

## Sobolev spaces with non-isotropic dilations and square functions of Marcinkiewicz type

by

SHUICHI SATO (Kanazawa)

**Abstract.** We consider the weighted Sobolev spaces associated with non-isotropic dilations of Calderón–Torchinsky and characterize the spaces by the square functions of Marcinkiewicz type including those defined with repeated uses of averaging operation.

**1. Introduction.** Let  $B(x, t)$  be a ball in  $\mathbb{R}^n$  with radius  $t$  centered at  $x$ . For  $0 < \alpha < 2$  let

$$(1.1) \quad V_\alpha(f)(x) = \left( \int_0^\infty \left| f(x) - \int_{B(x,t)} f(y) dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

where  $\int_{B(x,t)} f(y) dy$  denotes  $|B(x, t)|^{-1} \int_{B(x,t)} f(y) dy$  and  $|B(x, t)|$  the Lebesgue measure. In [1] the operator  $V_1$  was used to characterize the Sobolev space  $W^{1,p}(\mathbb{R}^n)$  as follows.

**THEOREM A.** *Let  $1 < p < \infty$ . Then  $f$  belongs to  $W^{1,p}(\mathbb{R}^n)$  if and only if  $f \in L^p(\mathbb{R}^n)$  and  $V_1(f) \in L^p(\mathbb{R}^n)$ ; furthermore,*

$$\|V_1(f)\|_p \simeq \|\nabla f\|_p,$$

which means that there exist positive constants  $c_1, c_2$  independent of  $f$  such that

$$c_1 \|V_1(f)\|_p \leq \|\nabla f\|_p \leq c_2 \|V_1(f)\|_p.$$

Let  $\mathcal{S}(\mathbb{R}^n)$  be the Schwartz class of rapidly decreasing smooth functions on  $\mathbb{R}^n$ . Define

$$\mathcal{S}_0(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : \hat{f} \text{ vanishes near the origin}\},$$

---

2020 *Mathematics Subject Classification*: Primary 42B25; Secondary 46E35.

*Key words and phrases*: Littlewood–Paley functions, Marcinkiewicz function, Fourier multipliers, Sobolev spaces.

Received 19 August 2021.

Published online 24 June 2022.

where the Fourier transform  $\hat{f}$  is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx, \quad \langle x, \xi \rangle = \sum_{k=1}^n x_k \xi_k.$$

We also write  $\mathcal{F}(f)$  for  $\hat{f}$ . For  $0 < \alpha < n$ ,  $n \geq 2$ , let  $I_\alpha$  be the Riesz potential operator defined by

$$(1.2) \quad \mathcal{F}(I_\alpha(f))(\xi) = (2\pi|\xi|)^{-\alpha} \hat{f}(\xi), \quad f \in \mathcal{S}_0$$

(see [28, Chap. V]). Let

$$(1.3) \quad S_\alpha(f)(x) = \left( \int_0^\infty \left| I_\alpha(f)(x) - \int_{B(x,t)} I_\alpha(f)(y) dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

Then we also find the following result in [1].

**THEOREM B.** *Let  $0 < \alpha < 2$  and  $1 < p < \infty$ . Then*

$$\|S_\alpha(f)\|_p \simeq \|f\|_p.$$

Theorem A can be derived from this result with  $\alpha = 1$  when  $n \geq 2$ .

The operator  $S_\alpha$  is a kind of Littlewood–Paley operator. Let  $\psi \in L^1(\mathbb{R}^n)$  satisfy

$$(1.4) \quad \int_{\mathbb{R}^n} \psi(x) dx = 0.$$

Put  $\psi_t(x) = t^{-n} \psi(t^{-1}x)$ . Then the Littlewood–Paley function on  $\mathbb{R}^n$  is defined by

$$(1.5) \quad g_\psi(f)(x) = \left( \int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

We can see that  $S_\alpha(f) = g_{\psi^{(\alpha)}}(f)$ , where

$$(1.6) \quad \psi^{(\alpha)}(x) = L_\alpha(x) - \Phi * L_\alpha(x),$$

with

$$L_\alpha(x) = \tau(\alpha) |x|^{\alpha-n}, \quad \tau(\alpha) = \frac{\Gamma(n/2 - \alpha/2)}{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}$$

and  $\Phi = \chi_0$ ,  $\chi_0 = |B(0, 1)|^{-1} \chi_{B(0,1)}$  ( $\chi_E$  denotes the characteristic function of a set  $E$ ). We note that  $\mathcal{F}(L_\alpha)(\xi) = (2\pi|\xi|)^{-\alpha}$ ,  $0 < \alpha < n$ .

The square function  $S_1(f)$  is closely related to the Marcinkiewicz function on  $\mathbb{R}^1$ , which is defined by

$$\mu(f)(x) = \left( \int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where  $F(x) = \int_{-\infty}^x f(y) dy$  for  $f \in \mathcal{S}(\mathbb{R})$ . It is known that

$$(1.7) \quad \|\mu(f)\|_p \simeq \|f\|_p$$

for  $1 < p < \infty$ . Also, we consider a variant of  $\mu(f)$  which can be regarded as an analogue of  $S_1$  in the one-dimensional case:

$$\nu(f)(x) = \left( \int_0^{\infty} |F(x) - F * \Phi_t(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where  $\Phi = (1/2)\chi_{[-1,1]}$ . It is known that

$$\mu(f) = g_{\psi}(f) \quad \text{with} \quad \psi(x) = \chi_{[-1,1]}(x) \operatorname{sgn}(x).$$

By inspection, we see that  $\nu(f) = g_{\psi^{(0)}}(f)$ , where  $\psi^{(0)}(x) = (1/2)\psi(x) - (1/2)\psi^{(1)}(x)$  with  $\psi^{(1)}(x) = x\chi_{[-1,1]}(x)$ . This would indicate that the square functions  $\mu(f)$  and  $\nu(f)$  are intimately related. For the Marcinkiewicz function we refer to [14], Zygmund [32], Waterman [31].

An interesting feature of Theorem A is that it suggests the possibility of defining the Sobolev space analogous to  $W^{1,p}(\mathbb{R}^n)$  in metric measure spaces in a reasonable way. In this note, we shall extend Theorem A to the case of weighted Sobolev spaces with parabolic metrics of Calderón–Torchinsky [3, 4].

Let  $P$  be an  $n \times n$  real matrix,  $n \geq 2$ , such that

$$(1.8) \quad \langle Px, x \rangle \geq \langle x, x \rangle \quad \text{for all } x \in \mathbb{R}^n.$$

A dilation group  $\{\delta_t\}_{t>0}$  on  $\mathbb{R}^n$  is defined by  $\delta_t = t^P = \exp((\log t)P)$ .

It is known that  $|\delta_t x| = \langle \delta_t x, \delta_t x \rangle^{1/2}$  is strictly increasing as a function of  $t$  on  $(0, \infty)$  when  $x \neq 0$ . Let  $\rho(x)$ ,  $x \neq 0$ , be the unique positive real number  $t$  such that  $|\delta_{t^{-1}} x| = 1$ , and let  $\rho(0) = 0$ . Then the norm function  $\rho$  is continuous on  $\mathbb{R}^n$  and infinitely differentiable in  $\mathbb{R}^n \setminus \{0\}$  and satisfies  $\rho(A_t x) = t\rho(x)$ ,  $t > 0$ ,  $x \in \mathbb{R}^n$ . We have the following properties of  $\rho(x)$  (see [3, 5]):

- (1)  $\rho(-x) = \rho(x)$  for all  $x \in \mathbb{R}^n$ ;
- (2)  $\rho(x + y) \leq \rho(x) + \rho(y)$  for all  $x, y \in \mathbb{R}^n$ ;
- (3)  $\rho(x) \leq 1$  if and only if  $|x| \leq 1$ ;
- (4)  $c_1 \rho(x)^{\tau_1} \leq |x| \leq \rho(x)$  when  $|x| \leq 1$  for some  $c_1, \tau_1 > 0$ ;
- (5)  $\rho(x) \leq |x| \leq c_2 \rho(x)^{\tau_2}$  when  $|x| \geq 1$  for some  $c_2, \tau_2 > 0$ .

Moreover,

- (a)  $|\delta_t x| \geq t|x|$  for all  $x \in \mathbb{R}^n$  and  $t \geq 1$ ;
- (b)  $|\delta_t x| \leq t|x|$  for all  $x \in \mathbb{R}^n$  and  $0 < t \leq 1$ .

Let  $\delta_t^*$  denote the adjoint of  $\delta_t$ . Then we can also consider a norm function  $\rho^*(x)$  associated with the dilation group  $\{\delta_t^*\}_{t>0}$ , and we have properties of

$\rho^*(x)$  and  $\delta_t^*$  analogous to those of  $\rho(x)$  and  $\delta_t$  above. It is known that a polar coordinates expression for the Lebesgue measure

$$(1.9) \quad \int_{\mathbb{R}^n} f(x) dx = \int_0^\infty \int_{S^{n-1}} f(\delta_t \theta) t^{\gamma-1} s(\theta) d\sigma(\theta) dt$$

holds, where  $\gamma = \text{trace } P$  and  $s$  is a strictly positive  $C^\infty$  function on  $S^{n-1} = \{|x| = 1\}$  and  $d\sigma$  is the Lebesgue surface measure on  $S^{n-1}$  (see [7, 16, 29]). We note that the condition (1.8) implies that all eigenvalues of  $P$  have real parts greater than or equal to 1 (see [3, pp. 3–4], [13, p. 137]). So we have  $\gamma \geq n$ .

Let

$$(1.10) \quad B(x, t) = \{y \in \mathbb{R}^n : \rho(x - y) < t\}$$

be a ball with respect to  $\rho$  (a  $\rho$ -ball) in  $\mathbb{R}^n$  with radius  $t$  centered at  $x$ . We say that a weight function  $w$  belongs to the Muckenhoupt class  $A_p$ ,  $1 < p < \infty$ , if

$$[w]_{A_p} = \sup_B \left( |B|^{-1} \int_B w(x) dx \right) \left( |B|^{-1} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all  $\rho$ -balls  $B$  in  $\mathbb{R}^n$ . The Hardy–Littlewood maximal operator  $M$  is defined as

$$M(f)(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| dy,$$

where the supremum is taken over all  $\rho$ -balls  $B$  in  $\mathbb{R}^n$  containing  $x$ . The class  $A_1$  is defined to be the family of weight functions  $w$  such that  $M(w) \leq Cw$  almost everywhere; the infimum of all such  $C$  will be denoted by  $[w]_{A_1}$ . We denote by  $L_w^p$  (or  $L^p(w)$ ) the weighted  $L^p$  space with the norm defined as

$$\|f\|_{L_w^p} = \|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}.$$

See [2, 6, 9, 30] for results related to the weight class  $A_p$ . The following results are known and useful.

PROPOSITION 1.1. *Let  $1 < p < \infty$  and  $w \in A_p$ .*

- (i) *The space  $\mathcal{S}_0$  is dense in  $L_w^p$ .*
- (ii) *The maximal operator  $M$  is bounded on  $L_w^p$ .*
- (iii) *If  $\varphi \in \mathcal{S}$ , then  $\sup_{t>0} |f * \varphi_t| \leq CM(f)$ . Here and in what follows  $\varphi_t(x) = t^{-\gamma} \varphi(\delta_t^{-1} x)$ .*
- (iv)  *$\mathcal{F}(g * \varphi_t)(\xi) = \hat{g}(\xi) \hat{\varphi}(\delta_t^* \xi)$  for  $g, \varphi \in \mathcal{S}$ .*

Let  $\beta \in \mathbb{R}$  and define the Riesz potential operator  $\mathcal{I}_\beta$  associated with the dilations  $\delta_t^*$  by

$$(1.11) \quad \mathcal{F}(\mathcal{I}_\beta(f))(\xi) = \rho^*(\xi)^{-\beta} \hat{f}(\xi)$$

for  $f \in \mathcal{S}_0$ . Let  $1 < p < \infty$ ,  $\alpha > 0$  and  $w \in A_p$ . Define the weighted parabolic Sobolev space  $W_w^{\alpha,p}$  by

$$(1.12) \quad W_w^{\alpha,p} = \{f \in L_w^p : f = \mathcal{I}_\alpha(g) \text{ for some } g \in L_w^p\},$$

where  $f = \mathcal{I}_\alpha(g)$  means that

$$\int_{\mathbb{R}^n} f(x)h(x) dx = \int_{\mathbb{R}^n} g(x)\mathcal{I}_\alpha(h) dx \quad \text{for all } h \in \mathcal{S}_0.$$

We note that the function  $g \in L_w^p$  is uniquely determined by  $f$ , since  $\mathcal{I}_\alpha$  is a bijection on  $\mathcal{S}_0$  and  $\mathcal{S}_0$  is dense in  $L^{p'}(w^{-p'/p})$ , the dual space of  $L^p(w)$ , where  $1/p + 1/p' = 1$ . We write  $g = \mathcal{I}_{-\alpha}(f)$ . For  $f \in W_w^{\alpha,p}$  we define

$$(1.13) \quad \|f\|_{p,\alpha,w} = \|f\|_{p,w} + \|\mathcal{I}_{-\alpha}(f)\|_{p,w}.$$

We have analogues of Theorems A and B in the case of non-isotropic dilations  $\delta_t$  with weights. Let  $B(x,t)$  be as in (1.10) and

$$(1.14) \quad B_\alpha(f)(x) = \left( \int_0^\infty \left| f(x) - \int_{B(x,t)} f(y) dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad \alpha > 0.$$

**THEOREM 1.2.** *Suppose that  $1 < p < \infty$ ,  $w \in A_p$  and  $0 < \alpha < 2$ . Then  $f \in W_w^{\alpha,p}$  if and only if  $f \in L_w^p$  and  $B_\alpha(f) \in L_w^p$ ; moreover,*

$$\|\mathcal{I}_{-\alpha}(f)\|_{p,w} \simeq \|B_\alpha(f)\|_{p,w}.$$

Let

$$(1.15) \quad C_\alpha(f)(x) = \left( \int_0^\infty \left| \mathcal{I}_\alpha(f)(x) - \int_{B(x,t)} \mathcal{I}_\alpha(f)(y) dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}.$$

Then Theorem 1.2 can be derived from the following result.

**THEOREM 1.3.** *Let  $1 < p < \infty$ ,  $w \in A_p$ ,  $0 < \alpha < 2$  and let  $C_\alpha$  be as in (1.15). Then*

$$\|C_\alpha(f)\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

The range of  $\alpha$  in Theorem 1.2 will be extended in Theorem 4.2 by considering square functions with repeated uses of averaging operation  $\int_B f$ .

We consider square functions generalizing  $B_\alpha$  and  $C_\alpha$  in (1.14) and (1.15). Let  $\Phi$  be a bounded function on  $\mathbb{R}^n$  with compact support. We say that  $\Phi \in \mathcal{M}^\alpha$ ,  $\alpha \geq 0$ , if  $\Phi$  satisfies

- (i)  $\int_{\mathbb{R}^n} \Phi(x) dx = 1$ ;
- (ii) if  $\alpha \geq 1$ , then

$$(1.16) \quad \int_{\mathbb{R}^n} \Phi(x)x^a dx = 0 \quad \text{for all multi-indices } a \text{ with } 1 \leq |a| \leq [\alpha],$$

where  $x^a = x_1^{a_1} \dots x_n^{a_n}$  with  $a = (a_1, \dots, a_n)$ ,  $|a| = a_1 + \dots + a_n$ ,  $a_j \in \mathbb{Z}$ ,  $a_j \geq 0$ ,  $1 \leq j \leq n$ , and  $[\alpha] = \max \{k \in \mathbb{Z} : k \leq \alpha\}$ .

We note that  $\mathcal{M}^\alpha \subset \mathcal{M}^\beta$  if  $\alpha \geq \beta$  and  $\mathcal{M}^\alpha = \mathcal{M}^j$  if  $j \leq \alpha < j + 1$ ,  $j \geq 0$ ,  $j \in \mathbb{Z}$ . If  $\Phi$  is even and  $1 \leq \alpha < 2$ , we have (1.16). In particular,  $\chi_0 = |B(0, 1)|^{-1} \chi_{B(0, 1)} \in \mathcal{M}^\alpha$  for  $0 \leq \alpha < 2$ .

Let  $\Phi \in \mathcal{M}^\alpha$  and

$$(1.17) \quad G_\alpha(f)(x) = \left( \int_0^\infty |f(x) - \Phi_t * f(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad \alpha > 0.$$

We note that if  $\Phi = \chi_0$  in (1.17), we get  $B_\alpha$  of (1.14). Also, let  $\Phi \in \mathcal{M}^\alpha$  and

$$(1.18) \quad H_\alpha(f)(x) = \left( \int_0^\infty |\mathcal{I}_\alpha(f)(x) - \Phi_t * \mathcal{I}_\alpha(f)(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad 0 < \alpha < \gamma.$$

If we set  $\Phi = \chi_0$  in (1.18), we get  $C_\alpha$  of (1.15) for  $0 < \alpha < 2$ .

We prove the following.

**THEOREM 1.4.** *Let  $H_\alpha$  be as in (1.18) and  $0 < \alpha < \gamma$ ,  $1 < p < \infty$ ,  $w \in A_p$ . Then*

$$\|H_\alpha(f)\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

Applying Theorem 1.4, we obtain the following.

**THEOREM 1.5.** *Suppose that  $1 < p < \infty$ ,  $w \in A_p$  and  $0 < \alpha < \gamma$ . Let  $G_\alpha$  be as in (1.17). Then  $f \in W_w^{\alpha,p}$  if and only if  $f \in L_w^p$  and  $G_\alpha(f) \in L_w^p$ ; furthermore,*

$$\|\mathcal{I}_{-\alpha}(f)\|_{p,w} \simeq \|G_\alpha(f)\|_{p,w}.$$

Theorems 1.2 and 1.3 follow from Theorems 1.5 and 1.4, respectively. The proofs of Theorems 1.4 and 1.5 will be given in Section 3. To prove Theorem 1.4, we consider the Littlewood–Paley functions

$$(1.19) \quad g_\psi(f)(x) = \left( \int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where  $\psi_t(x) = t^{-\gamma} \psi(\delta_t^{-1}x)$  with  $\psi \in L^1(\mathbb{R}^n)$  satisfying (1.4). Then we can see that  $H_\alpha(f) = g_{\psi^{(\alpha)}}$  for some  $\psi^{(\alpha)}$  analogous to the one in (1.6). We shall prove Theorem 1.4 by applying Theorem 2.1 below in Section 2, which is a result for parabolic Littlewood–Paley functions complementing the boundedness result given in [25] and generalizing [22, Corollary 2.11] to the case of non-isotropic dilations.

The proof of Theorem 2.1 will be completed by applying Theorem 2.8, which provides the estimates

$$(1.20) \quad \|f\|_{p,w} \leq C \|g_\psi(f)\|_{p,w}$$

under certain conditions. Theorem 2.8 is deduced from Corollary 2.7, which is a result on the invertibility of Fourier multipliers homogeneous of degree 0

with respect to  $\delta_t^*$  generalizing [22, Corollary 2.6] to the case of general homogeneity. Corollary 2.7 will follow from a more general result (Theorem 2.3).

Here we review some recent developments of the theory related to the results given in this note after the article [1] (see also the remarks at the end of this note).

Theorem A was generalized to the weighted Sobolev spaces in [10]. Also, Theorems A and B were extended to the weighted Sobolev spaces in [19] by applying a theorem of [17] for the boundedness of Littlewood–Paley functions  $g_\psi$  in (1.5) on the weighted  $L^p$  spaces, which is partly a special case of Theorem 2.1.

In [19] it was shown that the theorem of [17] is particularly suitable for handling the square functions in Theorem 1.4 for the case of the Euclidean structures (with the Euclidean norm and the ordinary dilation). Some results of [19] were generalized in [22] by introducing the function class  $\mathcal{M}^\alpha$  and by proving the weighted  $L^p$  norm equivalence between  $g_\psi(f)$  in (1.5) and  $f$ , part of which was not included in [17]; the estimates in (1.20) in the case of the Euclidean structures for a sufficiently large class of  $\psi$  and  $p \in (1, \infty)$ ,  $w \in A_p$  were absent from [17].

In [20] and [22], discrete parameter versions of Littlewood–Paley functions  $g_\psi(f)$  in (1.5) of the form

$$\Delta_\psi(f)(x) = \left( \sum_{k=-\infty}^{\infty} |f * \psi_{2^k}(x)|^2 \right)^{1/2}$$

are also applied to characterize Sobolev spaces. See also [10] and [21] for applications of the square function

$$D^\alpha(f)(x) = \left( \int_0^\infty \left| t^{-\alpha} \int_{S^{n-1}} (f(x - t\theta) - f(x)) d\sigma(\theta) \right|^2 \frac{dt}{t} \right)^{1/2}$$

in the theory of Sobolev spaces.

In Section 4, we shall establish another characterization of the Sobolev spaces  $W_w^{\alpha,p}$  similar to Theorem 1.2 (Theorem 4.2), which is novel even in the case of the Euclidean structures. In Theorem 1.2, the averaging operator  $\mathfrak{J}_B f$  is used to define the square function  $B_\alpha(f)$  in (1.14), which is applied to characterize  $W_w^{\alpha,p}$  for  $\alpha \in (0, 2)$ . In Theorem 4.2 we shall extend the range of  $\alpha$  by introducing square functions which are defined with repeated uses of the averaging operation.

Finally, in Section 5 we shall illustrate by example how the Sobolev spaces  $W_w^{\alpha,p}$  defined above can be characterized by distributional derivatives in some cases, by the arguments similar to the one in [28, Chap. V, proof of Theorem 3].

**2. Invertibility of Fourier multipliers homogeneous with respect to  $\delta_t^*$  and Littlewood–Paley operators.** We consider a majorant of  $\psi$  defined by

$$H_\psi(x) = h(\rho(x)) = \sup_{\rho(y) \geq \rho(x)} |\psi(y)|$$

and two seminorms  $B_\epsilon$  and  $D_u$  defined as

$$B_\epsilon(\psi) = \int_{|x|>1} |\psi(x)| |x|^\epsilon dx \quad \text{for } \epsilon > 0,$$

$$D_u(\psi) = \left( \int_{|x|<1} |\psi(x)|^u dx \right)^{1/u} \quad \text{for } u > 1.$$

In proving Theorem 1.4 we apply the following result.

**THEOREM 2.1.** *Suppose that  $\psi \in L^1(\mathbb{R}^n)$  satisfies (1.4). Let  $\epsilon > 0$ ,  $u > 1$  and  $C_j > 0$ ,  $1 \leq j \leq 3$ . Suppose that*

- (1)  $B_\epsilon(\psi) \leq C_1$ ;
- (2)  $D_u(\psi) \leq C_2$ ;
- (3)  $\|H_\psi\|_1 \leq C_3$ .

Then  $g_\psi$  defined by (1.19) is bounded on  $L_w^p$ :

$$(2.1) \quad \|g_\psi(f)\|_{p,w} \leq C \|f\|_{p,w} \quad \text{for all } p \in (1, \infty) \text{ and } w \in A_p,$$

where the constant  $C$  depends only on  $p$ ,  $w$ ,  $\epsilon$ ,  $u$  and  $C_j$ ,  $1 \leq j \leq 3$ , and does not otherwise depend on  $\psi$ . If we further assume the non-degeneracy condition

$$(2.2) \quad \sup_{t>0} |\hat{\psi}(\delta_t^* \xi)| > 0 \quad \text{for } \xi \neq 0,$$

then we also have the reverse inequality of (2.1) and hence

$$\|g_\psi(f)\|_{p,w} \simeq \|f\|_{p,w} \quad \text{for all } p \in (1, \infty) \text{ and } w \in A_p.$$

By [25, Theorem 1.1], which generalizes a result of [17] to the case of non-isotropic dilations, we have the boundedness (2.1) under conditions (1)–(3) of Theorem 2.1, and the quantitative property of the constant  $C$  specified follows by checking the proof in [25]. The proof of [25, Theorem 1.1] is based on estimates for oscillatory integrals in [18].

**REMARK 2.2.** If there exist positive numbers  $\sigma_1, \sigma_2$  such that

$$|\psi(x)| \leq C(1 + \rho(x)^{-1})^{\gamma - \sigma_1} (1 + \rho(x))^{-\gamma - \sigma_2} \quad \text{for all } x \in \mathbb{R}^n,$$

then conditions (1)–(3) of Theorem 2.1 are satisfied with some  $\epsilon$ ,  $u$  and  $C_j$ ,  $1 \leq j \leq 3$ . To see this, the formula (1.9) is useful.

To prove the reverse inequality of (2.1), we apply a result on the invertibility on weighted  $L^p$  spaces of Fourier multipliers homogeneous with



respect to  $\delta_t^*$ . Let  $m \in L^\infty(\mathbb{R}^n)$ ,  $w \in A_p$ ,  $1 < p < \infty$ . The Fourier multiplier operator  $T_m$  is defined by

$$(2.3) \quad T_m(f)(x) = \int_{\mathbb{R}^n} m(\xi) \hat{f}(\xi) e^{2\pi i(x,\xi)} d\xi.$$

We say that  $m$  is a *Fourier multiplier* for  $L_w^p$  and write  $m \in M_w^p$  (we also write  $M^p(w)$  for  $M_w^p$ ) if there exists a constant  $C > 0$  such that

$$(2.4) \quad \|T_m(f)\|_{p,w} \leq C \|f\|_{p,w} \quad \text{for all } f \in \mathcal{S}.$$

We define  $\|m\|_{M^p(w)}$  to be the infimum of the constants  $C$  satisfying (2.4). Since  $\mathcal{S}$  is dense in  $L_w^p$ , we have a unique extension of  $T_m$  to a bounded linear operator on  $L_w^p$  if  $m \in M_w^p$ . We observe that  $M^p(w) = M^{p'}(\tilde{w}^{-p'/p})$  by duality, where  $\tilde{w}(x) = w(-x)$ . See [12] for relevant results.

We need the following result generalizing [22, Theorem 2.5] to the case of non-isotropic dilations.

**THEOREM 2.3.** *Let  $m$  be a bounded function on  $\mathbb{R}^n$  which is continuous on  $\mathbb{R}^n \setminus \{0\}$ . Suppose that  $m$  is homogeneous of degree 0 with respect to  $\delta_t^*$  and that  $m(\xi) \neq 0$  for all  $\xi \in \mathbb{R}^n \setminus \{0\}$ . Also, suppose that  $m \in M_v^r$  for all  $r \in (1, \infty)$  and all  $v \in A_r$ . Let  $1 < p < \infty$ ,  $w \in A_p$  and let  $F(z)$  be holomorphic in  $D = \mathbb{C} \setminus \{0\}$ . Then  $F(m(\xi)) \in M_w^p$ .*

For  $m \in M_w^p$ ,  $1 < p < \infty$ ,  $w \in A_p$ , we consider the spectral radius operator

$$\rho_{p,w}(m) = \lim_{k \rightarrow \infty} \|m^k\|_{M^p(w)}^{1/k}.$$

To prove Theorem 2.3, we need the following.

**PROPOSITION 2.4.** *Suppose that  $1 < p < \infty$ ,  $w \in A_p$  and  $m \in L^\infty(\mathbb{R}^n)$ . Let  $m$  be homogeneous of degree 0 with respect to the dilations  $\delta_t^*$  and continuous on  $S^{n-1}$ . Assume that  $m \in M_v^r$  for all  $r \in (1, \infty)$  and all  $v \in A_r$ . Then, for any  $\epsilon > 0$ , there exists  $\ell \in M_w^p$  which is homogeneous of degree 0 with respect to  $\delta_t^*$  and in  $C^\infty(\mathbb{R}^n \setminus \{0\})$  such that  $\|m - \ell\|_\infty < \epsilon$  and  $\rho_{p,w}(m - \ell) < \epsilon$ .*

To prove Proposition 2.4, we apply the following lemmas.

**LEMMA 2.5.** *Let  $\eta \in C^\infty(\mathbb{R})$ ,  $\text{supp } \eta \subset [1, 2]$ ,  $\eta \geq 0$  and  $\int_0^\infty |\eta(t)|^2 dt/t = 1$ . Define a real function  $\psi$  in  $\mathcal{S}_0(\mathbb{R}^n)$  by  $\hat{\psi}(\xi) = \eta(\rho^*(\xi))$ . Then*

$$\|g_\psi(f)\|_{p,w} \simeq \|f\|_{p,w} \quad \text{for all } p \in (1, \infty) \text{ and } w \in A_p.$$

**LEMMA 2.6.** *Suppose that  $m \in L^\infty(\mathbb{R}^n)$ ,  $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$  and that  $m$  is homogeneous of degree 0 with respect to  $\delta_t^*$ . Then  $m \in M_w^p$  for all  $p \in (1, \infty)$  and  $w \in A_p$  and*

$$\|m\|_{M^p(w)} \leq C \sup_{1 \leq \rho^*(\xi) \leq 2, |a| \leq [\gamma] + 1} |(\partial_\xi)^a m(\xi)|$$

with a constant  $C$  independent of  $m$ , where  $(\partial_\xi)^a = (\partial/\partial\xi_1)^{a_1} \dots (\partial/\partial\xi_n)^{a_n}$  with  $a = (a_1, \dots, a_n)$ ,  $a_j \in \mathbb{Z}$ ,  $a_j \geq 0$ ,  $1 \leq j \leq n$ .

*Proof of Lemma 2.5.* By [25, Theorem 1.1] we see that  $\|g_\psi(f)\|_{p,w} \leq C\|f\|_{p,w}$  for all  $p \in (1, \infty)$  and  $w \in A_p$ . To prove the reverse inequality we note that  $\|g_\psi(f)\|_2 = \|f\|_2$ . Thus the polarization implies that for real-valued  $f, h \in \mathcal{S}$ ,

$$\begin{aligned} 4 \int_{\mathbb{R}^n} f(x)h(x) dx &= \int_{\mathbb{R}^n} (f(x) + h(x))^2 dx - \int_{\mathbb{R}^n} (f(x) - h(x))^2 dx \\ &= \int_{\mathbb{R}^n} (g_\psi(f + h)(x))^2 dx - \int_{\mathbb{R}^n} (g_\psi(f - h)(x))^2 dx \\ &= 4 \int_{\mathbb{R}^n} \int_0^\infty f * \psi_t(x) h * \psi_t(x) \frac{dt}{t} dx. \end{aligned}$$

Therefore, by the inequalities of Schwarz and Hölder we have

$$\left| \int_{\mathbb{R}^n} f(x)h(x) dx \right| \leq \|g_\psi(f)\|_{p,w} \|g_\psi(h)\|_{p',w^{-p'/p}} \leq C \|g_\psi(f)\|_{p,w} \|h\|_{p',w^{-p'/p}}.$$

Taking the supremum over  $h$  with  $\|h\|_{p',w^{-p'/p}} \leq 1$ , we find that  $\|f\|_{p,w} \leq C\|g_\psi(f)\|_{p,w}$ , from which we can derive the desired estimates for complex-valued functions. ■

*Proof of Lemma 2.6.* Let  $\psi$  be as in Lemma 2.5 and define  $\psi_m$  by  $\mathcal{F}(\psi_m)(\xi) = \hat{\psi}(\xi)m(\xi)$ . Then  $g_\psi(T_m f) = g_{\psi_m}(f)$ . So, by Lemma 2.5 for  $w \in A_p$ ,  $1 < p < \infty$ , we have

$$(2.5) \quad \|T_m f\|_{p,w} \leq C \|g_\psi(T_m f)\|_{p,w} = C \|g_{\psi_m}(f)\|_{p,w}.$$

Since  $\psi_m \in \mathcal{S}_0$ ,  $g_{\psi_m}$  is bounded on  $L_w^p$ . To specify the operator bounds, we apply the estimates (2.1). It is sufficient to observe the following estimates:

$$(2.6) \quad |\psi_m(x)| = \left| \int_{\mathbb{R}^n} \hat{\psi}(\xi)m(\xi)e^{2\pi i\langle x,\xi \rangle} d\xi \right| \leq C(1 + |x|)^{-[\gamma]-1} \sup_{1 \leq \rho^*(\xi) \leq 2, |a| \leq [\gamma]+1} |(\partial_\xi)^a m(\xi)|,$$

which follows by integration by parts, with the constant  $C$  independent of  $m$ . Combining (2.5), (2.6) and the estimates (2.1), we obtain the conclusion. ■

*Proof of Proposition 2.4.* As in [11], [22], we take a sequence  $\{\varphi_j\}_{j=1}^\infty$  of functions on the orthogonal group  $O(n)$  with the following properties:

- (1) each  $\varphi_j$  is infinitely differentiable, non-negative and  $\int_{O(n)} \varphi_j(A) dA = 1$ , where  $dA$  is the Haar measure on  $O(n)$ ;
- (2) for any neighborhood  $U$  of the identity of  $O(n)$ , there exists a positive integer  $N$  such that  $\text{supp}(\varphi_j) \subset U$  for  $j \geq N$ .

For  $\xi \in S^{n-1}$ , let

$$\tilde{m}_j(\xi) = \int_{O(n)} m(A\xi)\varphi_j(A) dA.$$

Then  $\tilde{m}_j$  is  $C^\infty$  on  $S^{n-1}$  (see [11, pp. 123–124]). For  $\xi \in \mathbb{R}^n \setminus \{0\}$ , let

$$m_j(\xi) = \tilde{m}_j(\delta_{\rho^*(\xi)}^{-1}\xi).$$

Then  $m_j$  is homogeneous of degree 0 with respect to  $\delta_t^*$ ,  $m_j \in C^\infty(\mathbb{R}^n \setminus \{0\})$  and  $m_j = \tilde{m}_j$  on  $S^{n-1}$ .

We prove

$$(2.7) \quad \rho_{r,v}(m_j) \leq \|m\|_\infty, \quad r \in (1, \infty), v \in A_r.$$

For this it suffices to show that

$$\|m_j^k\|_{M^r(v)} \leq C_j k^{[\gamma]+1} \|m\|_\infty^k,$$

where  $C_j$  is independent of  $k$ . This follows by Lemma 2.6, since

$$\sup_{1 \leq \rho^*(\xi) \leq 2, |a| \leq [\gamma]+1} |(\partial_\xi)^a m_j(\xi)^k| \leq C_j k^{[\gamma]+1} \|m\|_\infty^k.$$

To see this, it is helpful to refer to [11, pp. 123–124].

Since  $m_j \rightarrow m$  as  $j \rightarrow \infty$  uniformly on  $S^{n-1}$ , we can take  $\ell = m_j$  for  $j$  large enough to get  $\|m - \ell\|_\infty < \epsilon$ . Let  $p \in (1, \infty)$ ,  $w \in A_p$ . Confirming that a result analogous to [22, Proposition 2.2] holds true in the setting of non-isotropic dilations, we can find  $r > 1$ ,  $s > 1$  and  $\theta \in (0, 1)$  such that  $w^s \in A_r$  and

$$\|(m - m_j)^k\|_{M^p(w)} \leq \|(m - m_j)^k\|_\infty^{1-\theta} \|(m - m_j)^k\|_{M^r(w^s)}^\theta.$$

Thus

$$\rho_{p,w}(m - m_j) \leq \|m - m_j\|_\infty^{1-\theta} \rho_{r,w^s}(m - m_j)^\theta.$$

Since

$$\rho_{r,w^s}(m - m_j) \leq \rho_{r,w^s}(m) + \rho_{r,w^s}(m_j)$$

(see Riesz–Nagy [15, p. 426]), it follows that

$$\begin{aligned} \rho_{p,w}(m - m_j) &\leq \|m - m_j\|_\infty^{1-\theta} (\rho_{r,w^s}(m) + \rho_{r,w^s}(m_j))^\theta \\ &\leq \|m - m_j\|_\infty^{1-\theta} (\rho_{r,w^s}(m) + \|m\|_\infty)^\theta, \end{aligned}$$

where the last inequality follows from (2.7). Since  $\|m - m_j\|_\infty \rightarrow 0$  as  $j \rightarrow \infty$ , for a given  $\epsilon > 0$ , taking  $\ell = m_j$  with  $j$  large enough, we have  $\rho_{p,w}(m - \ell) < \epsilon$  and  $\|m - \ell\|_\infty < \epsilon$ . ■

*Proof of Theorem 2.3.* The proof is similar to that of [22, Theorem 2.5]. Let

$$\epsilon_0 = \frac{1}{4} \min_{\xi \in S^{n-1}} |m(\xi)|.$$

Applying Proposition 2.4, we can find  $\ell \in M_w^p$  which is homogeneous of degree 0 with respect to  $\delta_t^*$  and belongs to  $C^\infty(\mathbb{R}^n \setminus \{0\})$  such that  $\|m - \ell\|_\infty < \epsilon_0$  and  $\rho_{p,w}(m - \ell) < \epsilon_0$ . Let  $C : \ell(\xi) + 2\epsilon_0 e^{i\theta}$ ,  $0 \leq \theta \leq 2\pi$ , be a circle in  $D$ . Apply Cauchy's formula to get

$$(2.8) \quad F(m(\xi)) = \frac{1}{2\pi i} \int_C \frac{F(\zeta)}{\zeta - m(\xi)} d\zeta = \frac{\epsilon_0}{\pi} \int_0^{2\pi} \frac{F(\ell(\xi) + 2\epsilon_0 e^{i\theta})}{2\epsilon_0 e^{i\theta} + \ell(\xi) - m(\xi)} e^{i\theta} d\theta$$

for  $\xi \in \mathbb{R}^n \setminus \{0\}$ . We expand the integrand in the last integral into a power series by using

$$(2.9) \quad \frac{e^{i\theta}}{2\epsilon_0 e^{i\theta} + \ell(\xi) - m(\xi)} = \frac{1}{2\epsilon_0} \sum_{k=0}^{\infty} \left( \frac{m(\xi) - \ell(\xi)}{2\epsilon_0 e^{i\theta}} \right)^k,$$

where the series converges uniformly in  $\theta \in [0, 2\pi]$  since

$$\left| \frac{m(\xi) - \ell(\xi)}{2\epsilon_0 e^{i\theta}} \right| \leq \frac{1}{2}.$$

Substituting (2.9) in (2.8), we have

$$(2.10) \quad F(m(\xi)) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \left( \frac{m(\xi) - \ell(\xi)}{2\epsilon_0} \right)^k N_k(\xi),$$

where

$$N_k(\xi) = \int_0^{2\pi} F(\ell(\xi) + 2\epsilon_0 e^{i\theta}) e^{-ik\theta} d\theta$$

and the series on the right hand side of (2.10) converges uniformly in  $\xi \in \mathbb{R}^n \setminus \{0\}$ , since

$$\left| \frac{m(\xi) - \ell(\xi)}{2\epsilon_0} \right| \leq \frac{1}{2}, \quad \epsilon_0 \leq |\ell(\xi) + 2\epsilon_0 e^{i\theta}| \leq \|m\|_\infty + 3\epsilon_0.$$

Also,  $N_k(\xi)$  is homogeneous of degree 0 with respect to  $\delta_t^*$  and infinitely differentiable in  $\mathbb{R}^n \setminus \{0\}$  and

$$\sup_{1 \leq \rho^*(\xi) \leq 2, |a| \leq [\gamma] + 1} |(\partial_\xi)^a N_k(\xi)| \leq C$$

with  $C$  independent of  $k$ . Therefore, by Lemma 2.6 we have  $\|N_k\|_{M^p(w)} \leq C$  with a constant  $C$  independent of  $k$ . Thus we see that

$$\sum_{k=0}^{\infty} (2\epsilon_0)^{-k} \|(m - \ell)^k\|_{M^p(w)} \|N_k\|_{M^p(w)} \leq C \sum_{k=0}^{\infty} (2\epsilon_0)^{-k} \|(m - \ell)^k\|_{M^p(w)},$$

and the last series converges since  $\|(m - \ell)^k\|_{M^p(w)} \leq \epsilon_0^k$  if  $k$  is sufficiently large. From this and (2.10) we can infer that  $F(m) \in M_w^p$ . ■

By Theorem 2.3 in particular we have the following.

**COROLLARY 2.7.** *Let  $1 < p < \infty$  and  $w \in A_p$ . Suppose that  $m$  is homogeneous of degree 0 with respect to  $\delta_t^*$  and that  $m \in M_v^r$  for all  $r \in (1, \infty)$  and all  $v \in A_r$ . Assume further that  $m$  is continuous on  $S^{n-1}$  and does not vanish there. Then  $m^{-1} \in M_w^p$ .*

*Proof.* Take  $F(z) = 1/z$  in Theorem 2.3. ■

Applying Corollary 2.7 in the theory of Littlewood–Paley functions, we can prove the following.

**THEOREM 2.8.** *Let  $\psi \in L^1(\mathbb{R}^n)$  satisfy (1.4). Suppose that  $\|g_\psi(f)\|_{r,v} \leq C_{r,v}\|f\|_{r,v}$ ,  $f \in \mathcal{S}$ , for all  $r \in (1, \infty)$  and all  $v \in A_r$  and that  $m(\xi) = \int_0^\infty |\hat{\psi}(\delta_t^* \xi)|^2 dt/t$  is continuous and strictly positive on  $S^{n-1}$ . Let  $f \in \mathcal{S}$ . Then*

$$\|f\|_{p,w} \leq C_{p,w} \|g_\psi(f)\|_{p,w}$$

for all  $p \in (1, \infty)$  and all  $w \in A_p$ .

To prove Theorem 2.8, we also need the following lemma.

**LEMMA 2.9.** *Suppose that  $\|g_\psi(f)\|_{r,v} \leq C_{r,v}\|f\|_{r,v}$ ,  $f \in \mathcal{S}$ , for all  $r \in (1, \infty)$  and all  $v \in A_r$ . If  $m(\xi)$  is defined as in Theorem 2.8 and  $1 < p < \infty$ ,  $w \in A_p$ , then  $m \in M_w^p$ .*

*Proof.* For  $\epsilon \in (0, 1)$ , let

$$\Psi^{(\epsilon)}(x) = \int_{\epsilon}^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t(x+y) \bar{\psi}_t(y) dy \frac{dt}{t},$$

where  $\bar{\psi}_t$  denotes the complex conjugate. We note that

$$\mathcal{F}(\Psi^{(\epsilon)})(\xi) = \int_{\epsilon}^{\epsilon^{-1}} \hat{\psi}(\delta_t^* t \xi) \hat{\bar{\psi}}(-\delta_t^* \xi) \frac{dt}{t} = \int_{\epsilon}^{\epsilon^{-1}} |\hat{\psi}(\delta_t^* \xi)|^2 \frac{dt}{t} =: m^{(\epsilon)}(\xi).$$

Therefore  $\Psi^{(\epsilon)} * f = T_{m^{(\epsilon)}} f$ . We observe that

$$\begin{aligned} \Psi^{(\epsilon)} * f(x) &= \int_{\epsilon}^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t * f(y) \bar{\psi}_t(y-x) dy \frac{dt}{t}; \\ \int_{\mathbb{R}^n} \Psi^{(\epsilon)} * f(x) h(x) dx &= \int_{\epsilon}^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t * f(y) \bar{\psi}_t * h(y) dy \frac{dt}{t} \end{aligned}$$

for  $f, h \in \mathcal{S}$ . Thus by the inequalities of Schwarz and Hölder we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \Psi^{(\epsilon)} * f(x) h(x) dx \right| &\leq \int_{\mathbb{R}^n} g_\psi(f)(y) g_\psi(\bar{h})(y) dy \\ &\leq \|g_\psi(f)\|_{p,w} \|g_\psi(\bar{h})\|_{p',w^{-p'/p}} \\ &\leq C \|g_\psi(f)\|_{p,w} \|h\|_{p',w^{-p'/p}}. \end{aligned}$$

Taking the supremum over functions  $h$  with  $\|h\|_{p', w^{-p'/p}} \leq 1$ , we have

$$\|T_{m(\epsilon)} f\|_{p,w} \leq C \|g_\psi(f)\|_{p,w}.$$

Letting  $\epsilon \rightarrow 0$  and noting  $m^{(\epsilon)} \rightarrow m$ , we have

$$(2.11) \quad \|T_m f\|_{p,w} \leq C \|g_\psi(f)\|_{p,w}.$$

Since  $\|g_\psi(f)\|_{p,w} \leq C \|f\|_{p,w}$ , we see that  $m \in M_w^p$ . ■

*Proof of Theorem 2.8.* Let  $m$  be as in Theorem 2.8. Then by Lemma 2.9,  $m \in M_w^p$  for all  $p \in (1, \infty)$  and  $w \in A_p$ . So we can apply Corollary 2.7 to  $m$  to conclude that  $m^{-1} \in M_w^p$  if  $1 < p < \infty$ ,  $w \in A_p$  and hence by (2.11),

$$\|f\|_{p,w} = \|T_{m^{-1}} T_m f\|_{p,w} \leq C \|T_m f\|_{p,w} \leq C \|g_\psi(f)\|_{p,w}$$

for  $f \in \mathcal{S}$ , which implies the conclusion. ■

*Proof of Theorem 2.1.* It remains to prove the reverse inequality of (2.1). If  $m(\xi) = \int_0^\infty |\hat{\psi}(\delta_t^* \xi)|^2 dt/t$ , then by the non-degeneracy (2.2) we have  $m(\xi) \neq 0$  for  $\xi \neq 0$ . Therefore, by Theorem 2.8 we only have to show that  $m$  is continuous on  $S^{n-1}$ . In [25], it has been shown that

$$\int_{2^k}^{2^{k+1}} |\hat{\psi}(\delta_t^* \xi)|^2 \frac{dt}{t} \leq C \min(|\delta_{2^k}^* \xi|^\epsilon, |\delta_{2^k}^* \xi|^{-\epsilon})$$

for  $\xi \in S^{n-1}$  and  $k \in \mathbb{Z}$  with some  $\epsilon > 0$  (see [25, Lemmas 3.1 and 3.3]). By analogues for  $\delta_t^*$  of (a), (b) for  $\delta_t$  in Section 1, it follows that

$$\int_{2^k}^{2^{k+1}} |\hat{\psi}(\delta_t^* \xi)|^2 \frac{dt}{t} \leq C \min(2^{k\epsilon}, 2^{-k\epsilon}).$$

This implies that

$$\int_\epsilon^{\epsilon^{-1}} |\hat{\psi}(\delta_t^* \xi)|^2 \frac{dt}{t} \rightarrow \int_0^\infty |\hat{\psi}(\delta_t^* \xi)|^2 \frac{dt}{t} \quad \text{as } \epsilon \rightarrow 0$$

uniformly in  $\xi \in S^{n-1}$ . We note that  $\int_\epsilon^{\epsilon^{-1}} |\hat{\psi}(\delta_t^* \xi)|^2 dt/t$  is continuous on  $S^{n-1}$  for each fixed  $\epsilon > 0$ . Thus the continuity of  $m$  on  $S^{n-1}$  follows by uniform convergence. ■

REMARK 2.10. Let  $\psi^{(j)} \in L^1(\mathbb{R}^n)$  for  $j = 1, 2, \dots, \ell$ . Suppose that  $\psi^{(j)}$  satisfies (1.4) and (1)–(3) of Theorem 2.1 for every  $j$ ,  $1 \leq j \leq \ell$ . Let

$$\Psi(x) = (\psi^{(1)}(x), \dots, \psi^{(\ell)}(x)),$$

$$\Psi_t(x) = (\psi_t^{(1)}(x), \dots, \psi_t^{(\ell)}(x)), \quad \mathcal{F}(\Psi_t)(\xi) = (\mathcal{F}(\psi_t^{(1)})(\xi), \dots, \mathcal{F}(\psi_t^{(\ell)})(\xi)).$$

We further assume that

$$(2.12) \quad \sup_{t>0} |\mathcal{F}(\Psi_t)(\xi)| = \sup_{t>0} \left( \sum_{j=1}^{\ell} |\mathcal{F}(\psi^{(j)})(\delta_t^* \xi)|^2 \right)^{1/2} > 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Let

$$f * \Psi_t(x) = (f * \psi_t^{(1)}(x), \dots, f * \psi_t^{(\ell)}(x))$$

and

$$g_{\Psi}(f)(x) = \left( \int_0^{\infty} |f * \Psi_t(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad |f * \Psi_t(x)| = \left( \sum_{j=1}^{\ell} |f * \psi_t^{(j)}(x)|^2 \right)^{1/2}.$$

Then by Theorem 2.1 we have  $\|g_{\Psi}(f)\|_{p,w} \leq C\|f\|_{p,w}$ . We can also prove the reverse inequality by adapting the arguments given above when  $\ell = 1$  for the present situation, applying the non-degeneracy (2.12). Thus

$$(2.13) \quad \|g_{\Psi}(f)\|_{p,w} \simeq \|f\|_{p,w}.$$

EXAMPLE. We give an example in the case of the Euclidean structures  $(\rho(x) = |x|, \delta_t(x) = tx)$  for which we can apply Remark 2.10 to get the norm equivalence in (2.13). Let  $P_t(x)$  be the Poisson kernel on the upper half-space  $\mathbb{R}^n \times (0, \infty)$  defined by

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}} = \int_{\mathbb{R}^n} e^{-2\pi t|\xi|} e^{2\pi i \langle x, \xi \rangle} d\xi.$$

Let  $\psi^{(j)}(x) = (\partial/\partial x_j)P_1(x)$ ,  $1 \leq j \leq n$ . Then

$$\mathcal{F}(\psi^{(j)})(\xi) = 2\pi i \xi_j e^{-2\pi|\xi|}.$$

We can see that all the requirements in Remark 2.10 for  $\psi^{(j)}$ ,  $1 \leq j \leq n$ , needed in the proof of (2.13) are fulfilled; in particular, (2.12) follows from

$$|\mathcal{F}(\Psi_t)(\xi)| = 2\pi t|\xi| e^{-2\pi t|\xi|}.$$

Thus we have (2.13) for  $\Psi = ((\partial/\partial x_1)P_1, \dots, (\partial/\partial x_n)P_1)$ .

**3. Proofs of Theorems 1.4 and 1.5.** We apply the following estimates in proving Theorem 1.4.

LEMMA 3.1. *Let  $F$  be a function in  $C^\infty(\mathbb{R}^n \setminus \{0\})$  which is homogeneous of degree  $d$  with respect to  $\delta_t$ . Then for  $\rho(x) \geq 1$  we have*

$$|(\partial_x)^a F(x)| \leq C_a \rho(x)^{d-|a|}$$

for all multi-indices  $a$  with a positive constant  $C_a$  independent of  $x$ .

*Proof.* We write  $\delta_t = (\delta_{ij}(t))$ ,  $1 \leq i, j \leq n$ . We have  $t^d F(x) = F(\delta_t x)$ . Differentiating both sides by using the chain rule on the right hand side, we

have

$$t^d(\partial_x)^a F(x) = \left[ \left( \prod_{j=1}^n \left( \sum_{i=1}^n \delta_{ij}(t) \partial / \partial x_i \right)^{a_j} \right) F \right] (\delta_t x).$$

Substituting  $t = \rho(x)^{-1}$  in this equation, we have

$$|(\partial_x)^a F(x)| \leq C \left( \sup_{|b|=|a|, \rho(x)=1} |(\partial_x)^b F(x)| \right) \left( \sup_{1 \leq i \leq n, 1 \leq j \leq n} \delta_{ij}(\rho(x)^{-1}) \right)^{|a|} \rho(x)^d.$$

This implies what we need, since  $|\delta_{ij}(t)| \leq Ct$  for  $0 < t \leq 1$  by (b) of Section 1. ■

*Proof of Theorem 1.4.* Let  $0 < \alpha < \gamma$  and  $\mathcal{L}_\alpha = \mathcal{F}^{-1}(\rho^*(\xi)^{-\alpha})$ . Then  $\mathcal{L}_\alpha$  is homogeneous of degree  $\alpha - \gamma$  with respect to  $\delta_t$  and belongs to  $C^\infty(\mathbb{R}^n \setminus \{0\})$  (see [4, pp. 162–165]). Let  $\psi^{(\alpha)} = \mathcal{L}_\alpha - \mathcal{L}_\alpha * \Phi$ . Then  $H_\alpha(f) = g_{\psi^{(\alpha)}}(f)$ .

We easily see that

$$(3.1) \quad |\psi^{(\alpha)}(x)| \leq C\rho(x)^{\alpha-\gamma} \quad \text{for } \rho(x) \leq 2.$$

Since

$$\psi^{(\alpha)}(x) = \int_{\mathbb{R}^n} (\mathcal{L}_\alpha(x) - \mathcal{L}_\alpha(x-y))\Phi(y) dy,$$

and

$$|(\partial_x)^a \mathcal{L}_\alpha(x)| \leq C_a \rho(x)^{\alpha-\gamma-|a|} \quad \text{for } \rho(x) \geq 2$$

for all multi-indices  $a$  by Lemma 3.1, using Taylor's formula with (1.16) and noting that  $\Phi$  is compactly supported, we see that

$$(3.2) \quad |\psi^{(\alpha)}(x)| \leq C\rho(x)^{\alpha-\gamma-[\alpha]-1} \quad \text{for } \rho(x) \geq 2,$$

where  $\alpha - \gamma - [\alpha] - 1 < -\gamma$ . By (3.1), (3.2) and (1.9) it follows that  $\psi^{(\alpha)} \in L^1$  (see Remark 2.2). Also, we have

$$|\mathcal{F}(\psi^{(\alpha)})(\xi)| = |\rho^*(\xi)^{-\alpha}(1 - \hat{\Phi}(\xi))| \leq C\rho^*(\xi)^{-\alpha}|\xi|^{[\alpha]+1} \leq C\rho^*(\xi)^{-\alpha+[\alpha]+1}$$

for  $\rho^*(\xi) \leq 1$  by the analogue for  $\rho^*$  of (4) for  $\rho$  of Section 1. So we have  $\mathcal{F}(\psi^{(\alpha)})(0) = 0$ , i.e.,  $\int \psi^{(\alpha)} = 0$ ; combining this with (3.1), (3.2) and (5) for  $\rho$  of Section 1 we see that conditions (1)–(3) of Theorem 2.1 are satisfied for  $\psi^{(\alpha)}$ . Further, it is easy to see that

$$\sup_{t>0} |\mathcal{F}(\psi^{(\alpha)})(\delta_t^* \xi)| > 0$$

for  $\xi \neq 0$ . Thus all the assumptions of Theorem 2.1 are fulfilled for  $\psi^{(\alpha)}$  and the conclusion of Theorem 1.4 follows by applying Theorem 2.1 to  $g_{\psi^{(\alpha)}}$ . ■

REMARK 3.2. If  $\psi^{(\alpha)}$  is as in (1.6), in the case of the Euclidean norm and the ordinary dilation, to prove  $\|f\|_{p,w} \leq C\|g_{\psi^{(\alpha)}}(f)\|_{p,w}$ ,  $0 < \alpha < 2$ ,  $1 < p < \infty$ ,  $w \in A_p$ , we can also apply the polarization technique as in the



proof of Lemma 2.5 (see also [19]) instead of using Theorem 2.1 with the non-degeneracy condition (2.2), which is applicable in a more general situation of Theorem 1.4. This is the case because  $\mathcal{F}(\psi^{(\alpha)})$  is a radial function.

To prove Theorem 1.5 we prepare the following lemmas.

LEMMA 3.3. *Let  $1 < p < \infty$ ,  $w \in A_p$  and  $f \in L_w^p$ . For a positive integer  $m$ , let  $f_{(m)} = f\chi_{E_m}$ , where*

$$E_m = \{x \in \mathbb{R}^n : |x| \leq m, |f(x)| \leq m\}.$$

*Then  $f_{(m)} \rightarrow f$  almost everywhere and in  $L_w^p$  as  $m \rightarrow \infty$ .*

LEMMA 3.4. *Let  $p, w$  and  $f$  be as in Lemma 3.3. Let  $\varphi$  be an infinitely differentiable, non-negative function on  $\mathbb{R}^n$  such that  $\varphi(\xi) = 1$  for  $\rho^*(\xi) \leq 1$ ,  $\text{supp}(\varphi) \subset \{\rho^*(\xi) \leq 2\}$  and  $\varphi(\xi) = \varphi_0(\rho^*(\xi))$  for some  $\varphi_0$  on  $\mathbb{R}$ . Define  $\zeta^{(\epsilon)} \in \mathcal{S}_0$  by*

$$\zeta^{(\epsilon)}(\xi) = \varphi(\delta_\epsilon^* \xi) - \varphi(\delta_{\epsilon^{-1}}^* \xi), \quad \epsilon \in (0, 1/2).$$

*Note that  $\zeta^{(\epsilon)}(\xi) = \zeta^{(\epsilon/2)}(\xi)\zeta^{(\epsilon)}(\xi)$ . Let  $f^{(\epsilon)} = f * \mathcal{F}^{-1}(\zeta^{(\epsilon)})$ . Then  $f^{(\epsilon)} \rightarrow f$  almost everywhere and in  $L_w^p$  as  $\epsilon \rightarrow 0$ .*

*Proof of Lemma 3.3.* The pointwise convergence is obvious and the norm convergence follows from Lebesgue's dominated convergence theorem since  $|f_{(m)}| \leq |f|$ . ■

*Proof of Lemma 3.4.* If  $f \in \mathcal{S}$ , we easily see that  $f^{(\epsilon)} \rightarrow f$  pointwise as  $\epsilon \rightarrow 0$ . Therefore, for  $f \in L_w^p$ , we have

$$\begin{aligned} \left\| \limsup_{\epsilon \rightarrow 0} |f^{(\epsilon)} - f| \right\|_{p,w} &\leq \left\| \limsup_{\epsilon \rightarrow 0} |(f-h)^{(\epsilon)} - (f-h)| \right\|_{p,w} \\ &\leq C \|M(f-h)\|_{p,w} \leq C \|f-h\|_{p,w} \end{aligned}$$

for any  $h \in \mathcal{S}$ . As  $\mathcal{S}$  is dense in  $L_w^p$ , it follows that  $\limsup_{\epsilon \rightarrow 0} |f^{(\epsilon)}(x) - f(x)| = 0$  a.e., which implies pointwise convergence. Norm convergence follows from pointwise convergence and the dominated convergence theorem since  $|f^{(\epsilon)}| \leq CM(f) \in L_w^p$ . ■

*Proof of Theorem 1.5.* Define  $f_{m,\epsilon} = (f_{(m)})^{(\epsilon)}$  for  $f \in L_w^p$ . Then  $f_{m,\epsilon} \in \mathcal{S}_0$ . By Theorem 1.4, we see that

$$(3.3) \quad \|G_\alpha(f_{m,\epsilon})\|_{p,w} = \|H_\alpha(\mathcal{I}_{-\alpha} f_{m,\epsilon})\|_{p,w} \simeq \|\mathcal{I}_{-\alpha}^{(\epsilon/2)} f_{m,\epsilon}\|_{p,w},$$

where  $\mathcal{I}_\beta^{(\epsilon/2)}(f) = \mathcal{F}^{-1}(\zeta^{(\epsilon/2)}(\rho^*)^{-\beta}) * f$ ,  $\beta \in \mathbb{R}$ , for  $f \in L_w^p$  and we have used the equality  $\mathcal{I}_{-\alpha} f_{m,\epsilon} = \mathcal{I}_{-\alpha}^{(\epsilon/2)} f_{m,\epsilon}$ . Using Lemma 3.3, we see that  $f_{m,\epsilon} \rightarrow f^{(\epsilon)}$  in  $L_w^p$ , since

$$\|f_{m,\epsilon} - f^{(\epsilon)}\|_{p,w} \leq C \|M(f_{(m)} - f)\|_{p,w} \leq C \|f_{(m)} - f\|_{p,w}$$

and also  $f_{m,\epsilon} \rightarrow f^{(\epsilon)}$  pointwise, since

$$\begin{aligned} |f_{m,\epsilon}(x) - f^{(\epsilon)}(x)| &= \left| \int (f_{(m)}(y) - f(y)) \mathcal{F}^{-1}(\zeta^{(\epsilon)})(x-y) dy \right| \\ &\leq \|f_{(m)} - f\|_{p,w} \left( \int |\mathcal{F}^{-1}(\zeta^{(\epsilon)})(x-y)|^{p'} w(y)^{-p'/p} dy \right). \end{aligned}$$

Thus  $f_{m,\epsilon} - \Phi_t * f_{m,\epsilon} \rightarrow f^{(\epsilon)} - \Phi_t * f^{(\epsilon)}$  a.e. as  $m \rightarrow \infty$  and by (3.3) we have, via Fatou's lemma,

$$\begin{aligned} \|G_\alpha(f^{(\epsilon)})\|_{p,w} &\leq \liminf_{m \rightarrow \infty} \|G_\alpha(f_{m,\epsilon})\|_{p,w} \\ &\leq C \liminf_{m \rightarrow \infty} \|\mathcal{I}_{-\alpha}^{(\epsilon/2)} f_{m,\epsilon}\|_{p,w} = C \|\mathcal{I}_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)}\|_{p,w}, \end{aligned}$$

where the last equality follows since  $\mathcal{I}_{-\alpha}^{(\epsilon/2)}$  is bounded on  $L_w^p$ . Thus we see that  $G_\alpha(f^{(\epsilon)}) \in L_w^p$ . In fact, we also have the reverse inequality. To see this we first note that

$$(3.4) \quad \begin{aligned} \|G_\alpha(f^{(\epsilon)}) - G_\alpha(f_{m,\epsilon})\|_{p,w} &\leq \|G_\alpha(f^{(\epsilon)} - f_{m,\epsilon})\|_{p,w} \\ &= \|G_\alpha((f - f_{(m)})^{(\epsilon)})\|_{p,w}. \end{aligned}$$

Since

$$(f_{(k)} - f_{(m)})^{(\epsilon)} - \Phi_t * (f_{(k)} - f_{(m)})^{(\epsilon)} \rightarrow (f - f_{(m)})^{(\epsilon)} - \Phi_t * (f - f_{(m)})^{(\epsilon)} \quad \text{a.e.}$$

as  $k \rightarrow \infty$ , by Fatou's lemma we have

$$(3.5) \quad \|G_\alpha((f - f_{(m)})^{(\epsilon)})\|_{p,w} \leq \liminf_{k \rightarrow \infty} \|G_\alpha((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w}.$$

Since  $(f_{(k)} - f_{(m)})^{(\epsilon)} \in \mathcal{S}_0$ , by Theorem 1.4 we have

$$\begin{aligned} \|G_\alpha((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w} &\simeq \|\mathcal{I}_{-\alpha}((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w} \\ &= \|\mathcal{I}_{-\alpha}^{(\epsilon/2)}((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w}. \end{aligned}$$

Since  $f_{(m)} \rightarrow f$  in  $L_w^p$ , this implies that

$$\lim_{k,m \rightarrow \infty} \|G_\alpha((f_{(k)} - f_{(m)})^{(\epsilon)})\|_{p,w} = 0.$$

Thus by (3.4) and (3.5), it follows that  $G_\alpha(f_{m,\epsilon}) \rightarrow G_\alpha(f^{(\epsilon)})$  in  $L_w^p$  as  $m \rightarrow \infty$ . Therefore, letting  $m \rightarrow \infty$  in (3.3), we have

$$(3.6) \quad \|G_\alpha(f^{(\epsilon)})\|_{p,w} \simeq \|\mathcal{I}_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)}\|_{p,w}.$$

Suppose that  $f \in W_w^{\alpha,p}$  and let  $g = \mathcal{I}_{-\alpha}(f)$ . We show that

$$(3.7) \quad \mathcal{I}_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)} = g^{(\epsilon)}$$

as follows. For  $h \in \mathcal{S}_0$  we have

$$\begin{aligned} \int g^{(\epsilon)} \mathcal{I}_\alpha(h) dx &= \lim_{m \rightarrow \infty} \int g_{m,\epsilon} \mathcal{I}_\alpha(h) dx = \lim_{m \rightarrow \infty} \int \mathcal{I}_\alpha^{(\epsilon/2)}(g_{m,\epsilon}) h dx \\ &= \int \mathcal{I}_\alpha^{(\epsilon/2)}(g^{(\epsilon)}) h dx. \end{aligned}$$

Also,

$$\begin{aligned} \int g^{(\epsilon)} \mathcal{I}_\alpha(h) dx &= \lim_{m \rightarrow \infty} \int g_{m,\epsilon} \mathcal{I}_\alpha(h) dx \\ &= \lim_{m \rightarrow \infty} \int g_{(m)} \mathcal{I}_\alpha(h^{(\epsilon)}) dx = \int g \mathcal{I}_\alpha(h^{(\epsilon)}) dx. \end{aligned}$$

By the definition of  $g = \mathcal{I}_{-\alpha}(f)$ ,  $\int g \mathcal{I}_\alpha(h^{(\epsilon)}) dx = \int f h^{(\epsilon)} dx$ . Thus

$$\begin{aligned} \int g^{(\epsilon)} \mathcal{I}_\alpha(h) dx &= \int f h^{(\epsilon)} dx = \lim_{m \rightarrow \infty} \int f_{(m)} h^{(\epsilon)} dx \\ &= \lim_{m \rightarrow \infty} \int f_{m,\epsilon} h dx = \int f^{(\epsilon)} h dx. \end{aligned}$$

Therefore

$$\int \mathcal{I}_\alpha^{(\epsilon/2)}(g^{(\epsilon)}) h dx = \int f^{(\epsilon)} h dx \quad \text{for all } h \in \mathcal{S}_0,$$

which implies that  $\mathcal{I}_\alpha^{(\epsilon/2)}(g^{(\epsilon)}) = f^{(\epsilon)}$ . Since  $\mathcal{I}_\alpha^{(\epsilon/2)}$  and  $\mathcal{I}_{-\alpha}^{(\epsilon/2)}$  are bounded on  $L_w^p$  and the mapping  $f \mapsto f^{(\epsilon)}$  is also bounded on  $L_w^p$ , by Lemma 3.3 we see that

$$\begin{aligned} \mathcal{I}_{-\alpha}^{(\epsilon/2)}(f^{(\epsilon)}) &= \mathcal{I}_{-\alpha}^{(\epsilon/2)} \mathcal{I}_\alpha^{(\epsilon/2)}(g^{(\epsilon)}) = \lim_{m \rightarrow \infty} \mathcal{I}_{-\alpha}^{(\epsilon/2)} \mathcal{I}_\alpha^{(\epsilon/2)}(g_{m,\epsilon}) \\ &= \lim_{m \rightarrow \infty} g_{m,\epsilon} = g^{(\epsilon)}, \end{aligned}$$

which proves (3.7).

By (3.6) and (3.7), we have

$$\|G_\alpha(f^{(\epsilon)})\|_{p,w} \leq C \|g^{(\epsilon)}\|_{p,w} \leq C \|M(g)\|_{p,w} \leq C \|g\|_{p,w}.$$

Letting  $\epsilon \rightarrow 0$  and applying Lemma 3.4 and Fatou's lemma, we have

$$(3.8) \quad \|G_\alpha(f)\|_{p,w} \leq C \|\mathcal{I}_{-\alpha}(f)\|_{p,w}.$$

Conversely, let us assume that  $f \in L_w^p$  and  $G_\alpha(f) \in L_w^p$ . By Minkowski's inequality we see that

$$(3.9) \quad \|G_\alpha(f^{(\epsilon)})\|_{p,w} \leq C \|M(G_\alpha(f))\|_{p,w} \leq C \|G_\alpha(f)\|_{p,w}.$$

Applying (3.6) and (3.9), we find that

$$\sup_{\epsilon \in (0,1/2)} \|\mathcal{I}_{-\alpha}^{(\epsilon/2)} f^{(\epsilon)}\|_{p,w} \leq C \sup_{\epsilon \in (0,1/2)} \|G_\alpha(f^{(\epsilon)})\|_{p,w} \leq C \|G_\alpha(f)\|_{p,w}.$$

Therefore, there exist a sequence  $\{\epsilon_k\}$ ,  $0 < \epsilon_k < 1/2$ , and a function  $g \in L_w^p$  such that  $\epsilon_k \rightarrow 0$  and  $\mathcal{I}_{-\alpha}^{(\epsilon_k/2)} f^{(\epsilon_k)} \rightarrow g$  weakly in  $L_w^p$  as  $k \rightarrow \infty$  and

$$(3.10) \quad \|g\|_{p,w} \leq C \|G_\alpha(f)\|_{p,w}.$$

We now show that  $f = \mathcal{I}_\alpha g$ . By Lemma 3.4,  $f^{(\epsilon_k)} \rightarrow f$  in  $L_w^p$ . So, for  $h \in \mathcal{S}_0$  we have

$$\begin{aligned} \int_{\mathbb{R}^n} f h \, dx &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f^{(\epsilon_k)} h \, dx = \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} f_{m, \epsilon_k} h \, dx \\ &= \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} \mathcal{I}_{-\alpha}(f_{m, \epsilon_k}) \mathcal{I}_\alpha(h) \, dx \\ &= \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} \mathcal{I}_{-\alpha}^{(\epsilon_k/2)}(f_{m, \epsilon_k}) \mathcal{I}_\alpha(h) \, dx \\ &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \mathcal{I}_{-\alpha}^{(\epsilon_k/2)}(f^{(\epsilon_k)}) \mathcal{I}_\alpha(h) \, dx = \int_{\mathbb{R}^n} g \mathcal{I}_\alpha(h) \, dx. \end{aligned}$$

This implies that  $f = \mathcal{I}_\alpha g$  by definition. By (3.10) we have

$$\|\mathcal{I}_{-\alpha} f\|_{p,w} = \|g\|_{p,w} \leq C \|G_\alpha(f)\|_{p,w},$$

which combined with (3.8), completes the proof of Theorem 1.5. ■

**4. Characterization of  $W_w^{\alpha,p}$  by square functions defined by repeated averaging.** Let  $\Phi \in \mathcal{M}^1$ . Define  $\Lambda_t^j f(x)$ ,  $j \geq 1$ , by  $\Lambda_t^j f(x) = f * \Phi_t^{(j)}(x)$ , where

$$\Phi^{(1)}(x) = \Phi(x), \quad \Phi^{(j)}(x) = \overbrace{\Phi * \cdots * \Phi}^j(x), \quad j \geq 2.$$

We also write  $\Lambda_t f(x)$  for  $\Lambda_t^1 f(x)$ . Let  $I$  be the identity operator and  $k$  a positive integer. We consider

$$\begin{aligned} (4.1) \quad (I - \Lambda_t)^k f(x) &= f(x) + \sum_{j=1}^k (-1)^j \binom{k}{j} \Lambda_t^j f(x) \\ &= f(x) - K_t^{(k)} * f(x) = \int_{\mathbb{R}^n} (f(x) - f(x-y)) K_t^{(k)}(y) \, dy \end{aligned}$$

for appropriate functions  $f$ , where

$$(4.2) \quad K^{(k)}(x) = - \sum_{j=1}^k (-1)^j \binom{k}{j} \Phi^{(j)}(x),$$

and we have used the equation

$$(4.3) \quad \int_{\mathbb{R}^n} K^{(k)}(x) \, dx = - \sum_{j=1}^k (-1)^j \binom{k}{j} = 1.$$

Define

$$(4.4) \quad E_\alpha^{(k)}(f)(x) = \left( \int_0^\infty |(I - \Lambda_t)^k f(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad \alpha > 0.$$

If  $\Phi = \chi_0 = |B(0, 1)|^{-1} \chi_{B(0,1)}$  and  $k = 2$  in (4.4), we have

$$E_\alpha^{(2)}(f)(x) = \left( \int_0^\infty \left| f(x) - 2 \int_{B(x,t)} f(y) dy + \int_{B(x,t)} (f)_{B(y,t)} dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

where  $(f)_{B(y,t)} = \int_{B(y,t)} f$ . Also, let

$$(4.5) \quad U_\alpha^{(k)}(f)(x) = \left( \int_0^\infty |(I - \Lambda_t)^k \mathcal{I}_\alpha(f)(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

where  $0 < \alpha < \gamma$ ,  $f \in \mathcal{S}_0$ . Using (4.1), we can rewrite  $E_\alpha^{(k)}(f)$  in (4.4) and  $U_\alpha^{(k)}(f)$  in (4.5) as follows:

$$(4.6) \quad E_\alpha^{(k)}(f)(x) = \left( \int_0^\infty |f(x) - K_t^{(k)} * f(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

$$(4.7) \quad U_\alpha^{(k)}(f)(x) = \left( \int_0^\infty |\mathcal{I}_\alpha(f)(x) - K_t^{(k)} * \mathcal{I}_\alpha(f)(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

where  $K^{(k)}$  is as in (4.2).

As applications of Theorems 1.4 and 1.5 we have the following theorems.

**THEOREM 4.1.** *Let  $0 < \alpha < \min(2k, \gamma)$ ,  $1 < p < \infty$ ,  $w \in A_p$  and let  $U_\alpha^{(k)}$  be as in (4.5). Then*

$$\|U_\alpha^{(k)}(f)\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

**THEOREM 4.2.** *Let  $1 < p < \infty$ ,  $w \in A_p$  and  $0 < \alpha < \min(2k, \gamma)$ . Let  $E_\alpha^{(k)}$  be as in (4.4). Then  $f \in W_w^{\alpha,p}$  if and only if  $f \in L_w^p$  and  $E_\alpha^{(k)}(f) \in L_w^p$ ; moreover,*

$$\|\mathcal{I}_{-\alpha}(f)\|_{p,w} \simeq \|E_\alpha^{(k)}(f)\|_{p,w}.$$

*Proofs of Theorems 4.1 and 4.2.* Using the expressions of  $E_\alpha^{(k)}(f)$  and  $U_\alpha^{(k)}(f)$  in (4.6) and (4.7), we can derive Theorems 4.1 and 4.2 from Theorems 1.4 and 1.5, respectively, if  $K^{(k)} \in \mathcal{M}^{2k-1}$ , since then  $K^{(k)} \in \mathcal{M}^\alpha$  for  $\alpha \in (0, \min(2k, \gamma))$ .

To show that  $K^{(k)} \in \mathcal{M}^{2k-1}$ , first we easily see that  $K^{(k)}$  is bounded and compactly supported. Since we have already noted (4.3), it remains to show that

$$(4.8) \quad \int_{\mathbb{R}^n} y^\alpha K^{(k)}(y) dy = 0 \quad \text{if } 1 \leq |\alpha| < 2k.$$

This can be shown as follows. Since  $\Phi \in \mathcal{M}^1$ , we have  $\int y^\alpha \Phi(y) dy = 0$  for  $|\alpha| = 1$ , which implies that  $\partial_\xi^\alpha \hat{\Phi}(0) = 0$  for  $|\alpha| = 1$ . Thus near  $\xi = 0$ , we have

$$(4.9) \quad 1 - \mathcal{F}(K^{(k)})(\xi) = 1 + \sum_{j=1}^k (-1)^j \binom{k}{j} \hat{\Phi}(\xi)^j = (1 - \hat{\Phi}(\xi))^k = O(|\xi|^{2k}).$$

Also, by Taylor's formula we see that

$$(4.10) \quad \mathcal{F}(K^{(k)})(\xi) = 1 + \sum_{1 \leq |a| < 2k} C_a \xi^a \partial_\xi^a \mathcal{F}(K^{(k)})(0) + O(|\xi|^{2k}).$$

From (4.9) and (4.10) it follows that

$$\sum_{1 \leq |a| < 2k} C_a \xi^a \partial_\xi^a \mathcal{F}(K^{(k)})(0) = O(|\xi|^{2k}).$$

This implies that  $\partial_\xi^a \mathcal{F}(K^{(k)})(0) = 0$  for  $1 \leq |a| < 2k$ , and hence we have (4.8). ■

REMARK 4.3. In the definitions of  $E_\alpha^{(k)}$  and  $U_\alpha^{(k)}$  in (4.4) and (4.5), if we assume only that  $\Phi$  belongs to  $\mathcal{M}^0$ , then we have analogues of Theorems 4.1 and 4.2 for the range  $(0, \min(k, \gamma))$  of  $\alpha$ .

**5. The Sobolev spaces  $W_w^{\alpha,p}$  and distributional derivatives.** In  $\mathbb{R}^2$ , we consider  $P = \text{diag}(1, 2)$ ,  $\delta_t = \text{diag}(t, t^2)$ . Then  $\gamma = 3$  and

$$\rho(x_1, x_2) = \frac{1}{\sqrt{2}} \sqrt{x_1^2 + \sqrt{x_1^4 + 4x_2^2}},$$

$\rho^* = \rho$ ,  $\delta_t^* = \delta_t$ . Under this setting, let  $W_w^{\alpha,p}$  be the weighted Sobolev space on  $\mathbb{R}^2$  defined in Section 1 with  $0 < \alpha < 3$ ,  $1 < p < \infty$ ,  $w \in A_p$ . Then  $W_w^{2,p}$  can be characterized by using distributional derivatives as follows.

THEOREM 5.1. *Let  $f \in L_w^p$  with  $1 < p < \infty$ ,  $w \in A_p$ . Let  $(\partial/\partial x_1)^2 f$ ,  $\partial/\partial x_2 f$  be the distributional derivatives in  $\mathcal{S}'$  (the space of tempered distributions). Then  $f \in W_w^{2,p}$  if and only if  $(\partial/\partial x_1)^2 f \in L_w^p$  and  $\partial/\partial x_2 f \in L_w^p$ ; further,*

$$\|\mathcal{I}_{-\alpha}(f)\|_{2,w} \simeq \|(\partial/\partial x_1)^2 f\|_{p,w} + \|\partial/\partial x_2 f\|_{p,w}.$$

*Proof.* Suppose that  $f \in W_w^{2,p}$ . Let  $g = \mathcal{I}_{-2}(f) \in L_w^p$ . Then

$$(5.1) \quad \int f h \, dx = \int g \mathcal{I}_2(h) \, dx \quad \text{for all } h \in \mathcal{S}_0.$$

Let  $k(\xi) = -4\pi^2 \xi_1^2$ . Let  $g_{m,\epsilon} = g_{(m)} * \mathcal{F}^{-1}(\zeta^{(\epsilon)})$  be as in Section 3. Then by (5.1) we see that for  $h \in \mathcal{S}_0$ ,

$$(5.2) \quad \begin{aligned} \int f (\partial/\partial x_1)^2 h \, dx &= \int g \mathcal{I}_2((\partial/\partial x_1)^2 h) \, dx = \int g \mathcal{I}_2(T_k h) \, dx \\ &= \lim_{\epsilon \rightarrow 0} \lim_{m \rightarrow \infty} \int g_{m,\epsilon} \mathcal{I}_2(T_k h) \, dx = \lim_{\epsilon \rightarrow 0} \lim_{m \rightarrow \infty} \int T_{k(\rho^*)^{-2}}(g_{m,\epsilon}) h \, dx. \end{aligned}$$

Since  $k(\rho^*)^{-2}$  is homogeneous of degree 0 with respect to  $\delta_t^*$  and infinitely differentiable in  $\mathbb{R}^2 \setminus \{0\}$ , by Lemma 2.6 the multiplier operator  $T_{k(\rho^*)^{-2}}$  is

bounded on  $L_w^p$ . Thus  $T_{k(\rho^*)-2}(g_{m,\epsilon}) \rightarrow T_{k(\rho^*)-2}(g)$  in  $L_w^p$  as  $m \rightarrow \infty$  and  $\epsilon \rightarrow 0$  since  $g_{m,\epsilon} \rightarrow g$  in  $L_w^p$  as  $m \rightarrow \infty$  and  $\epsilon \rightarrow 0$ . Therefore, by (5.2) we have

$$(5.3) \quad \int f(\partial/\partial x_1)^2 h \, dx = \int T_{k(\rho^*)-2}(g) h \, dx \quad h \in \mathcal{S}_0,$$

which implies that

$$(5.4) \quad \int f(\partial/\partial x_1)^2 \psi \, dx = \int T_{k(\rho^*)-2}(g) \psi \, dx \quad \text{for all } \psi \in \mathcal{S}.$$

It follows that

$$(5.5) \quad (\partial/\partial x_1)^2 f = T_{k(\rho^*)-2}(g) \quad \text{in } \mathcal{S}'.$$

To see (5.4), substitute  $\psi - \mathcal{F}^{-1}(\varphi(\delta_\epsilon^{-1}\xi)\hat{\psi}(\xi))$  for  $h$  in (5.3), where  $\varphi$  is as in Lemma 3.4, and let  $\epsilon \rightarrow 0$ .

Let  $\ell(\xi) = 2\pi i\xi_2$ . Then, arguing similarly to the above and noting that  $\ell(\rho^*)^{-2}$  is homogeneous of degree 0 with respect to  $\delta_t^*$  and infinitely differentiable in  $\mathbb{R}^2 \setminus \{0\}$ , we see that  $T_{\ell(\rho^*)-2}(g) \in L_w^p$  and

$$-\int f\partial/\partial x_2 \psi \, dx = \int T_{\ell(\rho^*)-2}(g) \psi \, dx \quad \text{for all } \psi \in \mathcal{S},$$

which implies that

$$(5.6) \quad \partial/\partial x_2 f = T_{\ell(\rho^*)-2}(g) \quad \text{in } \mathcal{S}'.$$

Combining (5.5) and (5.6), we have

$$(5.7) \quad \|f(\partial/\partial x_1)^2 f\|_{p,w} + \|\partial/\partial x_2 f\|_{p,w} \leq C\|g\|_{p,w} = C\|\mathcal{I}_{-2}(f)\|_{p,w}.$$

Conversely, suppose that  $(\partial/\partial x_1)^2 f =: \Theta \in L_w^p$  and  $\partial/\partial x_2 f =: \Xi \in L_w^p$ . Then, for  $h \in \mathcal{S}_0$  we have

$$\int f(\partial/\partial x_1)^2 h \, dx = \int \Theta h \, dx, \quad -\int f\partial/\partial x_2 h \, dx = \int \Xi h \, dx,$$

and hence

$$(5.8) \quad \int f(T_k h - T_\ell h) \, dx = \int f((\partial/\partial x_1)^2 h - \partial/\partial x_2 h) \, dx = \int (\Theta + \Xi) h \, dx,$$

where  $k(\xi)$  and  $\ell(\xi)$  are as above. Let

$$N(\xi) = \frac{k(\xi) - \ell(\xi)}{\rho^*(\xi)^2} = \frac{-4\pi^2\xi_1^2 - 2\pi i\xi_2}{\rho^*(\xi)^2}.$$

Then, substituting  $\mathcal{I}_2(h)$  for  $h$  in (5.8), we have

$$(5.9) \quad \int f T_N h \, dx = \int (\Theta + \Xi) \mathcal{I}_2(h) \, dx.$$

We note that the functions  $N$  and  $\tilde{N}^{-1}$  are homogeneous of degree 0 with respect to  $\delta_t^*$  and infinitely differentiable in  $\mathbb{R}^2 \setminus \{0\}$ , where  $\tilde{N}(\xi) = N(-\xi)$ . So,  $T_{\tilde{N}^{-1}}$  is bounded on  $L_w^p$  by Lemma 2.6. Substituting  $T_{\tilde{N}^{-1}} h$  for  $h$  in (5.9), we have

$$(5.10) \quad \int f h \, dx = \int (\Theta + \Xi) T_{\tilde{N}^{-1}}(\mathcal{I}_2(h)) \, dx = \int T_{\tilde{N}^{-1}}(\Theta + \Xi) \mathcal{I}_2(h) \, dx,$$

where the last equality follows as (5.3), since  $T_{\tilde{N}-1}$  is bounded on  $L_w^p$ . By (5.10) we see that  $f \in W_w^{2,p}$  and

$$\mathcal{I}_{-2}(f) = T_{\tilde{N}-1}(\Theta + \Xi)$$

and

$$\|\mathcal{I}_{-2}(f)\|_{p,w} \leq C\|\Theta\|_{p,w} + C\|\Xi\|_{p,w} = C\|(\partial/\partial x_1)^2 f\|_{p,w} + C\|\partial/\partial x_2 f\|_{p,w},$$

which combined with (5.7) completes the proof of the theorem. ■

We conclude this note with two remarks.

**REMARK 5.2.** To characterize the (unweighted) Sobolev spaces  $W^{\alpha,p}$  we can also apply the square functions of Luzin area integral type instead of Littlewood–Paley function type (see [26]). In [23], certain  $(H^1)$  Sobolev spaces were characterized by using square functions of Luzin area integral type. The characterization of those Sobolev spaces by square functions of Littlewood–Paley type analogous to Theorem 1.5 is yet to be proved.

**REMARK 5.3.** Let us consider another square function of Marcinkiewicz type:

$$D_\alpha(f)(x) = \left( \int_{\mathbb{R}^n} |I_\alpha(f)(x+y) - I_\alpha(f)(x)|^2 |y|^{-n-2\alpha} dy \right)^{1/2},$$

where  $I_\alpha$  is as in (1.2). Let  $0 < \alpha < 1$  and  $p_0 = 2n/(n+2\alpha) > 1$ . Then it is known that the operator  $D_\alpha$  is bounded on  $L^p(\mathbb{R}^n)$  if  $p_0 < p < \infty$  ([27]) and that  $D_\alpha$  is of weak type  $(p_0, p_0)$  ([8]). In [24] analogues of these results were established in the case of dilations  $\delta_t = t^P$  when  $P$  is diagonal.

**Acknowledgements.** The author is partly supported by Grant-in-Aid for Scientific Research (C) No. 20K03651, Japan Society for Promotion of Science.

## References

- [1] R. Alabern, J. Mateu and J. Verdera, *A new characterization of Sobolev spaces on  $\mathbb{R}^n$* , Math. Ann. 354 (2012), 589–626.
- [2] A. P. Calderón, *Inequalities for the maximal function relative to a metric*, Studia Math. 57 (1976), 297–306.
- [3] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Adv. Math. 16 (1975), 1–64.
- [4] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution. II*, Adv. Math. 24 (1977), 101–171.
- [5] O. N. Capri, *On an inequality in the theory of parabolic  $H^p$  spaces*, Rev. Un. Mat. Argentina 32 (1985), 17–28.
- [6] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250.



- [7] E. B. Fabes and N. Rivière, *Singular integrals with mixed homogeneity*, *Studia Math.* 27 (1966), 19–38.
- [8] C. Fefferman, *Inequalities for strongly singular convolution operators*, *Acta Math.* 124 (1970), 9–36.
- [9] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.
- [10] P. Hajłasz and Z. Liu, *A Marcinkiewicz integral type characterization of the Sobolev space*, *Publ. Mat.* 61 (2017), 83–104.
- [11] L. Hörmander, *Estimates for translation invariant operators in  $L^p$  spaces*, *Acta Math.* 104 (1960), 93–140.
- [12] D. S. Kurz and R. L. Wheeden, *Results on weighted norm inequalities for multipliers*, *Trans. Amer. Math. Soc.* 255 (1979), 343–362.
- [13] P. D. Lax, *Linear Algebra*, Wiley, New York, 1997.
- [14] J. Marcinkiewicz, *Sur quelques intégrales du type de Dini*, *Ann. Soc. Polon. Math.* 17 (1938), 42–50.
- [15] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Ungar, New York, 1955.
- [16] N. Rivière, *Singular integrals and multiplier operators*, *Ark. Mat.* 9 (1971), 243–278.
- [17] S. Sato, *Remarks on square functions in the Littlewood–Paley theory*, *Bull. Austral. Math. Soc.* 58 (1998), 199–211.
- [18] S. Sato, *Estimates for singular integrals along surfaces of revolution*, *J. Austral. Math. Soc.* 86 (2009), 413–430.
- [19] S. Sato, *Littlewood–Paley operators and Sobolev spaces*, *Illinois J. Math.* 58 (2014), 1025–1039.
- [20] S. Sato, *Square functions related to integral of Marcinkiewicz and Sobolev spaces*, *Linear Nonlinear Anal.* 2 (2016), 237–252.
- [21] S. Sato, *Spherical square functions of Marcinkiewicz type with Riesz potentials*, *Arch. Math. (Basel)* 108 (2017), 415–426.
- [22] S. Sato, *Littlewood–Paley equivalence and homogeneous Fourier multipliers*, *Integral Equations Operator Theory* 87 (2017), 15–44.
- [23] S. Sato, *Characterization of  $H^1$  Sobolev spaces by square functions of Marcinkiewicz type*, *J. Fourier Anal. Appl.* 25, 842–873 (2019).
- [24] S. Sato, *Weak type estimates for functions of Marcinkiewicz type with fractional integrals of mixed homogeneity*, *Math. Scand.* 125 (2019), 135–162.
- [25] S. Sato, *Boundedness of Littlewood–Paley operators relative to non-isotropic dilations*, *Czechoslovak Math. J.* 69 (2019), 337–351.
- [26] S. Sato, F. Wang, D. Yang and W. Yuan, *Generalized Littlewood–Paley characterizations of fractional Sobolev spaces*, *Comm. Contemp. Math.* 20 (2018), no. 7, art. 1750077, 48 pp.
- [27] E. M. Stein, *The characterization of functions arising as potentials*, *Bull. Amer. Math. Soc.* 67 (1961), 102–104.
- [28] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.
- [29] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, *Bull. Amer. Math. Soc.* 84 (1978), 1239–1295.
- [30] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, *Lecture Notes in Math.* 1381, Springer, 1989.
- [31] D. Waterman, *On an integral of Marcinkiewicz*, *Trans. Amer. Math. Soc.* 91 (1959), 129–138.
- [32] A. Zygmund, *On certain integrals*, *Trans. Amer. Math. Soc.* 55 (1944), 170–204.

Shuichi Sato  
Department of Mathematics  
Faculty of Education  
Kanazawa University  
Kanazawa 920-1192, Japan  
E-mail: shuichipm@gmail.com