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Numerical range of the Foguel–Halmos operator

by

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Abstract. We study properties of the numerical range of the Foguel–Halmos operator $F_T = \left[\begin{smallmatrix} S^* & T \\ 0 & S \end{smallmatrix}\right]$ on $\ell^2 \oplus \ell^2$, where S is the simple unilateral shift and $T = \operatorname{diag}(a_1, a_2, \ldots)$ with $a_n = 1$ if $n = 3^k$ for some $k \geq 1$ and $a_n = 0$ otherwise. Among other things, we show that the numerical range $W(F_T)$ is neither open nor closed, and give lower and upper bounds for the numerical radius $w(F_T)$.

1. Introduction. A *Foguel operator* F_T is one of the form

$$\begin{bmatrix} S^* & T \\ 0 & S \end{bmatrix}, \tag{1.1}$$

where S is the (simple) unilateral shift $S(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$ on ℓ^2, and T is some other operator on ℓ^2. Such operators arise in the work of Foguel [3] in which he showed that there is a power-bounded operator which is not similar to a contraction, thus refuting a conjecture of Sz.-Nagy. Halmos [7] then modified Foguel's construction and introduced an operator of the form (1.1) with $T = \operatorname{diag}(a_1, a_2, \ldots)$, where $a_n = 1$ if $n = 3^k$ for some $k \geq 1$ and 0 otherwise. This we call the *Foguel–Halmos operator*. A generalized form of the Foguel operator also features prominently in Pisier's solution [11] to Halmos's polynomially bounded operator problem. More recently, Garcia [4] computed the norm of a Foguel operator and determined the spectrum of its modulus.

Note that the Foguel–Halmos operator F_T may be defined more generally as one with $T = \operatorname{diag}(a_1, a_2, \ldots)$ such that a_n is 1 or 0 depending on whether $n = n_k$, $k \geq 1$, or otherwise, where $\{n_k\}_{k=1}^{\infty}$ is lacunary (be it $n_{k+1}/n_k \geq r > 1$ for all k, $\underline{\lim}_k (n_{k+1}/n_k) > 1$, or $\lim_k (n_{k+1} - n_k) = \infty$). Here we restrict ourselves to the choice $n_k = 3^k$ for simplicity and definiteness.

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In this paper, we launch a study of the numerical range of the Foguel–Halmos operator F_T. Recall that the *numerical range* $W(A)$ of a (bounded linear) operator A on a complex Hilbert space H is the subset $\{\langle Ax, x\rangle : x \in H,\ \|x\| = 1\}$ of the complex plane, where $\langle\cdot,\cdot\rangle$ is the inner product in H and $\|\cdot\|$ its associated norm. One measure of the size of $W(A)$ is the *numerical radius* $w(A) = \sup\{|z| : z \in W(A)\}$ of A. Short of an exact description of $W(F_T)$, we derive as much information of it as possible. This includes its symmetry properties, topological properties, estimates of $w(F_T)$, and its generalized Crawford number.

Since properties of the numerical ranges of Foguel operators have been investigated in [5], we start in Section 2 below with a brief review of the relevant ones concerning the Foguel operators F_T with diagonal T. Then we move on to the building of a machinery useful for proving properties of $W(F_T)$ when T is a positive-semidefinite diagonal operator.

After such preparations, we formally study the numerical range of the Foguel–Halmos operator F_T in Section 3. We prove that $W(F_T)$ is neither open nor closed and $\overline{W(F_T)}$ is not an elliptic disc (Corollary 3.9), obtain upper and lower bounds for $w(F_T)$: $\sqrt{5}/2$ $(= 1.1180\ldots) < w(F_T) \le 1.1392\ldots$ (Theorem 3.11), and show that $W(F_T)$ contains the open circular disc $\{z \in \mathbb{C} : |z| < \sqrt{5}/2\}$ (Theorem 3.13). These are all based on a sequence of parameters $\{b_n\}_{n=1}^{\infty}$ of the matrix $T + S + S^*$ developed in Lemma 2.3 for the positive-semidefinite diagonal matrix T.

For an operator A, $\sigma(A)$ and $\sigma_e(A)$ denote its spectrum and essential spectrum, respectively, and $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ its *spectral radius.* We use $\operatorname{Re} A$ for its real part $(A + A^*)/2$. Further, A is *positive-semidefinite* (resp., *positive-definite*), denoted by $A \ge 0$ (resp., $A > 0$), if $\langle Ax, x\rangle \ge 0$ (resp., $\langle Ax, x\rangle > 0$) for all vectors x (resp., nonzero vectors x). If A and B are two Hermitian operators, then $A \le B$ means that $B - A \ge 0$. There is another partial ordering, not to be confused with the above one, for (finite or infinite) real matrices. A real matrix $A = [a_{ij}]_{i,j=1}^{n}$, $1 \le n \le \infty$, is *nonnegative*, denoted by $A \succcurlyeq 0$, if $a_{ij} \ge 0$ for all i and j. For two real matrices A and B, $A \preccurlyeq B$ means that $B - A \succcurlyeq 0$. For any (complex) matrix $A = [a_{ij}]_{i,j=1}^{n}$, $|A|$ denotes the nonnegative matrix $[|a_{ij}|]_{i,j=1}^{n}$. If A is any m-by-n matrix, then the n-by-m A^T is its *transpose*. The identity operator (resp., zero operator) is I (resp., 0) and the n-by-n identity matrix (resp., zero matrix) is I_n (resp., 0_n). We use ℓ^2 (resp., $\ell^2(\mathbb{Z})$) to denote the Hilbert space $\{(a_1, a_2, \ldots) : a_n \in \mathbb{C}$ for all n, $\sum_{n=1}^{\infty} |a_n|^2 < \infty\}$ (resp., $\{(\ldots, a_{-1}, a_0, a_1, \ldots) : a_n \in \mathbb{C}$ for all n, $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty\}$). For a set $\triangle$, $\triangle^*$ is its conjugate $\{\overline{z} : z \in \triangle\}$. We use $\mathbb{D}$ to denote the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the plane.

For properties of numerical ranges and numerical radii, the reader is referred to [8, Chapter 22] or [6]. For operator theory in general, see also [8]. The next proposition appeared as [5, Proposition 1.1], which we quote for easy reference.

PROPOSITION 1.1.

(a) *For any operator A, we have $w(A) = \sup\{\|\mathrm{Re}(\lambda A)\| : |\lambda| = 1\}$.*
(b) *If $A = [a_{ij}]_{i,j=1}^{\infty}$ and $A_n = [a_{ij}]_{i,j=1}^{n}$ for $1 \le n < \infty$, then $\overline{W(A)} = \overline{\bigcup_{n=1}^{\infty} W(A_n)}$ and $w(A_n)$ increases to $w(A)$.*
(c) *If A and B are (finite or infinite) matrices of the same size, $B \succcurlyeq 0$ and $|A| \preccurlyeq B$, then $w(A) \le w(B)$.*

2. Foguel operators with diagonal T. In this section, we build up a machinery which is useful for proving properties of $W(F_T)$ with T a positive-semidefinite diagonal operator.

For any $a \ge 2$, let $a_{\pm} = (a \pm \sqrt{a^2-4})/2$. It is easily seen that $1/a < a_- \le a_+ < a$, $1/a_{\pm} = a_{\mp}$, $a - a_{\pm} = a_{\mp}$, and $z^2 - az + 1 = (z-a_+)(z-a_-)$. The next lemma involves some basic inequalities, which will be used repeatedly in the discussions later on. Its easy proof we leave out.

LEMMA 2.1. *For any $a > 2$:*

(a) *if $a_+ < x < a$, then $x < 1/(a-x)$,*
(b) *if $a_- < x < a_+$, then $a_- < 1/(a-x) < x < a_+$,*
(c) *if $0 < x < a_-$, then $x < 1/(a-x) < a_-$, and*
(d) *if $x < a_+$, then $1/(a-x) < a_+$.*

The next proposition from [5, Lemma 3.1 and Proposition 3.2] provides a way to compute the numerical radius of F_T for diagonal T.

PROPOSITION 2.2. *Let $T = \mathrm{diag}(a_1, a_2, a_3, \ldots)$.*

(a) *If $|\lambda| = 1$, then λF_T and $\mathrm{Re}(\lambda F_T)$ are unitarily similar to $F_{T'}$ and $\frac{1}{2}\left[\begin{smallmatrix} 0 & C_\lambda \\ C_\lambda^* & 0 \end{smallmatrix}\right]$, respectively, where $T' = \mathrm{diag}(a_1, \lambda^2 a_2, \lambda^4 a_3, \ldots)$ and*

$$C_\lambda = \begin{bmatrix} \overline{a}_1 & 1 & & \\ 1 & \lambda^2 a_2 & 1 & \\ & 1 & \overline{\lambda}^4 \overline{a}_3 & \ddots \\ & & \ddots & \ddots \end{bmatrix}.$$

Hence $w(\mathrm{Re}(\lambda F_T)) = \|C_\lambda\|/2$ for $|\lambda| = 1$, $w(F_T) = \frac{1}{2}\sup\{\|C_\lambda\| : |\lambda| = 1\}$, and $W(F_T) = -W(F_T)$.

(b) *If the a_n's are all real, then $w(\mathrm{Re}\, F_T) = \|C\|/2$, where*

$$C = \begin{bmatrix} a_1 & 1 & & \\ 1 & a_2 & 1 & \\ & 1 & a_3 & \ddots \\ & & \ddots & \ddots \end{bmatrix}. \tag{2.1}$$

(c) *If the a_n's are all nonnegative, then $w(F_T) = w(\mathrm{Re}\, F_T) = \|C\|/2$.*

We now proceed to derive a sequence $\{b_n\}_{n=1}^{\infty}$ of parameters of the matrix C for the purpose of estimating the value of $w(C)$ $(= \|C\|)$.

LEMMA 2.3. *Let C be the matrix in* (2.1) *with $a_n \geq 0$ for all n. Then*

(a) $w(C) \geq 2$,

(b) *for any a satisfying $a \geq 2$, $w(C) \leq a$ if and only if the sequence $\{b_n\}_{n=1}^{\infty}$ defined by $b_1 = 1/(a - a_1)$ and $b_n = 1/(a - a_n - b_{n-1})$ for $n \geq 2$ is such that $b_n > 0$ for all n,*

(c) *under the conditions in* (b), *we have $1/a \leq b_1 \leq a_+$ and $1/a < b_n \leq a_+$ for all $n \geq 2$, and*

(d) *$w(C) = 2$ if and only if the b'_n's defined by $b'_1 = 1/(2 - a_1)$ and $b'_n = 1/(2 - a_n - b'_{n-1})$ for $n \geq 2$ are such that $1/2 \leq b'_n \leq 1$ for all n.*

Proof. (a) Since $C \succcurlyeq 2\,\mathrm{Re}\,S$, we have $w(C) \geq 2w(\mathrm{Re}\,S) = 2$ by Proposition 1.1(c).

(b) For each $n \geq 1$, let C_n denote the n-by-n principal submatrix

$$\begin{bmatrix} a_1 & 1 & & \\ 1 & a_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & a_n \end{bmatrix}$$

of C. Assume first that $w(C) \leq a$. Since $w(C_n) \leq w(C) \leq a$, we have $C_n \leq aI_n$ and thus $d_n \equiv \det(aI_n - C_n) \geq 0$ for all n. Note that $a_n < w(C_{n+1}) \leq w(C) \leq a$ for $n \geq 1$ by [2, Lemma 1.5]. We now prove by induction that $d_n > 0$ for all n. Indeed, we have $d_1 = a - a_1 > 0$ and

$$d_{n+1} = (a - a_{n+1})d_n - d_{n-1} \tag{2.2}$$

for $n \geq 1$ ($d_0 \equiv 1$) by expanding $\det(aI_{n+1} - C_{n+1})$ along the last row of the matrix $aI_{n+1} - C_{n+1}$. Assume that $d_j > 0$ for all j, $1 \leq j \leq n-1$ ($n \geq 2$). If $d_n = 0$, then $d_{n+1} = -d_{n-1} < 0$ from (2.2), which contradicts $d_{n+1} \geq 0$. Hence $d_n > 0$ for all n.

Let b_n, $n \geq 1$, be defined as asserted. We prove by induction that $b_1 \ldots b_n d_n = 1$ for all $n \geq 1$. For $n = 1$, we have $b_1 = 1/(a - a_1) > 0$ and $b_1 d_1 = (1/(a - a_1))(a - a_1) = 1$. If $n \geq 2$, then, by the induction hypothesis and (2.2), we have $b_1 \ldots b_{n-1} > 0$ and

$$a - a_n - b_{n-1} = (b_1 \ldots b_{n-1})((a - a_n)d_{n-1} - d_{n-2}) = b_1 \ldots b_{n-1} d_n > 0.$$

Therefore, $b_1 \ldots b_n d_n = b_n(a - a_n - b_{n-1}) = 1$ from the above and thus $b_n = 1/(a - a_n - b_{n-1}) > 0$ for all n.

Conversely, if $b_n > 0$ for all n, we prove that $d_n = 1/(b_1 \ldots b_n)$ by induction on $n \geq 1$. For $n = 1$, we have $d_1 = a - a_1 = 1/b_1$. If $n \geq 2$, then $1/b_n = a - a_n - b_{n-1} = b_1 \ldots b_{n-1} d_n$ as above. Thus $d_n = 1/(b_1 \ldots b_n) > 0$ for all $n \geq 1$. From these, we infer that $C_n \leq aI_n$ for all $n \geq 1$ and hence $C \leq aI$ or $w(C) \leq a$.

(c) Since $a_n \geq 0$ and $b_n > 0$ for all $n \geq 1$, we have $b_1 = 1/(a - a_1) \geq 1/a$ and $b_n = 1/(a - a_n - b_{n-1}) > 1/a$ for $n \geq 2$. In particular, this shows that $a - b_{n-1} \geq a - a_n - b_{n-1} > 0$ or $b_{n-1} < a$ for all $n \geq 2$. We next prove that $b_n \leq a_+$ for all $n \geq 1$. Indeed, if $b_{n_0} > a_+$ for some $n_0 \geq 1$, then, by induction, we have

$$b_{n+1} = \frac{1}{a - a_{n+1} - b_n} \geq \frac{1}{a - b_n} > b_n > a_+, \qquad n \geq n_0,$$

where $1/(a - b_n) > b_n$ follows from Lemma 2.1(a). This shows that b_n, $n \geq n_0$, is strictly increasing. Since it is also bounded above by a, b_n converges to, say, b ($> a_+$). Hence $a_n = a - (1/b_n) - b_{n-1}$ converges to

$$a - \frac{1}{b} - b = -\frac{1}{b}(b^2 - ab + 1) = -\frac{1}{b}(b - a_+)(b - a_-).$$

As $a_n \geq 0$ for all n, we have $a - (1/b) - b \geq 0$ or $(b - a_+)(b - a_-) \leq 0$. Thus $a_- \leq b \leq a_+$, which contradicts $b > a_+$. Hence $b_n \leq a_+$ for all $n \geq 1$.

(d) This follows easily from (a)–(c) above. ∎

We remark that if the a_n's are all real, then the above proof of part (b) is still valid.

LEMMA 2.4. *Let C be the matrix in* (2.1) *with real a_n's. Let $a = w(C)$ and let b_n, $n \geq 1$, be defined by $b_1 = 1/(a - a_1)$ and $b_n = 1/(a - a_n - b_{n-1})$ for $n \geq 2$. Then a is an eigenvalue of C if and only if $\sum_{n=1}^{\infty} 1/(b_1 \ldots b_n)^2 < \infty$.*

Proof. Assume that a is an eigenvalue of C. Let $x = (x_1, x_2, \ldots)$ be a unit vector in ℓ^2 such that $Cx = ax$. Then

$$(aI - C)x = \begin{bmatrix} a - a_1 & -1 & & \\ -1 & a - a_2 & -1 & \\ & -1 & a - a_3 & \ddots \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = 0,$$

from which we obtain $x_2 = (a - a_1)x_1$ and $x_n = (a - a_{n-1})x_{n-1} - x_{n-2}$ for $n \geq 3$. Replacing a_1 by $a - 1/b_1$ and a_{n-1} by $a - 1/b_{n-1} - b_{n-2}$, $n \geq 3$, yields $x_n = x_1/(b_1 \ldots b_{n-1})$, $n \geq 2$. Hence $x = (x_1, x_1/b_1, x_1/(b_1 b_2), \ldots)$. Since x is in ℓ^2, we obtain $\sum_{n=1}^{\infty} 1/(b_1 \ldots b_n)^2 < \infty$. The converse can be proved by reversing the above arguments. ∎

As an application of the preceding two lemmas, the next proposition gives information on $W(\operatorname{Re} F_T)$.

PROPOSITION 2.5. *If $T = \operatorname{diag}(a_1, a_2, \ldots)$ with $a_n \geq 0$ for all n and $w(F_T) = 1$, then $W(\operatorname{Re} F_T) = (-1, 1)$.*

Proof. Let C be the matrix in (2.1). We have $w(C) = \|C\| = 2w(F_T) = 2$ by Proposition 2.2(c). If 2 is an eigenvalue of C, then Lemma 2.4 says that $\sum_{n=1}^{\infty} 1/(b_1 \ldots b_n)^2 < \infty$, where $b_1 = 1/(2 - a_1)$ and $b_n = 1/(2 - a_n - b_{n-1})$

for $n \geq 2$. But, in this case, $b_n \leq 1$ by Lemma 2.3(d). This implies that $1/(b_1 \dots b_n)^2 \geq 1$ for all $n \geq 1$, contradicting the convergence of $\sum_{n=1}^{\infty} 1/(b_1 \dots b_n)^2$. Hence 2 cannot be an eigenvalue of C. On the other hand, if -2 is an eigenvalue of C, then let x be a unit vector in ℓ^2 such that $Cx = -2x$. From $-2 = \langle Cx, x\rangle \geq \langle (2\operatorname{Re} S)x, x\rangle \geq -2$ (since $C \geq 2\operatorname{Re} S$), we obtain $\langle (\operatorname{Re} S)x, x\rangle = -1$. It follows that -1 is an eigenvalue of $\operatorname{Re} S$, which is obviously false. Thus -2 cannot be an eigenvalue of C either. We conclude that $W(C) \subseteq (-2, 2)$. As $\operatorname{Re} F_T$ is unitarily equivalent to $(1/2)\left[\begin{smallmatrix} 0 & C \\ C & 0 \end{smallmatrix}\right]$ by Proposition 2.2(a) and the latter operator is unitarily equivalent to $(1/2)\left[\begin{smallmatrix} C & 0 \\ 0 & -C \end{smallmatrix}\right]$ via the unitary $(1/\sqrt{2})\left[\begin{smallmatrix} I & I \\ I & -I \end{smallmatrix}\right]$, we also have $W(\operatorname{Re} F_T) \subseteq (-1, 1)$. Together with $w(\operatorname{Re} F_T) = 1$, this implies that $W(\operatorname{Re} F_T) = (-1, 1)$. ∎

We end this section by considering $W(F_T)$ for a rank-one diagonal T.

Lemma 2.6. *If $T = \operatorname{diag}(0, \dots, 0, a, 0, \dots)$, where a is its nth ($n \geq 1$) diagonal, then $W(F_T)$ is a circular disc centered at the origin.*

Proof. By Proposition 2.2(a), for any λ, $|\lambda| = 1$, λF_T is unitarily similar to $F_{\lambda^{2(n-1)}T}$. The latter operator is easily seen to be unitarily similar to F_T. Thus $W(\lambda F_T) = W(F_T)$ for all λ, $|\lambda| = 1$, and our assertion follows. ∎

The next proposition gives a description of $W(F_T)$ when T is of the form $\operatorname{diag}(a, 0, 0, \dots)$.

Proposition 2.7. *If $T = \operatorname{diag}(a, 0, 0, \dots)$, then*

$$W(F_T) = \begin{cases} \mathbb{D} & \text{if } |a| \leq 1, \\ \{z \in \mathbb{C} : |z| \leq (|a| + (1/|a|))/2\} & \text{otherwise.} \end{cases}$$

Proof. As F_T is unitarily similar to $F_{\lambda T}$ for any λ, $|\lambda| = 1$, we may assume that $a \geq 0$. Then [5, Corollary 2.8] implies that

$$W(F_T) = W\left(\begin{bmatrix} S & T \\ 0 & S^* \end{bmatrix}\right) = W\left(\begin{bmatrix} S & T \\ 0 & S^* \end{bmatrix}^*\right) = W\left(\begin{bmatrix} S^* & 0 \\ T & S \end{bmatrix}\right).$$

We can represent $\left[\begin{smallmatrix} S^* & 0 \\ T & S \end{smallmatrix}\right]$ as

$$\left[\begin{array}{cccc|cccc} \ddots & & & & & & & \\ \ddots & 0 & & & & & 0 & \\ & 1 & 0 & & & & & \\ & & 1 & 0 & & & & \\ \hline & & & a & 0 & & & \\ & & 0 & & 1 & 0 & & \\ & 0 & & & & 1 & 0 & \\ \iddots & & & & & & \ddots & \ddots \end{array}\right],$$

that is, as a bilateral weighted shift with weights $\ldots,1,1,a,1,1,\ldots$. By [12, Theorem 4.9], its numerical range, which is also the numerical range of F_T, is given as asserted. ∎

If the nonzero diagonal a in $T = \operatorname{diag}(0,\ldots,0,a,0,\ldots)$ does not appear in the first position, then $W(F_T)$ behaves more or less as above except that the radius of the circular disc $W(F_T)$ is in general difficult to determine.

THEOREM 2.8. *Let $T_n = \operatorname{diag}(0,\ldots,0,a,0,\ldots)$, where a appears in the nth ($n \geq 2$) position. Then*

$$W(F_{T_n}) = \begin{cases} \mathbb{D} & \text{if } |a| \leq 1/n, \\ \{z \in \mathbb{C} : |z| \leq r_n\}, \text{ where } 1 < r_n < 1 + (|a| - (1/n))/2 & \text{otherwise.} \end{cases}$$

For the proof of the case $|a| > 1/n$, we need another lemma.

LEMMA 2.9. *Let C be the matrix in (2.1) with $a_n \geq 0$ for all n and $a_n = 0$ for $n \geq n_0 + 1$ ($n_0 \geq 1$). Let $a > 2$ and let b_n, $n \geq 1$, be defined by $b_1 = 1/(a - a_1)$ and $b_n = 1/(a - a_n - b_{n-1})$ for $n \geq 2$. Then $w(C) = a$ if and only if $b_n > 0$ for all $n \geq 1$ and $b_n = a_+$ for $n \geq n_0 + 1$. In this case, a is an eigenvalue of C.*

Proof. Assume that $w(C) = a > 2$ and $b_{n_0+1} \neq a_+$. By Lemma 2.3(c), we have $b_1 \geq 1/a$ and $1/a < b_n \leq a_+$ for $n \geq 2$. In particular, $b_n > 0$ for all n and $b_{n_0+1} < a_+$. We choose a', $2 < a' < a$, by continuity such that $b'_1 \equiv 1/(a' - a_1) > 0$, $b'_n \equiv 1/(a' - a_n - b'_{n-1}) > 0$ for $2 \leq n \leq n_0$, and $0 < 1/(a' - b'_{n_0}) < a'_+$. Note that $a' - b'_{n_0} > 1/a'_+ = a'_-$. Hence we may further choose a small $a'_{n_0+1} > 0$ such that $a' - a'_{n_0+1} - b'_{n_0} > a'_-$ and let

$$C' = \begin{bmatrix} a_1 & 1 & & & & & \\ 1 & a_2 & \ddots & & & & \\ & \ddots & \ddots & 1 & & & \\ & & 1 & a_{n_0} & 1 & & \\ & & & 1 & a'_{n_0+1} & 1 & \\ & & & & 1 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{bmatrix}.$$

Moreover, we let $b'_{n_0+1} = 1/(a' - a'_{n_0+1} - b'_{n_0})$ and $b'_n = 1/(a' - b'_{n-1})$ for $n \geq n_0 + 2$. Then $0 < b'_{n_0+1} < a'_+ < a'$. By Lemma 2.1(d) and induction, we obtain $0 < b'_n < a'_+ < a'$ for all $n \geq n_0 + 2$. Thus $b'_n > 0$ for $n \geq 1$. Hence $w(C') \leq a'$ by Lemma 2.3(b). On the other hand, we also have $C \preccurlyeq C'$, which implies, by Proposition 1.1(c), that $a = w(C) \leq w(C') \leq a'$, a contradiction. Thus we must have $b_{n_0+1} = a_+$. For $n \geq n_0 + 2$, we conclude by induction that $b_n = 1/(a - b_{n-1}) = 1/(a - a_+) = a_+$ as asserted.

Conversely, assume that the b_n's as defined satisfy the asserted conditions. Lemma 2.3(b) implies that $w(C) \leq a$. To prove $w(C) = a$, we assume otherwise that $w(C) < a$. Let a' be such that $w(C) < a' < a$. Then Lemma 2.3 says that $0 < b'_1 \equiv 1/(a' - a_1) \leq a'_+$ and $0 < b'_n \equiv 1/(a' - a_n - b'_{n-1}) \leq a'_+$ for all $n \geq 2$. We claim that $b_n < b'_n$ for $n \geq 1$. Indeed, we have

$$\begin{aligned} b_1 &= \frac{1}{a - a_1} < \frac{1}{a' - a_1} = b'_1, \\ b_n &= \frac{1}{a - a_n - b_{n-1}} < \frac{1}{a' - a_n - b'_{n-1}} = b'_n, \qquad n \geq 2, \end{aligned}$$

by induction. Thus

$$b_{n_0+1} = a_+ = \tfrac{1}{2}(a + \sqrt{a^2 - 4}) < b'_{n_0+1} \leq a'_+ = \tfrac{1}{2}(a' + \sqrt{a'^2 - 4}),$$

which contradicts the fact that $f(x) = x + \sqrt{x^2 - 4}$ is an increasing function on $(2, \infty)$. We conclude that $w(C) = a$.

To prove that a is an eigenvalue of C, by Lemma 2.4 we need to show that $\sum_{n=1}^{\infty} 1/(b_1 \ldots b_n)^2 < \infty$. This is indeed the case since $a_- < 1$ (because $a > 2$) and thus

$$\begin{aligned} \sum_{n=n_0+1}^{\infty} \frac{1}{(b_1 \ldots b_n)^2} &= \frac{1}{(b_1 \ldots b_{n_0})^2} \sum_{n=n_0+1}^{\infty} \frac{1}{(b_{n_0+1} \ldots b_n)^2} \\ &= \frac{1}{(b_1 \ldots b_{n_0})^2} \sum_{n=n_0+1}^{\infty} \frac{1}{a_+^{2(n-n_0)}} = \frac{1}{(b_1 \ldots b_{n_0})^2} \sum_{n=n_0+1}^{\infty} a_-^{2(n-n_0)} < \infty. \ \blacksquare \end{aligned}$$

Proof of Theorem 2.8. We may assume that $a > 0$. Let C be the matrix in (2.1) with $a_j = a$ if $j = n$ ($n \geq 2$) and $a_j = 0$ otherwise. If $a = 1/n$, then $b_1 = 1/(2 - a_1) = 1/2$ and

$$b_j = \frac{1}{2 - a_j - b_{j-1}} = \begin{cases} \frac{j}{j+1} & \text{if } 2 \leq j \leq n-1, \\ 1 & \text{if } j \geq n. \end{cases}$$

Thus Lemma 2.3(d) implies that $w(C) = 2$. More generally, if $0 < a < 1/n$, then, letting C' denote the matrix

$$\begin{bmatrix} 0 & 1 & & & & & & \\ 1 & \ddots & \ddots & & & & & \\ & \ddots & 0 & 1 & & & & \\ & & 1 & 1/n & 1 & & & \\ & & & 1 & 0 & 1 & & \\ & & & & 1 & 0 & \ddots & \\ & & & & & \ddots & \ddots & \end{bmatrix}$$

with $1/n$ as its nth diagonal, we have $2 = w(C') \geq w(C)$ by Proposition 1.1(c) (since $C' \succcurlyeq C$). Together with $w(C) \geq 2$ (by Lemma 2.3(a)), we obtain $w(C) = 2$ and thus $w(F_{T_n}) = \|C\|/2 = w(C)/2 = 1$ by Proposition 2.2(c). Moreover, in this case, $W(F_{T_n}) = \mathbb{D}$ follows from Lemma 2.6 and Proposition 2.5.

Next assume that $a > 1/n$. If $w(C) = 2$, then let $\{b_j\}_{j=1}^{\infty}$ be the sequence defined by $b_1 = 1/(2 - a_1)$ and $b_j = 1/(2 - a_j - b_{j-1})$ for $j \geq 2$. Then, as before, we have $b_j = j/(j+1)$ for $2 \leq j \leq n-1$ and $b_j > 1$ for $j \geq n$. The latter contradicts Lemma 2.3(d). Thus we have $w(C) > 2$. Then Lemma 2.9 implies that a is an eigenvalue of C. As $\operatorname{Re} F_{T_n}$ is unitarily similar to $\frac{1}{2}\left[\begin{smallmatrix} 0 & C \\ C & 0 \end{smallmatrix}\right]$ (by Proposition 2.2(a)) or to $\frac{1}{2}\left[\begin{smallmatrix} C & 0 \\ 0 & -C \end{smallmatrix}\right]$, we infer that $W(\operatorname{Re} F_{T_n}) = [-r_n, r_n]$, where $r_n = w(\operatorname{Re} F_{T_n}) = w(C)/2$, and hence $W(F_{T_n}) = \{z \in \mathbb{C} : |z| \leq r_n\}$ by Lemma 2.6. That $r_n > 1$ is a consequence of $w(C) > 2$.

On the other hand, since $w(C) \leq w(C') + w(C - C') = 2 + (a - 1/n)$, we obtain $r_n = w(C)/2 \leq 1 + (a - 1/n)/2$. To prove the strict inequality by contradiction, we assume that $r_n = 1 + (a - 1/n)/2$. Then $\|C\| = \|C'\| + \|C - C'\|$ by the above. [1, Theorem 2.1] implies that $\|C'\|\,\|C - C'\|$ is in $\overline{W(C'(C - C'))}$. Since $C'(C - C')$ is unitarily similar to

$$\left(a - \frac{1}{n}\right) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/n & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus 0,$$

where

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/n & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is unitarily similar to

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1/n \end{bmatrix} \quad \text{or to} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 1/n \end{bmatrix},$$

and $w\left(\left[\begin{smallmatrix} 0 & \sqrt{2} \\ 0 & 1/n \end{smallmatrix}\right]\right) = (1 + \sqrt{2n^2 + 1})/(2n) \leq 1$ for $n \geq 2$, we have

$$w(C'(C - C')) = \left(a - \frac{1}{n}\right) w\left(\begin{bmatrix} 0 & \sqrt{2} \\ 0 & 1/n \end{bmatrix}\right) \leq a - \frac{1}{n}.$$

But $\|C'\|\,\|C - C'\| = 2(a - 1/n) > a - 1/n$, and this shows that $\|C'\|\,\|C - C'\|$ cannot be in $\overline{W(C'(C - C'))}$. This contradiction implies $r_n < 1 + (a - 1/n)/2$ as asserted. ■

3. Foguel–Halmos operator. In this section, we concentrate on properties of the numerical range of the Foguel–Halmos operator. Recall that this is the Foguel operator $\left[\begin{smallmatrix} S^* & T \\ 0 & S \end{smallmatrix}\right]$ with $T = \operatorname{diag}(a_1, a_2, \ldots)$, where $a_n = 1$ or 0

depending on whether $n = 3^k$ for some $k \geq 1$ or otherwise. We start with some symmetry properties of its numerical range.

PROPOSITION 3.1. *If F_T is the Foguel–Halmos operator, then $W(F_T)$ is symmetric with respect to the line $y = x\tan(j\pi/12)$, $j = 0, \pm 1, \pm 2, \ldots$.*

Proof. Proposition 2.2(a) implies that, for any λ and η with $|\lambda| = |\eta| = 1$, λF_T is unitarily similar to $F_{T'}$, where

$$T' = \operatorname{diag}(0, 0, \eta\overline{\lambda}^4, 0, 0, 0, 0, 0, \eta\overline{\lambda}^{16}, 0, \ldots)$$

with $\eta\overline{\lambda}^{2(3^k-1)}$ as the 3^kth diagonal for $k \geq 1$. Hence if $\lambda = e^{-i\pi/6}$ and $\eta = e^{4\pi i/3}$, then $\eta\overline{\lambda}^{2(3^k-1)} = e^{\pi i(3^{k-1}+1)} = 1$ for all $k \geq 1$. This shows that $T' = T$ for such choices of λ and η. Therefore, $e^{-i\pi/6}F_T$ is unitarily similar to F_T.

Let $z = |z|e^{i\theta}$ (θ real) be in $W(F_T)$. Then $e^{-ij\pi/6}z = |z|e^{-i((j\pi/6)-\theta)}$, $j = 0, \pm 1, \pm 2, \ldots$, is in $W(F_T)$ by what was proved above and hence $e^{ij\pi/6}\overline{z} = |z|e^{i((j\pi/6)-\theta)}$ is also in $W(F_T)$ by [5, Corollary 2.8]. Reversing these arguments, we see that $e^{ij\pi/6}\overline{z}$ in $W(F_T)$ implies z in $W(F_T)$ for all j. This proves the asserted symmetries of $W(F_T)$. ∎

The preceding proposition can be rephrased as saying that $W(F_T)$ is invariant under the dihedral group

$$\mathcal{D}_{12} = \{I_2, A, A^2, \ldots, A^{11}, B, AB, A^2B, \ldots, A^{11}B\}$$

of order 24 with generators A and B, where $A = \left[\begin{smallmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{smallmatrix}\right]$ represents rotation by $\pi/6$ around the origin (a proper rotation since $\det A = 1$) and $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ the reflection with respect to the x-axis (an improper rotation since $\det B = -1$).

COROLLARY 3.2. *The numerical range of the Foguel–Halmos operator is symmetric with respect to the x-axis, y-axis and the lines $y = \pm x$.*

Proof. This follows from Proposition 3.1 by letting $j = 0, 6, 3$ and 9, respectively. ∎

The next proposition gives a preliminary estimate of the numerical radius of the Foguel–Halmos operator. It will be improved in Theorem 3.11.

PROPOSITION 3.3. *For the Foguel–Halmos operator F_T, we have $1 < w(F_T) \leq \sqrt{6}/2$.*

Proof. Since $w(F_T) = w(C)/2$ by Proposition 2.2(c), where C is as in (2.1) with $a_n = 1$ if $n = 3^k$ for some $k \geq 1$ and $a_n = 0$ otherwise, we need only prove that $2 < w(C) \leq \sqrt{6}$. By Lemma 2.3(a), we have $w(C) \geq 2$. On the other hand, the sequence b_n, $n \geq 1$, defined by $b_1 = 1/(2 - a_1)$ and $b_n = 1/(2 - a_n - b_{n-1})$ for $n \geq 2$ is such that $b_1 = 1/2$, $b_2 = 2/3$ and $b_3 = 3$. Hence $w(C) > 2$ by Lemma 2.3(d).

To prove $w(C) \leq \sqrt{6}$, note that $w(C) = \|C\| = \|C^2\|^{1/2}$ and $C^2 = C_1 + C_2$, where

$$C_1 = \begin{bmatrix} 0 & & & & & & & \\ & \boxed{\begin{matrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{matrix}} & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & \boxed{\begin{matrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{matrix}} & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & & & & & \ddots \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 1 & & \\ 0 & 1 & 0 & 1 & \\ 1 & 0 & 1 & 0 & \ddots \\ & 1 & 0 & 1 & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Hence

$$\begin{aligned} \|C^2\| &\leq \left\| \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right\| + \|I + 2\operatorname{Re}(S^2)\| \\ &= \rho\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right) + \rho(I + 2\operatorname{Re}(S^2)) \\ &= 3 + \big(1 + 2\rho(\operatorname{Re}(S^2))\big) = 4 + 2\rho(\operatorname{Re} S) = 6, \end{aligned}$$

where we used the unitary similarity of S^2 and $S \oplus S$. Therefore, $w(C) = \|C^2\|^{1/2} \leq \sqrt{6}$ as asserted. ■

The next theorem yields a better lower bound for the numerical radius of the Foguel–Halmos operator.

THEOREM 3.4. *For the Foguel–Halmos operator F_T, $W(\operatorname{Re} F_T)$ equals a closed interval $[-r, r]$ with $r = w(F_T) > \sqrt{5}/2$.*

The proof will be given after a series of lemmas. The first two are technical ones, which are in preparation for the main step later on. As in Section 2, we let $a_\pm = (a \pm \sqrt{a^2 - 4})/2$ for any $a \geq 2$.

LEMMA 3.5. *Let $a > 2$, $a_0 \geq a_-$, $b_1 = 1/a_0$, and $b_n = 1/(a - b_{n-1})$ for $n \geq 2$.*

(a) *If $a_0 = a_\pm$, then $b_n = a_\mp$ for all $n \geq 1$.*
(b) *If $a_- < a_0 < a_+$ (resp., $a_0 > a_+$), then $\{b_n\}_{n=1}^\infty$ is a strictly decreasing (resp., strictly increasing) sequence converging to a_-.*

(c) *If $a_- < a' < a_+$, then, for any $\varepsilon > 0$, there is an integer N such that $a_- < b_n < a_- + \varepsilon$ for all a_0 satisfying $a' \le a_0 < a_+$ and all $n \ge N$.*

Proof. (a) is trivial.

(b) Assume that $a_- < a_0 < a_+$. Then $a_- < b_1 = 1/a_0 < a_+$, which implies that $a_- < b_2 = 1/(a-b_1) < b_1 < a_+$ by Lemma 2.1(b). By induction, we obtain $a_- < b_{n+1} < b_n < a_+$ for all $n \ge 1$. Thus the sequence $\{b_n\}_{n=1}^{\infty}$ is strictly decreasing to, say, b. Since $a - b_{n-1} = 1/b_n$ converges to both $a - b$ and $1/b$, their equality yields $b = a_+$ or a_-. As the b_n's decrease to b and $a_- < b_n < a_+$ for all n, we must have $b = a_-$. The case $a_0 > a_+$ can be proved analogously using Lemma 2.1(c).

(c) Let $a_- < a' < a_+$ and $\varepsilon > 0$. Part (b) implies the existence of an N such that $a_- < b'_n < a_- + \varepsilon$ for all $n \ge N$, where b'_n, $n \ge 1$, is defined by $b'_1 = 1/a'$ and $b'_n = 1/(a - b'_{n-1})$ for $n \ge 2$. On the other hand, for any a_0 satisfying $a' \le a_0 < a_+$, we have $b_n > a_-$ for all $n \ge 1$ by (b). Moreover, $b_1 = 1/a_0 \le 1/a' = b'_1$ and $b_n = 1/(a - b_{n-1}) \le 1/(a - b'_{n-1}) = b'_n$ for $n \ge 2$ by induction. We conclude that $a_- < b_n < a_- + \varepsilon$ for all $n \ge N$. ∎

LEMMA 3.6. *Let $0 < s < 1$, $r \ge s$ and N be a positive integer. If c_n, $n \ge 1$, is such that $0 < c_n < r$ for all n and $j_k \equiv \#\{n \ge 1 : c_n \ge s, 3^k \le n < 3^{k+1}\} \le N$ for all $k > N$, then $\sum_{n=1}^{\infty}(c_1 \ldots c_n) < \infty$.*

Proof. We prove that $\sum_{n=3^{N+1}}^{\infty}(c_1 \ldots c_n) < \infty$. For $\ell > N$, we have

$$\prod_{n=3^\ell}^{3^{\ell+1}-1} c_n \le r^{j_\ell} s^{3^{\ell+1}-3^\ell-j_\ell} = \left(\frac{r}{s}\right)^{j_\ell} s^{2\cdot 3^\ell} \le \left(\frac{r}{s}\right)^{N} s^{2\cdot 3^\ell}$$

from our assumptions on the c_n's. Hence

$$\begin{aligned}\prod_{n=3^N}^{3^k-1} c_n &= \prod_{\ell=N}^{k-1}\Big(\prod_{n=3^\ell}^{3^{\ell+1}-1} c_n\Big) \\ &\le \prod_{\ell=N}^{k-1}\left(\frac{r}{s}\right)^{N} s^{2\cdot 3^\ell} = \left(\frac{r}{s}\right)^{N(k-N)} \cdot s^{2\cdot 3^N(1+3+\cdots+3^{k-N-1})} \\ &= \left(\frac{r}{s}\right)^{N(k-N)} \cdot s^{3^N(3^{k-N}-1)} = \left(\left(\frac{r}{s}\right)^{-N^2} \cdot s^{-3^N}\right)\left(\frac{r}{s}\right)^{Nk} s^{3^k}.\end{aligned} \tag{3.1}$$

On the other hand, for $k > N$ and $0 \le p < 3^{k+1} - 3^k$, we also have

$$c_{3^k} c_{3^k+1} \ldots c_{3^k+p} \le \begin{cases} r^{p+1} \\ r^{p+1} \\ r^{j_k} s^{p+1-j_k} \end{cases} \le \begin{cases} r^N & \text{if } 0 \le p < N \text{ and } r \ge 1, \\ r^{p+1} & \text{if } 0 \le p < N \text{ and } r < 1, \\ \left(\frac{r}{s}\right)^N s^{p+1} & \text{if } N \le p < 3^{k+1} - 3^k. \end{cases}$$

Hence

$$\text{(3.2)}\qquad \sum_{p=0}^{3^{k+1}-3^k-1} (c_{3^k} c_{3^k+1} \ldots c_{3^k+p})$$
$$\leq \begin{cases} Nr^N + \left(\frac{r}{s}\right)^N (s^{N+1} + s^{N+2} + \cdots) \\ (r + r^2 + \cdots) + \left(\frac{r}{s}\right)^N (s^{N+1} + s^{N+2} + \cdots) \end{cases} \leq \begin{cases} r^N\left(N + \frac{s}{1-s}\right) & \text{if } r \geq 1, \\ \frac{r}{1-r} + \frac{r^N s}{1-s} & \text{if } r < 1. \end{cases}$$

Let R_1 denote the bound at the right-most end of (3.2). Combining (3.1) and (3.2), we obtain

$$\sum_{n=3^k}^{3^{k+1}-1} (c_1 \ldots c_n)$$
$$= (c_1 \ldots c_{3^N-1})(c_{3^N} c_{3^N+1} \ldots c_{3^k-1}) \sum_{p=0}^{3^{k+1}-3^k-1} (c_{3^k} c_{3^k+1} \ldots c_{3^k+p})$$
$$\leq R_2 \left(\frac{r}{s}\right)^{Nk} s^{3^k},$$

where $R_2 = (c_1 c_2 \ldots c_{3^N-1})((r/s)^{-N^2} s^{-3^N})R_1$ is independent of k. It follows that

$$\sum_{n=3^{N+1}}^{\infty} (c_1 \ldots c_n) = \Big(\sum_{n=3^{N+1}}^{3^{N+2}-1} + \sum_{n=3^{N+2}}^{3^{N+3}-1} + \cdots\Big)(c_1 \ldots c_n) \leq R_2 \sum_{k=N+1}^{\infty} \left(\frac{r}{s}\right)^{Nk} s^{3^k}.$$

Note that

$$\left(\left(\frac{r}{s}\right)^{Nk} s^{3^k}\right)^{1/k} = \left(\frac{r}{s}\right)^N s^{3^k/k} \to 0$$

as $k \to \infty$ (since $0 < s < 1$). The root test and comparison test imply that $\sum_{n=3^{N+1}}^{\infty} (c_1 \ldots c_n) < \infty$ as asserted. ∎

The next lemma is the major step in our proving of Theorem 3.4.

LEMMA 3.7. *Let C be the matrix in (2.1) with $a_n = 1$ if $n = 3^k$ for some $k \geq 1$ and $a_n = 0$ otherwise. Let $a = w(C)$, $b_1 = 1/a$ and $b_n = 1/(a - a_n - b_{n-1})$ for $n \geq 2$. Then*

(a) *$a_- < b_n < a_+$ for all $n \geq 3$,*
(b) *$1 < b_{3^k} \leq b_{3^{k+1}}$ for all $k \geq 1$,*
(c) *$b_n \geq b_8$ for $n \geq 3$, and*
(d) *$\sum_{n=1}^{\infty} 1/(b_1 \ldots b_n) < \infty$.*

Proof. (a) By Lemma 2.3(c), we have $1/a < b_n \leq a_+$ for all $n \geq 2$. If $b_{n_0} = a_+$ for some n_0 and $3^k \leq n_0 < 3^{k+1}$ for some $k \geq 0$, then $b_{n_0} = b_{n_0+1} = \cdots = b_{3^{k+1}-1} = a_+$ and $b_{3^{k+1}} = 1/(a_- - 1) < 0$ (by Lemma 2.3(a)), which contradicts Lemma 2.3(b). Thus we have $b_n < a_+$ for all $n \geq 2$.

We next prove that $b_3 > 1$. Indeed, we have $b_2 = a/(a^2-1)$ and $b_3 = (a^2-1)/(a^3-a^2-2a+1)$. Let $p(x) = x^3-2x^2-2x+2$. As $p'(x) = 3x^2-4x-2$ equals 0 at $(2 \pm \sqrt{10})/3$, p is increasing on $[(2+\sqrt{10})/3, \infty)$. We infer from $(2+\sqrt{10})/3 < 2 < a \leq \sqrt{6}$ (by Proposition 3.3) and $p(\sqrt{6}) = 4\sqrt{6} - 10 < 0$ that $p(a) = a^3 - 2a^2 - 2a + 2 < 0$ or, equivalently, $b_3 > 1$.

We now proceed to show that $b_n > a_-$ for all $n \geq 3$. For $n = 3$, we have $b_3 > 1 > a_-$. Assume that $b_{3^k} > a_-$ for some $k \geq 1$. As $a > 2$, $a_- < b_{3^k} < a_+$ and $a_{3^k+1} = \cdots = a_{3^{k+1}-1} = 0$, we may apply Lemma 2.1(b) repeatedly to obtain

$$b_{3^k} > b_{3^k+1} > \cdots > b_{3^{k+1}-1} > a_-.$$

Hence

$$b_{3^{k+1}} = \frac{1}{a-1-b_{3^{k+1}-1}} > \frac{1}{a-1-a_-} > \frac{1}{a-a_-} = a_-.$$

By induction, we have $b_n > a_-$ for all $n \geq 3$.

(b) Since $b_3 > 1$ from (a), we need only check $b_{3^k} \leq b_{3^{k+1}}$ for all $k \geq 1$. Assume otherwise that $b_{3^k} > b_{3^{k+1}}$ for some k. By Lemma 2.3(b) and part (a) above, we have $b_n > 0$ for all $n \geq 1$ and $a_- < b_{3^{k+1}} < b_{3^k} < a_+$. By continuity, we can choose an a', $2 < a' < a$, such that the sequence b'_n, $n \geq 1$, defined by $b'_1 = 1/a'$ and $b'_n = 1/(a' - a_n - b'_{n-1})$ for $n \geq 2$, satisfies $b'_1, b'_2, \ldots, b'_{3^{k+1}-1} > 0$ and $a'_- < b'_{3^{k+1}} < b'_{3^k} < a'_+$. We check that $b'_n > 0$ for all n. For this, we need only show that $b'_n > 0$ for $n \geq 3^{k+1}$. Note that

$$b'_{3^k+1} = \frac{1}{a' - b'_{3^k}} > \frac{1}{a' - b'_{3^{k+1}}} = b'_{3^{k+1}+1}.$$

Similarly, $b'_{3^k+2} > b'_{3^{k+1}+2}, \ldots, b'_{3^{k+1}-1} = b'_{3^k+(3^{k+1}-3^k-1)} > b'_{3^{k+1}+(3^{k+1}-3^k-1)}$ (since $3^{k+1} + (3^{k+1} - 3^k - 1) \leq 3^{k+2} - 1$). On the other hand, since $a' > 2$ and $a'_- < b'_{3^{k+1}} < a'_+$, we also have

$$b'_{3^{k+1}} > b'_{3^{k+1}+1} > \cdots > b'_{3^{k+2}-1} > a'_-$$

by repeatedly applying Lemma 2.1(b). As $b'_{3^{k+1}+(3^{k+1}-3^k-1)}$ is one of the above b'_n's, we obtain $b'_{3^{k+1}-1} > b'_{3^{k+2}-1}$. It then follows that

$$\begin{aligned} a'_+ > b'_{3^{k+1}} &= \frac{1}{a'-1-b'_{3^{k+1}-1}} > \frac{1}{a'-1-b'_{3^{k+2}-1}} = b'_{3^{k+2}} \\ &> \frac{1}{a'-b'_{3^{k+2}-1}} > \frac{1}{a'-a'_-} = a'_-. \end{aligned}$$

Proceeding by induction, we obtain $a'_- < b'_{3^{\ell+2}} < b'_{3^{\ell+1}} < a'_+$ for all $\ell \geq k$. For any $n \geq 3^{k+1}$, there is some $k_0 \geq k+1$ such that $3^{k_0} \leq n < 3^{k_0+1}$. Since $a'_- < b'_{3^{k_0}} < a'_+$, we infer from Lemma 2.1(b) that

$$b'_{3^{k_0}} > b'_{3^{k_0}+1} > \cdots > b'_{3^{k_0+1}-1} > a'_-.$$

As b'_n is one of the above, we obtain $b'_n > a'_- > 0$ for all $n \geq 3^{k+1}$ and thus $b'_n > 0$ for all n. Lemma 2.3(b) then implies that $w(C) \leq a' < a$, which contradicts our assumption of $w(C) = a$. Thus $b_{3^k} \leq b_{3^{k+1}}$ for all $k \geq 1$.

(c) For $n \geq 3$, there is a $k_0 \geq 1$ such that $3^{k_0} \leq n < 3^{k_0+1}$. Since $a > 2$ and $a_- < b_n < a_+$ by (a), we infer from Lemma 2.1(b) that

$$b_{3^{k_0}} > b_{3^{k_0}+1} > \cdots > b_{3^{k_0+1}-1}. \tag{3.3}$$

On the other hand, (b) implies that $b_{3^{k_0+1}} \geq b_9$. Thus

$$\frac{1}{a-1-b_{3^{k_0+1}-1}} = b_{3^{k_0+1}} \geq b_9 = \frac{1}{a-1-b_8},$$

from which we obtain $b_{3^{k_0+1}-1} \geq b_8$. Hence $b_n \geq b_8$ from (3.3).

(d) This will be proved by invoking Lemma 3.6. Let $j_k = \#\{n \geq 1 : b_n \leq b_3,\ 3^k \leq n < 3^{k+1}\}$ for each $k \geq 0$. We check that there is an N such that $j_k \leq N$ for all $k > N$. Indeed, since $a_- < 1/b_3 < a_+$ from (a), we apply Lemma 3.5(c) with ε satisfying $0 < \varepsilon < b_8 - a_-$ to obtain an N such that the sequence b'_n, $n \geq 1$, defined by $b'_1 = 1/a'_0$ and $b'_n = 1/(a - b'_{n-1})$ for $n \geq 2$ is such that $a_- < b'_n < a_- + \varepsilon$ for all a'_0, $1/b_3 < a'_0 < a_+$, and all $n \geq N$. Then $a_- < b'_n < b_8$ for all such a'_0 and n. On the other hand, for each $k > N$, we also have

$$b_{3^k} \geq b_3 \geq b_8 \quad \text{and} \quad b_{3^k} > b_{3^k+1} > \cdots > b_{3^{k+1}-1} \geq b_8$$

by (b), (c) and Lemma 2.1(b). If $b_{3^{k+1}-1} \geq b_3 \geq b_8$, then $j_k \leq 1 \leq N$. Otherwise, there is an m, $1 \leq m < 3^{k+1} - 3^k - 1$, such that $b_{3^k+m-1} \geq b_3 > b_{3^k+m}$. Hence among the $3^{k+1} - 3^k$ many n's with $3^k \leq n < 3^{k+1}$, there are exactly m many n's satisfying $b_n \geq b_3$. This means that $j_k = 3^{k+1} - 3^k - m$. We now show that $j_k \leq N$. Indeed, if we choose a'_0 to be $1/b_{3^k+m}$, then $1/b_3 < a'_0 < a_+$ and hence

$$a_- < b'_n < b_8 \tag{3.4}$$

for all $n \geq N$. But $b'_1 = b_{3^k+m}, b'_2 = b_{3^k+m+1}, \ldots$ and $b'_{3^{k+1}-3^k-m-1} = b_{3^{k+1}-2}$. Thus if $j_k = 3^{k+1} - 3^k - m > N$, then we infer from (3.4) that $b_{3^{k+1}-2} = b'_{3^{k+1}-3^k-m-1} < b_8$, which contradicts (c). Hence we must have $j_k \leq N$. Let $c_n = 1/b_n$ for $n \geq 1$. Then $j_k = \#\{n \geq 1 : c_n \geq c_3, 3^k \leq n < 3^{k+1}\} \leq N$ for all $k > N$, where $0 < c_3 < 1$ (by (b)) and $0 < c_n \leq c_8$ for all $n \geq 3$ (by (c)). We conclude by Lemma 3.6 that $\sum_{n=1}^{\infty} 1/(b_1 \ldots b_n) = \sum_{n=1}^{\infty} (c_1 \ldots c_n) < \infty$. ∎

Proof of Theorem 3.4. Let C, a and b_n, $n \geq 1$, be as in Lemma 3.7. Then Lemma 3.7(d) implies that $\sum_{n=1}^{\infty} 1/(b_1 \ldots b_n)^2 < \infty$. Hence a is an eigenvalue of C by Lemma 2.4. We infer from Proposition 2.2 that $\operatorname{Re} F_T$ is unitarily similar to $\frac{1}{2}\left[\begin{smallmatrix} C & 0 \\ 0 & -C \end{smallmatrix}\right]$ and hence $W(\operatorname{Re} F_T) = [-a/2, a/2]$, where $a = w(C) = 2w(\operatorname{Re} F_T) = 2w(F_T)$. We now show that $a > \sqrt{5}$. Indeed, Lemma 3.7(a) says that $a_- < b_n < a_+$ for all $n \geq 3$. In particular, we have

$$a_- = \frac{1}{a_+} < \frac{1}{b_9} = a - 1 - b_8 < \frac{1}{a_-} = a_+$$

and

$$a_- = a - a_+ < a - b_8 < a - a_- = a_+.$$

Adding $a_- < a - b_8 - 1$ and $-a_+ < b_8 - a$ yields $a_- - a_+ < -1$ or $1 < a_+ - a_- = \sqrt{a^2 - 4}$. Thus $a > \sqrt{5}$ and $w(F_T) = a/2 > \sqrt{5}/2$. ∎

The next theorem gives the numerical range of $\mathrm{Re}(\lambda F_T)$ for another value of λ on the unit circle.

THEOREM 3.8. *If F_T is the Foguel–Halmos operator and $\lambda = e^{i\pi/12}$, then $W(\mathrm{Re}(\lambda F_T)) = (-\sqrt{5}/2, \sqrt{5}/2)$.*

COROLLARY 3.9. *For the Foguel–Halmos operator F_T, $W(F_T)$ is neither open nor closed, and $\overline{W(F_T)}$ is not a (closed) elliptic disc.*

Proof. As $W(\mathrm{Re}\, F_T)$ (resp., $W\big(\mathrm{Re}(e^{i\pi/12}F_T)\big)$) is a closed (resp., open) interval by Theorem 3.4 (resp., Theorem 3.8), $W(F_T)$ cannot be open nor closed. Moreover, $w(\mathrm{Re}\, F_T) > w\big(\mathrm{Re}(e^{i\pi/12}F_T)\big)$ implies that $\overline{W(F_T)}$ cannot be a circular disc centered at the origin. If $\overline{W(F_T)}$ is an elliptic disc, then the equality of $\overline{W(F_T)}$ and $-\overline{W(F_T)}$ (by Proposition 2.2(a)) implies that it must be centered at the origin. Then its symmetry with respect to the x-axis and the line $x = y$ (by Corollary 3.2) implies that it must be a circular disc, which has already been shown to be impossible. ∎

To prove Theorem 3.8, we need another lemma.

LEMMA 3.10. *For each $n \geq 1$, let*

$$B_n = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & \ddots & \ddots & & & & \\ & \ddots & 0 & 1 & & & \\ & & 1 & 1 & 1 & & \\ & & & 1 & 0 & \ddots & \\ & & & & \ddots & \ddots \end{bmatrix},$$

where the diagonal 1 appears in the nth position. Then $w(B_n)$ increases to $\sqrt{5}$ as n approaches infinity.

Proof. Since each B_n can be considered as a principal submatrix of B_{n+1}, the increasingness of $w(B_n)$ follows immediately. Fix an $n \geq 1$. Let $a_k = 1$ if $k = n$ and $a_k = 0$ otherwise, and let $b_1 = 1/(\sqrt{5} - a_1)$ and $b_k = 1/(\sqrt{5} - a_k - b_{k-1})$ for $k \geq 2$. We first show that $b_k > 0$ for all k.

Since $\sqrt{5} > 2$ and $1/\sqrt{5} < \sqrt{5}_- = (\sqrt{5}-1)/2$, we may apply Lemma 2.1(c) repeatedly to obtain $b_1 < \cdots < b_{n-1} < \sqrt{5}_-$. Note that

$$0 < b_1 < b_{n-1} = \frac{1}{\sqrt{5} - b_{n-2}} < b_n = \frac{1}{\sqrt{5} - 1 - b_{n-1}}$$
$$< \frac{1}{\sqrt{5} - 1 - \sqrt{5}_-} = \sqrt{5}_+.$$

If $b_n > \sqrt{5}_-$, then $\sqrt{5}_- < b_n < \sqrt{5}_+$ implies, by Lemma 2.1(b), that $\sqrt{5}_- < b_k < \sqrt{5}_+$ for all $k \geq n$. On the other hand, if $b_n \leq \sqrt{5}_-$, then $b_k \geq b_n$ for all $k > n$ by Lemma 2.1(c). In any case, we have $b_k > 0$ for all k. Thus $w(B_n) \leq \sqrt{5}$ by Lemma 2.3(b).

Let a be such that $2 < a < \sqrt{5}$. We now show that there is an n_0 such that $w(B_{n_0}) > a$. Assume otherwise that $w(B_n) \leq a$ for all $n \geq 1$. For each $n \geq 1$, let $b_{n,1} = 1/(a - a_1)$ and $b_{n,k} = 1/(a - a_k - b_{n,k-1})$ for $k \geq 2$, where $a_k = 1$ or 0 depending on whether $k = n$ or otherwise, and let $c_1 = 1/a$ and $c_k = 1/(a - c_{k-1})$ for $k \geq 2$. Then $c_1 < a_-$ and $c_1 < c_2 < \cdots < a_-$ by Lemma 2.1(c). Let $c = \lim_k c_k$ ($\leq a_-$). The convergence of $c_k = 1/(a - c_{k-1})$ yields $c = 1/(a-c)$ or $c^2 - ac + 1 = 0$. Thus $c = a_+$ or a_-. As $c \leq a_- < a_+$, we must have $c = a_-$. Since $2 < a < \sqrt{5}$, we have $a_+ - a_- = \sqrt{a^2 - 4} < 1$. Hence, for $0 < \varepsilon < 1 - (a_+ - a_-)$, there is a k_0 such that $a_- - c_{k_0 - 1} < \varepsilon$. Note that $b_{n,k} = c_k$ for $n > k \geq 1$. Hence $a_- - b_{k_0, k_0 - 1} < \varepsilon$, and therefore

$$b_{k_0,k_0} = \frac{1}{a - 1 - b_{k_0,k_0-1}} > \frac{1}{a - 1 - a_- + \varepsilon} = \frac{1}{a_+ - 1 + \varepsilon} > \frac{1}{a_-} = a_+.$$

But $w(B_{k_0}) \leq a$ implies that $b_{k_0,k} \leq a_+$ for all $k \geq 1$ by Lemma 2.3(c). This leads to a contradiction. Hence $w(B_{n_0}) > a$ for some n_0. Since this holds for all $2 < a < \sqrt{5}$, we conclude that there is an n_0 such that $w(B_{n_0})$ is arbitrarily close to $\sqrt{5}$. So $\lim_n w(B_n) = \sqrt{5}$ as asserted. ∎

Proof of Theorem 3.8. As shown in the proof of Proposition 3.1, for any λ and η satisfying $|\lambda| = |\eta| = 1$, λF_T is unitarily similar to $F_{T'}$, where $T' = \operatorname{diag}(0, 0, \eta\lambda^4, 0, 0, 0, 0, 0, \eta\lambda^{16}, 0, \ldots)$ with the nth diagonal of T' being $\eta\lambda^{2(3^k - 1)}$ if $n = 3^k$ ($k \geq 1$) and 0 if otherwise. If $\lambda = e^{i\pi/12}$ and $\eta = e^{i2\pi/3}$, then

$$\eta\lambda^{2(3^k-1)} = e^{i2\pi/3} \cdot e^{i(\pi/12)2(3^k-1)} = e^{i\pi(3^{k-1}+1)/2}.$$

Since $(3^{k-1}+1)/2$ and k have the same parity, the (3^k)th diagonal of T' equals $(-1)^k$. Hence, by Proposition 2.2(a), $\operatorname{Re}(\lambda F_T)$ is unitarily similar to $\frac{1}{2}\left[\begin{smallmatrix} C & 0 \\ 0 & -C \end{smallmatrix}\right]$, where C is as in (2.1) with $a_n = (-1)^k$ if $n = 3^k$ ($k \geq 1$) and $a_n = 0$ otherwise. Thus, in the following, we need only show that $W(C) = (-\sqrt{5}, \sqrt{5})$.

Let B_n, $n \geq 1$, be as in Lemma 3.10 and let $B_{n,m}$, $m \geq n$, be the m-by-m principal submatrix

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & \ddots & \ddots & & & & \\ & \ddots & 0 & 1 & & & \\ & & 1 & 1 & 1 & & \\ & & & 1 & 0 & \ddots & \\ & & & & \ddots & \ddots & 1 \\ & & & & & 1 & 0 \end{bmatrix}$$

of B_n, where the nth diagonal element of $B_{n,m}$ is equal to 1. We first show that $w(C) \geq \sqrt{5}$. Indeed, for any fixed B_n and any $\varepsilon > 0$, there is a large m such that $w(B_n) - w(B_{n,m}) < \varepsilon$ by Proposition 1.1(b). Since each $B_{n,m}$ is also a principal submatrix of C, we have $w(B_{n,m}) \leq w(C)$ by Proposition 1.1(b) again. Hence either $w(B_n) \leq w(C)$ or $0 < w(B_n) - w(C) < \varepsilon$. In any case, we have $w(B_n) \leq w(C)$ for all n. As $w(B_n)$ increases to $\sqrt{5}$ by Lemma 3.10, we obtain $\sqrt{5} \leq w(C)$.

We next check that $w(C) = \sqrt{5}$. For this, let $b_1 = 1/(\sqrt{5} - a_1)$ and $b_n = 1/(\sqrt{5} - a_n - b_{n-1})$ for $n \geq 2$. If it can be shown that $b_n > 0$ for all n, then by Lemma 2.3(b), which is also valid for real a_n's, we obtain $w(C) \leq \sqrt{5}$ and hence $w(C) = \sqrt{5}$.

We start by showing that $0 < b_n < \sqrt{5}_-$ for $n = 3^k$ with odd k. Indeed, let $k = 2\ell - 1$, $\ell \geq 1$. For the case $\ell = 1$, since $b_1 = 1/\sqrt{5} < \sqrt{5}_-$, we have $b_1 < b_2 < \sqrt{5}_-$ by Lemma 2.1(c) and hence $0 < b_3 = 1/(\sqrt{5}+1-b_2) < \sqrt{5}_-$. Next assume that $0 < b_{3^{2\ell-1}} < \sqrt{5}_-$ for some $\ell \geq 1$. Then

$$0 < b_{3^{2\ell-1}} < b_{3^{2\ell-1}+1} < \cdots < b_{3^{2\ell}-1} < \sqrt{5}_-$$

by Lemma 2.1(c) again. Thus

$$0 < b_{3^{2\ell}} = \frac{1}{\sqrt{5} - 1 - b_{3^{2\ell}-1}} < \frac{1}{\sqrt{5} - 1 - \sqrt{5}_-} = \frac{1}{\sqrt{5}_+ - 1} = \sqrt{5}_+.$$

We have three cases to consider:

(i) If $b_{3^{2\ell}} < \sqrt{5}_-$, then, by Lemma 2.1(c),

$$0 < b_{3^{2\ell}} < b_{3^{2\ell}+1} < \cdots < b_{3^{2\ell+1}-1} < \sqrt{5}_-$$

and hence

$$0 < b_{3^{2\ell+1}} = \frac{1}{\sqrt{5} + 1 - b_{3^{2\ell+1}-1}} < \frac{1}{\sqrt{5} - b_{3^{2\ell+1}-1}} < \frac{1}{\sqrt{5} - \sqrt{5}_-} = \sqrt{5}_-.$$

(ii) If $b_{3^{2\ell}} = \sqrt{5}_-$, then

$$b_{3^{2\ell}} = b_{3^{2\ell}+1} = \cdots = b_{3^{2\ell+1}-1} = \sqrt{5}_-$$

and hence

$$b_{3^{2\ell+1}} = \frac{1}{\sqrt{5}+1-b_{3^{2\ell+1}-1}} = \frac{1}{\sqrt{5}+1-\sqrt{5}_-} = \frac{1}{\sqrt{5}_+ + 1} < \sqrt{5}_-.$$

(iii) If $\sqrt{5}_- < b_{3^{2\ell}} < \sqrt{5}_+$, then, by Lemma 2.1(b), we have

$$\sqrt{5}_+ > b_{3^{2\ell}} > b_{3^{2\ell}+1} > \cdots > b_{3^{2\ell+1}-1} > \sqrt{5}_-$$

and hence

$$0 < b_{3^{2\ell+1}} = \frac{1}{\sqrt{5}+1-b_{3^{2\ell+1}-1}} < \frac{1}{\sqrt{5}+1-\sqrt{5}_+} = \frac{1}{\sqrt{5}_- + 1} = \sqrt{5}_-.$$

The above shows that we always have $b_{3^{2\ell+1}} < \sqrt{5}_-$. We infer by induction that, for all odd k, we have $0 < b_{3^k} < \sqrt{5}_-$ and

$$0 < b_n < \sqrt{5}_- \quad \text{for } 3^k \leq n < 3^{k+1}. \tag{3.5}$$

Now we consider what happens for even k. In this case, $k-1$ is odd and we have $0 < b_{3^{k-1}} < \sqrt{5}_-$ from the above. Hence

$$0 < b_{3^{k-1}} < b_{3^{k-1}+1} < \cdots < b_{3^k-1} < \sqrt{5}_-$$

by Lemma 2.1(c). Thus

$$0 < b_{3^k} = \frac{1}{\sqrt{5}-1-b_{3^k-1}} < \frac{1}{\sqrt{5}-1-\sqrt{5}_-} = \frac{1}{\sqrt{5}_+ - 1} = \sqrt{5}_+.$$

Moreover, for $3^k \leq n < 3^{k+1}$, we also have

$$0 < b_n < \sqrt{5}_+ \tag{3.6}$$

as before. Thus $b_n > 0$ for all n and therefore $w(C) = \sqrt{5}$ as asserted.

We now show that $\sqrt{5}$ is not an eigenvalue of C. Indeed, according to Lemma 2.4, we need only show that $\sum_{n=1}^{\infty} 1/(b_1 \ldots b_n)^2 = \infty$. By (3.5) and (3.6), we have $0 < b_n < \sqrt{5}_-$ (resp., $0 < b_n < \sqrt{5}_+$) for $3^k \leq n < 3^{k+1}$ with k odd (resp., k even). For any even k, let p_k be the number of those b_n's among $b_1, b_2, \ldots, b_{3^k-1}$ for which $b_n < \sqrt{5}_-$. The remaining b_n's satisfy $b_n < \sqrt{5}_+$. From what was proved above, we have $p_k > (3^k - 1) - p_k$. Hence, for even k, we have

$$b_1 \ldots b_{3^k-1} < \sqrt{5}_+^{3^k-1-p_k} \sqrt{5}_-^{p_k} = \sqrt{5}_-^{2p_k-(3^k-1)} < 1$$

since $\sqrt{5}_- < 1$ and $2p_k > 3^k - 1$. It follows that $\sum_{n=1}^{\infty} 1/(b_1 \ldots b_n)^2 = \infty$ and hence $\sqrt{5}$ is not an eigenvalue of C. Note that if $U = \operatorname{diag}(1, -1, 1, -1, \ldots)$, then

$$B \equiv U^*(-C)U = \begin{bmatrix} a_1' & 1 & & \\ 1 & a_2' & 1 & \\ & 1 & a_3' & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

with $a'_n = (-1)^{k+1}$ if $n = 3^k$ for some $k \geq 1$ and $a'_n = 0$ otherwise. A similar argument to the above shows that $\sqrt{5}$ is not an eigenvalue of B. Thus $-\sqrt{5}$ is not an eigenvalue of C. We conclude that $W(C) = (-\sqrt{5}, \sqrt{5})$. ∎

It seems difficult to give the exact value of the numerical radius of the Foguel–Halmos operator. We have already seen some estimates of it. For example, Proposition 3.3 gives both lower and upper bounds for its value and Theorem 3.4 a lower bound for it. The next theorem improves its upper bound.

THEOREM 3.11. *For the Foguel–Halmos operator F_T, we have $\sqrt{5}/2 < w(F_T) \leq 1.1392\ldots$.*

Note that $\sqrt{5}/2 = 1.1180\ldots$. Using some simple calculations, one of the referees of this paper suggests the improvement $w(F_T) > 1.124$ as follows: Let C be the matrix in (2.1) with $a_n = 1$ if $n = 3^k$ for some $k \geq 1$ and $a_n = 0$ otherwise, and let C_n be the n-by-n leading principal submatrix of C for $n \geq 1$. If $d_n(\lambda) = \det(\lambda I_n - C_n)$ denotes the characteristic polynomial of C_n, then we have $d_1(\lambda) = \lambda$, $d_2(\lambda) = \lambda^2 - 1$, and

$$d_n(\lambda) = \begin{cases} (\lambda - 1)d_{n-1}(\lambda) - d_{n-2}(\lambda) & \text{if } n = 3^k, \\ \lambda d_{n-1}(\lambda) - d_{n-2}(\lambda) & \text{if } n \neq 3^k. \end{cases}$$

Elementary calculations yield $d_{15}(2.249) = -0.0199$. Since $d_{15}(\lambda)$ is a polynomial of degree 15, we have $\lim_{\lambda\to\infty} d_{15}(\lambda) = \infty$, which shows that the largest zero of $d_{15}(\lambda)$ is strictly bigger than 2.249. Hence

$$w(F_T) = \tfrac{1}{2}w(C) \geq \tfrac{1}{2}w(C_{15}) = \tfrac{1}{2}\rho(C_{15}) > 1.124.$$

Further numerical computations along this line give the approximate value of $w(F_T)$ as $1.1246798\ldots$.

For the proof of Theorem 3.11, we need another lemma.

LEMMA 3.12. *Let $a_j \geq 0$ for $1 \leq j \leq k$, and let*

$$A_n = \begin{bmatrix} a_1 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & a_k & 1 & & & & \\ & & & \ddots & 1 & & & \\ & & & & a_1 & \ddots & & \\ & & & & & \ddots & 1 & \\ 1 & & & & & & a_k \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & a_k & 1 & & & \\ & & & a_1 & \ddots & & \\ & & & & \ddots & 1 & \\ & & & & & a_k & \ddots \\ & & & & & & \ddots \end{bmatrix},$$

where A_n is of size nk $(n \geq 1)$ and A is an infinite matrix. Then $w(A_1) = w(A_n) = w(A)$ for all $n \geq 1$.

Proof. To prove $w(A_1) \le w(A_n)$, let $x = (x_1, \ldots, x_k)$ be a unit vector in $\mathbb{C}^k$ with $x_j \ge 0$ for all j such that $w(A_1) = \langle A_1 x, x\rangle$ (cf. [10, Proposition 3.3]). Then

$$\begin{aligned} w(A_1) = \langle A_1 x, x\rangle &= \sum_{j=1}^{k} a_j x_j^2 + \sum_{j=1}^{k} x_j x_{j+1} \qquad (x_{k+1} \equiv x_1) \\ &= n \sum_{j=1}^{k} a_j \left(\frac{x_j}{\sqrt{n}}\right)^2 + n \sum_{j=1}^{k} \frac{x_j}{\sqrt{n}} \frac{x_{j+1}}{\sqrt{n}} = \langle A_n y, y\rangle \le w(A_n), \end{aligned}$$

where $y = (1/\sqrt{n})(x_1, \ldots, x_k, \ldots, x_1, \ldots, x_k)$ is also a unit vector in $\mathbb{C}^{nk}$. For the reverse inequality $w(A_n) \le w(A_1)$, let $x' = (x'_1, \ldots, x'_{nk})$ be a unit vector in $\mathbb{C}^{nk}$ with $x'_j \ge 0$ for all j such that $w(A_n) = \langle A_n x', x'\rangle$. Then

$$w(A_n) = \langle A_n x', x'\rangle = \sum_{j=1}^{k} a_j \Big(\sum_{\ell=0}^{n-1} x'^2_{\ell k+j}\Big) + \sum_{j=1}^{nk} x'_j x'_{j+1} \qquad (x'_{nk+1} \equiv x'_1).$$

Let $y'_j = (\sum_{\ell=0}^{n-1} x'^2_{\ell k+j})^{1/2}$ for $1 \le j \le k$ and $y' = (y'_1, \ldots, y'_k)$. Then y' is a unit vector in $\mathbb{C}^k$. We have

$$\begin{aligned} \sum_{j=1}^{nk} x'_j x'_{j+1} &= \sum_{j=1}^{k} \Big(\sum_{\ell=0}^{n-1} x'_{\ell k+j} x'_{\ell k+j+1}\Big) \\ &\le \sum_{j=1}^{k} \Big[\Big(\sum_{\ell=0}^{n-1} x'^2_{\ell k+j}\Big)^{1/2} \Big(\sum_{\ell=0}^{n-1} x'^2_{\ell k+j+1}\Big)^{1/2}\Big] = \sum_{j=1}^{k} y'_j y'_{j+1} \qquad (y'_{k+1} = y'_1). \end{aligned}$$

Hence

$$\langle A_n x', x'\rangle \le \sum_{j=1}^{k} a_j y'^2_j + \sum_{j=1}^{k} y'_j y'_{j+1} = \langle A_1 y', y'\rangle \le w(A_1).$$

From this, we obtain $w(A_n) \le w(A_1)$ and thus $w(A_1) = w(A_n)$ for all $n \ge 1$.

We now prove $w(A_1) = w(A)$. For $w(A_1) \le w(A)$, let $x = (x_1, \ldots, x_k)$ and $y = (1/\sqrt{n})(x_1, \ldots, x_k, \ldots, x_1, \ldots, x_k)$ be as before. If B_n is the (nk)-by-(nk) matrix

$$\begin{bmatrix} a_1 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & a_k & 1 & & & \\ & & & \ddots & 1 & & \\ & & & & a_1 & \ddots & \\ & & & & & \ddots & 1 \\ & & & & & & a_k \end{bmatrix},$$

then

$$w(A_1) = w(A_n) = \langle A_n y, y\rangle = \langle B_n y, y\rangle + \frac{1}{n}x_1x_k$$
$$\le w(B_n) + \frac{1}{n}x_1x_k \to w(A) \quad \text{as } n \to \infty.$$

Hence $w(A_1) \le w(A)$. To prove $w(A) \le w(A_1)$, note that

$$w(A) = \lim_n w(B_n) \le \lim_n w(A_n) = w(A_1),$$

where the inequality in the middle follows from $0_{nk} \preccurlyeq B_n \preccurlyeq A_n$ (cf. Proposition 1.1(c)). Thus $w(A_1) = w(A_n) = w(A)$ for all $n \ge 1$. ∎

Proof of Theorem 3.11. Let C be the matrix in (2.1) with $a_n = 1$ if $n = 3^k$ for some $k \ge 1$ and $a_n = 0$ otherwise. We first prove that $w(C) \le 2w(A_1)$, where A_1 is the 6-by-6 matrix

$$\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 1/2 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{bmatrix}.$$

Let $D = \operatorname{diag}(d_1, d_2, \ldots)$ on ℓ^2, where $d_n = 1/2$ if $n = 6k-3$ for some $k \ge 1$ and $d_n = 0$ otherwise, and let S be the (simple) unilateral shift on ℓ^2. By Lemma 3.12, we have $w(A_1) = w(D + S^*)$. Note that

$$\begin{aligned} w(D+S^*) &= \sup\{w(\operatorname{Re}(\lambda(D+S^*))) : |\lambda| = 1\} && \text{(by Proposition 1.1(a))} \\ &= w(\operatorname{Re}(D+S^*)) && \text{(by Proposition 1.1(c))} \\ &= \tfrac{1}{2}w(2D+S+S^*). \end{aligned}$$

Moreover, since $6\big((3^{k-1}+1)/2\big) - 3 = 3^k$ for any $k \ge 1$, we have $0 \preccurlyeq C \preccurlyeq 2D + S + S^*$ and hence $w(C) \le w(2D+S+S^*)$ by Proposition 1.1(c). Combining these, we obtain

$$w(C) \le w(2D+S+S^*) = 2w(D+S^*) = 2w(A_1).$$

By Proposition 1.1(a) and (c) again, we have $w(A_1) = w(\operatorname{Re} A_1) = \rho(\operatorname{Re} A_1)$. Since the characteristic polynomial of $\operatorname{Re} A_1$ is

$$\big(z^2 - \tfrac{1}{4}\big)\big(z^4 - \tfrac{1}{2}z^3 - \tfrac{5}{4}z^2 + \tfrac{3}{8}z + \tfrac{1}{4}\big),$$

the eigenvalues of $\operatorname{Re} A_1$ are $\pm 1/2$, $1.1392\ldots$, $0.6587\ldots$, $-0.3523\ldots$, and $-0.9456\ldots$. We conclude from Proposition 2.2(c) that

$$w(F_T) = \tfrac{1}{2}\|C\| \le w(A_1) = 1.1392\ldots.$$

That $w(F_T) > \sqrt{5}/2$ was proved in Theorem 3.4. ∎

As S^* is a restriction of any Foguel operator F_T, $W(F_T)$ always contains $W(S^*) = \mathbb{D}$. Our final result says that the numerical range of the Foguel–Halmos operator contains the larger open circular disc $\{z \in \mathbb{C} : |z| < \sqrt{5}/2\}$.

THEOREM 3.13. *For the Foguel–Halmos operator F_T, $W(F_T)$ contains $\{z \in \mathbb{C} : |z| < \sqrt{5}/2\}$.*

An easy consequence is the following corollary on the generalized Crawford number of F_T. Recall that the *generalized Crawford number* $C(A)$ of an operator A is $\min\{|z| : z \in \partial W(A)\}$.

COROLLARY 3.14. *For the Foguel–Halmos operator F_T, $C(F_T)$ equals $\frac{\sqrt{5}}{2}$.*

Proof. This is because $C(F_T) \le \sqrt{5}/2$ by Theorem 3.8 while $C(F_T) \ge \sqrt{5}/2$ by Theorem 3.13. ∎

The proof of Theorem 3.13 depends on the following lemma and a result of Weyl (cf. [9, Theorem 1.4.1]).

LEMMA 3.15. *If A_{2n+1} $(n \ge 0)$ is the $(2n+1)$-by-$(2n+1)$ matrix*

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & \ddots & \ddots & & & & \\ & \ddots & 0 & 1 & & & \\ & & 1 & 1 & 1 & & \\ & & & 1 & 0 & \ddots & \\ & & & & \ddots & \ddots & 1 \\ & & & & & 1 & 0 \end{bmatrix},$$

where 1 *appears as its $(n+1)$st diagonal, then $w(A_{2n+1})$ increases to $\sqrt{5}$ as n approaches infinity.*

Proof. Let B_n $(n \ge 1)$ be the infinite matrix in Lemma 3.10. Since A_{2n+1} is a principal submatrix of both A_{2n+3} and B_{n+1}, we can see that $w(A_{2n+1})$ increases and $w(A_{2n+1}) \le w(B_{n+1})$ for all n. Thus $\lim_n w(A_{2n+1}) \le \lim_n w(B_{n+1}) = \sqrt{5}$ by Lemma 3.10. On the other hand, for any fixed $m \ge 1$, let B_n' be the n-by-n matrix $(n \ge m)$

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & \ddots & \ddots & & & & \\ & \ddots & 0 & 1 & & & \\ & & 1 & 1 & 1 & & \\ & & & 1 & 0 & \ddots & \\ & & & & \ddots & \ddots & 1 \\ & & & & & 1 & 0 \end{bmatrix},$$

with 1 as its mth diagonal. As B'_n is a principal submatrix of B'_{n+1}, B_m and A_{2n+1}, we find that $w(B'_n)$ increases to $w(B_m)$ (by Proposition 1.1(b)) and $w(B'_n) \leq w(A_{2n+1})$ for all $n \geq m$. Hence $w(B_m) = \lim_n w(B'_n) \leq \lim_n w(A_{2n+1})$ for all m and, again by Lemma 3.10, $\sqrt{5} = \lim_m w(B_m) \leq \lim_n w(A_{2n+1}) \leq \sqrt{5}$. It follows that $\lim_n w(A_{2n+1}) = \sqrt{5}$. ∎

Proof of Theorem 3.13. We have to show that $w(\mathrm{Re}(\lambda F_T)) \geq \sqrt{5}/2$ for all λ with $|\lambda| = 1$. By Proposition 2.2(a), $w(\mathrm{Re}(\lambda F_T))$ is equal to $\|C_\lambda\|/2$, where $C_\lambda = D_\lambda + S + S^*$ with $D_\lambda = \mathrm{diag}(a_1, a_2, \ldots)$ and $a_n = \overline{\lambda}^{2(3^k-1)}$ if $n = 3^k$ ($k \geq 1$) and $a_n = 0$ otherwise. Note that [9, Theorem 1.4.1] says that the set $\{\theta \in [0, 2\pi] : \{2(3^k - 1)\theta/(2\pi) - [2(3^k - 1)\theta/(2\pi)] : k \geq 1\}$ is dense in $[0, 1]\}$ has Lebesgue measure 2π, where, for any real x, $[x]$ denotes its integer part. We choose one such θ and let k_j, $j \geq 1$, be a sequence of integers such that $2(3^{k_j} - 1)\theta/(2\pi) - [2(3^{k_j} - 1)\theta/(2\pi)] \to 0$ as $j \to \infty$. Then $e^{2(3^{k_j}-1)\theta i} \to 1$. Let $\lambda = e^{-i\theta}$ and

$$A'_{k_j} = \begin{bmatrix} 0 & 1 & & & & & & \\ 1 & \ddots & \ddots & & & & & \\ & \ddots & 0 & 1 & & & & \\ & & 1 & \overline{\lambda}^{2(3^{k_j}-1)} & 1 & & & \\ & & & 1 & 0 & \ddots & & \\ & & & & \ddots & \ddots & 1 \\ & & & & & 1 & 0 \end{bmatrix},$$

a matrix of size $2(3^{k_j} - 3^{k_j-1} - 1) + 1$ with $\overline{\lambda}^{2(3^{k_j}-1)}$ as its middle diagonal entry. Then A'_{k_j} is a principal submatrix of C_λ and hence $w(A'_{k_j}) \leq w(C_\lambda)$ for all k_j. When j is large, A'_{k_j} is very close to a matrix of the form in Lemma 3.15. Thus $\sqrt{5} = \lim_j w(A'_{k_j}) \leq w(C_\lambda)$. From our choice of θ and Weyl's result, this inequality holds for almost all points λ on the unit circle. Thus $w\big(\mathrm{Re}(\lambda F_T)\big) = w(C_\lambda)/2 \geq \sqrt{5}/2$ for all λ, $|\lambda| = 1$. ∎

We remark that, after some modifications, most of the results in this section on the Foguel–Halmos operator are also valid when, for some $m > 1$, $T = \mathrm{diag}(a_1, a_2, \ldots)$ is such that $a_n = 1$ if $n = m^k$ for some $k \geq 1$ and $a_n = 0$ otherwise.

We conclude this paper with the following question: What is the exact value of the numerical radius of the Foguel–Halmos operator? Upper and lower bounds of it have been given in Proposition 3.3 and Theorem 3.11.

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References

[1] M. Barraa and M. Boumazgour, *Inner derivations and norm equality*, Proc. Amer. Math. Soc. 130 (2002), 471–476.

[2] M. A. Dritschel and H. J. Woerdeman, *Model theory and linear extreme points in the numerical radius unit ball*, Mem. Amer. Math. Soc. 129 (1997), no. 615, viii+62 pp.

[3] S. R. Foguel, *A counterexample to a problem of Sz.-Nagy*, Proc. Amer. Math. Soc. 15 (1964), 788–790.

[4] S. R. Garcia, *The norm and modulus of a Foguel operator*, Indiana Univ. Math. J. 58 (2009), 2305–2315.

[5] H.-L. Gau, K.-Z. Wang and P. Y. Wu, *Numerical ranges of Foguel operators*, Linear Algebra Appl. 610 (2021), 766–784.

[6] K. E. Gustafson and D. K. M. Rao, *Numerical Range. The Field of Values of Linear Operators and Matrices*, Springer, New York, 1997.

[7] P. R. Halmos, *On Foguel's answer to Nagy's question*, Proc. Amer. Math. Soc. 15 (1964), 791–793.

[8] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer, New York, 1982.

[9] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Dover Publ., New York, 2006.

[10] C.-K. Li, B.-S. Tam and P. Y. Wu, *The numerical range of a nonnegative matrix*, Linear Algebra Appl. 350 (2002), 1–23.

[11] G. Pisier, *A polynomially bounded operator on Hilbert space which is not similar to a contraction*, J. Amer. Math. Soc. 10 (1997), 351–369.

[12] K.-Z. Wang and P. Y. Wu, *Numerical ranges of weighted shifts*, J. Math. Anal. Appl. 381 (2011), 897–909.

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