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# Essential normality of Bergman modules over intersections of complex ellipsoids

by

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**Abstract.** This paper studies the essential normality of Bergman modules over the intersection of complex ellipsoids, as well as their quotients by monomial ideals.

**1. Introduction and statement of results.** A commuting tuple $(T_1, \ldots, T_m)$ of operators, also called a multioperator, on a Hilbert space $\mathcal{H}$ is *essentially normal* if all of the commutators $[T_j, T_k^*]$, $j, k = 1, \ldots, m$, are compact. Alternatively, essential normality can be attributed to the Hilbert $\mathbb{C}[z_1, \ldots, z_m]$-module generated by $(T_1, \ldots, T_m)$, that is, $\mathcal{H}$ equipped with the module action $P(z_1, \ldots, z_m) \cdot f$, $P \in \mathbb{C}[z_1, \ldots, z_m]$, $f \in \mathcal{H}$, given by $P(T_1, \ldots, T_m)f$. Brown, Douglas and Fillmore [BDF73, BDF77, D80] classified essentially normal multioperators up to unitary equivalence. The complete classifier here is the odd $K$-homology functor $K_1$ from the category of compact metrizable spaces to the category of abelian groups. More precisely, for any compact subspace $X \subseteq \mathbb{C}^m$, the abelian group $K_1(X)$ classifies essentially normal multioperators with essential Taylor spectrum $X$ up to unitary equivalence; the elements of $K_1(X)$ are equivalence classes of C*-monomorphisms from $C(X)$ to the algebra of bounded operators on $\mathcal{H}$ modulo the ideal of compact operators, the so-called *Calkin algebra*.

A rich source of essentially normal multioperators is given by the Arveson conjecture, which we now elaborate. Consider the Bergman space $L_a^2(\Omega)$ of square-integrable analytic functions on a bounded strongly pseudoconvex domain $\Omega \subseteq \mathbb{C}^m$ with smooth boundary. The multiplication by polynomials makes the Bergman space a Hilbert $\mathbb{C}[z_1, \ldots, z_m]$-module. Boutet de Monvel's theory of generalized Toeplitz operators shows that this Hilbert module

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is essentially normal [B79, BG81]. Let $I \subseteq \mathbb{C}[z_1, \ldots, z_m]$ be a homogeneous ideal of the ring of polynomials. The quotient Hilbert space $\mathcal{Q}_I := L^2_a(\Omega)/\overline{I}$ has a natural Hilbert module structure given by

$$P \cdot (f + \overline{I}) = Pf + \overline{I}, \quad P \in \mathbb{C}[z_1, \ldots, z_m], \quad f \in L^2_a(\Omega).$$

Transporting this action to the orthogonal complement

$$I^\perp = L^2_a(\Omega) \ominus \overline{I} \cong \mathcal{Q}_I$$

makes $I^\perp$ a Hilbert module. Alternatively, the module structure of $I^\perp$ is given by the compression $T_p := P_{I^\perp} M_p|_{I^\perp}$ of multiplication operators $M_p : L^2_a(\Omega) \to L^2_a(\Omega)$, where $P_{I^\perp}$ is the orthogonal projection in $L^2_a(\Omega)$ onto $I^\perp$. Arveson [A02, A05], based on his work on the model theory of spherical contractions in multivariate dilation theory, conjectured: *$I^\perp$ is essentially normal.* In other words, all commutators $[T_{z_j}, T^*_{z_k}]$, $j, k = 1, \ldots, m$, are compact. Arveson made his conjecture for the $m$-shift (or Drury–Arveson) space instead of the Bergman space; the use of Bergman spaces is due to Douglas [D06-1]. In the same paper, Douglas proposed the problem of the explicit computation of the element that $I^\perp$ represents in $K_1(X)$, where $X$, being the essential Taylor spectrum of $I^\perp$, can be canonically identified with the zero set $\{z \in \mathbb{C}^m : p(z) = 0, \forall p \in I\}$ of $I$ intersected with the unit sphere in $\mathbb{C}^m \cong \mathbb{R}^{2m}$ [GW08, Theorem 5.1]. This is the so-called *Douglas' index problem.* A summary of results about this conjecture/problem is given in [A15, Vol. 2, Chapter 41], [GW20]. In particular, when $\Omega$ is the unit ball and $I$ is monomial, Arveson's conjecture is proved in [A05, D06-2, DJTY18], and Douglas' index problem is answered in [DJTY18].

In another direction, the essential normality of Bergman modules over domains other than strongly pseudoconvex ones has been heavily studied in the literature. Here are some of the results:

- The Bergman module over a non-pseudoconvex, complete Reinhardt domain is not essentially normal [CS85].
- The Bergman module over any bounded, connected, planar domain is essentially normal [ACM82].
- The Bergman module over the polydiscs of dimension $> 1$ is not essentially normal. More generally, the Bergman module over bounded symmetric domains of rank $> 1$ is not essentially normal. This makes the spectral theory and index theory of Toeplitz operators more complicated on these domains [U84, U96], [Z19, Chapter 4].
- The essential normality of the Bergman module on a bounded, pseudoconvex domain is equivalent to the compactness of the $\overline{\partial}$-Neumann operator $N_1$ on $(0, 1)$-forms with $L^2$ coefficients [CD97, FS01, S89, SSU89]. Several sufficient conditions for this are given in the literature [C84-1, C84-2, HI97, M02]. For example, strongly pseudoconvex domains, domains of fi-

nite type, and pseudoconvex domains with real-analytic boundary have compact $N_1$.

- The Bergman module on a pseudoconvex, complete Reinhardt domain in $\mathbb{C}^2$ is essentially normal if and only if the boundary of the domain contains no one-dimensional holomorphic component [SSU89].
- The Bergman module over complex ellipsoids of the form

$$\Big\{\sum_{j=1}^{m} |z_j|^{2p_j} < 1\Big\} \subseteq \mathbb{C}^m, \qquad p_j > 0, \tag{1}$$

is essentially normal [CS85, CM85].

This article is about the essential normality of Bergman modules and their quotients over the intersection of complex ellipsoids of the form (1). Note that ellipsoids are pseudoconvex (because they are logarithmically convex, complete Reinhard domains [R86, Theorem 3.28]), hence so are their finite intersections [R86, p. 97]. Some intersections of ellipsoids are not essentially normal, such as the polydisks of dimension $> 1$; however:

Theorem 1. *Let $J \geq 1$, $K \geq 1$, $L_1, \ldots, L_K \geq 0$ be integers. The Bergman module over the domain $\Omega \subseteq \mathbb{C}^{J+L_1+\cdots+L_K}$ given by*

$$\Big\{\sum_{j=1}^{J} |z_j|^{2p_j} + \sum_{l=1}^{L_k} |w_{kl}|^{2q_{kl}} < 1 : k = 1, \ldots, K\Big\}, \qquad p_j, q_{kl} > 0, \tag{2}$$

*is essentially normal.*

This theorem will be proved in Section 2.

Remark 2. There is a finer version of essential normality: A Hilbert module over $\mathbb{C}[z_1, \ldots, z_m]$ is *$p$-essentially normal*, $0 < p < \infty$, if all of the commutators $[T_j, T_k^*]$, $j, k = 1, \ldots, m$, are *Schatten $p$-summable* (that is, $|[T_j, T_k^*]|^p$ is trace class), where $T_j$ is the module action corresponding to the coordinate function $z_j$. The $p$-essential normality of the Bergman module over the unit ball, and more generally, over ellipsoids of the form (1), is studied in [AFJP91, J22, KLR97]. In these cases, the Bergman module is $p$-essentially normal exactly when $p$ is strictly larger than a certain number. This cut-off value reflects the boundary geometry of the domain. It is interesting to study the $p$-essential normality of the Bergman space over domains of the form (2).

Next, we study the essential normality of quotients of the Bergman module over domains of the form (2) by monomial ideals.

Theorem 3. *Let $\Omega$ be a domain of the form* (2)*, and let $I$ be a monomial ideal.*

(a) *There are a positive integer $k$, essentially normal Hilbert $\mathbb{C}[z_1, \dots, z_m]$-modules $\mathcal{A}_0 := L_a^2(\Omega)$, $\mathcal{A}_1, \dots, \mathcal{A}_k$, and Hilbert $\mathbb{C}[z_1, \dots, z_m]$-module morphisms $\Psi_q : \mathcal{A}_q \to \mathcal{A}_{q+1}$, $q = 0, \dots, k-1$, such that the sequence*

$$0 \to \overline{I} \hookrightarrow \mathcal{A}_0 \xrightarrow{\Psi_0} \mathcal{A}_1 \xrightarrow{\Psi_1} \cdots \xrightarrow{\Psi_{k-1}} \mathcal{A}_k \to 0 \tag{3}$$

*is exact. Here, $\overline{I}$ denotes the closure of $I$ in the subspace topology of the Hilbert space $L_a^2(\Omega)$.*

(b) *$I^\perp$ is essentially normal.*

(c) *For each $q$, let $\sigma_e^q$ be the essential Taylor spectrum of the Hilbert module $\mathcal{A}_q$, and let $\alpha_q$ be the C*-monomorphism from $C(\sigma_e^q)$ to the Calkin algebra of $\mathcal{A}_q$ induced by essential normality. Then, in the group $K_1(\bigcup_{j=1}^k \sigma_e^j)$, the equivalence class induced by the essential normality of $I^\perp$ is given by the formula $\sum_{q=1}^k (-1)^{q-1}[\alpha_q]$.*

This theorem is proved in Section 3.

**2. Proof of Theorem 1.** Since $\Omega$ is a complete Reinhardt domain, polynomials are dense in $L_a^2(\Omega)$ with respect to the topology of uniform convergence on compact subsets [R86, p. 47]. Then a standard shrinking argument ([Z05, p. 43], [DS04, p. 11]) shows that the normalized monomials

$$b_{\alpha,\beta} := \frac{z^\alpha w^\beta}{\sqrt{\omega(\alpha,\beta)}} = \frac{z^\alpha \prod_{k=1}^K w_k^{2\beta_k}}{\sqrt{\omega(\alpha,\beta_1,\dots,\beta_K)}} = \frac{\prod_{j=1}^J z_j^{\alpha_j} \prod_{k=1}^K \prod_{l=1}^{L_k} w_{kl}^{\beta_{kl}}}{\sqrt{\omega(\alpha,\beta_1,\dots,\beta_K)}},$$

with $\alpha$ ranging over $\mathbb{N}^J$ and $\beta = (\beta_1, \dots, \beta_K)$ ranging over $\mathbb{N}^{L_1+\cdots+L_K}$ constitute an orthonormal basis for the Hilbert space $L_a^2(\Omega)$.

LEMMA 4. *The norm of the monomials in the Bergman space $L_a^2(\Omega)$ is given by*

$$\begin{aligned} \omega(\alpha,\beta) &:= \|z^\alpha w^\beta\|^2_{L_a^2(\Omega)} \\ &= \frac{2^K \pi^{J+L_1+\cdots+L_K}}{\prod p_j \prod q_{kl}} \frac{1}{\prod_{k=1}^K \left|\frac{2\beta_k+2}{q_k}\right|} \\ &\quad \times B\left(\left|\frac{\alpha+1}{p}\right|, \left|\frac{\beta+1}{q}\right| + 1\right) B\left(\frac{\alpha+1}{p}\right) \prod_{k=1}^K B\left(\frac{\beta_k+1}{q_k}\right), \end{aligned} \tag{4}$$

*where*

$$\frac{\alpha+1}{p} = \left(\frac{\alpha_1+1}{p_1}, \dots, \frac{\alpha_J+1}{p_J}\right), \qquad \frac{\beta_k+1}{q_k} = \left(\frac{\beta_{k1}+1}{q_{k1}}, \dots, \frac{\beta_{kL_k}+1}{q_{kL_k}}\right),$$

$$\left|\frac{\alpha+1}{p}\right| = \sum_{j=1}^J \frac{\alpha_j+1}{p_j}, \qquad \left|\frac{\beta_k+1}{q_k}\right| = \sum_{l=1}^{L_k} \frac{\beta_{kl}+1}{q_{k1}}, \qquad \left|\frac{\beta+1}{q}\right| = \sum_{k=1}^K \left|\frac{\beta_k+1}{q_k}\right|,$$

*and*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

$$B\left(\frac{\alpha+1}{p}\right) = \frac{\prod_{j=1}^J \Gamma\left(\frac{\alpha_j+1}{p_j}\right)}{\Gamma\left(\sum_{j=1}^J \frac{\alpha_j+1}{p_j}\right)}$$

*are multivariable Beta functions.*

*Proof.* (Compare [B16-1, B16-2, B17].) Using polar coordinates $z_j = x_j e^{\sqrt{-1}\,\theta_j}$, $w_{kl} = y_{kl} e^{\sqrt{-1}\,\varphi_{kl}}$, we have

$$\omega(\alpha,\beta) = (2\pi)^{J+L_1+\cdots+L_K} \times \int\limits_{x\in\mathbb{R}_+^J,\, y_k\in\mathbb{R}_+^{L_k},\, x^{2p}+y_k^{2q_k}<1,\, k=1,\ldots,K} x^{2\alpha+1} y^{2\beta+1}\,dx \prod_{k=1}^K dy_k,$$

where $dx = \prod_{j=1}^J dx_j$, $dy_k = \prod_{l=1}^{L_k} dy_{kl}$ are Lebesgue measures. After the change of variables $X_j := x_j^p$, $Y_{kl} := y_{kl}^{q_{kl}}$, we have

$$\omega(\alpha,\beta) = \frac{(2\pi)^{J+L_1+\cdots+L_K}}{\prod p_j \prod q_{kl}} \times \int\limits_{X\in\mathbb{R}_+^J,\, Y_k\in\mathbb{R}_+^{L_k},\, X^2+Y_k^2<1,\, k=1,\ldots,K} X^{\frac{2\alpha+2}{p}-1} Y^{\frac{2\beta+2}{q}-1}\,dX \prod_{k=1}^K dY_k.$$

Changing to the spherical coordinates $X = r\xi$, $Y_k = s_k\eta_k$, where $r, s_k$ are positive reals and $\xi, \eta_k$ live respectively on the unit spheres $\mathbb{S}^{J-1} \subseteq \mathbb{R}^J$, $\mathbb{S}^{L_k-1} \subseteq \mathbb{R}^{L_k}$, we have

$$\omega(\alpha,\beta) = \frac{(2\pi)^{J+L_1+\cdots+L_K}}{\prod p_j \prod q_{kl}} \times \int\limits_{r,s_k\in\mathbb{R}_+,\, r^2+s_k^2<1,\, k=1,\ldots,K} r^{|\frac{2\alpha+2}{p}|-1} \prod_{k=1}^K s_k^{|\frac{2\beta_k+2}{q_k}|-1}\,dr \prod_{k=1}^K ds_k \times \int\limits_{\xi\in\mathbb{S}_+^{J-1},\, \eta_k\in\mathbb{S}_+^{L_k-1},\, k=1,\ldots,K} \xi^{\frac{2\alpha+2}{p}-1} \eta^{\frac{2\beta+2}{q}-1}\,d\sigma_J(\xi) \prod_{k=1}^K d\sigma_k(\eta_k),$$

where $S_+^{J-1} := \mathbb{S}^{J-1} \cap \mathbb{R}_+^J$, and $d\sigma_J$ is the Riemannian density that $dX$ induces on $\mathbb{S}^{J-1}$, and similarly for others. The first integral is given by

$$\int_0^1 \int_0^{\sqrt{1-r^2}} \cdots \int_0^{\sqrt{1-r^2}} r^{\left|\frac{2\alpha+2}{p}\right|-1} \prod_{k=1}^K s_k^{\left|\frac{2\beta_k+2}{q_k}\right|-1} \, dr \prod_{k=1}^K ds_k$$
$$= \frac{1}{\prod_{k=1}^K \left|\frac{2\beta_k+2}{q_k}\right|} \int_0^1 r^{\left|\frac{2\alpha+2}{p}\right|-1} (1-r^2)^{\left|\frac{\beta+1}{q}\right|} \, dr$$
$$= \frac{1}{2\prod_{k=1}^K \left|\frac{2\beta_k+2}{q_k}\right|} B\left(\left|\frac{\alpha+1}{p}\right|, \left|\frac{\beta+1}{q}\right| + 1\right).$$

The second integral can be computed using the following famous formula [AAR99, Section 1.8], [Z05, p. 13]:

$$\int_{x\in\mathbb{S}_+^{m-1}} x^\alpha \, d\sigma_m(x) = 2^{1-m} B\left(\frac{\alpha+1}{2}\right).$$

This proves Lemma 4. ∎

Lemma 5. *For positive real numbers $a, b$, and a real variable $x$, we have*

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} x^{b-a} = 1 + \frac{(a-b)(a+b-1)}{2x} + \frac{(a-b)(a-b-1)(3(a+b-1)^2 - a + b - 1)}{24x^2} + O(x^{-3}),$$
$$\frac{\Gamma(x+a)^2}{\Gamma(x)\Gamma(x+2a)} = 1 - \frac{a^2}{x} + \frac{a^2(a^2+2a-1)}{2x^2} + O(x^{-3}) \quad \textit{as } x \to \infty.$$

*Proof.* The first formula is proved in [TE51], [AAR99, Appendix C]. The second formula follows immediately from the first one. ∎

Next, we show that the Bergman module $L_a^2(\Omega)$ is essentially normal. Let $M_{z_j}, M_{w_{kl}} \in B(L_a^2(\Omega))$ be the multiplication by the coordinate functions $z_j, w_{kl}$. Since these operators commute with each other, according to the Fuglede–Putnam theorem, it suffices to verify that each $M_{z_j}, M_{w_{kl}}$ is essentially normal. A straightforward computation shows that

$$[M_{z_1}, M_{z_1}^*](b_{\alpha,\beta}) = \lambda b_{\alpha,\beta}, \quad \forall \alpha \in \mathbb{N}^J, \forall \beta = (\beta_1, \ldots, \beta_K) \in \mathbb{N}^{L_1+\cdots+L_K},$$

where $\lambda = \lambda' - \lambda''$ and

$$\lambda' = \frac{\omega(\alpha,\beta)}{\omega(\alpha_1 - 1, \alpha_2, \ldots, \alpha_J, \beta)}, \quad \lambda'' = \frac{\omega(\alpha_1 + 1, \alpha_2, \ldots, \alpha_J, \beta)}{\omega(\alpha,\beta)},$$

and $\lambda'$ is set to be zero when $\alpha_1 = 0$. We need to check that $\lambda \to 0$ when the norm of $(\alpha, \beta)$ (say the $l^1$-norm $|\alpha| + |\beta|$) tends to infinity. By formula (4), we have

$$\lambda' = \begin{cases} \dfrac{\Gamma\left(\frac{\alpha_1+1}{p_1}\right)}{\Gamma\left(\frac{\alpha_1}{p_1}\right)} \dfrac{\Gamma\left(\frac{\alpha_1}{p_1} + A\right)}{\Gamma\left(\frac{\alpha_1+1}{p_1} + A\right)} & \text{if } \alpha_1 > 0, \\ 0 & \text{if } \alpha_1 = 0, \end{cases}$$

where

$$A := 1 + \sum_{j=2}^{J} \frac{\alpha_j + 1}{p_j} + \left|\frac{\beta + 1}{q}\right|.$$

Note that $\lambda''$ has the same expression as $\lambda'$ after replacing $\alpha_1$ by $\alpha_1 + 1$. According to Lemma 5, when $\alpha_1$ is bounded and $A \to \infty$, both $\lambda'$ and $\lambda''$ are dominated by $A^{-1/p_1}$, so $\lambda \to 0$. Next, assume that $\alpha_1 \to \infty$. We write $\lambda = \lambda'(1 - \lambda''/\lambda')$, where

$$\frac{\lambda''}{\lambda'} = \frac{\Gamma\big(\frac{\alpha_1+2}{p_1}\big)\Gamma\big(\frac{\alpha_1}{p_1}\big)}{\Gamma\big(\frac{\alpha_1}{p_1}\big)^2} \frac{\Gamma\big(\frac{\alpha_1+1}{p_1} + A\big)^2}{\Gamma\big(\frac{\alpha_1+1}{p_1} + A\big)\Gamma\big(\frac{\alpha_1}{p_1} + A\big)}.$$

According to Lemma 5, as $\alpha_1 \to \infty$, $\lambda'$ is bounded and $\lambda''/\lambda' \to 1$. Therefore, $\lambda \to 0$.

It remains to verify that $M_{w_{11}}$ is also essentially normal. We have

$$[M_{w_{11}}, M_{w_{11}}^*](b_{\alpha,\beta}) = \mu b_{\alpha,\beta}, \quad \forall \alpha \in \mathbb{N}^J, \forall \beta = (\beta_1, \ldots, \beta_K) \in \mathbb{N}^{L_1+\cdots+L_K},$$

where $\mu = \mu' - \mu''$ and

$$\mu' = \frac{\omega(\alpha, \beta)}{\omega(\alpha, \beta_{11} - 1, \beta_{12}, \ldots, \beta_{1L_1}, \ldots, \beta_{KL_K})},$$
$$\mu'' = \frac{\omega(\alpha, \beta_{11} + 1, \beta_{12}, \ldots, \beta_{1L_1}, \ldots, \beta_{KL_K})}{\omega(\alpha, \beta)},$$

and $\mu'$ is set to be zero when $\beta_{11} = 0$. We need to check that $\mu \to 0$ when $|\alpha| + |\beta| \to \infty$. By formula (4), we have

$$\mu' = \begin{cases} \dfrac{\Gamma\big(\frac{\beta_{11}+1}{q_{11}}\big)}{\Gamma\big(\frac{\beta_{11}}{q_{11}}\big)} \dfrac{\Gamma\big(\frac{\beta_{11}}{q_{11}} + B\big)}{\Gamma\big(\frac{\beta_{11}+1}{q_{11}} + B\big)} \dfrac{\Gamma\big(\frac{\beta_{11}+1}{q_{11}} + B + C\big)}{\Gamma\big(\frac{\beta_{11}}{q_{11}} + B + C\big)} \dfrac{\Gamma\big(\frac{\beta_{11}}{q_{11}} + B + C + D\big)}{\Gamma\big(\frac{\beta_{11}+1}{q_{11}} + B + C + D\big)} \\ \hfill \text{if } \beta_{11} > 0, \\ 0 \quad \text{if } \beta_{11} = 0, \end{cases}$$

where

$$B := 1 + \sum_{l=2}^{L_1} \frac{\beta_{1l} + 1}{q_{1l}}, \qquad C := \sum_{l=2}^{L_1} \frac{\beta_{1l} + 1}{q_{1l}} + \sum_{k=2}^{K} \left|\frac{\beta_k + 1}{q_k}\right|,$$
$$D := \sum_{l=2}^{L_1} \frac{\beta_{1l} + 1}{q_{1l}} + \sum_{k=2}^{K} \left|\frac{\beta_k + 1}{q_k}\right| + \left|\frac{\alpha + 1}{p}\right|.$$

Note that $\mu''$ has the same expression as $\mu'$ after replacing $\beta_{11}$ by $\beta_{11} + 1$. According to Lemma 5, when $\beta_{11}$ is bounded and $B + C + D \to \infty$, both $\mu'$ and $\mu''$ tend to zero, so $\mu \to 0$. Next, assume $\beta_{11} \to \infty$. We write $\mu =$

$\mu'(1-\mu''/\mu')$, where

$$\frac{\mu''}{\mu'} = PQ, \qquad P = \frac{\Gamma\big(\frac{\beta_{11}+2}{q_{11}}\big)\Gamma\big(\frac{\beta_{11}}{q_{11}}\big)}{\Gamma\big(\frac{\beta_{11}}{q_{11}}\big)^2}\,\frac{\Gamma\big(\frac{\beta_{11}+1}{q_{11}}+B\big)^2}{\Gamma\big(\frac{\beta_{11}+1}{q_{11}}+B\big)\Gamma\big(\frac{\beta_{11}}{q_{11}}+B\big)},$$

$$Q = \frac{\Gamma\big(\frac{\beta_{11}+2}{q_{11}}+B+C\big)\Gamma\big(\frac{\beta_{11}}{q_{11}}+B+C\big)}{\Gamma\big(\frac{\beta_{11}}{q_{11}}+B+C\big)^2} \times \frac{\Gamma\big(\frac{\beta_{11}+1}{q_{11}}+B+C+D\big)^2}{\Gamma\big(\frac{\beta_{11}+1}{q_{11}}+B+C+D\big)\Gamma\big(\frac{\beta_{11}}{q_{11}}+B+C+D\big)}.$$

According to Lemma 5, as $\beta_{11}\to\infty$, $\mu'$ is bounded and $\mu''/\mu'\to 1$. Therefore, $\mu\to 0$. This finishes the proof of Theorem 1.

**3. Proof of Theorem 3.** The theory developed in [DJTY18] for the case of the unit ball is applicable for the domain $\Omega$ (given in (2)) and therefore gives the resolution (3). Since the construction of this resolution is lengthy, we bring it in several steps.

STEP I: *Some notation.* Let the complex variables $\{\zeta_1,\dots,\zeta_m\}$ be an enumeration of $\{z_j : j=1,\dots,J\}\cup\{w_{kl} : k=1,\dots,K,\ l=1,\dots,L_k\}$. We will use the notation

$$\zeta^{\mathfrak{n}} := \frac{\zeta_1^{n^1}\cdots\zeta_m^{n^m}}{\sqrt{\omega(\mathfrak{n})}}, \qquad \mathfrak{n}=(n^1,\dots,n^m)\in\mathbb{N}^m,$$

for the elements of the monomial orthonormal basis of $L^2_a(\Omega)$. Given a positive integer $q$, let $S_q(m)$ denote the set of all *$q$-shuffles* of the set $\{1,\dots,m\}$, that is,

$$S_q(m) := \{\mathfrak{j} := (j^1,\dots,j^q)\in\mathbb{Z}^q : 1\le j^1<j^2<\dots<j^q\le m\}.$$

Whenever necessary, we identify shuffles in $S_q(m)$ with subsets of $\{1,\dots,m\}$ of size $q$. This enables us to talk about the union, intersection, etc. of shuffles of $\{1,\dots,m\}$ with themselves and with other subsets of $\{1,\dots,m\}$.

STEP II: *Boxes and their associated Hilbert modules.* To each shuffle $\mathfrak{j}=(j^1,\dots,j^q)\in S_q(m)$ and $\mathfrak{b}=(b^1,\dots,b^q)\in\mathbb{N}^q$, we associate the *box*

$$\mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}} := \{(n^1,\dots,n^m)\in\mathbb{N}^m : n^{j^i}\le b^i \text{ for } i=1,\dots,q\},$$

and to each box $\mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}$, we associate the Hilbert space

$$\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}} := L^2_a(\Omega)\ominus\overline{\langle\zeta_{j^1}^{b^1+1},\dots,\zeta_{j^q}^{b^q+1}\rangle}$$

consisting of all functions $X=\sum_{\mathfrak{n}\in\mathbb{N}^m}X_{\mathfrak{n}}\zeta^{\mathfrak{n}}\in L^2_a(\Omega)$ such that $X_{\mathfrak{n}}=0$ for every $\mathfrak{n}\in\mathbb{N}^m\setminus\mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}$. An element $X\in\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}}$ has the Taylor expansion $X=\sum X_{n^1\dots n^m}\zeta^{\mathfrak{n}}$ with summation over $n^{j^1}\le b^1,\dots,n^{j^q}\le b^q$. The general construction in the Introduction of the orthogonal complements of polynomial ideals makes $\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}}$ a Hilbert $\mathbb{C}[\zeta_1, \dots, \zeta_m]$-module. More explicitly, the action of the coordinate function $\zeta_i$ on $\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}}$ is given by the operator $T_{\zeta_i}^{\mathfrak{j},\mathfrak{b}} \in B(\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}})$ defined by

$$T_{\zeta_i}^{\mathfrak{j},\mathfrak{b}}(\zeta^{\mathfrak{n}}) := \begin{cases} \zeta_i z^{\mathfrak{n}} & \text{if } (n^1, \dots, n^{i-1}, n^i + 1, n^{i+1}, \dots, n^m) \in \mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}, \\ 0 & \text{otherwise.} \end{cases}$$

A fundamental fact is that each $\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}}$ is essentially normal. To prove this, according to the argument given in [DJTY18, Proposition 2.3], it suffices to verify that the ratio

$$\frac{\omega(n^1, \dots, n^{l-1}, b^l + 1, n^{l+1}, \dots, n^m)}{\omega(n^1, \dots, n^{l-1}, b^l, n^{l+1}, \dots n^m)},$$

with $l$ and $b_l$ fixed, approaches zero when the norm of

$$(n^1, \dots, n^{l-1}, b^l, n^{l+1}, \dots, n^m)$$

tends to infinity. This was verified during the proof of the essential normality of $L_a^2(\Omega)$ in Theorem 1.

Step III: *The construction of modules $\mathcal{A}_q$ in the resolution* (3). Let the ideal $I \subseteq \mathbb{C}[\zeta_1, \dots, \zeta_m]$ be generated by distinct monomials

$$\zeta^{\alpha_i}, \qquad \alpha_i := (\alpha_i^1, \dots, \alpha_i^m) \in \mathbb{N}^m,\ i = 1, \dots, l.$$

Let $\mathsf{C}(I) \subseteq \mathbb{N}^m$ be the set of the exponents of those monomials which do not belong to $I$. Note that the set of monomials belonging to $I$ is a basis of $I$ as a complex vector space [HH11, Theorem 1.1.2]. Also note that a monomial $u$ belongs to $I$ if and only if there is a monomial $v$ such that $u = vz^{\alpha_i}$ for some $i = 1, \dots, l$ [HH11, Proposition 1.1.5]. In other words, $\zeta_1^{n^1} \cdots \zeta_m^{n^m} \in \mathsf{C}(I)$ if and only if for every $i = 1, \dots, l$ there exists $s_i \in \{1, \dots, m\}$ such that $n^{s_i} < \alpha_i^{s_i}$. Consider the finite collection

$$S(\alpha_1, \dots, \alpha_l) := \{1, \dots, m\}^l$$

of $l$-tuples $\mathfrak{s} = (s_1, \dots, s_l)$ of integers such that $1 \le s_i \le m$ for every $i$. Given $\mathfrak{s}$, let $\mathfrak{j}_{\mathfrak{s}}$ be the shuffle associated to the set $\{s_1, \dots, s_l\}$. For each $j \in \mathfrak{j}_{\mathfrak{s}}$, let $b_j$ be the minimum of all $\alpha_i^{s_i} - 1$, $i = 1, \dots, l$, such that $s_i = j$. Set $\mathfrak{b}_{\mathfrak{s}} := (b_j)_{j \in \mathfrak{j}_{\mathfrak{s}}}$. It is not hard to show that $\mathsf{C}(I)$ is the union of the boxes $\mathbf{B}_{\mathfrak{j}_{\mathfrak{s}}}^{\mathfrak{b}_{\mathfrak{s}}}$, $\mathfrak{s} \in S(\alpha_1, \dots, \alpha_l)$. From now on, fix a finite collection of boxes

$$\mathbf{B}_{\mathfrak{j}_i}^{\mathfrak{b}_i}, \qquad i = 1, \dots, k,$$

such that their union equals $\mathsf{C}(I)$. Given $I \subseteq \{1, \dots, k\}$ (note that we are using the symbol $I$ for two purposes), let

$$\mathbf{B}_{\mathfrak{j}_I}^{\mathfrak{b}_I} := \bigcap_{i \in I} \mathbf{B}_{\mathfrak{j}_i}^{\mathfrak{b}_i}.$$

(Note that intersections of boxes are again boxes.) Each box $\mathbf{B}_{\mathfrak{j}_I}^{\mathfrak{b}_I}$ has a corresponding Hilbert module $\mathcal{H}_{\mathfrak{j}_I}^{\mathfrak{b}_I}$ as introduced in Step II. For each $q = 1, \ldots, k$, set

$$\mathcal{A}_q := \bigoplus_{I \in S_q(k)} \mathcal{H}_{\mathfrak{j}_I}^{\mathfrak{b}_I}, \qquad \mathcal{A}_0 := L_a^2(\Omega).$$

Note that each Hilbert space $\mathcal{A}_q$ has a Hilbert $\mathbb{C}[\zeta_1, \ldots, \zeta_m]$-module structure coming from the $\mathbb{C}[\zeta_1, \ldots, \zeta_m]$-module structures on its direct summands. Since Hilbert modules associated to boxes are essentially normal, it follows that each $\mathcal{A}_q$ is also essentially normal [D06-1, Theorem 2.2].

Step IV: *The construction of maps $\Psi_q$ in the resolution* (3). Thinking of the elements of $S_{q+1}(k)$ as the subsets $I_{q+1} \subseteq \{1, \ldots, k\}$ of size $q+1$, define the maps $f_{q+1}^i : S_{q+1}(k) \to S_q(k)$, $i = 1, \ldots, q+1$, by setting $f_{q+1}^i(I_{q+1})$ to be the subset of $\{1, \ldots, k\}$ obtained by dropping the $i$th smallest element in $I_{q+1}$. The map $\Psi_q : \mathcal{A}_q \to \mathcal{A}_{q+1}$ is defined by sending $X = \sum_{I_q \in S_q(k)} X^{I_q} \in \mathcal{A}_q$, $X^{I_q} \in \mathcal{H}_{\mathfrak{j}_{I_q}}^{\mathfrak{b}_{I_q}}$, to $Y = \sum_{I_{q+1} \in S_{q+1}(k)} Y^{I_{q+1}} \in \mathcal{A}_{q+1}$, $Y^{I_{q+1}} \in \mathcal{H}_{\mathfrak{j}_{I_{q+1}}}^{\mathfrak{b}_{I_{q+1}}}$, given by

$$(Y^{I_{q+1}})_{\mathfrak{n}} = \begin{cases} \sum_{i=1}^{q+1} (-1)^{i-1} (X^{f_{q+1}^i(I_{q+1})})_{\mathfrak{n}} & \text{if } \mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{I_{q+1}}}^{\mathfrak{b}_{I_{q+1}}}, \\ 0 & \text{otherwise.} \end{cases}$$

The arguments in [DJTY18] prove that the sequence (3) just constructed is a long exact sequence of Hilbert $\mathbb{C}[\zeta_1, \ldots, \zeta_m]$-modules and bounded module maps between them. This completes the proof of Theorem 3(a).

Set

$$\mathcal{A}_q^- := \operatorname{Im}(\Psi_{q-1}) = \operatorname{Ker}(\Psi_q), \qquad q = 1, \ldots, k-1.$$

The long exact sequence (3) can be decomposed into short exact sequences

$$(5) \qquad 0 \to \overline{I} \hookrightarrow \mathcal{A}_0 \xrightarrow{\Psi_0} \mathcal{A}_1^- \to 0,$$
$$0 \to \mathcal{A}_q^- \hookrightarrow \mathcal{A}_q \xrightarrow{\Psi_q} \mathcal{A}_{q+1}^- \to 0, \qquad q = 1, \ldots, k-2,$$
$$0 \to \mathcal{A}_{k-1}^- \hookrightarrow \mathcal{A}_{k-1} \xrightarrow{\Psi_{k-1}} \mathcal{A}_k \to 0.$$

A fundamental fact proved by Arveson [A05, Theorem 4.3] as well as Douglas [D06-2, Theorem 2.1], [D06-1, Theorem 2.2] says that in a short exact sequence of Hilbert modules, the essential normality of the middle and either of the other two modules implies the essential normality of the remaining. Applying this fact repeatedly on our short exact sequences in (5) implies that $\overline{I}$ is essentially normal. Applying the fact one more time to the short exact sequence

$$0 \to \overline{I} \to L_a^2(\Omega) \to L_a^2(\Omega)/\overline{I} \to 0$$

implies that $L_a^2(\Omega)/\overline{I} \cong I^\perp$ is essentially normal. This finishes the proof of Theorem 3(b).

Let $\alpha_q$ (respectively, $\alpha_q^-$) be the C*-monomorphism from $C(\sigma_e^q)$ to the Calkin algebra of $\mathcal{A}_q$ (respectively, from $C(\sigma_e^{q-})$ to the Calkin algebra of $\mathcal{A}_q^-$) induced by essential normality. Corollary 3.9 in [DJTY18] applied to the last two short exact sequences in (5) gives the canonical identifications

$$[\alpha_{k-1}] = [\alpha_{k-1}^-] + [\alpha_k] \in K_1(\sigma_e^{k-1}), \qquad [\alpha_{k-2}] = [\alpha_{k-2}^-] + [\alpha_{k-1}^-] \in K_1(\sigma_e^{k-2}).$$

Pushing forward these equations into $K_1(\sigma_e^{k-1} \cup \sigma_e^{k-2})$ by inclusion maps $\sigma_e^{k-1}, \sigma_e^{k-2} \hookrightarrow \sigma_e^{k-1} \cup \sigma_e^{k-2}$ gives

$$[\alpha_{k-2}^-] = [\alpha_{k-2}] - [\alpha_{k-1}] + [\alpha_k] \in K_1(\sigma_e^{k-1} \cup \sigma_e^{k-2}).$$

Continuing this argument, we have

$$[\alpha_1^-] = [\alpha_1] - [\alpha_2] + \cdots + (-1)^{k-1}[\alpha_k] \in K_1(\sigma_e^1 \cup \cdots \cup \sigma_e^k). \tag{6}$$

On the other hand, the short exact sequence

$$0 \to \overline{I} \to L_a^2(\Omega) \to \mathcal{A}_1^- \to 0$$

establishes a natural Hilbert module isomorphism between $\mathcal{A}_1^-$ and $L_a^2(\Omega)/\overline{I} \cong I^\perp$, hence the equivalence class represented by $I^\perp$ equals $[\alpha_1^-]$ according to [DTY16, Proposition 4.4]. This, together with (6), gives the index formula in Theorem 3(c). The proof of Theorem 3 is thus complete.

## References


[A15] D. Alpay (ed.), *Operator Theory*, Springer, Basel, 2015.

[AAR99] G. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.

[AFJP91] J. Arazy, S. Fisher, S. Janson and J. Peetre, *Membership of Hankel operators on the ball in unitary ideals*, J. London Math. Soc. (2) 43 (1991), 485–508.

[A02] W. Arveson, *The Dirac operator of a commuting d-tuple*, J. Funct. Anal. 89 (2002), 53–79.

[A05] W. Arveson, *p-Summable commutators in dimension d*, J. Operator Theory 54 (2005), 101–117.

[ACM82] S. Axler, J. Conway and G. McDonald, *Toeplitz operators on Bergman spaces*, Canad. J. Math. 34 (1982), 466–483.

[B16-1] T. Beberok, *The Bergman kernel for intersection of two complex ellipsoids*, Bull. Korean Math. Soc. 53 (2016), 1291–1308.

[B16-2] T. Beberok, *Lu Qi-Keng's problem for intersection of two complex ellipsoids*, Complex Anal. Operator Theory 10 (2016), 943–951.

[B17] T. Beberok, *The Bergman kernel: explicit formulas, deflation, Lu Qi-Keng problem and Jacobi polynomials*, Czechoslovak Math. J. 67 (142) (2017), 537–549.

[B79] L. Boutet de Monvel, *On the index of Toeplitz operators of several complex variables*, Invent. Math. 50 (1979), 249–272.

[BG81] L. Boutet de Monvel and V. Guillemin, *The Spectral Theory of Toeplitz Operators*, Ann. of Math. Stud. 99, Princeton Univ. Press, Princeton, NJ, and Univ. of Tokyo Press, Tokyo, 1981.

[BDF73] L. Brown, R. Douglas and P. Fillmore, *Unitary equivalence modulo the compact operators and extensions of $C^*$-algebras*, in: Proc. Conf. on Operator Theory (Halifax, NS, 1973), Lecture Notes in Math. 345, Springer, Berlin, 1973, 58–128.

[BDF77] L. Brown, R. Douglas and P. Fillmore, *Extensions of $C^*$-algebras and $K$-homology*, Ann. of Math. 105 (1977), 265–324.

[C84-1] D. Catlin, *Global regularity of the $\overline{\partial}$-Neumann problem*, in: Complex Analysis of Several Variables (Madison, WI, 1982), Proc. Sympos. Pure Math. 41, Amer. Math. Soc., Providence, RI, 1984, 39–49.

[C84-2] D. Catlin, *Boundary invariants of pseudoconvex domains*, Ann. of Math. 120 (1984), 529–586.

[CD97] D. Catlin and J. D'Angelo, *Positivity conditions for bihomogeneous polynomials*, Math. Res. Lett. 4 (1997), 555–567.

[CM85] R. Curto and P. Muhly, *$C^*$-algebras of multiplication operators on Bergman spaces*, J. Funct. Anal. 64 (1985), 315–329.

[CS85] R. Curto and N. Salinas, *Spectral properties of cyclic subnormal $m$-tuples*, Amer. J. Math. 107 (1985), 113–138.

[D80] R. Douglas, *$C^*$-algebra Extensions and $K$-homology*, Ann. of Math. Stud. 95, Princeton Univ. Press, Princeton, NJ, and Univ. of Tokyo Press, Tokyo, 1980.

[D06-1] R. Douglas, *A new kind of index theorem*, in: Analysis, Geometry and Topology of Elliptic Operators, World Sci., Hackensack, NJ, 2006, 369–382.

[D06-2] R. Douglas, *Essentially reductive Hilbert modules*, J. Operator Theory 55 (2006), 117–133.

[DJTY18] R. Douglas, M. Jabbari, X. Tang and G. Yu, *A new index theorem for monomial ideals by resolutions*, J. Funct. Anal. 275 (2018), 735–760.

[DTY16] R. Douglas, X. Tang and G. Yu, *An analytic Grothendieck Riemann Roch theorem*, Adv. Math. 294 (2016), 307–331.

[DS04] P. Duren and A. Schuster, *Bergman Spaces*, Amer. Math. Soc., Providence, RI, 2004.

[FS01] S. Fu and E. Straube, *Compactness in the $\overline{\partial}$-Neumann problem*, in: Complex Analysis and Geometry, J. McNeal (ed.), Ohio State Univ. Math. Res. Inst. Publ. 9, de Gruyter, Berlin, 2001, 141–160.

[GW08] K. Guo and K. Wang, *Essentially normal Hilbert modules and $K$-homology*, Math. Ann. 340 (2008), 907–934.

[GW20] K. Guo and Y. Wang, *A survey on the Arveson–Douglas conjecture*, in: Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology: Ronald G. Douglas Memorial Volume, R. Curto et al. (eds.), Birkhäuser, Cham, 2020, 289–311.

[HI97] G. Henkin and A. Iordan, *Compactness of the Neumann operator for hyperconvex domains with non-smooth B-regular boundary*, Math. Ann. 307 (1997), 151–168.

[HH11] J. Herzog and T. Hibi, *Monomial Ideals*, Springer, London, 2011.

[J22] M. Jabbari, *$p$-Summable commutators in Bergman spaces on egg domains*, Complex Anal. Operator Theory 16 (2022), no. 2, art. 21, 28 pp.

[KLR97] S. Krantz, S.-Y. Li and R. Rochberg, *The effect of boundary geometry on Hankel operators belonging to the trace ideals of Bergman spaces*, Integral Equations Operator Theory 28 (1997), 196–213.

[M02] J. McNeal, *A sufficient condition for compactness of the $\overline{\partial}$-Neumann operator*, J. Funct. Anal. 195 (2002), 190–205.

[R86] R. M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer, New York, 1986.

[S89] N. Salinas, *The $\overline{\partial}$-formalism and the $C^*$-algebra of the Bergman $n$-tuple*, J. Operator Theory 22 (1989), 325–343.

[SSU89] N. Salinas, A. Sheu and H. Upmeier, *Toeplitz operators on pseudoconvex domains and foliation $C^*$-algebras*, Ann. of Math. (2) 130 (1989), 531–565.

[TE51] F. Tricomi and A. Erdélyi, *The asymptotic expansion of a ratio of gamma functions*, Pacific J. Math. 1 (1951), 133–142.

[U84] H. Upmeier, *Toeplitz operators on bounded symmetric domains*, Ann. of Math. 119 (1984), 549–576.

[U96] H. Upmeier, *Toeplitz Operators and Index Theory in Several Complex Variables*, Birkhäuser, Basel, 1996.

[Z05] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer, New York, 2005.

[Z19] K. Zhu (ed.), *Handbook of Analytic Operator Theory*, CRC Press, Boca Raton, FL, 2019.

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