+r)^{\alpha}}-\frac{1}{(\lfloor X\rfloor+n)^{\alpha}}.$$

Thus, by the monotonicity of $t\mapsto 1/t^{\alpha}$ and the fact that $X+r$ and $\lfloor X\rfloor+n$ are both contained in $[X+n-1,X+n]$, the right hand side of (3.1) is at least

$$\sum_{n=1}^{\infty}\left(\frac{1}{(X+n)^{\alpha}}-\frac{1}{(X+n-1)^{\alpha}}\right)=-\frac{1}{X^{\alpha}}.\ \blacksquare$$

The following lemma estimates asymptotically some sums even when they are empty.

Lemma 3.2. *Let $X>0$ and $\alpha>0$. If $0<\delta\leq 1$, we have*

$$\sum_{n\leq X}\frac{1}{n}=\log(X)+\gamma+O^{*}\left(\frac{\varDelta_1^{\delta}}{X^{\delta}}\right);\tag{3.2}$$

if $\max\{0,\alpha-1\}<\delta\leq\alpha$ *and* $\alpha\neq 1$, *we have*

$$\sum_{n\leq X}\frac{1}{n^{\alpha}}=\zeta(\alpha)-\frac{1}{(\alpha-1)X^{\alpha-1}}+O^{*}\left(\frac{\varDelta_{\alpha}^{\delta}}{X^{\delta}}\right),\tag{3.3}$$

where $\varDelta_1^{\delta}=\max\left\{\gamma,\frac{1}{\delta e^{\gamma\delta+1}}\right\}$ *and, for* $\alpha\neq 1$,

$$\varDelta_{\alpha}^{\delta}=\begin{cases}\max\left\{1,\left(\frac{1}{\delta^{\delta}}\left(\frac{(\delta-\alpha+1)}{|\zeta(\alpha)(\alpha-1)|}\right)^{\delta-\alpha+1}\right)^{\frac{1}{\alpha-1}},\zeta(\alpha)-\frac{1}{\alpha-1}\right\} & \textit{if } \delta\neq\alpha,\\ 1 & \textit{if } \delta=\alpha.\end{cases}$$

Proof. By [8, Lemma 2.1] and Lemma 3.1, for $X>0$ we have

$$\sum_{n\leq X}\frac{1}{n}=\log(X)+\gamma+O^{*}\left(\frac{\gamma}{X}\right),\tag{3.4}$$

$$\sum_{n\leq X}\frac{1}{n^{\alpha}}=\zeta(\alpha)-\frac{1}{(\alpha-1)X^{\alpha-1}}+O^{*}\left(\frac{1}{X^{\alpha}}\right)\quad\text{if } \alpha>0 \text{ and } \alpha\neq 1,\tag{3.5}$$

respectively. Thus, if $X\geq 1$, the result holds trivially as $\delta'\mapsto X^{\delta'}$ is increasing and $\delta<\alpha$. Otherwise, when $0<X<1$ the above summations are empty; write $X=1/Y$ with $Y>1$ and observe first that the function $f:1\leq Y\mapsto\frac{\log(Y)-\gamma}{Y^{\delta}}$ has a single critical point at $y_0=e^{1/\delta+\gamma}>1$ taking the value $f(y_0)=\frac{1}{\delta e^{\gamma\delta+1}}>0$. As $f(1)=-\gamma$ and $\lim_{Y\to\infty}f(Y)=0$, f is increasing in $[1, y_0]$ and decreasing in $[y_0, \infty)$, and hence $\sup_{\{Y>1\}} |f(Y)| = \max\left\{\gamma, \frac{1}{\delta e^{\gamma\delta+1}}\right\}$.

Secondly, by [3, Cor. 1.14], we have $\zeta(\alpha) > \frac{1}{\alpha-1}$ and $\zeta(\alpha)(\alpha-1) > 0$ for all $\alpha \geq 0$ and $\alpha \neq 1$. Moreover, the function $g : 0 < Y \mapsto \frac{1}{Y^\delta}\left(\zeta(\alpha) - \frac{Y^{\alpha-1}}{\alpha-1}\right)$ has a critical point y_0 satisfying $y_0^{\alpha-1} = \frac{\zeta(\alpha)(\alpha-1)\delta}{\delta-\alpha+1} > 0$, since $\delta > \alpha - 1$ and $\delta > 0$ and in this case, we have $\lim_{Y\to\infty} g(Y) = 0$, and thus $|g|$ is decreasing in $[y_0, \infty)$. We conclude that $\max_{[y_0,\infty)} |g(Y)| = |g(y_0)|$, where

$$|g(y_0)| = \left(\frac{1}{\delta^\delta}\left(\frac{(\delta-\alpha+1)}{|\zeta(\alpha)(\alpha-1)|}\right)^{\delta-\alpha+1}\right)^{\frac{1}{\alpha-1}}.$$

If $y_0 \leq 1$, then $|g(1)| = g(1) \leq |g(y_0)|$ and $\sup_{\{Y>1\}} |g(Y)| = g(1)$; otherwise, if $y_0 > 1$, as g is also monotonic between 1 and y_0, we deduce that $\sup_{\{Y>1\}} |g(Y)| = \max\{g(1), |g(y_0)|\}$, which gives us the desired result. ■

It is important to point out that for $\alpha > 1$, it would be possible to give an error expression even if $\delta = \alpha - 1 > 0$, whereas if $\delta < \alpha - 1$, then $|g|$ would be unbounded in $[1, \infty)$.

On the other hand, as pointed out at the beginning of §3, it is essential to have an estimation of the above summations when they are actually empty, that is, when $X \in (0, 1)$. Indeed, this will provide regularity for some sum conditions during the proof of Theorem 3.3 that otherwise would impose some variables to be at least 1 and some sums to be non-empty. It should be expected, though, that the fact of imposing regularity conditions, or rather asking for estimations of sums up to $X > 0$, will worsen a bit the constants in the error terms involved; for instance, when $\alpha = 1$ and when we are restricted to the range $X \geq 1$, the value of $\gamma = 0.57721\ldots$ given in (3.4) can be improved to $2(\log(2) + \gamma - 1) = 0.54072\ldots$ (refer to [8, Lemma 2.1]).

3.2. The convolution method. The following theorem will help us to state Corollary 3.4. Although inspired by [6, Lemma 3.2], it is presented in a much general framework, in an attempt to understand and deduce with ease the order of averages of sufficiently regular square-free supported arithmetic functions. By "sufficiently regular", we mean an arithmetic function having a specific constant dominant term on all sufficiently large prime numbers. It turns out that it is precisely the regularity of an arithmetic function that helps one to derive the asymptotic expansion of its average by the method of convolution.

THEOREM 3.3. *Let q be a positive integer and let X, α, β be real numbers such that $X > 0$, $\beta > 1$ and $\beta > \alpha > 1/2$. Consider a multiplicative function $f : \mathbb{Z}^+ \to \mathbb{C}$ such that $f(p) = 1/p^\alpha + O(1/p^\beta)$ for every sufficiently large prime number p coprime to q. Then for any real number $\delta > 0$ such that* $\max\{0, \alpha - 1\} < \delta < \min\{\beta - 1, \alpha - 1/2\}$ *we have*

$$\sum_{\substack{\ell\leq X\\(\ell,q)=1}}\mu^2(\ell)f(\ell)=F_\alpha^q(X)+O^*\left(\Delta_\alpha^\delta\frac{\kappa_{\alpha-\delta}(q)}{q^{\alpha-\delta}}\cdot\frac{\overline{H}_f^{\,q}(-\delta)}{X^\delta}\right),$$

where, if $\alpha\neq 1$,

$$F_\alpha^q(X)=\frac{H_f^q(0)\zeta(\alpha)\varphi_\alpha(q)}{q^\alpha}-\frac{H_f^q(1-\alpha)\varphi(q)}{(\alpha-1)q}\frac{1}{X^{\alpha-1}},$$

and if $f(p)=-1$ *for some prime number* p, *then* $F_1^q(X)=-\sum_d\frac{h_f^q(d)\log(d)}{d^\alpha}$, *whereas if* $f(p)\neq -1$ *for any prime number* p, *then*

$$F_1^q(X)=\frac{H_f^q(0)\varphi(q)}{q}\left(\log(X)+T_f^q+\gamma+\sum_{p|q}\frac{\log(p)}{p-1}\right),$$

$$T_f^q=\sum_{p\nmid q}\frac{\log(p)(1-(p-2)f(p))}{(f(p)+1)(p-1)}.$$

Here, Δ_α^δ *is defined as in Lemma* 3.2 *and* $H_f^q:\{s\in\mathbb{C}:\Re(s)>1/2-\alpha\}\to\mathbb{C}$ *is an analytic function satisfying*

$$H_f^q(s)=\prod_{p\nmid q}\left(1-\frac{1-f(p)p^\alpha}{p^{s+\alpha}}-\frac{f(p)}{p^{2s+\alpha}}\right)=\sum_{\substack{d\\(d,q)=1}}\frac{h_f^q(d)}{d^{s+\alpha}},$$

$$\overline{H}_f^{\,q}(s)=\prod_{p\nmid q}\left(1+\frac{|1-f(p)p^\alpha|}{p^{\Re(s)+\alpha}}+\frac{|f(p)|}{p^{2\Re(s)+\alpha}}\right)=\sum_{\substack{d\\(d,q)=1}}\frac{|h_f^q(d)|}{d^{\Re(s)+\alpha}}.$$

Proof. By the asymptotic condition on f in the statement, the Dirichlet series D_f^q associated with $\ell\mapsto\mu^2(\ell)f(\ell)\mathbb{1}_q(\ell)$, where $\mathbb{1}_q$ is defined as the multiplicative function $\ell\mapsto\mathbb{1}_{\{(\ell,q)=1\}}(\ell)$, converges absolutely for any $s\in\mathbb{C}$ with $\Re(s)>1-\alpha$. Thus, in the set $\{s\in\mathbb{C}:\Re(s)>1-\alpha\}$, the equality

$$D_f^q(s)=\sum_{\substack{\ell\\(\ell,q)=1}}\frac{\mu^2(\ell)f(\ell)}{\ell^s}=\prod_{p\nmid q}\left(1+\frac{f(p)}{p^s}\right)\tag{3.6}$$

holds and the function $s\mapsto\zeta(s+\alpha)$ can be expressed by an Euler product. For any s such that $\Re(s)>1-\alpha$, we have

$$\begin{aligned}\frac{D_f^q(s)}{\zeta(s+\alpha)}&=\prod_{p\nmid q}\left(1+\frac{f(p)}{p^s}\right)\left(1-\frac{1}{p^{s+\alpha}}\right)\cdot\prod_{p|q}\left(1-\frac{1}{p^{s+\alpha}}\right)\\&=\frac{\varphi_{s+\alpha}(q)}{q^{s+\alpha}}\cdot\prod_{p\nmid q}\left(1-\frac{1-f(p)p^\alpha}{p^{s+\alpha}}-\frac{f(p)}{p^{2s+\alpha}}\right)=\frac{\varphi_{s+\alpha}(q)}{q^{s+\alpha}}\cdot H_f^q(s).\end{aligned}$$

Also, we have $\frac{1-f(p)p^\alpha}{p^{s+\alpha}}=O\big(\frac{1}{p^{\Re(s)+\beta}}\big)$ and $\frac{f(p)}{p^{2s+\alpha}}=O\big(\frac{1}{p^{2\Re(s)+2\alpha}}\big)$. Since $\beta>\alpha$, H can be extended analytically from $\{s\in\mathbb{C}:\Re(s)>1-\alpha\}$ onto $\{s\in\mathbb{C}:\Re(s)>\max\{1-\beta,1/2-\alpha\}\}$. Further, as $0>1-\beta$ and $0>1/2-\alpha$, $H_f^q(0)$ exists, and if $f(p)\neq-1$ for any prime number p, it is different from 0, since each factor defining it can be expressed as $(1+f(p))(1-1/p^\alpha)$ and $\alpha\neq0$.

Now, the formal equality

$$D_f^q(s)=H_f^q(s)\cdot\prod_{p\nmid q}\left(1+\frac{1}{p^{s+\alpha}}+\frac{1}{p^{2(s+\alpha)}}+\cdots\right)$$

hides the convolution product

$$\ell^\alpha\mu^2(\ell)f(\ell)\mathbb{1}_{(\ell,q)=1}(\ell)=(h_f^q\star\mathbb{1}_q)(\ell)=\sum_{d|\ell}h_f^q(d)\mathbb{1}_q(\ell/d), \tag{3.7}$$

where h is a multiplicative function defined on the prime numbers as

$$h_f^q(p)=(f(p)p^\alpha-1)\cdot\mathbb{1}_q(p),\qquad h_f^q(p^2)=-f(p)p^\alpha\cdot\mathbb{1}_q(p),$$
$$h_f^q(p^k)=0,\qquad k>2. \tag{3.8}$$

Therefore, from (3.7) we conclude that

$$\begin{aligned}S_f(X):=\sum_{\substack{\ell\leq X\\(\ell,q)=1}}\mu^2(\ell)f(\ell)&=\sum_{\ell\leq X}\frac{(h_f^q\star\mathbb{1}_q)(\ell)}{\ell^\alpha}=\sum_d\frac{h_f^q(d)}{d^\alpha}\sum_{\substack{e\leq X/d\\(e,q)=1}}\frac{1}{e^\alpha}\\&=\sum_d\frac{h_f^q(d)}{d^\alpha}\sum_{e\leq X/d}\frac{1}{e^\alpha}\sum_{d'|e,\,d'|q}\mu(d')=\sum_d\frac{h_f^q(d)}{d^\alpha}\sum_{d'|q}\frac{\mu(d')}{d'^\alpha}\sum_{e\leq X/dd'}\frac{1}{e^\alpha},\end{aligned} \tag{3.9}$$

where there are no upper bounds on the variables d and d' in the outer sums above, their being encoded by the innermost sum of (3.9), which, in order to continue our analysis, we must estimate regardless of whether or not it is empty; Lemma 3.2 allows us to handle this situation.

Hence, as $\max\{0,\alpha-1\}<\delta<\min\{\beta-1,\alpha-1/2\}<\alpha$, we infer that the second sum in (3.9) can be expressed as

$$\begin{aligned}&\sum_{d'|q}\frac{\mu(d')}{d'^\alpha}\sum_{e\leq X/dd'}\frac{1}{e^\alpha}\\&\quad=\sum_{d'|q}\frac{\mu(d')}{d'^\alpha}\left(\zeta(\alpha)-\frac{(dd')^{\alpha-1}}{(\alpha-1)X^{\alpha-1}}+O^*\left(\varDelta_\alpha^\delta\frac{(dd')^\delta}{X^\delta}\right)\right)\\&\quad=\frac{\zeta(\alpha)\varphi_\alpha(q)}{q^\alpha}-\frac{\varphi(q)}{(\alpha-1)q}\cdot\frac{d^{\alpha-1}}{X^{\alpha-1}}+O^*\left(\varDelta_\alpha^\delta\frac{\kappa_{\alpha-\delta}(q)}{q^{\alpha-\delta}}\cdot\frac{d^\delta}{X^\delta}\right)\end{aligned} \tag{3.10}$$

if $\alpha \neq 1$, and

$$\sum_{d'|q}\frac{\mu(d')}{d'^\alpha}\sum_{e\leq\frac{X}{dd'}}\frac{1}{e^\alpha}=\sum_{d'|q}\frac{\mu(d')}{d'^\alpha}\left(\log\left(\frac{X}{dd'}\right)+\gamma+O^*\left(\frac{\mathit{\Delta}_1^\delta(dd')^\delta}{X^\delta}\right)\right) \tag{3.11}$$

$$=\frac{\varphi_\alpha(q)}{q^\alpha}\left(\log\left(\frac{X}{d}\right)+\gamma\right)-\sum_{d'|q}\frac{\mu(d')\log(d')}{d'^\alpha}+O^*\left(\frac{\mathit{\Delta}_1^\delta\kappa_{\alpha-\delta}(q)}{q^{\alpha-\delta}}\cdot\frac{d^\delta}{X^\delta}\right)$$

$$=\frac{\varphi_\alpha(q)}{q^\alpha}\left(\log\left(\frac{X}{d}\right)+\gamma+\sum_{p|q}\frac{\log(p)}{p^\alpha-1}\right)+O^*\left(\frac{\mathit{\Delta}_1^\delta\kappa_{\alpha-\delta}(q)}{q^{\alpha-\delta}}\cdot\frac{d^\delta}{X^\delta}\right)$$

if $\alpha = 1$, where we have used

$$-\sum_{d'|q}\frac{\mu(d')\log(d')}{d'^\alpha}=\left(\frac{\varphi_{s+\alpha}(q)}{q^{s+\alpha}}\right)'_{s=0}=\frac{\varphi_\alpha(q)}{q^\alpha}\sum_{p|q}\frac{\log(p)}{p^\alpha-1}. \tag{3.12}$$

On the other hand, observe that $H_f^q(1-\alpha)$ and $\overline{H}_f^q(-\delta)$ are well-defined, as $\min\{1-\alpha,-\delta\}>\max\{1-\beta,1/2-\alpha\}$. Therefore, from (3.9), the sum $S_f(X)$ can be estimated by

$$\sum_d\frac{h_f^q(d)}{d^\alpha}\left(\frac{\zeta(\alpha)\varphi_\alpha(q)}{q^\alpha}-\frac{\varphi(q)}{(\alpha-1)q}\cdot\frac{d^{\alpha-1}}{X^{\alpha-1}}+O^*\left(\frac{\mathit{\Delta}_\alpha^\delta\kappa_{\alpha-\delta}(q)}{q^{\alpha-\delta}}\cdot\frac{d^\delta}{X^\delta}\right)\right) \tag{3.13}$$

$$=H_f^q(0)\ \frac{\zeta(\alpha)\varphi_\alpha(q)}{q^\alpha}-\frac{\varphi(q)}{(\alpha-1)q}\cdot\frac{H_f^q(1-\alpha)}{X^{\alpha-1}}+O^*\left(\frac{\mathit{\Delta}_\alpha^\delta\kappa_{\alpha-\delta}(q)}{q^{\alpha-\delta}}\cdot\frac{\overline{H}_f^q(-\delta)}{X^\delta}\right)$$

if $\alpha \neq 1$, by using (3.10), or

(3.14)

$$\sum_d\frac{h_f^q(d)}{d^\alpha}\left(\frac{\varphi_\alpha(q)}{q^\alpha}\left(\log\left(\frac{X}{d}\right)+\gamma+\sum_{p|q}\frac{\log(p)}{p^\alpha-1}\right)+O^*\left(\frac{\mathit{\Delta}_1^\delta\kappa_{\alpha-\delta}(q)}{q^{\alpha-\delta}}\cdot\frac{d^\delta}{X^\delta}\right)\right)$$

$$=H_f^q(0)\ \frac{\varphi_\alpha(q)}{q^\alpha}\left(\log(X)+\gamma+\sum_{p|q}\frac{\log(p)}{p^\alpha-1}\right)+H_f^{q\prime}(0)$$

$$+O^*\left(\frac{\mathit{\Delta}_1^\delta\kappa_{\alpha-\delta}(q)}{q^{\alpha-\delta}}\cdot\frac{\overline{H}_f^q(-\delta)}{X^\delta}\right)$$

if $\alpha = 1$, by using (3.11) and $-\sum_d\frac{h_f^q(d)\log(d)}{d^\alpha}=H_f^{q\prime}(0)$. The result is thus obtained by noticing that if $H_f^q(0)\neq 0$, then $\frac{H_f^{q\prime}(0)}{H_f^q(0)}$ equals

$$\left(\prod_{p\nmid q}\left(1-\frac{1-f(p)p^\alpha}{p^{s+\alpha}}-\frac{f(p)}{p^{2s+\alpha}}\right)\right)'_{s=0}=\sum_{p\nmid q}\frac{\log(p)(1-f(p)p^\alpha+2f(p))}{(f(p)+1)(p^\alpha-1)}.\ \blacksquare$$

Corollary 3.4. *Let $X > 0$ and $q \in \mathbb{Z}_{>0}$. Then*

$$(3.15)\qquad (\mathbf{a})\ \sum_{\substack{\ell\leq X\\(\ell,q)=1}}\frac{\mu^2(\ell)}{\varphi(\ell)}=\frac{\varphi(q)}{q}(\log(X)+\mathfrak{a}_q)+O^*\left(\frac{7.36\cdot\mathcal{A}_q}{X^{1/3}}\right),$$

$$(3.16)\qquad (\mathbf{b})\ \sum_{\substack{\ell\leq X\\(\ell,q)=1}}\frac{\mu^2(\ell)}{\ell}=\frac{6}{\pi^2}\frac{q}{\kappa(q)}(\log(X)+\mathfrak{b}_q)+O^*\left(\frac{2.554\cdot\mathcal{B}_q}{X^{1/3}}\right),$$

where

$$\mathcal{A}_q=\prod_{p|q}\left(1+\frac{p-p^{1/3}-2}{(p-1)p^{2/3}+p^{1/3}+1}\right),\qquad \mathcal{B}_q=\prod_{p|q}\left(1+\frac{p^{2/3}-1}{p^{4/3}+1}\right),$$

and

$$\mathfrak{a}_q=\sum_p\frac{\log(p)}{p(p-1)}+\gamma+\sum_{p|q}\frac{\log(p)}{p},\qquad \sum_p\frac{\log(p)}{p(p-1)}+\gamma=1.33258228\ldots,$$

$$\mathfrak{b}_q=\sum_p\frac{2\log(p)}{p^2-1}+\gamma+\sum_{p|q}\frac{\log(p)}{p+1},\qquad \sum_p\frac{2\log(p)}{p^2-1}+\gamma=1.71713766\ldots.$$

Proof. For case (**a**) (respectively (**b**)), apply Theorem 3.3 with $f(p)=\frac{1}{\varphi(p)}=\frac{1}{p-1}$ (respectively $f(p)=1/p$), $\alpha=1$, $\beta=2$ and $0\leq\delta=1/3<1/2$.

The infinite products that enter the main and error terms as well as the infinite summation in the main term can be estimated by using a rigorous implementation of interval arithmetic, and some techniques for accelerating convergence. ■

Remarks. The conditions $\alpha>1/2$ and $\beta>1$ in Theorem 3.3 are necessary to ensure the existence of $H_f^q(0)$. Nonetheless, we can derive an analogous result for any multiplicative arithmetic function f satisfying $f(p)=1/p^\alpha+O(1/p^\beta)$ for every sufficiently large prime number p coprime to q, where $\alpha\leq1/2$ and $\beta>\alpha$ by using Theorem 3.3 and summation by parts. In this case, there will be no secondary term and the error term will be $O(X^{1-\alpha-\delta})$ for any $0<\delta<\min\{\beta-\alpha,1/2\}$.

With Theorem 3.3 at our disposal, the asymptotic estimation of averages $S_f(X)$ satisfying the conditions of that theorem becomes an automatized, but not uninteresting task, which involves a choice of parameters in each case: a value for δ and a precision value in order to obtain a rigorous estimation of some infinite products.

In general, we have freedom to choose the error term parameter δ described in §3 but some choices are not optimal. For instance, if $\alpha=1$, then in terms of Theorem 3.3 and Lemma 3.2, $\Delta_1^\delta\to\infty$ as $\delta\to0^+$. Since $\overline{H}_f^q(-\delta)$ converges, that makes the expression $\Delta_1^\delta\overline{H}_f^q(-\delta)$ tend to ∞ as well, thus providing a numerically unacceptable value. On the other hand, when $\delta \to (1/2)^-$, the infinite product given by $\overline{H}_f^q(-\delta)$ tends to ∞, whereas $\Delta_1^\delta \to \Delta_1^{1/2}$, thus bounded, so that the expression $\Delta_1^\delta \overline{H}_f^q(-\delta)$ becomes too big to be practical. One looks for a value of δ not too close to the endpoints of $(0, 1/2)$, and in almost all cases it seems acceptable to set $\delta = 1/3$.

A natural question is whether we can improve on the error estimation in Theorem 3.3, with a different method, for the exponent $\delta = \min\{\beta - 1, \alpha - 1/2\}$. If $\beta - \alpha > 1/2$, then $\delta = \alpha - 1/2$ and the answer is given in §4: it is affirmative and constitutes our main result. We also provide explicit estimations for those *critical exponents*.

Of the results above, the sum (3.15) is classical and it has been thoroughly studied by Ramaré and Akhilesh [8], by Ramaré [7, Thm. 3.1], [6, Lemma 3.4] and given in our simpler form by Helfgott [2, §6.1.1].

4. Improvements on the convolution method. During the proof of Theorem 3.3, it was crucial to have an empty sum estimation for the inner sum in (3.9) so that, thanks to the regularity on the variable d we find convergent main and error term coefficients, as shown in (3.13) and (3.14).

This general idea misses the fact that the function h_f^q defined in (3.8) vanishes on all non-cube-free numbers, and that the particular function $h_f^q : p, (p, q) = 1 \mapsto 1/p^\alpha$, with $\alpha > 1/2$, satisfies $h_f^q(p) = 0$. Moreover, the fact that that particular function is meaningful only on the squares of prime numbers, will allow us to achieve the critical exponent $\delta = 1/2$ if $\alpha = 1$, and $\delta = \alpha - 1/2$ if $\alpha \neq 1$ and $\alpha > 1/2$, when f is an arithmetic function satisfying the conditions of Theorem 3.3 with $\beta - \alpha > 1/2$.

4.1. A particular case. Let us see how we can improve the estimation (**b**) in Corollary 3.4.

LEMMA 4.1. *Let $X > 0$. Then*

$$\sum_{\ell \le X} \frac{\mu^2(\ell)}{\ell} = \frac{6}{\pi^2}(\log(X) + \mathfrak{b}_1) + O^*\left(\frac{1.044}{\sqrt{X}}\right), \tag{4.1}$$

$$\sum_{\substack{\ell \le X \\ (\ell,2)=1}} \frac{\mu^2(\ell)}{\ell} = \frac{4}{\pi^2}(\log(X) + \mathfrak{b}_2) + O^*\left(\frac{0.79}{\sqrt{X}}\right), \tag{4.2}$$

where $\mathfrak{b}_1 = \gamma + \sum_p \frac{2\log(p)}{p^2 - 1} = 1.71713766\ldots$, $\mathfrak{b}_2 = \mathfrak{b}_1 + \frac{\log(2)}{3} = 1.94818672\ldots$.

If we restrict ourselves to the range $X \ge 1$, then 1.044 *may be replaced by* 0.43 *and* 0.79 *may be replaced by* 0.407.

Proof. Equation (3.16) gives the main term of (4.2) and from that, we can conclude by summation by parts that for all $X \ge 1$, $\sum_{\ell \le X, (\ell,2)=1} \mu^2(\ell)/\ell$

equals

$$\frac{4(\log(X)+\mathfrak{b}_2)}{\pi^2} + \Bigg(\sum_{\substack{\ell\le X\\(\ell,2)=1}}\mu^2(\ell)-\frac{4}{\pi^2}X\Bigg)\frac{1}{X}-\int_X^\infty\Bigg(\sum_{\substack{\ell\le t\\(\ell,2)=1}}\mu^2(\ell)-\frac{4}{\pi^2}t\Bigg)\frac{dt}{t^2}. \tag{4.3}$$

Moreover, by [2, Lemma 5.2], we have

$$\sup_{\{X\ge 1573\}}\frac{1}{\sqrt{X}}\Bigg|\sum_{\substack{\ell\le X\\(\ell,2)=1}}\mu^2(\ell)-\frac{4}{\pi^2}X\Bigg|\le\frac{9}{70}, \tag{4.4}$$

so that, by (4.3),

$$\sum_{\substack{\ell\le X\\(\ell,2)=1}}\frac{\mu^2(\ell)}{\ell}=\frac{4}{\pi^2}(\log(X)+\mathfrak{b}_2)+O^*\left(\frac{27}{70}\frac{1}{\sqrt{X}}\right)\quad\text{if } X\ge 1573,$$

where $\frac{27}{70}=0.385\ldots$. We further verify by interval arithmetic that

$$\sup_{\{1\le X\le 1573\}}\sqrt{X}\Bigg|\sum_{\substack{\ell\le X\\(\ell,2)=1}}\frac{\mu^2(\ell)}{\ell}-\frac{4}{\pi^2}(\log(X)+\mathfrak{b}_2)\Bigg|\le 0.407, \tag{4.5}$$

the upper bound being almost achieved when $X\to 3^-$. On the other hand, [7, Cor. 1.2] tells us that

$$\sup_{\{X\ge 1\}}\sqrt{X}\Bigg|\sum_{\ell\le X}\frac{\mu^2(\ell)}{\ell}-\frac{6}{\pi^2}(\log(X)+\mathfrak{b}_1)\Bigg|\le 0.43. \tag{4.6}$$

Hence, by using (4.4)–(4.6), when $v\in\{1,2\}$ we have the bounds

$$\sup_{\{X\ge 1\}}\sqrt{X}\Bigg|\sum_{\substack{\ell\le X\\(\ell,v)=1}}\frac{\mu^2(\ell)}{\ell}-\frac{v}{\kappa(v)}\frac{6}{\pi^2}(\log(X)+\mathfrak{b}_v)\Bigg|\le\begin{cases}0.43 & \text{if } v=1,\\ 0.407 & \text{if } v=2.\end{cases} \tag{4.7}$$

In order to derive the result, it is sufficient to obtain bounds for (4.7) when $X\in(0,1)$, in which case the above summation vanishes. By defining $Y=1/X>1$ and $t_v: Y\mapsto\frac{6v(\log(Y)-\mathfrak{b}_v)}{\kappa(v)\pi^2\sqrt{Y}}$, we need to find $\sup_{\{Y>1\}}|t_v(Y)|$. By calculus, the function t_v has a critical point at $y_0=e^{2+\mathfrak{b}_v}$, with value $t_v(y_0)=\frac{12v}{\kappa(v)\pi^2e^{1+\mathfrak{b}_v/2}}$, and it is monotonic in $[1,y_0]$ and in $[y_0,\infty)$. As $\lim_{Y\to\infty}t_v(Y)=0$ and $t_v(y_0)>0$, we conclude that t_v is decreasing in $[y_0,\infty)$. Similarly, as $t_v(1) = -\frac{6v\mathfrak{b}_v}{\kappa(v)\pi^2} < 0$, t_v is increasing in $[1, y_0]$. Therefore

$$\sup_{\{0<X<1\}} \sqrt{X}\left|\sum_{\substack{\ell\leq X\\(\ell,v)=1}}\frac{\mu^2(\ell)}{\ell} - \frac{v}{\kappa(v)}\frac{6}{\pi^2}(\log(X)+\mathfrak{b}_v)\right| \tag{4.8}$$

$$= \max\{|t_v(1)|, |t_v(y_0)|\} = \frac{6v\mathfrak{b}_v}{\kappa(v)\pi^2} = \begin{cases} 1.044 & \text{if } v=1,\\ 0.79 & \text{if } v=2.\end{cases}$$

Finally, when either $v = 1$ or $v = 2$, the constant in the error term is obtained by taking the maximum between the bounds (4.7) and (4.8). ∎

LEMMA 4.2. *Let $X > 0$ and $\alpha > 1/2$. If $\alpha \neq 1$, then*

$$\sum_{\ell\leq X}\frac{\mu^2(\ell)}{\ell^\alpha} = \frac{\zeta(\alpha)}{\zeta(2\alpha)} - \frac{6}{(\alpha-1)\pi^2}\frac{1}{X^{\alpha-1}} + O^*\left(\frac{\mathrm{E}_\alpha^{(1)}}{X^{\alpha-1/2}}\right),$$

$$\sum_{\substack{\ell\leq X\\(\ell,2)=1}}\frac{\mu^2(\ell)}{\ell^\alpha} = \frac{2^\alpha}{(2^\alpha+1)}\frac{\zeta(\alpha)}{\zeta(2\alpha)} - \frac{4}{(\alpha-1)\pi^2}\frac{1}{X^{\alpha-1}} + O^*\left(\frac{\sqrt{2}}{\varphi_{1/2}(2)}\frac{\mathrm{E}_\alpha^{(2)}}{X^{\alpha-1/2}}\right),$$

where, for $v \in \{1, 2\}$, we have

$$\mathrm{E}_\alpha^{(v)} = \max\left\{\mathrm{D}_v\left(1+\frac{|\alpha-1|}{\alpha-1/2}\right), \frac{\varphi_{1/2}(v)}{\sqrt{v}}\left|\frac{v^\alpha}{\kappa_\alpha(v)}\frac{\zeta(\alpha)}{\zeta(2\alpha)} - \frac{v}{\kappa(v)}\frac{6}{(\alpha-1)\pi^2}\right|,\right.$$
$$\left.\frac{\varphi_{1/2}(v)}{\sqrt{v}}\frac{|\alpha-1|}{\alpha-1/2}\left(\frac{3\kappa_\alpha(v)\zeta(2\alpha)}{(\alpha-1/2)v^{\alpha-1}\kappa(v)\pi^2|\zeta(\alpha)(\alpha-1)|}\right)^{\frac{2}{\alpha-1}}\right\}$$

and $\mathrm{D}_1 = 0.43$, $\mathrm{D}_2 = 0.12$.

If $X \geq 1$, and $\alpha \neq 1$ then we can replace $\mathrm{E}_\alpha^{(v)}$ by $\mathrm{D}_v\big(1+\frac{|\alpha-1|}{\alpha-1/2}\big)$.

Proof. If $X \geq 1$, by summation by parts we can write $\sum_{\ell\leq X,\,(\ell,v)=1}\frac{\mu^2(\ell)}{\ell^\alpha}$ in the form

$$\Bigg(\sum_{\substack{\ell\leq X\\(\ell,v)=1}}\frac{\mu^2(\ell)}{\ell} - \frac{v}{\kappa(v)}\frac{6(\log(X)+\mathfrak{b}_v)}{\pi^2}\Bigg)\frac{1}{X^{\alpha-1}} - \frac{v}{\kappa(v)}\frac{6}{(\alpha-1)\pi^2}\frac{1}{X^{\alpha-1}} \tag{4.9}$$
$$+\frac{v}{\kappa(v)}\frac{6(\mathfrak{b}_v(\alpha-1)+1)}{\pi^2(\alpha-1)} + (\alpha-1)\int_1^X\Bigg(\sum_{\substack{\ell\leq t\\(\ell,v)=1}}\frac{\mu^2(\ell)}{\ell} - \frac{v}{\kappa(v)}\frac{6(\log(t)+\mathfrak{b}_v)}{\pi^2}\Bigg)\frac{dt}{t^\alpha}.$$

By Theorem 3.3, when $\alpha > 1/2$, the main term in the asymptotic expression of the above summation is

$$\frac{v^\alpha}{\kappa_\alpha(v)}\frac{\zeta(\alpha)}{\zeta(2\alpha)} - \frac{v}{\kappa(v)}\frac{6}{(\alpha-1)\pi^2}\frac{1}{X^{\alpha-1}}.$$

By using Lemma 4.1 and by making $X \to \infty$, we conclude from (4.9) that

$$\frac{v}{\kappa(v)}\frac{6(\mathfrak{b}(\alpha-1)+1)}{\pi^2(\alpha-1)}+(\alpha-1)\int_1^\infty\Bigg(\sum_{\substack{\ell\leq t\\(\ell,v)=1}}\frac{\mu^2(\ell)}{\ell}-\frac{6}{\pi^2}(\log(t)+\mathfrak{b}_v)\Bigg)\frac{dt}{t^\alpha}$$
$$=\frac{v^\alpha}{\kappa_\alpha(v)}\frac{\zeta(\alpha)}{\zeta(2\alpha)}.$$

Further, by (4.7), we conclude that, for all $X\geq 1$, $\sum_{\ell\leq X,\,(\ell,v)=1}\frac{\mu^2(\ell)}{\ell^\alpha}$ is equal to

$$\frac{v^\alpha}{\kappa_\alpha(v)}\frac{\zeta(\alpha)}{\zeta(2\alpha)}-\frac{v}{\kappa(v)}\frac{6}{(\alpha-1)\pi^2}\frac{1}{X^{\alpha-1}}$$
$$+O^*\Bigg(\frac{\sqrt{v}\,\mathrm{D}_v}{\varphi_{1/2}(v)}\Bigg(1+\frac{|\alpha-1|}{\alpha-1/2}\Bigg)\frac{1}{X^{\alpha-1/2}}\Bigg),$$

where $\mathrm{D}_1=0.43$ and $\frac{\varphi_{1/2}(2)}{\sqrt{2}}0.407\leq\mathrm{D}_2=0.12$.

Suppose now that $X\in(0,1)$. Define

$$g:0<X\mapsto\frac{v^{\alpha-1}\kappa(v)\pi^2\zeta(\alpha)(\alpha-1)}{6\kappa_\alpha(v)\zeta(2\alpha)}X^{\alpha-1/2}-\sqrt{X}.$$

By [3, Cor. 1.14] we have $1<\zeta(\alpha)(\alpha-1)<\alpha$. If $\alpha>1$, we infer that $\frac{\zeta(\alpha)(\alpha-1)}{\zeta(2\alpha)}>\frac{1}{\zeta(2)}$. As $\frac{v^{\alpha-1}\kappa(v)}{\kappa_\alpha(v)}=\frac{1+1/v}{1+1/v^\alpha}>1$ we conclude that $g(1)>0$ and g has a critical point x_0 satisfying

$$0<x_0^{\alpha-1}=\frac{3\kappa_\alpha(v)\zeta(2\alpha)}{(\alpha-1/2)v^{\alpha-1}\kappa(v)\pi^2|\zeta(\alpha)(\alpha-1)|}<1,$$

with value $g(x_0)=\frac{1-\alpha}{\alpha-1/2}\sqrt{x_0}<0$. As $g(0)=0$, we conclude that if $\alpha>1$, then $\sup_{\{0<X<1\}}|g(X)|=\max\{g(1),|g(x_0)|\}$.

On the other hand, if $1/2<\alpha<1$, then $2\alpha-1<1$, $\zeta(\alpha)(\alpha-1)<\alpha<1<\zeta(2\alpha)/\zeta(2)$ and $v^{\alpha-1}\kappa(v)/\kappa_\alpha(v)<1$, so that $g(1)<0$. Moreover, the critical point x_0 of g satisfies $x_0^{1-\alpha}<1$, so that $x_0<1$, and $g(x_0)>0$. Therefore, if $1/2<\alpha<1$, then $\sup_{\{0<X<1\}}|g(X)|=\max\{|g(1)|,g(x_0)\}$.

All in all, we derive

$$\sup_{\{0<X<1\}}X^{\alpha-1/2}\Bigg|\sum_{\substack{\ell\leq X\\(\ell,v)=1}}\frac{\mu^2(\ell)}{\ell^\alpha}-\frac{v^\alpha}{\kappa_\alpha(v)}\frac{\zeta(\alpha)}{\zeta(2\alpha)}+\frac{v}{\kappa(v)}\frac{6}{(\alpha-1)\pi^2}\frac{1}{X^{\alpha-1}}\Bigg|$$
$$=\frac{v}{\kappa(v)}\frac{6}{|\alpha-1|\pi^2}\max\{|g(1)|,|g(x_0)|\},\tag{4.10}$$

where

$$\frac{\varphi_{1/2}(v)}{\sqrt{v}}\,\frac{v}{\kappa(v)}\,\frac{6|g(1)|}{|\alpha-1|\pi^2}=\frac{\varphi_{1/2}(v)}{\sqrt{v}}\left|\frac{v^\alpha}{\kappa_\alpha(v)}\,\frac{\zeta(\alpha)}{\zeta(2\alpha)}-\frac{v}{\kappa(v)}\,\frac{6}{(\alpha-1)\pi^2}\right|,$$

$$\frac{\varphi_{1/2}(v)}{\sqrt{v}}\,\frac{v}{\kappa(v)}\,\frac{6|g(x_0)|}{|\alpha-1|\pi^2}$$

$$=\frac{\varphi_{1/2}(v)}{\sqrt{v}}\,\frac{|\alpha-1|}{\alpha-1/2}\left(\frac{3\kappa_\alpha(v)\zeta(2\alpha)}{(\alpha-1/2)v^{\alpha-1}\kappa(v)\pi^2|\zeta(\alpha)(\alpha-1)|}\right)^{\frac{2}{\alpha-1}}.$$

The result is obtained by defining $\mathrm{E}_\alpha^{(v)}$, $v\in\{1,2\}$, as the maximum between $\mathrm{D}_v\big(1+\frac{|\alpha-1|}{\alpha-1/2}\big)$ and the expression (4.10). ∎

Remark 4.3. In order to obtain a similar error shape in Lemma 4.1 to the one in Lemma 4.2, we extend the definition of $\mathrm{E}_\alpha^{(v)}$, $v\in\{1,2\}$, for $\alpha>1/2$, $\alpha\neq 1$, to the case $\alpha=1$ by defining $\mathrm{E}_1^{(1)}=1.044$ and, upon observing that $\frac{\varphi_{1/2}(2)}{\sqrt{2}}0.79\leq 0.232$, defining $\mathrm{E}_1^{(2)}=0.232$.

Lemma 4.4. *Let $X>0$ and $q\in\mathbb{Z}_{>0}$. Then $\sum_{\ell\leq X,\,(\ell,q)}\frac{\mu^2(\ell)}{\ell}$ equals*

$$\frac{q}{\kappa(q)}\,\frac{6}{\pi^2}(\log(X)+\mathfrak{b}_q)+O^*\left(\frac{\sqrt{q}}{\varphi_{1/2}(q)}\,\frac{\mathrm{E}_1^{(1)}\prod_{2|q}\frac{\mathrm{E}_1^{(2)}}{\mathrm{E}_1^{(1)}}}{\sqrt{X}}\right),$$

where $\mathfrak{b}_q$ is defined in Lemma 3.4; if $\alpha>1/2$, $\alpha\neq 1$, then $\sum_{\ell\leq X,\,(\ell,q)=1}\frac{\mu^2(\ell)}{\ell^\alpha}$ equals

$$\frac{q^\alpha}{\kappa_\alpha(q)}\,\frac{\zeta(\alpha)}{\zeta(2\alpha)}-\frac{q}{\kappa(q)}\,\frac{6}{(\alpha-1)\pi^2}\,\frac{1}{X^{\alpha-1}}+O^*\left(\frac{\sqrt{q}}{\varphi_{1/2}(q)}\,\frac{\mathrm{E}_\alpha^{(1)}\prod_{2|q}\frac{\mathrm{E}_\alpha^{(2)}}{\mathrm{E}_\alpha^{(1)}}}{X^{\alpha-1/2}}\right),$$

where $\mathrm{E}_\alpha^{(v)}$, $v\in\{1,2\}$, is defined as in Lemma 4.2.

Proof. Proceed as in [3, Lemma 2.17]. Define $\mathcal{D}_r=\{p \text{ prime}: p\,|\,d\Rightarrow p\,|\,r\}\subset\mathbb{Z}_{\geq 0}$. Consider $v\in\{1,2\}$ and write $q=v^k r$, $k\in\mathbb{Z}_{>0}$, with $(v,r)=1$ (where $k=0$ if $v=1$). Then for all $s\in\mathbb{C}$ with $\Re(s)>1-\alpha$, we have the identity

$$\sum_{\substack{\ell\\(\ell,q)=1}}\frac{\mu^2(\ell)}{\ell^{s+\alpha}}=\prod_{p|r}\left(1+\frac{1}{p^{s+\alpha}}\right)^{-1}\cdot\sum_{\substack{\ell\\(\ell,v)=1}}\frac{\mu^2(\ell)}{\ell^{s+\alpha}}=\sum_{\substack{d\\d\in\mathcal{D}_r}}\frac{\lambda(d)}{d^{s+\alpha}}\cdot\sum_{\substack{e\\(e,v)=1}}\frac{\mu^2(e)}{e^{s+\alpha}},$$

where λ corresponds to the Liouville function, the completely multiplicative function taking the value -1 at every prime number. Hence

$$\sum_{\substack{\ell\leq X\\(\ell,q)=1}}\frac{\mu^2(\ell)}{\ell^\alpha}=\sum_{\substack{d\\d\in\mathcal{D}_r}}\frac{\lambda(d)}{d^\alpha}\sum_{\substack{e\leq X/d\\(e,v)=1}}\frac{\mu^2(e)}{e^\alpha},\tag{4.11}$$

which, as in Lemma 3.2, does not require the condition $\{d \leq X\}$. We are thus considering an infinite range of values of d for the above outer sum, which can be estimated as long as the inner sum is expressed asymptotically with an error term valid even when it is empty plus the fact that the series of error terms for this expression, formed by the outer sum, converges.

If $\alpha = 1$, by using Lemma 4.1 in (4.11) we derive the same main term as the one given in Corollary 3.4(**b**), but with a better error term magnitude, since $\sum_{\ell \leq X,\,(\ell,q)=1} \frac{\mu^2(\ell)}{\ell}$ can be written as

$$\sum_{\substack{d\\ d\in\mathcal{D}_r}} \frac{\lambda(d)}{d}\left(\frac{6}{\pi^2}\,\frac{v}{\kappa(v)}\left(\log\left(\frac{X}{d}\right)+\mathfrak{b}_v\right)+O^*\left(\frac{\sqrt{v}}{\varphi_{1/2}(v)}\,\frac{\mathrm{E}_1^{(v)}\sqrt{d}}{\sqrt{X}}\right)\right)$$

$$= \frac{vr}{\kappa(vr)}\,\frac{6}{\pi^2}\,(\log(X)+\mathfrak{b}_v) - \frac{v}{\kappa(v)}\,\frac{6}{\pi^2}\sum_{\substack{d\\ d\in\mathcal{D}_r}}\frac{\lambda(d)\log(d)}{d}$$

$$+\,O^*\left(\frac{\sqrt{v}}{\varphi_{1/2}(v)}\sum_{\substack{d\\ d\in\mathcal{D}_r}}\frac{\mathrm{E}_1^{(v)}}{\sqrt{d}}\cdot\frac{1}{\sqrt{X}}\right)$$

$$= \frac{q}{\kappa(q)}\,\frac{6}{\pi^2}(\log(X)+\mathfrak{b}_q)+O^*\left(\frac{\sqrt{q}}{\varphi_{1/2}(q)}\,\frac{\mathrm{E}_1^{(1)}\prod_{2|q}\frac{\mathrm{E}_1^{(2)}}{\mathrm{E}_1^{(1)}}}{\sqrt{X}}\right),$$

where we have used

$$\sum_{\substack{d\\ d\in\mathcal{D}_r}}\frac{-\lambda(d)\log(d)}{d} = \frac{r}{\kappa(r)}\left(\sum_{\substack{d\\ d\in\mathcal{D}_r}}\frac{\lambda(d)}{d^s}\right)^{-1}_{s=1}\cdot\left(\sum_{\substack{d\\ d\in\mathcal{D}_r}}\frac{\lambda(d)}{d^s}\right)'_{s=1}$$

$$= \frac{r}{\kappa(r)}\sum_{p|r}\left[\left(\left(1+\frac{1}{p^s}\right)^{-1}\right)'\left(1+\frac{1}{p^s}\right)\right]_{s=1} = \frac{r}{\kappa(r)}\sum_{p|r}\frac{\log(p)}{p+1},$$

and

$$\frac{vr}{\kappa(vr)}=\frac{q}{\kappa(q)},\quad \frac{\sqrt{vr}}{\varphi_{1/2}(vr)}=\frac{\sqrt{q}}{\varphi_{1/2}(q)},\quad \sum_{p|v}\frac{\log(p)}{p+1}+\sum_{p|r}\frac{\log(p)}{p+1}=\sum_{p|q}\frac{\log(p)}{p+1}.$$

Finally, if $\alpha \neq 1$, then by applying Lemma 4.2 in (4.11) and by noticing that $\frac{(vr)^\alpha}{\kappa_\alpha(vr)} = \frac{q^\alpha}{\kappa_\alpha(q)}$, we find that $\sum_{\ell\leq X,\,(\ell,q)=1}\frac{\mu^2(\ell)}{\ell^\alpha}$ can be expressed as

$$\sum_{\substack{d\\ d\in\mathcal{D}_r}}\frac{\lambda(d)}{d^\alpha}\left(\frac{v^\alpha}{\kappa_\alpha(v)}\,\frac{\zeta(\alpha)}{\zeta(2\alpha)}-\frac{v}{\kappa(v)}\,\frac{6}{(\alpha-1)\pi^2}\,\frac{d^{\alpha-1}}{X^{\alpha-1}}+O^*\left(\frac{\sqrt{v}}{\varphi_{1/2}(v)}\,\frac{\mathrm{E}_\alpha^{(v)}d^{\alpha-1/2}}{X^{\alpha-1/2}}\right)\right)$$

$$= \frac{q^\alpha}{\kappa_\alpha(q)}\,\frac{\zeta(\alpha)}{\zeta(2\alpha)} - \frac{q}{\kappa(q)}\,\frac{6}{(\alpha-1)\pi^2}\,\frac{1}{X^{\alpha-1}}+O^*\left(\frac{\sqrt{q}}{\varphi_{1/2}(q)}\,\frac{\mathrm{E}_\alpha^{(1)}\prod_{2|q}\frac{\mathrm{E}_\alpha^{(2)}}{\mathrm{E}_\alpha^{(1)}}}{X^{\alpha-1/2}}\right),$$

which again has the expected main term according to Theorem 3.3 but an error term of lower order. ■

Let us recall that the requirement of empty sum estimation, as in Lemma 3.2, worsens a bit the error term constants with respect to the ones under the condition $X \geq 1$, say, as shown in Lemmas 4.1 and 4.2, but we gain regularity in our expressions in the variable d. It is precisely that regularity that allows us to derive the coprimality restrictions products in a simpler manner: for example, we deduce immediately that $\sum_{d\in\mathcal{D}_r}\lambda(d)/d = r/\kappa(r)$, whereas the condition $X/d \geq 1$ would require analyzing $\sum_{d\leq X,\, d\in\mathcal{D}_r}\lambda(d)/d$ or, rather, $\sum_{d>X,\, d\in\mathcal{D}_r}\lambda(d)/d$. This last observation is a key difference from the work carried out in [8].

Corollary 4.5. *Let $X > 0$. Then*

$$\sum_{\substack{\ell>X\\(\ell,q)=1}}\frac{\mu^2(\ell)}{\ell^2} = \frac{q}{\kappa(q)}\frac{6}{\pi^2}\frac{1}{X} + O^*\left(\frac{\sqrt{q}}{\varphi_{1/2}(q)}\frac{0.912}{X^{3/2}}\right) \quad \text{if } 2\nmid q,$$
$$= \frac{q}{\kappa(q)}\frac{6}{\pi^2}\frac{1}{X} + O^*\left(\frac{\sqrt{q}}{\varphi_{1/2}(q)}\frac{0.238}{X^{3/2}}\right) \quad \text{if } 2\,|\,q.$$

Proof. By applying Lemma 4.4 with $\alpha = 2$, we have

$$\sum_{\substack{\ell\leq X\\(\ell,q)=1}}\frac{\mu^2(\ell)}{\ell^2} = \frac{q^2}{\kappa_2(q)}\frac{\zeta(2)}{\zeta(4)} - \frac{q}{\kappa(q)}\frac{6}{\pi^2}\frac{1}{X} + O^*\left(\frac{\sqrt{q}}{\varphi_{1/2}(q)}\frac{\mathrm{E}_2^{(1)}\prod_{2|q}\frac{\mathrm{E}_2^{(2)}}{\mathrm{E}_2^{(1)}}}{X^{3/2}}\right),$$

where, for $v \in \{1,2\}$, $\mathrm{E}_2^{(v)}$ is defined as

$$\max\left\{\frac{5\mathrm{D}_v}{3}, \frac{\varphi_{1/2}(v)}{\sqrt{v}}\left|\frac{v^2}{\kappa_2(v)}\frac{\zeta(2)}{\zeta(4)} - \frac{v}{\kappa(v)}\frac{6}{\pi^2}\right|, \frac{\varphi_{1/2}(v)}{\sqrt{v}}\frac{2}{3}\left(\frac{2\kappa_2(v)\zeta(4)}{v\kappa(v)\pi^2\zeta(2)}\right)^2\right\}$$
$$\leq \begin{cases} 0.912 & \text{if } v = 1,\\ 0.238 & \text{if } v = 2.\end{cases}$$

We obtain the result by observing that

$$\sum_{\substack{\ell>X\\(\ell,q)=1}}\frac{\mu^2(\ell)}{\ell^2} = \frac{q^2}{\kappa_2(q)}\frac{\zeta(2)}{\zeta(4)} - \sum_{\substack{\ell\leq X\\(\ell,q)=1}}\frac{\mu^2(\ell)}{\ell^2}. \;■$$

4.2. Achieving the critical exponent. We present a new method to achieve the critical exponent for estimations of averages of the form studied in Theorem 3.3 provided that the difference between β and α defined therein is strictly bigger than 1/2: in this case, we go to the edge of the special form of the convolution method given in §3.2; moreover, no extra conditions on β

are needed but $\beta-\alpha>1/2$. Nonetheless, if $\beta-\alpha\leq 1/2$, then we should still refer to Theorem 3.3 and its choice of parameter (or indirectly to it, as shown by summation by parts in Theorem 4.6(**B**, **C**).

THEOREM 4.6. *Let $X>0$ be a real number and q a positive integer. Consider a multiplicative function $f:\mathbb{Z}^+\to\mathbb{C}$ such that for every sufficiently large prime number p satisfying $(p,q)=1$, we have $f(p)=1/p^{\alpha}+O(1/p^{\beta})$, where α, β are real numbers satisfying $\beta>\alpha$ and $\beta-\alpha>1/2$.*

(**A**) *If $\alpha>1/2$ then*

$$S_f(X):=\sum_{\substack{\ell\leq X\\ (\ell,q)=1}}\mu^2(\ell)f(\ell)=F_\alpha^q(X)+O^*\left(\mathrm{p}_\alpha(q)\cdot\frac{\mathrm{w}_\alpha^q\mathrm{P}_\alpha}{X^{\alpha-1/2}}\right),$$

where $F_\alpha^q(X)$ is defined as in Theorem 3.3, and if $2\,|\,q$, then $\mathrm{w}_\alpha^q=\mathrm{E}_\alpha^{(2)}$, whereas if $2\nmid q$ then

$$\mathrm{w}_\alpha^q=\left(\frac{\sqrt{2}-1}{\sqrt{2}-1+|2^\alpha f(2)-1|}\right)\left(\mathrm{E}_\alpha^{(1)}+\frac{|2^\alpha f(2)-1|\mathrm{E}_\alpha^{(2)}}{\varphi_{1/2}(2)}\right).$$

Here $\mathrm{E}_\alpha^{(v)}$, $v\in\{1,2\}$, is defined in Lemma 4.2 and Remark 4.3, and we have

$$\mathrm{p}_\alpha(q)=\prod_{p|q}\left(1+\frac{1-|f(p)p^\alpha-1|}{\sqrt{p}-1+|f(p)p^\alpha-1|}\right),\qquad \mathrm{P}_\alpha=\prod_{p}\left(1+\frac{|f(p)p^\alpha-1|}{\sqrt{p}-1}\right),$$

for all α, where P_α is a convergent infinite product.

(**B**) *If $\alpha<1/2$ then $S_f(X)$ can be expressed as*

$$\frac{H_{f'}^q(0)\varphi(q)}{(1-\alpha)q}X^{1-\alpha}+O^*\left(\mathrm{p}_\alpha(q)\cdot\left(1+\frac{2-2\alpha}{1-2\alpha}\right)\mathrm{w'}_\alpha^q\mathrm{P}_\alpha X^{1/2-\alpha}\right),$$

where $\mathrm{p}_\alpha(q)$ and P_α are as in (**A**) *and for $\alpha\leq 1/2$,*

$$H_{f'}^q(0)=\prod_{p\nmid q}\left(1-\frac{p^{1-\alpha}-f(p)p+f(p)}{p^{2-\alpha}}\right),$$

$$\mathrm{w'}_\alpha^q=\begin{cases}\mathrm{E}_1^{(2)}=0.232 & \text{if } 2\,|\,q,\\ \left(\frac{\sqrt{2}-1}{\sqrt{2}-1+|2^\alpha f(2)-1|}\right)\left(\mathrm{E}_1^{(1)}+\frac{|2^\alpha f(2)-1|\mathrm{E}_1^{(2)}}{\varphi_{1/2}(2)}\right) & \text{if } 2\nmid q.\end{cases}$$

(**C**) *If $\alpha=1/2$ then $S_f(X)$ can be written as*

$$\frac{H_{f'}^q(0)\varphi(q)}{(1-\alpha)q}X^{1-\alpha}+O^*\big(\mathrm{C}+\mathrm{p}_\alpha(q)\mathrm{w'}_\alpha^q\mathrm{P}_\alpha\big(1+\tfrac{1}{2}\log(X)\big)\big),$$

where $\mathrm{p}_\alpha(q)$ *and* P_α *are as in* (**A**), $H^q_{f'}(0)$ *and* $\mathrm{w'}^q_\alpha$ *are as in* (**B**) *and*

$$\mathrm{C}=\left|\frac{H^q_{f'}(0)\varphi(q)}{q}\left(\sum_{p\nmid q}\frac{\log(p)(\sqrt{p}-(p-2)f(p))}{(f(p)+\sqrt{p})(p-1)}+\gamma+\sum_{p|q}\frac{\log(p)}{p-1}-2\right)\right|.$$

Proof. Let us derive (**A**). Consider the arithmetic function i_f defined on each prime as $p\mapsto f(p)p^\alpha-1$. Observe that

$$S_f(X)=\sum_{\substack{\ell\leq X\\(\ell,q)=1}}\frac{\mu^2(\ell)}{\ell^\alpha}\cdot f(\ell)\ell^\alpha=\sum_{\substack{\ell\leq X\\(\ell,q)=1}}\frac{\mu^2(\ell)}{\ell^\alpha}\cdot\prod_{p|\ell}(1+i_f(p))$$
$$=\sum_{\substack{\ell\leq X\\(\ell,q)=1}}\frac{\mu^2(\ell)}{\ell^\alpha}\sum_{d|\ell}\mu^2(d)i_f(d)=\sum_{\substack{d\\(d,q)=1}}\frac{\mu^2(d)i_f(d)}{d^\alpha}\sum_{\substack{e\leq X/d\\(e,qd)=1}}\frac{\mu^2(e)}{e^\alpha},\tag{4.12}$$

where we have not imposed upper bounds on d.

In order to continue our estimation, we must be able to estimate the innermost summation in the right hand side of (4.12) regardless of whether or not it is empty, so that their remainder terms converge upon effecting the corresponding outer summation. As $\alpha>1/2$, this situation can be treated with the help of Lemma 4.4; we distinguish two cases.

(**i**) $2\,|\,q$. Then continuing from (4.12), along with the ideas of the proof of Theorem 3.3 and Lemma 4.4, it is not difficult to see, as expected, that for all $\alpha>1/2$, the main term of $S_f(X)$ is $F^q_\alpha(X)$. As for the error term, it corresponds to

$$\sum_{\substack{d\\(d,q)=1}}\frac{\mu^2(d)|i_f(d)|}{d^\alpha}O^*\left(\frac{\sqrt{qd}}{\varphi_{1/2}(qd)}\frac{\mathrm{E}^{(2)}_\alpha\ d^{\alpha-1/2}}{X^{\alpha-1/2}}\right)$$
$$=O^*\left(\frac{\sqrt{q}}{\varphi_{1/2}(q)}\prod_{p\nmid q}\left(1+\frac{|i_f(p)|}{\sqrt{p}-1}\right)\cdot\frac{\mathrm{E}^{(2)}_\alpha}{X^{\alpha-1/2}}\right),$$

where, for any $\alpha>1/2$, $\frac{\sqrt{q}}{\varphi_{1/2}(q)}\prod_{p\nmid q}(1+\frac{|i_f(p)|}{\sqrt{p}-1})$ may be expressed as

$$\frac{\sqrt{q}}{\varphi_{1/2}(q)}\prod_{p|q}\left(1+\frac{|f(p)p^\alpha-1|}{\sqrt{p}-1}\right)^{-1}\cdot\mathrm{P}_\alpha=\mathrm{p}_\alpha(q)\cdot\mathrm{P}_\alpha,$$

where $\mathrm{p}_\alpha(q)$ and P_α are defined in the statement. Observe that P_α converges, because $\frac{|i_f(p)|}{\sqrt{p}-1}=\frac{|f(p)p^\alpha-1|}{\sqrt{p}-1}=O\big(\frac{1}{p^{\beta-\alpha+1/2}}\big)$ and $\beta-\alpha+1/2>1$.

(**ii**) $2 \nmid q$. Then we can write (4.12) as

$$\sum_{\substack{d\\(d,2q)=1}} \frac{\mu^2(d)i_f(d)}{d^\alpha} \sum_{\substack{e\leq X/d\\(e,qd)=1}} \frac{\mu^2(e)}{e^\alpha} + \frac{i_f(2)}{2^\alpha} \sum_{\substack{d\\(d,2q)=1}} \frac{\mu^2(d)i_f(d)}{d^\alpha} \sum_{\substack{e\leq X/(2d)\\(e,2qd)=1}} \frac{\mu^2(e)}{e^\alpha}$$
$$=: S_\alpha^q(X) + \frac{i_f(2)}{2^\alpha} T_\alpha^q(X).$$

Again, it is not difficult to see that, for any $\alpha > 1/2$, the main term of $S_\alpha^q(X) + \frac{i_f(2)}{2^\alpha} T_\alpha^q(X)$ is $F_\alpha^q(X)$, defined in Theorem 3.3. On the other hand, the error term of $S_1^q(X) + \frac{i_f(2)}{2} T_1^q(X)$ can be expressed as

$$\sum_{\substack{d\\(d,2q)=1}} \frac{\mu^2(d)|i_f(d|)}{d^\alpha} O^*\left(\frac{\sqrt{qd}}{\varphi_{1/2}(qd)} \frac{\mathrm{E}_\alpha^{(1)} d^{\alpha-1/2}}{X^{\alpha-1/2}}\right)$$
$$+ \frac{|i_f(2)|}{2^\alpha} \sum_{\substack{d\\(d,2q)=1}} \frac{\mu^2(d)|i_f(d)|}{d^\alpha} O^*\left(\frac{\sqrt{2qd}}{\varphi_{1/2}(2qd)} \frac{\mathrm{E}_\alpha^{(2)}(2d)^{\alpha-1/2}}{X^{\alpha-1/2}}\right)$$
$$= O^*\left(\frac{\sqrt{q}}{\varphi_{1/2}(q)} \prod_{p\nmid 2q}\left(1 + \frac{|i_f(p)|}{\sqrt{p}-1}\right)\left(\mathrm{E}_\alpha^{(1)} + \frac{|i_f(2)|\mathrm{E}_\alpha^{(2)}}{\varphi_{1/2}(2)}\right) \cdot \frac{1}{X^{\alpha-1/2}}\right)$$
$$= O^*\left(\mathrm{p}_\alpha(q)\left(\frac{\sqrt{2}-1}{\sqrt{2}-1+|2^\alpha f(2)-1|}\right)\left(\mathrm{E}_\alpha^{(1)} + \frac{|2^\alpha f(2)-1|\mathrm{E}_\alpha^{(2)}}{\varphi_{1/2}(2)}\right) \cdot \frac{\mathrm{P}_\alpha}{X^{\alpha-1/2}}\right),$$

whence the first case.

The condition $\alpha > 1/2$ in case (**A**) is necessary, as we have used Lemma 4.4. Nonetheless, we can readily derive an analogous result for cases (**B**) and (**C**). Indeed, we can write $f(p) = p^{1-\alpha} f'(p)$, where the sum $A(t) = \sum_{\ell\leq t,\,(\ell,q)=1} \mu^2(\ell) f'(\ell)$ can be estimated as in case (**A**) with $\alpha' = 1$, $\beta' = 1-\alpha+\beta$. We can then estimate $S_f(X) = \sum_{\ell\leq X,\,(\ell,q)=1} \mu^2(\ell) f'(\ell)\ell^{1-\alpha}$ by summation by parts, obtaining the result. ∎

Note that the error term improvement from Theorem 3.3, when $\alpha = 1/2$ and under the conditions of Theorem 4.6, is of logarithmic nature with respect to $O(X^{1/2-\delta})$ for any $\delta \in (0, 1/2)$.

Concerning the error term in Theorem 4.6, in some particular cases one can do much better in terms of error constants. For instance, it is known by [2, Lemmas 5.1, 5.2] that if $f(p) = 1$ and $v \in \{1, 2\}$, then for any $X > 0$,

$$\sum_{\substack{\ell\leq X\\(\ell,v)=1}} \mu^2(\ell) = \frac{6}{\pi^2} \frac{v}{\kappa(v)} X + O^*(\mathrm{H}_v \sqrt{X}), \tag{4.13}$$

where

$$\mathrm{H}_v = \begin{cases} \sqrt{3}(1-6/\pi^2) & \text{if } v=1,\\ 1-4/\pi^2 & \text{if } v=2,\end{cases} \tag{4.14}$$

whereas Theorem 4.6 provides only an explicit error term of the form $O^*\big(\frac{\sqrt{q}}{\varphi_{1/2}(q)}\cdot 3.132\sqrt{X}\big)$.

4.3. Consequences

LEMMA 4.7. *Let $X>0$. Then the sum $\sum_{\ell\le X,\,(\ell,q)=1}\mu^2(\ell)/\varphi(\ell)$ may be estimated as*

(4.15)
$$\frac{\varphi(q)}{q}(\log(X)+\mathfrak{a}_q)+O^*\left(\prod_{p|q}\left(1+\frac{p-2}{p^{3/2}-p-\sqrt{p}+2}\right)\cdot\frac{4.4\prod_{2|q}0.493}{\sqrt{X}}\right),$$

where $\mathfrak{a}_q$ is defined in Corollary 3.4.

Proof. We already know the main term of the asymptotic expression of the above sum, thanks to Corollary 3.4(**a**); obtaining it again from the proof of Theorem 4.6 is an exercise. On the other hand, by Theorem 4.6 with $f(p)=\frac{1}{p-1}$, $\alpha=1$, $\beta=2$, its error term can be expressed as $O^*\big(\mathrm{p}(q)\cdot\frac{\mathrm{w}^q\mathrm{P}}{\sqrt{X}}\big)$, where

$$\mathrm{p}(q)=\prod_{p|q}\left(1+\frac{p-2}{p^{3/2}-p-\sqrt{p}+2}\right),$$

$$\mathrm{P}=\prod_{p}\left(1+\frac{1}{(p-1)(\sqrt{p}-1)}\right)\in[9.37522, 9.3753],$$

$$\mathrm{w}^q=\begin{cases}0.231 & \text{if } 2\,|\,q,\\ \big(1-\frac{1}{\sqrt{2}}\big)\big(\mathrm{E}_1^{(1)}+\frac{\mathrm{E}_1^{(2)}}{\varphi_{1/2}(2)}\big)=0.469\ldots & \text{if } 2\nmid q\end{cases}\le 0.47\prod_{2|q}0.493,$$

and where $\mathrm{E}_1^{(v)}$, $v\in\{1,2\}$, are defined in §4.1. ∎

When there are no coprimality conditions, we have obtained an error constant equal to 4.4, under the condition $X>0$. Ramaré and Akhilesh [8, Thm. 1.2] have given the constant 3.95 under the condition $X\ge 1$, later improved by Ramaré himself [7] to 2.44 under the condition $X>1$. From these last two bounds, it is not difficult to extend the range of estimation to $X>0$, as we have done for example throughout Lemma 3.2, and these bounds continue to be better than the value 4.4.

However, the above lemma considerably improves [8, Thm. 1.1] when coprimality conditions given by $q\ge 2$ are involved. For example, we have

$$
\begin{aligned}
2.169 \cdot \mathrm{p}(2) &\leq 2.169 \leq 4.955 \leq 5.9 \cdot \jmath(2), \\
4.4 \cdot \mathrm{p}(3) &\leq 6.186 \leq 7.221 \leq 5.9 \cdot \jmath(3), \\
4.4 \cdot \mathrm{p}(5) &\leq 6.621 \leq 7.679 \leq 5.9 \cdot \jmath(5), \\
2.169 \cdot \mathrm{p}(6) &\leq 3.049 \leq 6.066 \leq 5.9 \cdot \jmath(6), \\
2.169 \cdot \mathrm{p}(10) &\leq 3.263 \leq 6.451 \leq 5.9 \cdot \jmath(10), \\
2.169 \cdot \mathrm{p}(14) &\leq 3.166 \leq 6.424 \leq 5.9 \cdot \jmath(14),
\end{aligned} \tag{4.16}
$$

where $\jmath$ is the error term arithmetic function defined in [8, Thm. 1.1] as $2 \mapsto \frac{21}{25}$ and $3 \leq p \mapsto 1 + \frac{p-2}{p^{3/2}-\sqrt{p}+1}$. Furthermore, the estimation given in Lemma 4.7 is better than the one in [8, Thm. 1.1] for all $q = p$ prime. Indeed, we observe in (4.16) that it is better when $p \in \{2, 3, 5\}$; now, since

$$
\begin{aligned}
\frac{p-2}{p^{3/2}-p-\sqrt{p}+2} &< \frac{1}{\sqrt{p}} \qquad \text{for all } p \geq 3, \\
\frac{p-2}{p^{3/2}-\sqrt{p}+1} &> \frac{1}{2\sqrt{p}} \qquad \text{for all } p \geq 5,
\end{aligned}
$$

we have, for all $p \geq 3$,

$$
4.4 \cdot \mathrm{p}(p) \leq 4.4 \cdot \left(1 + \frac{1}{\sqrt{p}}\right) \leq 5.9 \cdot \left(1 + \frac{1}{2\sqrt{p}}\right) \leq 5.9 \cdot \jmath(p),
$$

whence the conclusion.

As a final remark, observe that that the main contribution to the product P in Lemma 4.7 is precisely when $p = 2$. This is the reason why, in the present work, we have distinguished between the cases when q is odd and when it is even. Further, as the second main contribution to the product P is given by its factor at $p = 3$ (the subsequent factors when $p > 3$ being rather small, as $\frac{1}{\sqrt{p}-1} < 1$), the interested reader may study the behavior of the error term bounds given in Theorem 4.6, and therefore the error term in Lemma 4.7, by distinguishing whether or not $(6, q) = 1$: this will require extending Lemma 4.1 to the cases $(3, q) = 1$ and, by using the inclusion-exclusion principle, to the case $(6, q) = 1$; afterwards, the analysis will continue exactly as in the current version of Theorem 4.6.

REFERENCES

[1] J. Büthe, *A Brun–Titchmarsh inequality for weighted sums over prime numbers*, Acta Arith. 166 (2014), 289–299.

[2] H. Helfgott, *The Ternary Goldbach Conjecture*, Ann. of Math. Stud., to appear; https://webusers.imj-rg.fr/~harald.helfgott/anglais/book.html (version 09/2019).

[3] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory: I. Classical Theory*, Cambridge Univ. Press, 2007.

[4] Y. Motohashi, *Primes in arithmetic progressions*, Invent. Math. 44 (1978), 163–178.

[5] D. J. Platt, *Numerical computations concerning the GRH*, Math. Comp. 85 (2016), 3009–3027.
[6] O. Ramaré, *On Šnirel'man's constant*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 22 (1995), 645–706.
[7] O. Ramaré, *Explicit average orders: news and problems*, in: Number Theory Week 2017, Banach Center Publ. 118, Inst. Math., Polish Acad. Sci., Warszawa, 2019, 153–176.
[8] O. Ramaré and P. Akhilesh, *Explicit averages of non-negative multiplicative functions: going beyond the main term*, Colloq. Math. 147 (2017), 275–313.
[9] D. R. Ward, *Some series involving Euler's function*, J. London Math. Soc. 2, October (1927), 210–214.

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