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A NOTE ON GORENSTEIN AC-PROJECTIVE AND GORENSTEIN AC-FLAT MODULES

BY

JUNPENG WANG (Lanzhou)

Abstract. We establish some relationships between Gorenstein AC-projective modules and Gorenstein AC-flat modules. As applications, we obtain some new characterizations of (weak) global dimension of any ring, and obtain the compactly-generatedness of some relative derived categories.

1. Introduction. As a relative homological algebra, Gorenstein homological algebra was established by Enochs, Jenda and Torrecillas [8, 9], and developed rapidly during the past several years (see [13, 19, 11, 4, 12, 15] etc.). In order that Gorenstein homological algebra should work for any ring, Bravo, Gillespie and Hovey [4] introduced the notions of Gorenstein AC-projective and Gorestein AC-injective modules and established two new hereditary abelian model structures with respect to such Gorenstein AC-modules. Recently, Bravo, Estrada and Iacob [3] introduced the notion of Gorenstein AC-flat modules. On the other hand, in order to show that any ring is GF-closed, Šaroch and Šťovíček [17] introduced the notion of projectively coresolved Gorenstein flat modules (PGF modules for short). From the definitions, it is easy to see that the following inclusions hold:

$$\begin{aligned}\{\text{projective modules}\} &\subseteq \{\text{Gorenstein AC-projective modules}\}\\ &\subseteq \{\text{PGF modules}\}\end{aligned}$$

and

$$\begin{aligned}\{\text{flat modules}\} &\subseteq \{\text{Gorenstein AC-flat modules}\}\\ &\subseteq \{\text{Gorenstein flat modules}\}.\end{aligned}$$

Thus, it is natural to ask

Question A. For what rings do we have

$$\{\text{Gorenstein AC-projective modules}\} = \{\text{projective modules}\}?$$

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Question B. For what rings do we have

$$\{\text{Gorenstein AC-projective modules}\} = \{\text{PGF modules}\}?$$

Question C. For what rings do we have

$$\{\text{Gorenstein AC-flat modules}\} = \{\text{flat modules}\}?$$

Question D. For what rings do we have

$$\{\text{Gorenstein AC-flat modules}\} = \{\text{Gorenstein flat modules}\}?$$

The main goal of this article is to investigate some relationships among such questions and give a partial answer to each of them. Note that Question A can be related to the notion of *strongly CM-free rings* in [7], and Question B was also considered by Šaroch and Št'ovíček [17, pp. 18, 27].

Our main results, stated below, show that Questions A and C (resp. Questions B and D) are equivalent.

Theorem 1.1. *Let R be a ring. Then the following are equivalent:*

(1) *Every projectively coresolved Gorenstein flat left R-module is Gorenstein AC-projective.*
(2) *Every Gorenstein flat left R-module is Gorenstein AC-flat.*

Theorem 1.2. *Let R be a ring. Then the following are equivalent:*

(1) *Every Gorenstein AC-projective left R-module is projective.*
(2) *Every Gorenstein AC-projective left R-module has finite level dimension.*
(3) *Every Gorenstein AC-flat left R-module is flat.*

As an immediate consequence of Theorem 1.2, we have

Corollary 1.3. *Let R be a ring with* $\text{Lev-lgldim}(R) < \infty$*, that is, with the supremum of the level dimensions of all left R-modules being finite. Then the Gorenstein AC-projective and the projective left R-modules coincide, as also do the Gorenstein AC-flat and the flat left R-modules.*

Now we give two applications of Corollary 1.3. Firstly, a new characterization of (weak) global dimension of a ring is established in the following corollary, where $\text{lgldim}(R)$ (resp. $\text{wgldim}(R)$) denotes the left global dimension (resp. weak global dimension) of a ring R; and $\text{ACGP-lgldim}(R)$ (resp. $\text{ACGF-lgldim}(R)$) denotes the supremum of the Gorenstein AC-projective (resp. Gorenstein AC-flat) dimensions of all left R-modules.

Corollary 1.4. *Let R be a ring. Then*

$$\text{lgldim}(R) = \max\{\text{Lev-lgldim}(R), \text{ACGP-lgldim}(R)\},$$
$$\text{wgldim}(R) = \max\{\text{Lev-lgldim}(R), \text{ACGF-lgldim}(R)\}.$$

As an example of triangulated categories, the usual derived category of left R-modules, $\mathbf{D}(R\text{-Mod})$, has been used effectively in (Gorenstein) homological algebra. Intending to close a gap of the corresponding version of derived categories in Gorenstein homological algebra, Gao and Zhang [11] introduced and studied the relative derived category with respect to Gorenstein projective modules (Gorenstein derived category for short), $\mathbf{D}_{\mathcal{GP}}(R\text{-Mod})$. One can naturally consider the relative derived category with respect to Gorenstein AC-projective modules, $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod})$. One of the important features of $\mathbf{D}(R\text{-Mod})$ is that it is compactly generated for any ring R. However, $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod})$ and $\mathbf{D}_{\mathcal{GP}}(R\text{-Mod})$ may fail to have this property, even if R is a QF ring (see Remark 4.6).

As another application of Corollary 1.3, the following corollary gives a sufficient condition for $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod})$ to be compactly generated.

Corollary 1.5. *Let R be a ring with finite left global level dimension. Then $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod})$ is compactly generated.*

2. Preliminaries. Throughout this article, all rings R are assumed to be associative rings with identity and all modules are unitary. By an "R-module" or just a "module" we always mean a left R-module unless otherwise stated.

In this section we mainly recall some notions and facts which will be used in the paper. Let R be a ring. As usual, denote by R-Mod the category of all R-modules; by $\mathcal{P}$ (resp. $\mathcal{F}$) the subcategory of R-Mod consisting of all projective (resp. flat) modules; and by $\mathrm{pd}_R(M)$ (resp. $\mathrm{fd}_R(M)$) the projective (resp. flat) dimension of a left or right R-module M.

2.1. Complexes and dimensions. For every complex C of modules

$$C = \cdots \to C^{m-1} \xrightarrow{\delta_C^{m-1}} C^m \xrightarrow{\delta_C^m} C^{m+1} \xrightarrow{\delta_C^{m+1}} \cdots,$$

the mth *cycle* (resp. mth *boundary*, mth *homology*) of C is defined as $\mathrm{Ker}(\delta_C^m)$ (resp. $\mathrm{Im}(\delta_C^{m-1})$, $\mathrm{Ker}(\delta_C^m)/\mathrm{Im}(\delta_C^{m-1})$) and denoted by $\mathrm{Z}^m(C)$ (resp. $\mathrm{B}^m(C)$, $\mathrm{H}^m(C)$). A complex C is said to be *acyclic* or *exact* if all its homologies $\mathrm{H}^m(C)$ are zero.

Let $\mathcal{X}$ be a subcategory of R-Mod and M an R-module. Then the *$\mathcal{X}$-projective dimension* of M, denoted by $\mathcal{X}\text{-}\mathrm{pd}_R(M)$, is defined as follows:

$$\begin{aligned}\mathcal{X}\text{-}\mathrm{pd}_R(M) = \inf\{n \in \mathbb{N} \mid {}&\text{there is an exact complex of } R\text{-modules} \\ &0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0, \text{ where each } X_i \text{ is in } \mathcal{X}\}.\end{aligned}$$

If no such exact sequence exists, then we set $\mathcal{X}\text{-}\mathrm{pd}_R(M) = \infty$.

Recall from [4] that an R-module M is of *type FP_∞* if there exists an exact complex of R-modules

$$\cdots \to P_1 \to P_0 \to M \to 0$$

where each P_i is finitely generated and projective; and a right R-module M (resp. an R-module N) is *absolutely clean* (resp. *level*) if $\mathrm{Ext}^1_R(T, M) = 0$ (resp. $\mathrm{Tor}^R_1(T, N) = 0$) for all right R-modules T of type FP_∞.

We denote by $\mathcal{L}\mathrm{ev}$ (resp. $\mathcal{A}\mathrm{bsC}$) the subcategory of all level (resp. absolutely clean) R-modules, and by $\mathrm{Lev}\text{-}\mathrm{pd}_R(M)$ the $\mathcal{L}\mathrm{ev}$-projective dimension (level dimension for short) of M.

2.2. Gorenstein homological modules and dimensions. An R-module M is said to be *Gorenstein projective* [8] (resp. *Gorenstein AC-projective* [4]) if there exists a $\mathrm{Hom}_R(-, \mathcal{P})$-exact (resp. $\mathrm{Hom}_R(-, \mathcal{L}\mathrm{ev})$-exact) exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective R-modules such that $M \cong \mathrm{Im}(P_0 \to P^0)$.

An R-module M is said to be *Gorenstein flat* [9] (resp. *Gorenstein AC-flat* [3]) if there exists an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of flat R-modules such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and that remains exact whenever the functor $I \otimes_R -$ is applied to it for any injective (resp. absolutely clean) right R-module I.

An R-module M is said to be *projectively coresolved Gorenstein flat* [17] if there exists an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective R-modules such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and that remains exact whenever the functor $I \otimes_R -$ is applied to it for any injective right R-module I.

We denote by $\mathcal{GP}$ (resp. $\mathcal{GF}$, $\mathrm{AC}\mathcal{GP}$, $\mathrm{AC}\mathcal{GF}$, $\mathcal{PGF}$) the subcategory of R-Mod consisting of all Gorenstein projective (resp. Gorenstein flat, Gorenstein AC-projective, Gorenstein AC-flat, projectively coresolved Gorenstein flat) R-modules. We denote by $\mathrm{Gpd}_R(M)$ (resp. $\mathrm{Gfd}_R(M)$, $\mathrm{AC}\text{-}\mathrm{Gpd}_R(M)$, $\mathrm{AC}\text{-}\mathrm{Gfd}_R(M)$) the Gorenstein projective (resp. Gorenstein flat, Gorenstein AC-projective, Gorenstein AC-flat) dimension of an R-module M, that is, $\mathcal{GP}$-projective (resp. $\mathcal{GF}$-projective, $\mathrm{AC}\mathcal{GP}$-projective, $\mathrm{AC}\mathcal{GF}$-projective) dimension of M.

It follows from [10, Example 2.21(2)] that the subcategory $\mathrm{AC}\mathcal{GF}$ is closed under extensions (actually it is projectively resolving). Then using arguments analogous to those in [2, proof of Theorem 2.11], one gets the following lemma.

LEMMA 2.1. *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence of* R-*modules. Then*

$$\mathrm{AC}\text{-}\mathrm{Gfd}_R(A) \leq \max\{\mathrm{AC}\text{-}\mathrm{Gfd}_R(B), \mathrm{AC}\text{-}\mathrm{Gfd}_R(C) - 1\}$$

($\mathrm{AC}\text{-}\mathrm{Gfd}_R(C) - 1$ *should be interpreted as* 0 *if* $\mathrm{AC}\text{-}\mathrm{Gfd}_R(C) = 0$).

REMARK 2.2. (1) By [4, Theorem A.6] an R-module M is Gorenstein AC-projective if and only if there exists an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of projective R-modules such that $M \cong \mathrm{Im}(F_0 \to F^0)$

and that remains exact whenever the functor $I \otimes_R -$ is applied to it for any absolutely clean right R-module I. It follows that AC$\mathcal{GP} \subseteq \mathcal{PGF}$ and AC$\mathcal{GP} \subseteq$ AC$\mathcal{GF}$.

(2) Trivially $\mathcal{PGF} \subseteq \mathcal{GF}$ and AC$\mathcal{GF} \subseteq \mathcal{GF}$. Also, $\mathcal{PGF} \subseteq \mathcal{GP}$ by [17, Theorem 4.4].

(3) We know from [17, Theorem 4.9 and its proof] that $\mathcal{PGF} \cap \mathcal{PGF}^{\perp} = \mathcal{P}$, and from [14, Lemma 6 and Corollary 1] that all modules M with finite flat dimension are in $\mathcal{PGF}^{\perp}$. Thus, any projectively coresolved Gorenstein flat module with finite projective or flat dimension is projective.

2.3. Cotorsion pairs. Let $\mathcal{X}, \mathcal{Y}$ be subcategories of R-modules. A pair $(\mathcal{X}, \mathcal{Y})$ is called a *cotorsion pair* if $\mathcal{X}^{\perp} = \mathcal{Y}$ and $\mathcal{X} = {}^{\perp}\mathcal{Y}$. Here $\mathcal{X}^{\perp} = \{A \in R\text{-Mod} \mid \mathrm{Ext}^1_R(X, A) = 0, \forall X \in \mathcal{X}\}$, and we define ${}^{\perp}\mathcal{Y}$ similarly. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be *hereditary* if $\mathrm{Ext}^n_R(X, Y) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $n \geq 1$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is *complete* if for any R-module A, there are exact sequences $0 \to Y \to X \to A \to 0$ and $0 \to A \to Y' \to X' \to 0$ with $X, X' \in \mathcal{X}$ and $Y, Y' \in \mathcal{Y}$.

3. Gorenstein AC-projective modules vs. projectively coresolved Gorenstein flat modules. We start with the following lemma, which is proved in [10, Theorem 2.14 and Example 2.2(3)].

LEMMA 3.1. *The following conditions are equivalent for an R-module M:*

(1) *M is Gorenstein AC-flat.*
(2) *There is a short exact sequence $0 \to M \to F \to N \to 0$ of R-modules with $F \in \mathcal{F}$ and $N \in$ AC$\mathcal{GP}$.*

The next lemma can be viewed as a continuation of Lemma 3.1.

LEMMA 3.2. *Let M be an R-module and $n \geq 0$ an integer. Then the following conditions are equivalent*:

(1) AC-Gfd$_R(M) \leq n$.
(2) *There is an exact sequence $0 \to M \to F \to N \to 0$ of R-modules with $\mathrm{fd}_R(F) \leq n$ and $N \in$ AC$\mathcal{GP}$.*

Proof. (1)⇒(2). We use induction on n. The case $n = 0$ holds by Lemma 3.1. Now let $n > 0$. Consider a short exact sequence $0 \to K \to H \to M \to 0$ of R-modules with H flat. Then Lemma 2.1 yields AC-Gfd$(K) \leq n-1$, and so by induction, there is a short exact sequence $0 \to K \to H' \to G \to 0$ of R-modules with $\mathrm{fd}_R(H') \leq n-1$ and $G \in$ AC$\mathcal{GP}$. Consider the pushout diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & H & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H' & \longrightarrow & H'' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G & = & G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Note that AC$\mathcal{GF}$ is closed under extensions by [10, Example 2.21(2)]. It follows that H'' is in AC$\mathcal{GF}$ since so are H and G in the middle column. Hence, by Lemma 3.1, there is a short exact sequence $0 \to H'' \to L \to N \to 0$ of R-modules with $L \in \mathcal{F}$ and $N \in$ AC$\mathcal{GP}$. Now we obtain another pushout diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H' & \longrightarrow & H'' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H' & \longrightarrow & L & \longrightarrow & F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N & = & N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

Since, in the middle row, $L \in \mathcal{F}$ and $\mathrm{fd}_R(H') \leq n-1$, it follows that $\mathrm{fd}_R(F) \leq n$. So condition (2) is established via the right non-zero column.

(2)⇒(1) follows from Lemma 2.1 since AC-$\mathrm{Gfd}_R(F) \leq \mathrm{fd}_R(F) \leq n$ and AC-$\mathrm{Gfd}_R(N) = 0$. ■

LEMMA 3.3. *Let M be an R-module and $n \geq 0$ an integer. Then the following conditions are equivalent:*

(1) AC-$\mathrm{Gpd}_R(M) \leq n$.
(2) *There is an exact sequence $0 \to M \to F \to N \to 0$ of R-modules with $\mathrm{pd}_R(F) \leq n$ and $N \in$ AC$\mathcal{GP}$.*

Proof. Apply the arguments analogous to those in the proof of Lemma 3.2. ■

We are now in a position to prove the main result of this section. It shows that Questions A and C (from the introduction) are equivalent.

THEOREM 3.4. *Let R be a ring. Then the following are equivalent:*

(1) *Every projectively coresolved Gorenstein flat R-module is Gorenstein AC-projective.*
(2) *Every projectively coresolved Gorenstein flat R-module has finite Gorenstein AC-projective dimension.*
(3) *Every Gorenstein flat R-module is Gorenstein AC-flat.*
(4) *Every Gorenstein flat R-module has finite Gorenstein AC-flat dimension.*

Proof. It is trivial that (1)⇒(2) and (3)⇒(4).

(1)⇒(3). Let M be a Gorenstein flat R-module. Then [17, Theorem 3.11] tells us that there is a short exact sequence $0 \to M \to F \to G \to 0$ of R-modules with $F \in \mathcal{F}$ and $G \in \mathcal{PGF}$. By (1), G is necessarily in AC$\mathcal{GP}$. Thus, M is Gorenstein AC-flat by Lemma 3.1.

(4)⇒(1). Let M be a projectively coresolved Gorenstein flat R-module. Then M is Gorenstein flat, and so by (4), there exists a nonnegative integer n such that $\text{AC-Gfd}_R(M) = n$. It follows from Lemma 3.2 that there is a short exact sequence $0 \to M \to F \to G \to 0$ of R-modules with $\text{fd}_R(F) < \infty$ and $G \in \text{AC}\mathcal{GP}$. Since the subcategory $\mathcal{PGF}$ is closed under extensions (via [17, Theorem 4.9]) and $\text{AC}\mathcal{GP} \subseteq \mathcal{PGF}$, F is in $\mathcal{PGF}$ and admits finite flat dimension. Thus, F is projective (see Remark 2.2(3)), and of course Gorenstein AC-projective. As the subcategory AC$\mathcal{GP}$ is projectively resolving (via [12, Fact 10.2]), M is Gorenstein AC-projective by the above short exact sequence.

(2)⇒(1). Apply the arguments used in the proof of (4)⇒(1). ∎

COROLLARY 3.5. *Let R be a right coherent ring or a ring with*

$$\sup\{\text{AC-Gfd}_R(M) \mid M \text{ is an } R\text{-module}\} < \infty.$$

Then $\mathcal{PGF} = \text{AC}\mathcal{GP}$.

Proof. The second case is an immediate consequence of Theorem 3.4. Now let R be right coherent. By [4, Corollary 2.9], the FP-injective and the absolutely clean right R-modules coincide. It follows from [16, Lemma 2.8] and the definition of Gorenstein AC-flat modules that $\mathcal{GF} = \text{AC}\mathcal{GF}$. Thus $\mathcal{PGF} = \text{AC}\mathcal{GP}$ by Theorem 3.4. ∎

COROLLARY 3.6. *Let R be a ring. Then the following are equivalent:*

(1′) *Every Gorenstein projective R-module is Gorenstein AC-projective.*
(2′) *Every Gorenstein projective R-module has finite Gorenstein AC-projective dimension.*
(3′) *Every Gorenstein projective R-module is Gorenstein flat and every Gorenstein flat R-module is Gorenstein AC-flat.*
(4′) *Every Gorenstein projective R-module is Gorenstein flat and every Gorenstein flat R-module has finite Gorenstein AC-flat dimension.*

In particular, $\mathcal{PGF} = AC\mathcal{GP} = \mathcal{GP}$ *if* $\sup\{\text{AC-Gfd}_R(M) \mid M$ *is an* R*-module*$\} < \infty$.

Proof. We know from [14, Theorem 3] that $\mathcal{GP} = \mathcal{PGF}$ if and only if $\mathcal{GP} \subseteq \mathcal{GF}$. It follows that conditions (1′), (3′) and (4′) are equivalent to the modified ones with "projectively coresolved Gorenstein flat" instead of "Gorenstein projective", respectively. Note that the proofs of (1)⇔(3)⇔(4) in Theorem 3.4 imply that such modified conditions are equivalent, and so (1′)⇔(3′)⇔(4′). Meanwhile, (1)⇒(2) is clear. To see (2)⇒(1), let M be a Gorenstein projective R-module. By (2), M admits $\text{AC-Gfd}_R(M) = n < \infty$. Then Lemma 3.3 yields a short exact sequence $0 \to M \to P \to G \to 0$ of R-modules with $\text{pd}_R(P) < \infty$ and $G \in \text{AC}\mathcal{GP}$. We conclude that $P \in \mathcal{GP}$ and $\text{pd}_R(P) < \infty$ since $M \in \mathcal{GP}$ and $G \in \text{AC}\mathcal{GP} \subseteq \mathcal{GP}$. Thus, [13, Proposition 2.27] gives $P \in \mathcal{P}$, which implies that M is necessarily Gorenstein AC-projective by the above short exact sequence and the projectively resolving property of AC$\mathcal{GP}$.

Now let $\sup\{\text{AC-Gfd}_R(M) \mid M \text{ is an } R\text{-module}\} < \infty$. Since $\text{AC}\mathcal{GP} \subseteq \mathcal{PGF} \subseteq \mathcal{GP}$, in order to see $\text{AC}\mathcal{GP} = \mathcal{PGF} = \mathcal{GP}$, it suffices to show $\mathcal{GP} \subseteq \mathcal{GF}$ by (1′)⇔(4′). Indeed, one has

$$\begin{aligned}
\sup\{\text{fd}_R(M) \mid M \text{ is an injective right } R\text{-module}\}& \\
&\le \sup\{\text{Gfd}_R(M) \mid M \text{ is an } R\text{-module}\}\\
&\le \sup\{\text{AC-Gfd}_R(M) \mid M \text{ is an } R\text{-module}\} < \infty,
\end{aligned}$$

where the first inequality follows from [6, Lemma 5.1], the second one is clear and the last one is by assumption. Then $\mathcal{PGF} = \mathcal{GP}$ by [14, Proposition 9 and its proof], and so $\mathcal{GP} \subseteq \mathcal{GF}$, as desired. ■

Remark 3.7. We note that condition (3′) (hence any of the conditions) in Corollary 3.6 is equivalent to the following one:

(3″) Every Gorenstein projective R-module is Gorenstein AC-flat.

Indeed, (3′)⇒(3″) is clear. To see the converse, suppose (3″) holds true. Then, of course, every Gorenstein projective R-module is Gorenstein flat. We must show that every Gorenstein flat R-module M is Gorenstein AC-flat. Indeed, [17, Theorem 4.11] yields a short exact sequence $0 \to M \to F \to G \to 0$ of R-modules with $F \in \mathcal{F}$ and $G \in \mathcal{PGF} \subseteq \mathcal{GP}$. By assumption, G is necessarily in AC$\mathcal{GF}$. As the subcategoryAC$\mathcal{GF}$ is projectively resolving (via [10, Example 2.21(2)]), one has $M \in \text{AC}\mathcal{GF}$ by applying that property of AC$\mathcal{GF}$ to the above short exact sequence.

However, to the best of our knowledge, it is not clear whether condition (4′) of Corollary 3.6 is equivalent to the one that every Gorenstein projective R-module has finite Gorenstein AC-flat dimension.

4. Gorenstein AC-projective modules vs. projective modules. In this section, we prove that Questions B and D (from the introduction) are equivalent (see Theorem 4.1), and give some applications.

THEOREM 4.1. *Let R be a ring. Then the following are equivalent:*

(1) *Every Gorenstein AC-projective R-module is projective.*
(2) *Every Gorenstein AC-projective R-module has finite projective or flat dimension.*
(3) *Every Gorenstein AC-projective R-module has finite level dimension.*
(4) *Every Gorenstein AC-flat R-module is flat.*
(5) *Every Gorenstein AC-flat R-module has finite flat dimension.*
(6) *Every Gorenstein AC-flat R-module has finite level dimension.*

Proof. It is trivial that (1)⇒(2)⇒(3) and (4)⇒(5)⇒(6)⇒(3).

(1)⇒(4). Let M be a Gorenstein AC-flat R-module. Then by Lemma 3.1, there exists a short exact sequence $0 \to M \to F \to G \to 0$ of R-modules with F flat and G Gorenstein AC-projective. By (1), G is necessarily projective. Hence, the above sequence is split, and so M is flat.

(3)⇒(1). Firstly, by the definition, it is routine to see that $\mathrm{Ext}_R^{i\geq 1}(G, L) = 0$ for all Gorenstein AC-projective R-modules G and all level R-modules L (one can view it as the Gorenstein AC-projective version of [13, Proposition 2.3]). Now let M be a Gorenstein AC-projective R-module with finite level dimension. Then, by the definition, there exists a short exact sequence of R-modules $0 \to M \to P \to G \to 0$ with P projective and G Gorenstein AC-projective. Thus, $\mathrm{Ext}_R^1(G, M) = 0$, and so the above sequence is split. Therefore, M is projective. ∎

We write $\text{Lev-lgldim}(R) = \sup\{\text{Lev-pd}_R(M) \mid M \text{ is an } R\text{-module}\}$. As an immediate consequence of Theorem 4.1, we have

COROLLARY 4.2. *Let R be a ring with* $\text{Lev-lgldim}(R) < \infty$. *Then* $\mathrm{AC}\mathcal{GP} = \mathcal{P}$ *and* $\mathrm{AC}\mathcal{GF} = \mathcal{F}$.

In the remaining part of the paper, we give some applications of Corollary 4.2. Let us denote by $\mathrm{lgldim}(R)$ (resp. $\mathrm{wgldim}(R)$) the left global dimension (resp. weak global dimension) of a ring R. Put

$$\text{ACGP-lgldim}(R) = \sup\{\text{AC-Gpd}_R(M) \mid M \text{ is an } R\text{-module}\},$$
$$\text{ACGF-lgldim}(R) = \sup\{\text{AC-Gfd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Then it is trivial that $\mathrm{lgldim}(R) \geq \max\{\text{Lev-lgldim}(R), \text{ACGP-lgldim}(R)\}$ and $\mathrm{wgl\,dim}(R) \geq \max\{\text{Lev-lgldim}(R), \text{ACGF-lgldim}(R)\}$. As a first application of Corollary 4.2, the following corollary shows that these inequalities are in fact equalities.

Corollary 4.3. *Let R be a ring. Then*

$$\operatorname{lgldim}(R) = \max\{\text{Lev-lgldim}(R), \text{ACGP-lgldim}(R)\},$$
$$\operatorname{wgldim}(R) = \max\{\text{Lev-lgldim}(R), \text{ACGF-lgldim}(R)\}.$$

Proof. It suffices to prove the relevant $\leq$ inequalities. Let M be any R-module. For the first inequality, we assume that $\max\{\text{Lev-lgldim}(R), \text{ACGP-lgldim}(R)\} = m < \infty$. Then Corollary 4.2 gives that $\mathrm{AC}\mathcal{GP} = \mathcal{P}$, and so $\mathrm{pd}_R(M) = \text{AC-Gpd}_R(M) \leq m$. The second inequality can be proved similarly. ∎

In order to give another application of Corollary 4.2, we recall some notions and facts.

Let $f : X \to Y$ be a morphism of complexes of R-modules. Recall that f is a *quasi-isomorphism* if it induces an isomorphism $\mathrm{H}^i(f) : \mathrm{H}^i(X) \to \mathrm{H}^i(Y)$, $\mathrm{H}^i(f)(x + \mathrm{B}^i(X)) = f^i(x) + \mathrm{B}^i(Y)$ for all $i \in \mathbb{Z}$. We say that f is a *Gorenstein AC-quasi-isomorphism* if $\mathrm{Hom}_R(G, f)$ is a quasi-isomorphism for all $G \in \mathrm{AC}\mathcal{GP}$.

As is well known, $\mathbf{D}(R\text{-Mod})$, the usual derived category of R-modules, is defined as the Verdier quotient $\mathrm{K}(R\text{-Mod})/\mathrm{K}_{\mathrm{ac}}(R\text{-Mod})$, where $\mathrm{K}(R\text{-Mod})$ denotes the usual homotopy category of R-modules, and $\mathrm{K}_{\mathrm{ac}}(R\text{-Mod})$ its thick subcategory consisting of all acyclic complexes of R-modules. So the isomorphisms in $\mathbf{D}(R\text{-Mod})$ are quasi-isomorphisms. If we replace $\mathrm{K}_{\mathrm{ac}}(R\text{-Mod})$ with $\mathrm{K}_{\mathrm{ac}}^{\mathrm{AC}\mathcal{GP}}(R\text{-Mod})$, the thick subcategory of $\mathrm{K}(R\text{-Mod})$ consisting of all $\mathrm{AC}\mathcal{GP}$-acyclic complexes of R-modules (recall that a complex C is $\mathrm{AC}\mathcal{GP}$*-acyclic* if $\mathrm{Hom}_R(G, C)$ is exact for all $G \in \mathrm{AC}\mathcal{GP}$), then we obtain the definition of $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod})$, the relative derived category with respect to Gorenstein AC-projective R-modules (see [5, Section 3] for general relative derived categories). It is easy to see that the isomorphisms in $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod})$ are Gorenstein AC-quasi-isomorphisms.

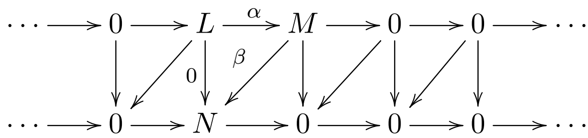
We define $\mathbf{D}_{\mathcal{GP}}(R\text{-Mod})$, the relative derived category with respect to Gorenstein projective R-modules (Gorenstein derived category for short), in a similar way (see Gao and Zhang [11] for details). According to [11, Corollary 2.5 and remarks], $\mathbf{D}_{\mathcal{GP}}(R\text{-Mod}) = \mathbf{D}(R\text{-Mod})$ if and only if $\mathcal{GP} = \mathcal{P}$. In the following we consider a Gorenstein AC-version for this result.

Lemma 4.4. *Let R be a ring. Then the following are equivalent:*

(1) $\mathrm{AC}\mathcal{GP} = \mathcal{P}$.
(2) $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod}) = \mathbf{D}(R\text{-Mod})$

Proof. (1)⇒(2) is clear as a complex C is acyclic if and only if $\mathrm{Hom}_R(P, C)$ is acyclic for any $P \in \mathcal{P}$.

(2)⇒(1). Let G be a Gorenstein AC-projective R-module. We have to show that G is projective. To this end, let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a short exact sequence of R-modules. Then β induces a trivial quasi-isomorphism



which is also a Gorenstein AC-quasi-isomorphism since $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod}) = \mathbf{D}(R\text{-Mod})$. Therefore, $0 \to \mathrm{Hom}_R(G, L) \to \mathrm{Hom}_R(G, M) \to \mathrm{Hom}_R(G, N) \to 0$ is acyclic, and hence G is projective. ■

Let $\mathcal{G}$ be a triangulated category with small coproducts. Recall that an object C of $\mathcal{G}$ is *compact* if for each collection $\{Y_j \mid j \in J\}$ of objects of $\mathcal{G}$, the canonical morphism

$$\coprod_{j\in J} \mathrm{Hom}_{\mathcal{G}}(C, Y_j) \to \mathrm{Hom}_{\mathcal{G}}\Big(C, \coprod_{j\in J} Y_j\Big)$$

is an isomorphism. The category $\mathcal{G}$ is *compactly generated* if there exists a small set $\mathcal{U} \subseteq \mathcal{G}$ of compact objects such that for each $0 \neq Y \in \mathcal{G}$ there is a morphism $0 \neq f : \Sigma^m U \to Y$ for some $U \in \mathcal{U}$ and $m \in \mathbb{Z}$, where Σ denotes the autofunctor of $\mathcal{G}$.

It is well-known that $\mathbf{D}(R\text{-Mod})$ is compactly generated for any ring R. Corollary 4.2 and Lemma 4.4 yield

Corollary 4.5. *Let R be a ring with* $\sup\{\text{Lev-pd}_R(M) \mid M$ *is an R-module*$\} < \infty$. *Then $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod})$ is compactly generated.*

We denote by $\mathrm{K}(\mathcal{P})$ (resp. $\mathrm{K}(\mathcal{GP})$, $\mathrm{K}(\mathrm{AC}\mathcal{GP})$) the homotopy subcategory of projective (resp. Gorenstein projective, Gorenstein AC-projective) R-modules.

Remark 4.6. We note that $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod})$ (or $\mathbf{D}_{\mathcal{GP}}(R\text{-Mod})$) may not be compactly generated over a general ring R. For example, let R be a QF ring which is not left pure semisimple (for the existence of such a ring, see [1, Example 1]). Then all R-modules are Gorenstein AC-projective (and Gorenstein projective). Hence it is a routine matter to prove that there are triangle equivalences $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod}) \simeq \mathrm{K}(\mathrm{AC}\mathcal{GP}) = \mathrm{K}(R\text{-Mod})$ and $\mathbf{D}_{\mathcal{GP}}(R\text{-Mod}) \simeq \mathrm{K}(\mathcal{GP}) = \mathrm{K}(R\text{-Mod})$. According to [18, Proposition 2.6], $\mathrm{K}(R\text{-Mod})$ is not compactly generated. Thus, neither of the equivalent triangulated categories $\mathbf{D}_{\mathrm{AC}\mathcal{GP}}(R\text{-Mod}) \simeq \mathrm{K}(R\text{-Mod}) \simeq \mathbf{D}_{\mathcal{GP}}(R\text{-Mod})$ is compactly generated.

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Junpeng Wang
Department of Mathematics
Northwest Normal University
730070 Lanzhou, P.R. China
E-mail: wangjunpeng1218@163.com