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On continuum-wise minimality

by

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Abstract. We say that a homeomorphism of a compact metric space is cw-minimal if all the proper closed invariant subsets have dimension zero. This concept was previously considered by H. Kato. We explore this notion and provide examples. We give sufficient conditions for the existence of cw-minimal subsets and we prove several characterizations. We show that cw-minimal systems are transitive and either minimal or sensitive if the space is locally connected. A subset is said to be mindual if it intersects every minimal subset. We show that every cw-minimal subset contains a closed, zero-dimensional mindual set.

1. Introduction. This article is motivated by a result on minimal sets of hyperbolic diffeomorphisms which states that they have dimension zero. It was first proved by R. Bowen [8] in 1970 for hyperbolic dynamics. In 1979 R. Mañé [20] extended the result to expansive homeomorphisms.

In this article we consider some results and ideas of H. Kato. In [14, Theorem 5.3] he generalized the Bowen–Mañé Theorem assuming a weaker form of expansivity called *cw-expansivity*. See [19, 5] for more on this and the corresponding problem in the context of flows. Let (M, dist) denote a compact metric space and $f\colon M \to M$ a homeomorphism. We say that f is *cw-expansive* if there is $\xi > 0$ such that if $C \subset M$ is connected and $\mathrm{diam}(f^i(C)) \leq \xi$ for all $i \in \mathbb{Z}$ then C is a singleton. Here, *cw* means *continuum-wise* where a *continuum* is a non-empty closed and connected subset. We say that f is *minimal* if every orbit is dense in M. A non-empty closed invariant subset $A \subset M$ is called *minimal* whenever $f\colon A \to A$ is minimal. In [16, Lemma 2.6] it is proved that every cw-expansive homeomorphism of a compact metric space of positive dimension has infinitely many minimal sets (in §2.1 we recall some facts about topological dimension). In [16] Kato

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considered a property that we call 'cw-minimality'. An invariant subset is *cw-minimal* if it has positive dimension and every proper closed invariant subset has dimension zero. In [17, Theorem 3.1] it is shown that every cw-expansive homeomorphism has a cw-minimal subset $Y \subset M$ and $f\colon Y \to Y$ is two-sided strongly chaotic in the sense of Ruelle–Takens (i.e. two-sided transitive and two-sided strongly sensitive). Also, in [16, Theorem 2.7] it is shown that such $f\colon Y \to Y$ is weakly chaotic in the sense of Devaney (i.e. sensitive, transitive and the union of minimal subsets is dense in Y). This proves that several chaotic properties of cw-expansive homeomorphisms are concentrated on cw-minimal subsets. Examples of cw-minimal homeomorphisms which are cw-expansive are pseudo-Anosov diffeomorphisms of closed surfaces and the two-solenoid [15, Proposition 2.4].

In this article we explore the notion of cw-minimality independently of cw-expansivity. Examples of cw-minimal homeomorphisms which are not cw-expansive are minimal homeomorphisms of compact metric spaces of positive topological dimension.

Let us describe the contents of the article. §2 gives some topological and dynamical preliminaries. We recall some properties of superior limits of set sequences and we introduce a uniform version which is closely related with minimality and cw-minimality as we will see in Proposition 2.18 and Theorem 3.10, respectively.

In §3 we study cw-minimality. §3.2 concerns the minimal subsets contained in a cw-minimal system. In §3.3 we give sufficient conditions for the existence of a cw-minimal subset. In §3.4 and §3.6 we give some characterizations of cw-minimality. In Theorem 3.17 it is shown that a homeomorphism of a compact space of positive dimension is cw-minimal if and only if the set of non-transitive points is zero-dimensional. In particular, cw-minimality for such spaces implies transitivity. In §3.5 we introduce the notion of 'cw-isolated set' and in Theorem 3.16 we prove that they admit arbitrarily small extensions to isolated sets. This is applied in Corollary 3.18 to conclude that if f is cw-minimal then the set of non-transitive points is a countable increasing union of dynamically isolated zero-dimensional closed invariant subsets. In Theorem 3.27 we show that on a locally connected compact metric space of positive dimension every cw-minimal homeomorphism is either minimal or sensitive.

§4 is devoted to the analysis of the known examples. The simplest are pseudo-Anosov maps of surfaces and minimal systems on spaces of positive dimension. In Theorem 4.2 we obtain more examples by applying Kato's notion of 'continuum-wise full expansivity'. In §4.5 we show that there are cw-minimal homeomorphisms with only one minimal subset which is a fixed point.

As we said above, cw-expansive homeomorphisms of compact metric spaces of positive topological dimension have infinitely many minimal subsets. Trivially, this means that no finite set intersects every minimal subset. This result is based on [16, Lemma 2.6] where it is shown that no zero-dimensional closed invariant subset intersects every minimal subset. From this viewpoint it is natural to ask whether this property is still true for non-invariant subsets. In §5 we introduce a definition: we say that $R \subset M$ is a *mindual set* for f if $\omega(x) \cap R \neq \emptyset$ for every $x \in M$. This condition is equivalent to meeting every minimal subset. In §5 we develop some results concerning this kind of sets. We complement [16, Lemma 2.6] by proving Theorem 5.4: if $f\colon M \to M$ is cw-minimal then M contains a closed, zero-dimensional mindual set.

2. Preliminaries. In this section we start by recalling some properties of topological dimension and superior limits of set sequences. Also, we introduce a uniform notion of this limit which is closely related to minimality and cw-minimality as we will see in Proposition 2.18 and Theorem 3.10, respectively. In §2.5 we state known facts about transitivity which will be used in §3.

2.1. Topological dimension. We consider the topological dimension as defined in [13] and denoted by dim. We will only need to distinguish vanishing from positive dimension. A space X is *zero-dimensional*, $\dim(X) = 0$, if X has a basis of its topology made up of open sets with empty boundary, i.e. clopen sets. Topological dimension is then defined inductively so that it is always either a non-negative integer or infinity. Thus, $\dim(X) > 0$ implies $\dim(X) \geq 1$. We say that X is *totally disconnected* if every connected subset of X is a singleton. For future reference we recall some results.

Remark 2.1. A locally compact metric space X is totally disconnected if and only if $\dim(X) = 0$ (see [13, Remark 1, p. 22]). Thus, for such a space X, $\dim(X) > 0$ if and only if X contains a non-trivial continuum.

Theorem 2.2 ([13, Sum Theorem for 0-dimensional Sets]). *A space which is the countable sum of zero-dimensional closed subsets has dimension zero.*

2.2. Expansivity. Throughout the article $f\colon M \to M$ will denote a homeomorphism of the compact metric space (M, dist). We recall the motivation of this research.

Theorem 2.3 (Bowen–Mañé–Kato). *If f is cw-expansive then every minimal subset is zero-dimensional.*

A careful reading of Mañé–Kato's proofs (which are essentially different from Bowen's arguments) reveals that in fact if f is cw-expansive and M

has positive dimension then

$$\text{there is } \varepsilon > 0 \text{ such that if } U \subset M \text{ is open and } \operatorname{diam}(U) < \varepsilon, \text{ then there is } x \in M \text{ such that } f^n(x) \notin U \text{ for all } n \geq 0. \tag{2.1}$$

In Proposition 2.4 we will see that (2.1) is strictly stronger than non-minimality. We need the following usual definition: the *ω-limit set* of $x \in M$ is defined as

$$\omega(x) = \{y \in M : \exists n_k \to \infty,\ f^{n_k}(x) \to y \text{ as } k \to \infty\}. \tag{2.2}$$

PROPOSITION 2.4. *Condition* (2.1) *is equivalent to the existence of at least two minimal subsets.*

Proof. Suppose that (2.1) is true. Then clearly f is not minimal; take a minimal subset $A \neq M$. From (2.1) we can take an open subset $U \subset M$ and $x \in M$ such that $U \cap A \neq \emptyset$ and $f^n(x) \notin U$ for all $n \geq 0$. Then the limit set $\omega(x)$ is disjoint from U. Since A intersects U it cannot be contained in $\omega(x)$, and as A is minimal we conclude that $\omega(x) \cap A = \emptyset$. Thus, we can take a minimal subset $B \subset \omega(x)$. This proves that M has at least two minimal subsets.

To prove the converse suppose that $A, B \subset M$ are minimal and disjoint. Take $\varepsilon > 0$ such that $\varepsilon < \operatorname{dist}(a, b)$ for all $a \in A$ and all $b \in B$. If $U \subset M$ with $\operatorname{diam}(U) < \varepsilon$ then U cannot intersect both A and B. Thus, taking $x \in A$ or $x \in B$ we conclude that condition (2.1) is true. ∎

In the light of Proposition 2.4 we can say that in fact the Mañé–Kato argument proves that if f is cw-expansive and M has positive dimension then M contains at least two minimal subsets. Notice that this result immediately implies Theorem 2.1, for if A is a minimal subset of M, then the restriction of f to A is itself a cw-expansive system containing only one minimal subset. Hence, A is zero-dimensional.

LEMMA 2.5 ([16, Lemma 2.6]). *If f is a cw-expansive homeomorphism of a compact metric space of positive dimension and $A \subset M$ is closed, invariant and zero-dimensional then there is a minimal subset $B \subset M$ which is disjoint from A.*

Lemma 2.5 implies the existence of infinitely many minimal subsets, as we will see in the next corollary. The *mincenter* [3] of f is the closure of the union of all minimal subsets.

COROLLARY 2.6. *If f is cw-expansive and M has positive dimension then*

(1) *the mincenter has positive dimension,*
(2) *M contains infinitely many minimal subsets, and*
(3) *if $A \subset M$ is finite then there is a minimal subset $B \subset M$ which is disjoint from A.*

Proof. The mincenter is closed and invariant. If in addition it had dimension zero we could apply Lemma 2.5 to obtain a minimal subset disjoint from the mincenter, a contradiction.

By Theorem 2.3 each minimal subset is zero-dimensional. If there were only finitely many minimal subsets then the mincenter would be zero-dimensional as a finite union of zero-dimensional subsets. This contradicts (1).

If $A \subset M$ is finite then it cannot intersect each minimal subset. ∎

To our knowledge, no cw-expansive homeomorphism of a compact space of positive dimension contains just countably many minimal subsets.

2.3. Topological limit sets. Suppose that $X_n \subset M$, $n \geq 1$, is a sequence of subsets of M. Following [12, §2-16], the set of all points $y \in M$ such that every open set containing y intersects infinitely many sets X_n is denoted as $\limsup X_n$.

Remark 2.7. We have

$$\limsup X_n = \bigcap_{m=1}^{\infty} \operatorname{clos}\Big(\bigcup_{n=m}^{\infty} X_n\Big).$$

Therefore, $\limsup X_n$ is closed. Also, $\limsup X_n = \limsup \operatorname{clos}(X_n)$.

Remark 2.8. It is clear that the lim sup is independent of the first set of the sequence, thus

$$\limsup X_n = \bigcap_{m=1}^{\infty} \operatorname{clos}\Big(\bigcup_{n=m}^{\infty} X_{n+k}\Big) \quad \text{for any } k \geq 0.$$

Remark 2.9. If M is a compact metric space, the subsets X_n are closed and a subsequence of $\{X_n\}$ converges to A in the Hausdorff distance, then $A \subset \limsup X_n$.

Remark 2.10. If $X_n \subset Y_n$ for all $n \geq 1$ then $\limsup X_n \subset \limsup Y_n$.

Remark 2.11. If $\operatorname{diam}(X_n) \to 0$ as $n \to \infty$ then for any sequence $x_n \in X_n$,

$$\limsup X_n = \limsup \{x_n\}.$$

Proposition 2.12. *If M is compact and $C_n \subset M$ is a sequence of continua such that $\limsup C_n$ is zero-dimensional then $\operatorname{diam}(C_n) \to 0$ as $n \to \infty$.*

Proof. If $\limsup \operatorname{diam}(C_n) > \varepsilon$ for some $\varepsilon > 0$ then there is a subsequence $\{C_{n_i}\}$ of sets of diameter greater than ε, which converges to a continuum A in the Hausdorff distance and $\operatorname{diam}(A) \geq \varepsilon$. Since $A \subset \limsup C_n$ we conclude that $\limsup C_n$ has positive dimension. ∎

Suppose that $\limsup X_n = Y$. We say that this limit is *uniform* if for all $\varepsilon > 0$ there is N such that

$$Y \subset B_\varepsilon\Big(\bigcup_{i=0}^{N} X_{m+i}\Big) \tag{2.3}$$

for all $m \geq 0$; we then write $Y = \operatorname{u\,lim\,sup} X_n$. The following proposition gives an alternative definition in terms of the Hausdorff distance dist_H. We refer the reader to [22] for the definition and main properties of the Hausdorff distance.

PROPOSITION 2.13. *The following equivalences hold:*

- $Y = \limsup X_n$ *if and only if for all* $\varepsilon > 0$ *there is* $n_0 \geq 1$ *such that*

$$\mathrm{dist}_H\Big(Y, \bigcup_{i=n}^{\infty} X_i\Big) < \varepsilon, \qquad \forall n \geq n_0.$$

- $Y = \operatorname{u\,lim\,sup} X_n$ *if and only if for all* $\varepsilon > 0$ *there are* $n_0, m \geq 1$ *such that*

$$\mathrm{dist}_H\Big(Y, \bigcup_{i=n}^{n+m} X_i\Big) < \varepsilon, \qquad \forall n \geq n_0.$$

This follows directly from the definitions.

2.4. Dynamical limit sets. Let $f\colon M \to M$ be a homeomorphism of the compact metric space with $\dim(M) \geq 1$. For $X \subset M$ define its *ω-limit set* as

$$\omega(X) = \limsup_{n\geq 0} f^n(X).$$

For $X = \{x\}$ a singleton, $\omega(X)$ is the usual ω-limit set denoted by $\omega(x)$, as defined in (2.2). In what follows we collect some elementary properties of ω-limit sets. As usual, α-limits are defined by considering the inverse of the homeomorphism, and the results for ω-limits also hold for α-limits.

From Remark 2.10, if $X \subset Y$ then $\omega(X) \subset \omega(Y)$. This implies that

$$\bigcup_{x\in X} \omega(x) \subseteq \omega(X). \tag{2.4}$$

REMARK 2.14. If $\mathrm{diam}(f^n(X)) \to 0$ as $n \to \infty$ then equality holds in (2.4) and $\omega(X) = \omega(x)$ for all $x \in X$. This follows from Remark 2.11.

Let us show that equality may not hold in (2.4) by considering the homeomorphism

$$f\colon [0,1] \to [0,1], \qquad f(x) = x^2. \tag{2.5}$$

Take $X = [0,1]$. It is clear that $\omega(1) = \{1\}$, $\omega(x) = \{0\}$ for all $x \in [0,1)$ and $\omega([0,1]) = [0,1]$.

PROPOSITION 2.15. *For every* $X \subset M$ *the set* $\omega(X)$ *is closed, f-invariant,* $\omega(X) = \omega(\operatorname{clos}(X))$ *and* $\omega(X) = \omega(f^i(X))$ *for all* $i \in \mathbb{Z}$.

Proof. Since $\omega(X)$ is a lim sup, it is closed and $\omega(X) = \omega(\operatorname{clos}(X))$ (see Remark 2.7). We used that $f^n(\operatorname{clos}(X)) = \operatorname{clos}(f^n(X))$ because f is a homeomorphism. From Remark 2.8 we know that $\omega(X)$ is f-invariant and $\omega(X) = \omega(f^i(X))$ for all $i \in \mathbb{Z}$. ■

REMARK 2.16. In [18, proof of Proposition 2.4] Kato considers the set

$$Z = \bigcap_{m=1}^{\infty} f^m\Big(\operatorname{clos}\Big(\bigcup_{n=1}^{\infty} f^n(X)\Big)\Big).$$

Let us show that $Z = \omega(X)$:

$$Z = \bigcap_{m=1}^{\infty} f^m\Big(\operatorname{clos}\Big(\bigcup_{n=1}^{\infty} f^n(X)\Big)\Big) = \bigcap_{m=1}^{\infty} \Big(\operatorname{clos}\Big(\bigcup_{n=1}^{\infty} f^{m+n}(X)\Big)\Big) = \omega(X).$$

PROPOSITION 2.17. *If* $C \subset M$ *is connected and* $\dim(\omega(C)) = 0$ *then*

$$\lim_{n\to\infty} \operatorname{diam}(f^n(C)) = 0.$$

Proof. This follows directly from Proposition 2.12. ■

The next proposition is related to a classical result due to G. D. Birkhoff [7, pp. 199–200].

PROPOSITION 2.18. *If* $x \in X \subset M$ *and* $\operatorname{diam}(f^n(X)) \to 0$ *as* $n \to \infty$ *then the following statements are equivalent:*

(1) $\omega(x)$ *is minimal,*
(2) $\operatorname{u}\limsup f^n(x) = \omega(x)$,
(3) $\operatorname{u}\limsup f^n(X) = \omega(x)$.

Proof. (1)⇒(2). Suppose that $\omega(x)$ is minimal and take $y \in \omega(x)$ and $n_k \to \infty$ such that $f^{n_k}(x) \to y$. Assuming that the limit is not uniform and applying Proposition 2.13 we have $\varepsilon > 0$ and $m_k \to \infty$ such that $\operatorname{dist}_H(\omega(x), \{f^{m_k+i}(x) : 0 \le i \le n_k\}) \ge \varepsilon$. This implies $\operatorname{dist}_H(\omega(x), \{f^i(y) : 0 \le i\}) \ge \varepsilon$ and $\operatorname{dist}_H(\omega(x), \omega(y)) \ge \varepsilon$. This is a contradiction because, as $\omega(x)$ is minimal and $y \in \omega(x)$, we have $\omega(y) = \omega(x)$.

(2)⇒(3). By Remark 2.14, $\limsup f^n(X) = \omega(x)$. The uniformity of this limit follows by the definition (recall (2.3)).

(3)⇒(1). Suppose that $\omega(x)$ is not minimal. Then there is a proper, closed, invariant subset $A \subset \omega(x)$. Take $\varepsilon = \operatorname{dist}_H(A, \omega(x))/2$. By continuity of f and the fact that $\operatorname{diam}(f^n(X)) \to 0$ as $n \to \infty$, for a given $n \ge 1$ we can take m arbitrarily large such that $f^m(X), \dots, f^{m+n}(X) \subset B_\varepsilon(A)$. Thus, $\operatorname{dist}_H(\omega(x), f^m(X) \cup \dots \cup f^{m+n}(X)) \ge \varepsilon$. From Proposition 2.13 we conclude that the limit is not uniform. ■

2.5. Transitivity. In this section we recall some known facts about transitivity, following [4, 23]. A subset A is invariant (resp. positively invariant, negatively invariant) when $f(A) = A$ (resp. $f(A) \subset A$, $f^{-1}(A) \subset A$). For a homeomorphism, A is invariant if and only if it is both positively and negatively invariant. Given $A, B \subset M$ define $N(A,B) = \{i \in \mathbb{Z} : f^i(A) \cap B \neq \emptyset\}$ and $N_+(A,B) = N(A,B) \cap \mathbb{Z}_+$ where $\mathbb{Z}_+$ is the set of non-negative integers.

The orbit of $x \in M$ is $\operatorname{orb}(x) = \{f^i(x) : i \in \mathbb{Z}\}$. Notice that $\operatorname{clos}(\operatorname{orb}(x)) = \operatorname{orb}(x) \cup \omega(x) \cup \alpha(x)$. Define

$$\begin{aligned} \operatorname{Trans}(f) &= \{x \in M : \operatorname{clos}(\operatorname{orb}(x)) = M\}, \\ \operatorname{Trans}_+(f) &= \{x \in M : \omega(x) = M\}, \\ \operatorname{Trans}_-(f) &= \{x \in M : \alpha(x) = M\}. \end{aligned}$$

The system is *transitive* when $\operatorname{Trans}(f)$ is not empty, and is *minimal* when $\operatorname{Trans}(f)$ is the whole M.

Proposition 2.19 ([23, Theorem 5.8]). *If $\mathcal{B}$ is a countable basis of non-empty open subsets then* $\operatorname{Trans}(f) = \bigcap_{U \in \mathcal{B}} \bigcup_{i \in \mathbb{Z}} f^i(U)$ *and the following are equivalent:*

- *f is transitive,*
- *$N(U,V) \neq \emptyset$ for every pair of non-empty open subsets $U, V \subset M$,*
- *every non-empty, open, invariant subset of M is dense in M.*

In that case, $\operatorname{Trans}(f)$ *is a dense G_δ subset of M.*

Clearly, f is transitive if and only if f^{-1} is transitive.

Proposition 2.20. *If f is transitive and $x \in M$ is an isolated point then the set of isolated points is the open dense set* $\operatorname{orb}(x) = \operatorname{Trans}(f)$*. If in addition M has infinitely many points then* $\operatorname{Trans}_+(f) = \operatorname{Trans}_-(f) = \emptyset$.

The proof of Proposition 2.20 is obvious. We say that M is *perfect* if it contains no isolated points.

Proposition 2.21 ([4, Theorem 1.4 and Proposition 4.7]). *If $\mathcal{B}$ is a countable basis of non-empty open subsets then*

$$\operatorname{Trans}_+(f) = \bigcap_{U \in \mathcal{B}} \bigcup_{i \in \mathbb{Z}_+} f^{-i}(U).$$

If M is perfect then the following are equivalent:

- *f is transitive,*
- *$N_+(U,V) \neq \emptyset$ for every pair of non-empty open subsets $U, V \subset M$,*
- *$N_+(U,V)$ is infinite for every pair of non-empty open subsets $U, V \subset M$,*
- *every non-empty, open, negatively invariant subset of M is dense in M.*

In that case, $\operatorname{Trans}_+(f)$ *is a dense G_δ subset of M.*

Proposition 2.22. *If f is transitive on M perfect then*

$$\mathrm{Trans}(f) = \mathrm{Trans}_+(f) \cup \mathrm{Trans}_-(f).$$

Proof. If x is a transitive point then

$$M = \mathrm{clos}(\mathrm{orb}(x)) = \mathrm{orb}(x) \cup \omega(x) \cup \alpha(x).$$

By the Baire Category Theorem applied to the perfect space M, the complement of the countable set $\mathrm{orb}(x)$ is dense in M. That is, $\omega(x) \cup \alpha(x)$ is a closed, dense subset of M and so equals M. By the Baire Category Theorem again, either $\omega(x)$ or $\alpha(x)$ has a non-empty interior. Suppose $U \subset \omega(x)$. Since $\mathrm{orb}(x)$ is dense, there exists $i \in \mathbb{Z}$ such that $f^i(x) \in U$. Since $\omega(x)$ is closed and invariant, $M = \mathrm{clos}(\mathrm{orb}(x)) = \mathrm{clos}(\mathrm{orb}(f^i(x))) \subset \omega(x)$ and so $x \in \mathrm{Trans}_+(f)$. Similarly, if $U \subset \alpha(x)$ then $x \in \mathrm{Trans}_-(f)$. ∎

3. Cw-minimality. Let us explain the general idea of 'cw-properties' of dynamical systems. First we illustrate it with the notions of expansivity. For a homeomorphism $f\colon M \to M$, $x \in M$ and $\varepsilon > 0$ consider the *dynamical ball*

$$\Gamma_\varepsilon(x) = \{y \in M : \mathrm{dist}(f^i(x), f^i(y)) \le \varepsilon \text{ for all } i \in \mathbb{Z}\}.$$

Expansivity means that for some ε the dynamical balls are as trivial as they can be, namely, $\Gamma_\varepsilon(x) = \{x\}$ for all $x \in M$. But if we relax the meaning of *trivial* we obtain other notions of expansivity, for example:

- f is defined to be *entropy expansive* [9] if there is $\varepsilon > 0$ such that $h(\Gamma_\varepsilon(x)) = 0$ for all $x \in M$, where h denotes topological entropy,
- f is cw-expansive if and only if there is $\varepsilon > 0$ such that $\dim(\Gamma_\varepsilon(x)) = 0$ for all $x \in M$ [6, Lemma 2.2],
- for a measure μ, f is *μ-expansive* [21] if there is $\varepsilon > 0$ such that $\mu(\Gamma_\varepsilon(x)) = 0$ for all $x \in M$.

In this article we consider minimality, which can be defined as: each closed invariant subset different from M is trivial, i.e. empty. If we say that *each closed invariant subset different from M is zero-dimensional* then we get cw-minimality. Following this idea we will consider cw-isolated sets in §3.5 and positive open-wise expansivity in §3.7 (usually known as *sensitivity*).

We start in §3.1 by deducing direct properties. §3.2 concerns minimal subsets of a cw-minimal set. In §3.3 we consider the problem of existence of cw-minimal subsets. In §3.4 and §3.6 we give some characterizations of cw-minimality. As a generalization of the notion of isolated set, in §3.5 we consider cw-isolated sets. In §3.7 we show that cw-minimal systems are either minimal or sensitive.

3.1. Definition and some properties. Consider the set

$$\mathcal{I}^+(f) = \{X \subset M : X \text{ is } f\text{-invariant, closed and } \dim(X) \ge 1\},$$

ordered by inclusion. The minimal members of $\mathcal{I}^+(f)$ will be called *cw-minimal*. Equivalently, $X \subset M$ is cw-minimal if it is closed, f-invariant, $\dim(X) \geq 1$ and for all $Y \subset X$ closed and invariant we have either $\dim(Y) = 0$ or $Y = X$. If M is cw-minimal we also say that f is cw-minimal. Note that every minimal homeomorphism on a space of positive dimension is cw-minimal.

REMARK 3.1. If $\dim(M) = 0$ then $\mathcal{I}^+(f)$ is empty and there are no cw-minimal subsets. Therefore, we will assume that $\dim(M) > 0$, i.e., M contains a non-trivial continuum.

REMARK 3.2. For minimal subsets $X, Y \subset M$ we have either $X \cap Y = \emptyset$ or $X = Y$. For cw-minimality we have this related property: if $X, Y \subset M$ are distinct and cw-minimal then either $\dim(X \cap Y) = 0$ or $X \cap Y = \emptyset$.

REMARK 3.3. If f is cw-minimal and C is a non-trivial continuum with $\alpha(C) \neq M$ then $\dim(\alpha(C)) = 0$ and $\operatorname{diam}(f^{-n}(C)) \to 0$ as $n \to \infty$. Indeed, $\dim(\alpha(C)) = 0$ because $\alpha(C)$ is a closed invariant subset and we simply apply the definition of cw-minimality; and $\operatorname{diam}(f^{-n}(C)) \to 0$ as $n \to \infty$ from Proposition 2.17.

PROPOSITION 3.4. *If f is cw-minimal then M is perfect.*

Proof. Suppose that $x \in M$ is an isolated point, that is, $\{x\}$ is open. Then $\operatorname{orb}(x)$ is open and $M \setminus \operatorname{orb}(x)$ is closed and invariant. Since $\dim(M) > 0$, we have $\dim(M \setminus \operatorname{orb}(x)) > 0$ by Theorem 2.2, which contradicts cw-minimality. ■

For future reference we state the following known result.

PROPOSITION 3.5 ([16, Theorem 2.7] and [17, Theorem 3.1]). *If $f : M \to M$ is cw-expansive, cw-minimal and* $\dim(M) > 0$ *then f is weakly chaotic in the sense of Devaney* (*sensitive, transitive and the union of minimal subsets is dense in M*).

We will see that cw-minimality (even without cw-expansivity) implies transitivity (Theorem 3.17). In §4.5 we will see a homeomorphism on the two-dimensional torus which is cw-minimal with only one minimal subset (a fixed point). A cw-minimal homeomorphism may not be sensitive: consider for instance an irrational rotation of the circle. However, in §3.7 we will see that on a locally connected space a cw-minimal homeomorphism which is not minimal has to be sensitive.

3.2. Minimal subsets. As we said before, minimal subsets of cw-expansive homeomorphisms are zero-dimensional. Thus, they are finite (periodic orbits) or Cantor sets. Following [16, Question 3.6] we emphasize that, to the author's knowledge, it is still not known whether every cw-expansive (even expansive) homeomorphism of a compact metric space (even a compact manifold) of positive dimension contains at least one minimal subset of each type (periodic orbits and Cantor sets). This is the case in all the examples known by the author. It is interesting to note that some advances were obtained in [18, Theorem 2.1] for one-dimensional compact spaces. In [1] the authors constructed a cw-expansive homeomorphism of a torus with infinitely many fixed points.

In [18, Proposition 2.6] it is shown that if $f\colon M \to M$ is cw-expansive, cw-minimal and $\dim(M) > 0$, then there is a sequence of minimal subsets $X_n \subset M$ such that $X_n \to M$ in the Hausdorff metric. In the next result we generalize this proposition, and we prove a converse.

PROPOSITION 3.6. *For a homeomorphism $f\colon M \to M$ of a compact metric space the following statements are equivalent:*

(1) *f is transitive and the union of minimal subsets is dense in M,*
(2) *there is a sequence of minimal subsets $X_n \subset M$ such that $X_n \to M$ in the Hausdorff metric.*

Proof. (1)⇒(2). Given $\varepsilon > 0$ take a transitive point $x \in M$ and $n \geq 1$ such that $\{x, f(x), \dots, f^n(x)\}$ is $\varepsilon/2$-dense, i.e. for every $y \in M$ there is $i \in \{0, \dots, n\}$ such that $\mathrm{dist}(y, f^i(x)) < \varepsilon/2$. Take $\delta > 0$ such that if $z \in M$ and $\mathrm{dist}(z, x) < \delta$ then $\mathrm{dist}(f^i(x), f^i(z)) < \varepsilon/2$ for all $i = 0, \dots, n$. Since minimal subsets are dense, we can take a minimal subset X with a point $z \in X$ such that $\mathrm{dist}(x, z) < \delta$. This implies that X is ε-dense. As $\varepsilon > 0$ is arbitrary, the proof of (1)⇒(2) is complete.

(2)⇒(1). We only have to show that f is transitive. Let $U, V \subset M$ be open non-empty subsets. From the hypothesis we can take a minimal subset X_n intersecting U and V. As X_n is minimal, for each $x \in X_n \cap U$ there is $m \geq 1$ such that $f^m(x) \in V$. Thus, $f^m(U) \cap V \neq \emptyset$. The transitivity now follows from Proposition 2.19. ∎

3.3. Existence of cw-minimal subsets. If we take as f the identity homeomorphism of the interval $[0, 1]$, we see that cw-minimal sets may not exist. In [17, Theorem 3.1] Kato proved that $\mathcal{I}^+(f)$ has minimal elements provided that f is cw-expansive. In what follows we will generalize this result giving a different proof (Kato uses Brouwer's Reduction Theorem). Define

$$\mathrm{mesh}(A) = \sup\{\mathrm{diam}(C) : C \subset A \text{ connected}\}.$$

PROPOSITION 3.7. *If there is $\delta > 0$ such that* $\mathrm{mesh}(X) \geq \delta$ *for every closed invariant subset $X \subset M$ of positive dimension then M contains a cw-minimal set.*

Proof. It is easy to see that if $A_n \to A$ in the Hausdorff distance then $\mathrm{mesh}(A) \geq \limsup \mathrm{mesh}(A_n)$. Therefore, the set

$$M_\delta = \{A \subset M : \mathrm{mesh}(A) \geq \delta,\ A \text{ closed and invariant}\}$$

is closed. Let μ be a size function defined on the compact subsets of M (see [22]). By definition, it is continuous and strictly increasing with respect to inclusion. Thus, μ has a minimum at some member of M_δ, which has to be cw-minimal. ∎

Notice that cw-expansive homeomorphisms satisfy the hypothesis of Proposition 3.7. To give an application of this proposition we introduce a definition. We say that a closed invariant set is *semisimple* if every point of this set is contained in a minimal subset.

COROLLARY 3.8. *If $f\colon M \to M$ is cw-expansive with semisimple mincenter then* $\dim(M) = 0$.

Proof. Arguing by contraposition, suppose that M has positive dimension. From Proposition 3.7 we know that there is a cw-minimal subset $X \subset M$. By Proposition 3.5 we know that $f\colon X \to X$ is transitive and X is contained in the mincenter. If we take $x \in X$ with orbit dense in X, we conclude that x does not belong to any minimal subset, contradicting the mincenter being semisimple. ∎

3.4. Characterizations of cw-minimality. We will give two characterizations of cw-minimality. The first one is essentially contained in [17, proof of Theorem 3.1, condition (∗)].

PROPOSITION 3.9. *For a closed f-invariant subset $X \subset M$ with* $\dim(X) \geq 1$*, the following statements are equivalent:*

(1) *X is cw-minimal,*
(2) *for every non-empty $A \subset X$, either $\omega(A) = X$ or $\dim(\omega(A)) = 0$,*
(3) *for every non-empty $A \subset X$ with $\dim(A) \geq 1$, either $\omega(A) = X$ or $\dim(\omega(A)) = 0$.*

Proof. (1)⇒(2) follows from the fact that $\omega(A)$ is closed and f-invariant. It is obvious that (2)⇒(3). To prove that (3)⇒(1) suppose that $Y \subset X$ is closed, f-invariant and $\dim(Y) \geq 1$. Applying the hypothesis for $A = Y$ we conclude that $\omega(Y) = X$. Since Y is f-invariant we have $\omega(Y) = Y$, thus $Y = X$. ∎

The next result is related to Proposition 2.18 as it express cw-minimality in terms of uniform convergence.

THEOREM 3.10. *A homeomorphism $f\colon M \to M$ is cw-minimal if and only if for every closed subset $C \subset M$ satisfying*

$$\inf_{n\geq 0} \operatorname{mesh}(f^n(C)) > 0 \tag{3.1}$$

we have

$$\operatorname{u}\limsup f^n(C) = M. \tag{3.2}$$

Proof. (Direct) Towards a contradiction suppose that there are $\varepsilon > 0$ and a divergent sequence $m_N \to \infty$ such that for some $C \subset M$ satisfying (3.1),

$$M \neq B_\varepsilon\Big(\bigcup_{i=0}^{N} f^{m_N+i}(C)\Big) \quad \text{for all } N \geq 1.$$

For each $N \geq 1$ consider the compact set

$$D_N = \bigcup_{i=0}^{N} f^{m_N+i}(C).$$

Taking subsequences we can assume that $D_N \to D_*$ and $f^{m_N}(C) \to C_*$ in the Hausdorff distance, for some compact subsets $C_*, D_* \subset M$. We have $M \neq B_\varepsilon(D_*)$ and $\omega(C_*) \neq M$. From (3.1) we see that $\dim(\omega(C_*)) \geq 1$. Thus, as f is cw-minimal, we conclude that $\omega(C_*) = M$. This contradiction proves the direct part.

(Converse) Suppose that f is not cw-minimal. Then there is a closed invariant proper subset $C \subset M$ of positive dimension. It is clear that C satisfies (3.1). Since $C \neq M$ and it is invariant, we have $\operatorname{u\,lim\,sup} f^n(C) \neq M$ and the proof is complete. ∎

See §4.1 for an example showing that (3.1) is necessary for (3.2) to hold in the direct part of Theorem 3.10.

3.5. Cw-isolated sets. Let $X \subset M$ be a closed invariant subset. We say that X is (*dynamically*) *isolated* [10] if there is an open neighborhood V of X such that if $f^i(x) \in \operatorname{clos}(V)$ for all $i \in \mathbb{Z}$ then $x \in X$ (equivalently, $X = \bigcap_{i\in\mathbb{Z}} f^i(\operatorname{clos}(V))$). We say that X is *cw-isolated* if there is a neighborhood U of X such that if $C \subset M$ is a non-trivial continuum and $f^i(C) \subset \operatorname{clos}(U)$ for all $i \in \mathbb{Z}$, then $C \subset X$. In this case we say that U is a *cw-isolating neighborhood* of X.

Remark 3.11. Every isolated set is cw-isolated.

Proposition 3.12. *A closed invariant subset X is cw-isolated with cw-isolating neighborhood U if and only if $Z \setminus X$ is zero-dimensional, where $Z = \bigcap_{i\in\mathbb{Z}} f^i(\operatorname{clos}(U))$.*

Proof. As $Z \setminus X$ is locally compact we can apply Remark 2.1. To prove the direct part notice that $C \subset Z \setminus X$ means $f^i(C) \subset \operatorname{clos}(U)$ for all $i \in \mathbb{Z}$ and $C \cap X = \emptyset$. To prove the converse, suppose that $f^i(C) \subset \operatorname{clos}(U)$ for all $i \in \mathbb{Z}$ where C is a non-trivial continuum not contained in X. As X is closed there is a non-trivial subcontinuum $D \subset C$ which is disjoint from X. Thus, $D \subset Z \setminus X$, completing the proof. ∎

We say that a homeomorphism $f\colon M \to M$ is *expansive* if there is $\xi > 0$ such that $\operatorname{dist}(f^i(x), f^i(y)) \leq \xi$ for all $i \in \mathbb{Z}$ implies $x = y$.

Remark 3.13. For an expansive homeomorphism every finite invariant subset is isolated.

Proposition 3.14. *For a cw-expansive homeomorphism every closed invariant subset of dimension zero is cw-isolated.*

Proof. Suppose that $X \subset M$ is closed, invariant and zero-dimensional and let ξ be a cw-expansivity constant. Consider open subsets $U_1, \dots, U_n$, pairwise disjoint and covering X with $\operatorname{diam}(U_j) < \xi$ for all $j = 1, \dots, n$. We will show that $U = U_1 \cup \dots \cup U_n$ is a cw-isolating neighborhood. From the construction, if a continuum is contained in U then it is contained in a unique U_j. Thus, if $f^i(C) \subset U$ for all $i \in \mathbb{Z}$, where C is a continuum, then $\operatorname{diam}(f^i(C)) < \xi$ for all $i \in \mathbb{Z}$, and cw-expansivity implies that C is a singleton. This finishes the proof. ∎

Proposition 3.15. *For a cw-minimal homeomorphism every proper closed invariant subset is cw-isolated.*

Proof. Suppose that $X \subset M$ is a proper closed and invariant subset. Cw-minimality implies that it is zero-dimensional. Take any open set U such that $X \subset U$ and $\operatorname{clos}(U) \neq M$. Consider $Z = \bigcap_{i \in \mathbb{Z}} f^i(\operatorname{clos}(U))$. As Z is closed, invariant and $Z \neq M$, cw-minimality implies that $\dim(Z) = 0$. The proof ends by applying Proposition 3.12. ∎

The next result allows us to find isolated and arbitrarily small extensions of any cw-isolated subset. The boundary of a subset V will be denoted by ∂V.

Theorem 3.16. *If X is cw-isolated then for every neighborhood U of X there is an isolated subset Y such that $X \subset Y \subset U$. If in addition* $\dim(X) = 0$ *then* $\dim(Y) = 0$.

Proof. Let U be an open cw-isolating neighborhood of X and define

$$Z = \bigcap_{i \in \mathbb{Z}} f^i(\operatorname{clos}(U)).$$

Take an open set V such that $X \subset V$ and $\operatorname{clos}(V) \subset U$. By Proposition 3.12 we know that $Z \setminus X$ is zero-dimensional, and consequently $T = Z \setminus V$ is compact and zero-dimensional. For each $x \in T \cap \partial V$ there is an open neighborhood W_x of x such that $\partial W_x \cap T = \emptyset$ and $\operatorname{clos}(W_x) \subset U$. As $T \cap \partial V$ is compact we can take finitely many such neighborhoods, $W_1, \dots, W_n$, covering $T \cap \partial V$. In this way, $V' = V \cup W_1 \cup \dots \cup W_n$ satisfies $\partial V' \cap Z = \emptyset$ and $X \subset V' \subset U$. Define $Y = \bigcap_{i \in \mathbb{Z}} f^i(\operatorname{clos}(V'))$. It is clear that $X \subset Y$. We will prove that Y is isolated with isolating neighborhood V'. For this purpose it only remains to show that $Y \subset V'$. Since $Y \subset Z$ and $\partial V' \cap Z = \emptyset$, we conclude that $Y \subset V'$, which ends the proof. ∎

3.6. More characterizations. The next result is partially contained in [16, 17] but here we do not assume cw-expansivity.

THEOREM 3.17. *Assume that* $\dim(M) > 0$. *The following are equivalent:*

(1) f *is cw-minimal,*
(2) *for every non-trivial continuum* $A \subset M$ *and every non-empty open* $U \subset M$, $N(A, U) \neq \emptyset$,
(3) *every non-empty, open, invariant subset of* M *meets every non-trivial continuum* $A \subset M$,
(4) *for every non-empty, open, invariant subset* U *and every non-trivial continuum* $A \subset M$ *the intersection* $A \cap U$ *is dense in* A,
(5) *for every non-trivial continuum* $A \subset M$, $\mathrm{Trans}(f) \cap A$ *is a dense* G_δ *subset of* A,
(6) f *is transitive and the set* $M \setminus \mathrm{Trans}(f)$ *is zero-dimensional.*

Proof. (1)⇒(2). For any non-trivial continuum $A \subset M$ the closure of $\bigcup_{i\in\mathbb{Z}} f^i(A)$ is invariant and has positive dimension and so equals M. Hence, any open set meets $\bigcup_{i\in\mathbb{Z}} f^i(A)$.

(2)⇒(3). If $i \in N(A, U)$ then there exists $x \in A$ such that $f^i(x) \in U$. So if U is invariant then $x \in U \cap A$.

(3)⇒(4). If $x \in A$ and $\varepsilon > 0$ then there exists a continuum $B \subset A$ with $x \in B$ and $0 < \mathrm{diam}(B) < \varepsilon$. Since U meets B it follows that $U \cap A$ is dense in A.

(4)⇒(5). By Proposition 2.19, if $\mathcal{B}$ is a countable basis of non-empty open subsets then

$$A \cap \mathrm{Trans}(f) = \bigcap_{U\in\mathcal{B}} \Big[A \cap \bigcup_{i\in\mathbb{Z}} f^i(U)\Big]$$

is a dense G_δ subset of A.

(5)⇒(6). Since $\dim(M) > 0$, there exists a non-trivial continuum $A \subset M$. Thus, $A \cap \mathrm{Trans}(f)$ is not empty and f is transitive. Hence, the F_σ set $M \setminus \mathrm{Trans}(f)$ contains no non-trivial continuum and so by Theorem 2.2, $M \setminus \mathrm{Trans}(f)$ is zero-dimensional.

(6)⇒(1). If X is a closed invariant subset of M with $\dim(X) > 0$ then (6) implies that X meets $\mathrm{Trans}(f)$. If $x \in \mathrm{Trans}(f) \cap X$ then $M = \mathrm{clos}(\mathrm{orb}(x)) \subset X$ because X is closed and invariant. ■

COROLLARY 3.18. *A system* f *is cw-minimal if and only if* $M \setminus \mathrm{Trans}(f)$ *is a countable increasing union of dynamically isolated zero-dimensional closed invariant subsets.*

Proof. (Direct) For each $x \in M \setminus \mathrm{Trans}(f)$ the closure of the orbit of x is a proper closed invariant set and so is zero-dimensional. By Proposition 3.15 and Theorem 3.16 there exists a proper open $U_x \subset M$ which contains clos(orb(x)) and such that

$$\mathrm{clos}(\mathrm{orb}(x)) \subset K_x = \bigcup_{i\in\mathbb{Z}} f^i(\mathrm{clos}(U_x)) \subset U_x.$$

By the Lindelöf Theorem [12, Theorem 2-44] we can choose a countable open subcover $\{U_{x_n}\}$ of $M \setminus \mathrm{Trans}(f)$. Each $y \in M \setminus \mathrm{Trans}(f)$ is contained in some U_{x_n} and so $\mathrm{clos}(\mathrm{orb}(y)) \subset K_{x_n}$. Since K_{x_n} is a proper closed invariant set, it is contained in $M \setminus \mathrm{Trans}(f)$. Hence $M \setminus \mathrm{Trans}(f)$ is the union of the K_{x_n}'s.

To get an increasing union, count the K_{x_n}'s as $K_1, K_2, \ldots$. Now inductively let $L_1 = K_1$ and we construct L_{k+1} dynamically isolated and containing $K_1, \ldots, K_{k+1}$ by applying Theorem 3.16 to $L_k \cup K_{k+1}$.

(Converse) From Theorem 2.2 we know that the set of non-transitive points is zero-dimensional. Thus, the converse follows by Theorem 3.17. ■

Corollary 3.19. *If f is cw-minimal with $\dim(M) > 0$ then for any non-empty open subset U of M, the union of the non-trivial continua contained in U is dense in U. In particular, U has positive dimension.*

Proof. Let V be a non-empty open subset of U and let A be a non-trivial continuum in M. From Theorem 3.17 we know that $N(A, V)$ is not empty. If $i \in N(A, V)$ then $f^i(A)$ is a non-trivial continuum with a point $x \in f^i(A) \cap V$. Assume that the $\varepsilon > 0$ ball around x is contained in V. There exists a continuum B with $x \in B$, $0 < \mathrm{diam}(B) < \varepsilon$ and $B \subset f^i(A)$. Hence, $B \subset V$. As V was arbitrary, the union of such B's is dense in U. ■

The next example shows that Corollary 3.19 requires cw-minimality. It is an example of a transitive homeomorphism on a space of positive dimension with a dense, invariant open subset which is zero-dimensional.

Example 3.20. Let K be a Cantor set and σ the shift homeomorphism on the product $K^{\mathbb{Z}}$ (which is itself a Cantor set). Define $d\colon K \to K^{\mathbb{Z}}$ so that $d(c)_i = c$ for all $i \in \mathbb{Z}$. That is, d is an embedding onto the set of fixed points of σ. Let C be an arbitrary non-trivial continuum and let $q\colon K \to C$ be an onto map (see for instance [12, Theorem 3-28]). Define M to be the quotient space of $K^{\mathbb{Z}} \cup C$ with $d(c)$ identified with $q(c)$. Thus, $\sigma \cup \mathrm{id}_C$ factors to a transitive map f on M, since $K^{\mathbb{Z}} \setminus d(K)$ is a dense open invariant subset of dimension zero.

In contrast with open sets and transitivity (Proposition 2.21), it is not true that cw-minimality implies $N_+(C, U) \neq \emptyset$ for all subcontinua C and all non-empty open sets U. For example, take as C a stable arc of a periodic point of an Anosov diffeomorphism of the two-dimensional torus, as in §4.1.

Lemma 3.21. *If f is cw-minimal and C is a non-trivial continuum with $\alpha(C) \neq M$ then*

- $\mathrm{Trans}_-(f) \cap C = \emptyset$,
- *for every non-trivial subcontinuum $D \subset C$, $\mathrm{Trans}_+(f) \cap D$ is a dense G_δ subset of D and $\omega(D) = M$.*

Proof. For each $x \in C$ we have $\alpha(x) \subset \alpha(C) \neq M$, thus $\mathrm{Trans}_-(f) \cap C = \emptyset$. If $D \subset C$ is a non-trivial continuum then $\mathrm{Trans}(f) \cap D$ is a dense G_δ subset of D (Theorem 3.17). From Proposition 3.4 we know that M is perfect, thus we can apply Proposition 2.22 to conclude that

$$\mathrm{Trans}(f) \cap D = \mathrm{Trans}_+(f) \cup \mathrm{Trans}_-(f) \cap D = \mathrm{Trans}_+(f) \cap D.$$

Thus, $\mathrm{Trans}_+(f) \cap D$ is a dense G_δ subset of D. For each $x \in D \cap \mathrm{Trans}_+(f)$ we have $M = \omega(x) \subset \omega(C)$. ∎

Corollary 3.22. *A homeomorphism f is cw-minimal if and only if for every non-trivial continuum $C \subset M$, either $\omega(C) = M$ or $\alpha(C) = M$.*

Proof. The result is vacuous if $\dim(M) = 0$ and so we may assume that M has positive dimension. If f is cw-minimal and C is a non-trivial continuum with $\alpha(C) \neq M$ then $\dim(\alpha(C)) = 0$. Applying Lemma 3.21 we conclude that $\omega(C) = M$.

To prove the converse suppose that $X \subset M$ is closed, invariant and has positive dimension. Take a non-trivial continuum $C \subset X$. If $\omega(C)$ or $\alpha(C)$ equals M then $X = M$ and f is cw-minimal. ∎

In the next remark we consider the case where $\omega(C)$ and $\alpha(C)$ both equal M.

Remark 3.23. If D, E are continua with $\alpha(D) \neq M$, $\omega(E) \neq M$, $D \cap E \neq \emptyset$ and $D \cap E$ is nowhere dense in $C = D \cup E$ then C is a continuum in M and both $\mathrm{Trans}_-(f)$ and $\mathrm{Trans}_+(f)$ meet C but neither is dense in C, although the union (which is $\mathrm{Trans}(f)$) is. In order to illustrate this situation consider an example: if 0 is a fixed point for a hyperbolic automorphism of the two-dimensional torus, let D be a small arc through 0 on the stable manifold and E be a small arc through 0 on the unstable manifold.

From Corollary 3.22 we trivially conclude that if f is cw-minimal and $C \subset M$ is a non-trivial continuum then either $\alpha(C)$ or $\omega(C)$ has positive dimension (as at least one of them equals M). Next we prove that this is also true for cw-expansivity.

Corollary 3.24. *If f is cw-expansive or cw-minimal and $C \subset M$ is a non-trivial continuum then either $\alpha(C)$ or $\omega(C)$ has positive dimension.*

Proof. The case of f cw-minimal follows from Corollary 3.22. Assume that f is cw-expansive with cw-expansivity constant $\xi > 0$. Suppose that C is a non-trivial continuum with $\alpha(C)$ and $\omega(C)$ zero-dimensional. By Proposition 2.17 we have $\mathrm{diam}(f^i(C)) \to 0$ as $i \to \pm\infty$. Take $n \geq 0$ such that

$\operatorname{diam}(f^i(C)) \le \xi$ for $|i| \ge n$ and a non-trivial subcontinuum $D \subset C$ such that $\operatorname{diam}(f^i(D)) \le \xi$ for $|i| \le n$. Thus, $\operatorname{diam}(f^i(D)) \le \xi$ for all $i \in \mathbb{Z}$. This contradicts cw-expansivity. ∎

3.7. Sensitivity. Following [16] we say that f is *sensitive* if there is $\xi > 0$ such that if $x \in M$ and U is any neighborhood of x in M, then there are $y \in U$ and $n \ge 0$ such that $\operatorname{dist}(f^n(x), f^n(y)) > \xi$. The next result will be used below, its proof is direct and allows us to see sensitivity as *positive open-wise expansivity.*

PROPOSITION 3.25. *The following are equivalent:*

(1) *f is sensitive,*
(2) *there is $\xi > 0$ such that*

$$W^s_\xi(x) = \{y \in M : \operatorname{dist}(f^n(x), f^n(y)) \le \xi \text{ for all } n \ge 0\}$$

has empty interior for all $x \in M$,
(3) *there is $\varepsilon > 0$ such that if $U \subset M$ is a non-empty open subset then there is $n \ge 0$ such that $\operatorname{diam}(f^n(U)) > \varepsilon$.*

LEMMA 3.26. *If f is cw-minimal, M is locally connected and $X \subset M$ is a proper closed invariant subset then there is a non-trivial continuum C such that $C \cap X \ne \emptyset$ and $\operatorname{diam}(f^n(C)) \to 0$ as $n \to \infty$.*

Proof. From Proposition 2.21 we can take $y \in \operatorname{Trans}_+(f)$ such that $y \notin X$. Take any $x \in X$ and $n_k \to \infty$ such that $f^{n_k}(y) \to x$. Since M is locally connected, we can take continua D_k such that $x, f^{n_k}(y) \in D_k$ and $\operatorname{diam}(D_k) \to 0$ as $k \to \infty$. By Proposition 3.15, X is cw-isolated. Take $\varepsilon > 0$ such that $B_\varepsilon(X)$ is a cw-isolating neighborhood of X and $y \notin B_\varepsilon(x)$.

Notice that $\operatorname{diam}(f^{-n_k}(D_k)) \ge \varepsilon$ because $y, f^{-n_k}(x) \in f^{-n_k}(D_k)$ and $f^{-n_k}(x) \in X$. Therefore, we can take $m_k \ge 0$ such that $\operatorname{diam}(f^{-n}(D_k)) \le \varepsilon$ for all $0 \le n \le m_k$ and $\operatorname{diam}(f^{-m_k}(D_k)) \ge \varepsilon$. As $\operatorname{diam}(D_k) \to 0$ the continuity of f implies $m_k \to \infty$. Define $E_k = f^{-m_k}(D_k)$. Taking a subsequence if needed we can assume that $E_k \to C$ in the Hausdorff distance. We find that C is a continuum, $\operatorname{diam}(C) \ge \varepsilon$, $\operatorname{diam}(f^n(C)) \le \varepsilon$ for all $n \ge 1$ (since $m_k \to \infty$) and $C \cap X \ne \emptyset$.

This implies that $f^n(C) \subset B_\varepsilon(X)$ for all $n \ge 1$ and $\omega(C) \subset B_\varepsilon(X) \ne M$. The cw-minimality implies that $\omega(C)$ is zero-dimensional. By Proposition 2.17 we conclude that $\operatorname{diam}(f^n(C)) \to 0$. ∎

THEOREM 3.27. *If M is locally connected, $\dim(M) > 0$ and f is cw-minimal then f is either minimal or sensitive.*

Proof. Suppose that M is not minimal; then there is a proper minimal subset $X \subset M$. As f is cw-minimal, X is zero-dimensional. From Proposition 3.15 we know that X is cw-isolated. As M is locally connected, by

Lemma 3.26 there is a continuum C such that $\omega(C) = X$ and $\alpha(C) = M$. Consider open subsets $V, W \subset M$ such that $X \subset W$ and $\text{clos}(V) \cap \text{clos}(W) = \emptyset$. Take $\varepsilon > 0$ such that $\text{dist}(v, w) > \varepsilon$ for all $v \in V$ and all $w \in W$. Consider $n_1 \geq 0$ such that $f^n(C) \subset W$ for all $n \geq n_1$.

To show that f is sensitive let $U \subset M$ be any non-empty open subset. As $\alpha(C) = M$, there is $n_2 \geq 0$ such that $f^{-n_2}(C) \cap U \neq \emptyset$. From Theorem 3.17 and Proposition 3.4 we know that f is transitive and M is perfect. By Proposition 2.21 we can take $n_3 \geq n_1 + n_2$ such that $f^{n_3}(U) \cap V \neq \emptyset$. Consider $x \in U \cap f^{-n_3}(V)$ and $y \in U \cap f^{-n_2}(C)$. Thus, $f^{n_3}(x) \in V$ and $f^{n_3}(y) \in f^{n_3-n_2}(C) \subset W$ (since $n_3 - n_2 \geq n_1$). Therefore, $\text{dist}(f^{n_3}(x), f^{n_3}(y)) > \varepsilon$ and so f is sensitive. ∎

4. Analysis of some examples. This section is devoted to providing examples illustrating our results. The pseudo-Anosov maps in §4.1 are interesting examples of cw-minimality since we can see and understand local stable sets and their associated invariant singular foliations. The minimal systems in §4.2 also constitute an interesting class of examples. However, we are interested in cw-minimality beyond minimality. In §4.3 we show that Anosov diffeomorphisms may not be cw-minimal and in §4.4 we consider Kato's continuum-wise full expansivity. We prove Theorem 4.2 to obtain some non-trivial examples of cw-minimal homeomorphisms. The examples in §4.5 are cw-minimal homeomorphisms with only one minimal subset.

4.1. Pseudo-Anosov homeomorphisms. In [15, Proposition 2.4] it is shown that Anosov diffeomorphisms of the two-torus are cw-minimal. Essentially the same proof shows that pseudo-Anosov homeomorphisms of higher genus surfaces are cw-minimal. These dynamics are known to be expansive. On the two-sphere a pseudo-Anosov homeomorphism with 1-prong singularities of the stable and unstable foliations is cw-minimal, cw-expansive but not expansive.

Considering these examples let us say some words about (3.1) in Theorem 3.10. Let C be a stable arc of an Anosov diffeomorphism f of the two-torus M. As we said, f is cw-minimal. Since C is stable, $\text{diam}(f^n(C)) \to 0$ as $n \to \infty$ and (3.1) is not true for the arc C. If C is contained in the stable manifold of a periodic point, we easily see that (3.2) does not hold. But even if C is contained in the stable manifold of a transitive point $x \in M$ (in which case $\omega(x) = M$), (3.2) cannot be true because f is not minimal and by Proposition 2.18 the limit in (3.2) cannot be uniform.

4.2. Minimal homeomorphisms. From the definitions it is easy to see that every minimal $X \subset M$ with $\dim(X) \geq 1$ is cw-minimal. Notice that by Theorem 2.3 these examples cannot be cw-expansive.

In [17, Remark 3.3] it is shown that the connected components of a cw-minimal and cw-expansive set have diameter bounded away from zero. Let us show that these hypotheses are necessary.

First, consider the example in [11] which is a minimal homeomorphism of a compact subset of the plane with positive dimension. As we said, this implies that it is cw-minimal but cannot be cw-expansive. The phase space contains non-trivial connected components of arbitrarily small diameter (the reader is referred to [11] for an explicit construction). This shows that the hypothesis of cw-expansivity is needed in [17, Remark 3.3].

Second, let us show that cw-minimality is also needed in [17, Remark 3.3]. Let $f\colon X \to X$ be an Anosov diffeomorphism of the two-torus and $g\colon Y \to Y$ a homeomorphism with two fixed points and a third orbit going from one of the fixed points to the other. Let $Z = X \times Y$ and $h\colon M \to M$ be the product $h = f \times g$. Consider a small wandering stable compact arc $\gamma \subset Z$ and define M as the closure of $\bigcup_{i\in\mathbb{Z}} h^i(\gamma)$. From the construction it is clear that the restriction $h\colon M \to M$ is cw-expansive and it contains arbitrarily small connected components, namely the positive iterates of γ.

4.3. Higher-dimensional Anosov diffeomorphisms. Let us show that in general Anosov diffeomorphisms need not be cw-minimal. Indeed, if f_i is an Anosov diffeomorphism on M_i, $i = 1, 2$, we can take the product $g = f_1 \times f_2$, a minimal subset $X \subset M_1$, and consider the proper, closed, g-invariant subset $X \times M_2$, which has positive dimension. This shows that the Anosov diffeomorphism g is not cw-minimal.

4.4. Continuum-wise full expansivity. Following [14, 15] we say that a homeomorphism $f\colon X \to X$ is *continuum-wise fully expansive* if for all $\varepsilon > 0$ and $\delta > 0$, there is $N \geq 1$ such that if A is a subcontinuum of X with diam $A > \delta$, then either $\operatorname{dist}_H(f^n(A), X) < \varepsilon$ for all $n > N$, or $\operatorname{dist}_H(f^{-n}(A), X) < \varepsilon$ for all $n > N$. Such systems will also be called *CF-expansive*. In the next result we collect some known properties of CF-expansivity.

PROPOSITION 4.1. *If $f\colon X \to X$ is CF-expansive then X is connected and f is topologically mixing, cw-minimal, cw-expansive, and for each non-degenerate subcontinuum $A \subset X$, either $f^n(A) \to X$ or $f^{-n}(A) \to X$ as $n \to \infty$.*

The topologically mixing part of Proposition 4.1 is shown in [14, Theorem 3.5]. The other properties are direct from the definitions. From §4.3 we know that the product of cw-minimal homeomorphisms may not be cw-minimal. The next result gives some conditions to obtain a cw-minimal product.

THEOREM 4.2. *Let $f\colon X \to X$ and $g\colon Y \to Y$ be homeomorphisms of compact metric spaces such that f is CF-expansive,* $\dim(X) > 0$, *and con-*

sider the product $h = f \times g$ defined on $Z = X \times Y$. Under these conditions, h is cw-minimal if and only if g is minimal and $\dim(Y) = 0$.

Proof. Suppose that h is cw-minimal. We will show that $\dim(Y) = 0$. As X is not a singleton and f is CF-expansive, there is a minimal subset $A \subset X$, $A \neq X$. The product $A \times Y \subset Z$ is h-invariant and closed. Since h is cw-minimal, this implies that $\dim(A \times Y) = 0$ and $\dim(Y) = 0$.

To prove that g is minimal take a closed g-invariant subset $B \subset Y$. The product $X \times B$ is closed, h-invariant and has positive dimension. Since h is cw-minimal, this implies that $B = Y$. This proves that g is minimal.

To prove the converse, assume that g is minimal and $\dim(Y) = 0$. Take $C \subset Z$ h-invariant, closed and of positive dimension. Since $\dim(Y) = 0$ there is a non-singleton continuum $D \subset X$ and a point $y \in Y$ such that $D \times \{y\} \subset C$. Since f is CF-expansive we can suppose that $f^n(D) \to X$ as $n \to \infty$ (the $-\infty$ case is analogous). This and the minimality of g imply that $\omega(D \times \{y\}) = X \times Y$ (the ω-limit with respect to h). As C is h-invariant, we conclude that $C = X \times Y$. This proves that h is cw-minimal. ∎

In the following remarks we give some examples and applications of the previous results.

REMARK 4.3. As the product of cw-expansive homeomorphisms is cw-expansive, if in Theorem 4.2 we know that g is cw-expansive then h is cw-expansive. Notice that if $\dim(Y) = 0$ then g is cw-expansive (trivially). If Y is disconnected then h is not CF-expansive (because only connected spaces admit CF-expansive homeomorphisms, recall Proposition 4.1). If g is minimal and Y has infinitely many points then h has no periodic point and is cw-expansive and cw-minimal. This remark is related to [17, Example 3.5].

Let us give another example, this time on a Peano continuum and with periodic points.

REMARK 4.4. Consider the product of an Anosov diffeomorphism of the two-dimensional torus and a periodic orbit of period 2. The phase space is a disjoint union of two tori which are permuted by the dynamics. Consider the quotient homeomorphism obtained by identifying two fixed points of the two components. In this way the phase space is now a Peano continuum. The quotient homeomorphism is expansive, cw-minimal but not CF-expansive.

4.5. Cw-minimality with trivial mincenter. We will construct a cw-minimal homeomorphism with a fixed point and no other minimal subset. Let M be smooth compact manifold with a non-vanishing vector field v generating a flow ϕ on M. Take any $p \in M$ and a non-negative smooth function $\rho\colon M \to \mathbb{R}$ vanishing only at p. Let $\psi\colon \mathbb{R} \times M \to M$ be the flow induced by the vector field ρv. Finally, let $f = \psi_1\colon M \to M$ be the time-1 homeomorphism induced by ψ. It is clear that p is a fixed point of f. We will show that if ϕ is minimal then f is cw-minimal and $\{p\}$ is the unique minimal subset.

We say that ψ is *topologically mixing* if for any non-empty open subsets $U, V \subset M$ there is $T > 0$ such that $\psi_t(U) \cap V \neq \emptyset$ for all $t > T$. The corresponding definition for a homeomorphism is analogous. Note that every topologically mixing system is transitive.

LEMMA 4.5. *If* $p \in \mathrm{Trans}_+(\phi) \cap \mathrm{Trans}_-(\phi)$ *then* ψ *and* f *are topologically mixing.*

In [2, Theorem 9.2] there is a proof of Lemma 4.5 for M the two-dimensional torus. However, with obvious modifications, that proof works for a higher-dimensional manifold.

PROPOSITION 4.6. *If* ϕ *is minimal then* $\mathrm{Trans}(f) = M \setminus \{p\}$. *Consequently,* f *is cw-minimal and* $\{p\}$ *is the unique minimal subset.*

Proof. Let $x \neq p$. As ϕ is minimal we know that $x \in \mathrm{Trans}_+(\psi) \cup \mathrm{Trans}_-(\psi)$. Suppose that $x \in \mathrm{Trans}_+(\psi)$ (the other case is analogous). By Lemma 4.5, f is transitive. Since M is perfect we can apply Proposition 2.21 to obtain $y \in \mathrm{Trans}_+(f)$. From the equalities

$$f^n(\psi_t(y)) = \psi_n(\psi_t(y)) = \psi_t(\psi_n(y)) = \psi_t(f^n(y))$$

and the fact that $\psi_t \colon M \to M$ is a homeomorphism for all $t \in \mathbb{R}$, we see that

$$\psi_t(y) \in \mathrm{Trans}_+(f) \quad \text{for all } t \in \mathbb{R}. \tag{4.1}$$

We will show that $\psi_s(y) \in \omega_f(x)$ for some $s \in [0, 1]$. Since $x \in \mathrm{Trans}_+(\psi)$, there is $t_n \to \infty$ such that $\psi_{t_n}(x) \to y$. Write $k_n = t_n + s_n \in \mathbb{Z}$ with $0 \leq s_n < 1$. Taking a subsequence we can assume that $s_n \to s \in [0, 1]$. Then

$$f^{k_n}(x) = \psi_{t_n+s_n}(x) = \psi_{s_n}(\psi_{t_n}(x)) \to \psi_s(y)$$

and $\psi_s(y) \in \omega_f(x)$.

Take a non-empty open subset $U \subset M$. From (4.1) there is $N_1 > 0$ such that $f^{N_1}(\psi_s(y)) \in U$ and we can take a small neighborhood V of $\psi_s(y)$ such that $f^{N_1}(V) \subset U$. Since $\psi_s(y) \in \omega_f(x)$, there is $N_2 > 0$ such that $f^{N_2}(x) \in V$. Thus, $f^{N_1+N_2}(x) \in U$. Therefore, the orbit of x is dense and $\mathrm{Trans}(f) = M \setminus \{p\}$. As $\{p\}$ is zero-dimensional, from Theorem 3.17 we conclude that f is cw-minimal. This also implies $\{p\}$ is the unique minimal subset (Proposition 2.4 can also be applied). ∎

5. Mindual sets. We say that a subset $R \subset M$ is a *mindual set* for f if $\omega(x) \cap R \neq \emptyset$ for every $x \in M$. In this section we will develop some results concerning this kind of sets. In Theorem 5.4 we show that every cw-minimal homeomorphism contains a zero-dimensional mindual set. Notice that mindual sets need not be either closed or invariant.

Proposition 5.1. *For $R \subset M$ the following statements are equivalent:*

(1) *R is mindual for f,*
(2) *R is mindual for f^{-1},*
(3) *$f^i(R)$ is mindual for f for every $i \in \mathbb{Z}$,*
(4) *$R \cap X \neq \emptyset$ for every minimal $X \subset M$,*
(5) *every closed $Q \supseteq R$ is mindual,*
(6) *$R \cap \operatorname{MinCen}(f)$ is mindual.*

Proof. (1)⇒(4). If X is minimal then $X = \omega(x)$ for all $x \in X$. Thus, $\omega(x) \cap R \neq \emptyset$ and R is mindual.

(4)⇒(1). This is true because every ω-limit set contains a minimal subset.

(1)⇔(2). This follows from the equivalence (4)⇔(1) since f and its inverse have the same minimal subsets.

Since ω-limit sets are invariant we find that (3)⇔(1).

The equivalence (1)⇔(5) is trivial. This yields (6)⇒(1).

(1)⇒(6). If X is minimal then $R \cap X \neq \emptyset$. This implies $[R \cap \operatorname{MinCen}(f)] \cap X \neq \emptyset$, because $X \subset \operatorname{MinCen}(f)$. Thus, $R \cap \operatorname{MinCen}(f)$ is mindual. ∎

Remark 5.2. To explain the name *mindual* consider the family generated by the minimal subsets $\mathcal{F} = \{A \subset M : A \text{ contains a minimal subset of } M\}$. The mindual sets are the members of the dual family $\mathcal{F}^* = \{B \subset M : B \cap A \neq \emptyset \text{ for all } A \in \mathcal{F}\}$. Notice that $B \in \mathcal{F}^*$ if and only if $B^c \notin \mathcal{F}$.

Corollary 5.3. *The mincenter, the non-wandering set and M are mindual sets. A closed invariant subset R is mindual if and only if it contains the mincenter.*

Proof. It is obvious that M is mindual. The mincenter is mindual from Proposition 5.1(6) with $R = M$. Since the mincenter is mindual and it is contained in the non-wandering set, from Proposition 5.1(5) we deduce that the non-wandering set is mindual.

Suppose that R is closed, invariant and mindual. As it intersects each minimal subset and it is invariant, R contains every minimal subset. Since R is closed, it contains the mincenter. The converse is trivial. ∎

Theorem 5.4. *If $f\colon M \to M$ is cw-minimal then M contains a closed mindual set of dimension zero.*

Proof. Take any $p \in M$ and define $U_n = \{x \in M : \operatorname{dist}(x, p) > 1/n\}$. Consider the maximal invariant subsets $A_n = \bigcap_{i\in\mathbb{Z}} f^i(\operatorname{clos} U_n)$. Each A_n is closed, invariant and $A_n \subset \operatorname{clos} U_n \neq M$. As f is cw-minimal, we conclude that each A_n is zero-dimensional. Define

$$E_1 = A_1, \quad E_{n+1} = E_n \cup (A_{n+1} \setminus U_n) \quad \text{for } n \geq 1. \tag{5.1}$$

Consider

$$R = \Big(\bigcup_{n\geq 1} E_n\Big) \cup \{p\}.$$

We will show that R is a closed, zero-dimensional mindual set. To see that R is closed notice that for each $N \geq 1$, the finite union $\bigcup_{n=1}^N E_n$ is closed and contained in U_N. As the complement of U_N is a ball converging to $p \in R$, we see that R is closed.

To prove that R is mindual take a minimal subset $X \subset M$. We will show that $X \cap R \neq \emptyset$. If $p \in X$ then $p \in X \cap R$. If $p \notin X$ then there is a smallest $m \geq 1$ such that $X \subset A_m$. If $m = 1$ then $X \subset A_1 = E_1$ and $X \subset R$. If $m > 1$ then X is not contained in A_{m-1}, hence not in U_{m-1} either. Thus, $X \cap (A_m \setminus U_{m-1}) \neq \emptyset$, which implies $X \cap E_m \neq \emptyset$ and $X \cap R \neq \emptyset$.

To prove that $\dim(R) = 0$ recall that each A_n is zero-dimensional. From (5.1) we know that $E_n \subset A_n$ for all $n \geq 1$. Therefore, each E_n has dimension zero. Since $E_{n+1} \setminus E_n$ is contained in the closed ball of radius $1/n$ centered at p, we conclude that $\dim(R) = 0$. ■

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