COLLOQUIUM MATHEMATICUM

Online First version

ON THE NUMBER OF τ-TILTING MODULES OVER THE AUSLANDER ALGEBRAS OF RADICAL SQUARE ZERO NAKAYAMA ALGEBRAS

BY

HANPENG GAO (Hefei), ZONGZHEN XIE (Nanjing) and
ZHAOYONG HUANG (Nanjing)

Abstract. Let Λ_n be a radical square zero Nakayama algebra with n simple modules and Γ_n the Auslander algebra of Λ_n. We calculate the number $|\tau\text{-tilt}\,\Gamma_n|$ of τ-tilting modules and the number $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ of support τ-tilting modules over Γ_n. In particular, we prove the recurrence relations

$$|\tau\text{-tilt}\,\Gamma_n| = 3|\tau\text{-tilt}\,\Gamma_{n-1}| + |\tau\text{-tilt}\,\Gamma_{n-2}|,$$
$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-2}|,$$

from which the exact values of $|\tau\text{-tilt}\,\Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ are derived.

1. Introduction. The starting point of tilting theory was the introduction of tilting modules over a hereditary algebra by Happel and Ringel [10]. Ever since, the study of tilting modules has been an important branch in the representation theory of finite-dimensional algebras.

In 2014, Adachi, Iyama and Reiten [1] introduced τ-tilting theory replacing the rigidity condition $\mathrm{Ext}^1_\Lambda(M,M) = 0$ for a tilting module by the weaker condition $\mathrm{Hom}_\Lambda(M,\tau M) = 0$ for a τ-tilting module, where Λ is a finite-dimensional algebra and τ is the Auslander–Reiten translation. The support τ-tilting modules are in bijection with some important objects in representation theory including functorially finite torsion classes introduced in [5], 2-term silting complexes introduced in [13], cluster-tilting objects in the cluster category and left finite semibricks introduced in [3]. Therefore, it is important to calculate the number of support τ-tilting modules over a given algebra.

For hereditary algebras, the (support) τ-tilting modules are exactly the (support) tilting modules. For algebras of Dynkin type, the numbers of these

---

2020 *Mathematics Subject Classification*: Primary 16G10; Secondary 05E10, 05A19.
Key words and phrases: τ-tilting modules, support τ-tilting modules, Nakayama algebras, Auslander algebras.
Received 17 January 2021.

DOI: 10.4064/cm8474-7-2021

modules were first calculated via cluster algebras [7], and later via representation theory [14]. In particular, over a hereditary algebra of type $\mathbb{A}_n$, the number of tilting modules is C_n and the number of support tilting modules is C_{n+1}, where C_i is the ith Catalan number $\frac{1}{i+1}\binom{2i}{i}$.

Recall from [4, V.3.2] that a finite-dimensional algebra is *Nakayama* if its quiver is one of the following:

$$\begin{tikzcd} A_n: & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n, & \widetilde{A}_n: & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n. \arrow[lllll, bend right] \end{tikzcd}$$

Adachi [2] gave a recurrence relation for the number of τ-tilting modules over Nakayama algebras of type A_n. Asai [3] also gave a recurrence relation for the number of support τ-tilting modules over Nakayama algebras KA_n/rad^r and $K\widetilde{A}_n/\mathrm{rad}^r$. More recently, Gao and Schiffler [9] extended the recurrence relation of Adachi to τ-tilting modules over $K\widetilde{A}_n/\mathrm{rad}^r$.

It was showed in [6] that the number of tilting modules over the Auslander algebra of $K[x]/(x^n)$ is $n!$. Kajita [12] calculated the number of tilting modules over the Auslander algebra of a hereditary algebra of Dynkin type. Iyama and Zhang [11] classified the support τ-tilting modules over the Auslander algebra of $K[x]/(x^n)$, and they also showed that there is a bijection between the set of support τ-tilting modules over the Auslander algebra of $K[x]/(x^n)$ and the symmetric group of degree n. More recently, Zhang [16] calculated the number of tilting modules over the Auslander algebra Γ_n of a radical square zero Nakayama algebra Λ_n. In particular, Zhang proved that the number of tilting modules over Γ_n is 2^{n-1} if Λ_n is of type A_n; and it is 2^n if Λ_n is of type $\widetilde{A}_n$.

In this paper, we calculate the number $|\tau\text{-tilt}\,\Gamma_n|$ of τ-tilting modules and the number $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ of support τ-tilting modules over the Auslander algebra Γ_n of a radical square zero Nakayama algebra Λ_n. Our result is as follows.

THEOREM 1.1 (Theorems 3.1, 3.5, 4.2 and 4.3). *Let Γ_n be the Auslander algebra of a radical square zero Nakayama algebra Λ_n.*

(1) *If Λ_n is of type A_n, then*

$$|\tau\text{-tilt}\,\Gamma_n| = \frac{(3+\sqrt{13})^n - (3-\sqrt{13})^n}{\sqrt{13}\cdot 2^n},$$

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = \frac{(3+2\sqrt{3})^n - (3-2\sqrt{3})^n}{2\sqrt{3}}.$$

(2) *If Λ_n is of type $\widetilde{A}_n$, then*

$$|\tau\text{-tilt}\,\Gamma_n| = \frac{(3+\sqrt{13})^n + (3-\sqrt{13})^n}{2^n},$$

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = (3+2\sqrt{3})^n + (3-2\sqrt{3})^n.$$

The paper is organized as follows. In Section 2, we fix some notations and recall several results about τ-tilting modules and Auslander algebras of radical square zero Nakayama algebras. In Section 3, we show that if Λ_n is of type A_n, then there are recurrence relations

$$|\tau\text{-tilt}\,\Gamma_n| = 3|\tau\text{-tilt}\,\Gamma_{n-1}| + |\tau\text{-tilt}\,\Gamma_{n-2}|,$$
$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-2}|.$$

In Section 4, we prove the same recurrence relations for Λ_n of type $\widetilde{A}_n$. From these recurrence relations the exact values of $|\tau\text{-tilt}\,\Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ are derived. Finally, we list the values of $|\tau\text{-tilt}\,\Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ for $1 \le n \le 8$ in a table in Section 5.

2. Preliminaries. Throughout this paper, all algebras are basic, connected, finite-dimensional K-algebras over an algebraically closed field K. For an algebra Λ, we denote by $\operatorname{mod}\Lambda$ the category of finitely generated right Λ-modules and by τ the Auslander–Reiten translation of Λ. We use P_i, I_i and S_i to denote the indecomposable projective, injective and simple modules of an algebra corresponding to the vertex i respectively. For any $i, j \in \{1, \dots, n\}$, we write $[i, j] = \{i, i+1, \dots, j\}$ if $i \le j$; otherwise, $[i, j] = \emptyset$. Let e_i be the primitive idempotent element of an algebra corresponding to the vertex i. We write $e_{[i,j]} := e_i + e_{i+1} + \cdots + e_j$.

For a module $M \in \operatorname{mod}\Lambda$, we write $|M|$ for the number of pairwise non-isomorphic indecomposable summands of M, and use $l(M)$ and $\operatorname{pd}_\Lambda M$ to denote the Loewy length and projective dimension of M respectively. For a finite set X, we let $|X|$ denote the cardinality of X. For two sets X_1 and X_2, $X_1 \amalg X_2$ stands for their disjoint union.

Definition 2.1 ([1, Definition 0.1]). Let Λ be an algebra and $M \in \operatorname{mod}\Lambda$. Then M is called

- *τ-rigid* if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$;
- *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$;
- *support τ-tilting* if it is a τ-tilting $\Lambda/\Lambda e\Lambda$-module for some idempotent e of Λ;
- *proper support τ-tilting* if it is support τ-tilting but not τ-tilting.

Recall that $M \in \operatorname{mod}\Lambda$ is called *sincere* if every simple Λ-module appears as a composition factor in M. It is well-known that the τ-tilting modules are exactly the sincere support τ-tilting modules [1, Proposition 2.2(a)].

We denote by $\tau\text{-tilt}\,\Lambda$ (respectively, $\mathrm{s}\tau\text{-tilt}\,\Lambda$, $\mathrm{ps}\tau\text{-tilt}\,\Lambda$) the set of isomorphism classes of basic τ-tilting (respectively, support τ-tilting, proper support τ-tilting) Λ-modules.

Set

$$\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\,\Lambda := \{M \in \mathrm{ps}\tau\text{-tilt}\,\Lambda \mid M \text{ has no projective direct summands}\}.$$

THEOREM 2.2 ([2, Theorem 2.6]). *Let Λ be a Nakayama algebra. Then there is a bijection between $\tau\text{-tilt}\,\Lambda$ and $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\,\Lambda$.*

The following result is useful.

PROPOSITION 2.3 ([2, Proposition 2.32]). *Let Λ be a Nakayama algebra of type A_n. Then each τ-tilting Λ-module has P_1 as a direct summand.*

As a consequence, we get

LEMMA 2.4. *Let Λ be a Nakayama algebra of type A_n. Then each support τ-tilting Λ-module which has $S_1, \dots, S_{l(P_1)}$ as composition factors has P_1 as a direct summand.*

Proof. Let M be a support τ-tilting Λ-module which has $S_1, \dots, S_{l(P_1)}$ as composition factors. If M is τ-tilting, then it has P_1 as a direct summand by Proposition 2.3. Now, assume that M has $S_1, \dots, S_{l(P_1)}, \dots, S_j$ as composition factors but not S_{j+1}. Let N be the maximal direct summand of M which only has $S_1, \dots, S_{l(P_1)}, \dots, S_j$ as composition factors. Then N is a τ-tilting $\Lambda/\langle e_{[j+1,n]}\rangle$-module. By Proposition 2.3, N has P_1 as a direct summand. ■

THEOREM 2.5 ([2, Theorem 2.33 and Corollary 2.34]). *Let Λ be a Nakayama algebra of type A_n. Then there are mutually inverse bijections*

$$\tau\text{-tilt}\,\Lambda \leftrightarrow \coprod_{i=1}^{l(P_1)} \tau\text{-tilt}(\Lambda/\langle e_i\rangle)$$

given by $\tau\text{-tilt}\,\Lambda \ni M \mapsto M/P_1$ and $N \mapsto N \oplus P_1 \in \tau\text{-tilt}\,\Lambda$. In particular,

$$|\tau\text{-tilt}\,\Lambda| = \sum_{i=1}^{l(P_1)} C_{i-1} \cdot |\tau\text{-tilt}(\Lambda/\langle e_{[1,i]}\rangle)|.$$

REMARK 2.6. Let Λ be a Nakayama algebra of type A_n. Then every τ-tilting Λ-module can be decomposed M as $M = P_1 \oplus N_1 \oplus N_2$ where N_1 is a maximal direct summand of M without S_1 as composition factors. Moreover, $N_1 \oplus N_2$ is a τ-tilting $\Lambda/\langle e_{j+1}\rangle$-module where $j := l(N_2)$ (see [2, proof of Theorem 2.33]).

An algebra Λ is of *finite representation type* if there are only finitely many indecomposable Λ-modules $X_1, \dots, X_m$ up to isomorphism. In this case, the endomorphism algebra $\mathrm{End}_\Lambda(\bigoplus_{i=1}^m X_i)$ is called the *Auslander algebra* of Λ.

By a straightforward calculation, we get the quiver of the Auslander algebra of radical square zero Nakayama algebras:

Proposition 2.7.

(1) *The Auslander algebra* Γ_n *of* $\Lambda_n := KA_n/\mathrm{rad}^2$ *is given by the quiver*

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1$$

with the relations $a_{2k-1}a_{2k} = 0$ *for* $1 \le k \le n-1$.

(2) *The Auslander algebra* Γ'_n *of* $\Lambda_n := K\widetilde{A}_n/\mathrm{rad}^2$ *is given by the quiver*

$$\begin{tikzcd} 1 \arrow[r, "a_1"'] & 2 \arrow[r, "a_2"'] & 3 \arrow[r, "a_3"'] & \cdots \arrow[r] & 2n-1 \arrow[r, "a_{2n-1}"'] & 2n \arrow[lllll, bend right, "a_{2n}"'] \end{tikzcd}$$

with the relations $a_{2k-1}a_{2k} = 0$ *for* $1 \le k \le n$.

3. The case for Γ_n. In this section, we will give formulas for $|\tau\text{-tilt}\,\Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$.

Let Δ_n be the algebra given by the quiver

$$0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1$$

with the relations $a_{2k-1}a_{2k} = 0$ for $1 \le k \le n-1$.

The following result gives a formula for $|\tau\text{-tilt}\,\Gamma_n|$.

Theorem 3.1. *We have*

$$|\tau\text{-tilt}\,\Gamma_n| = 3|\tau\text{-tilt}\,\Gamma_{n-1}| + |\tau\text{-tilt}\,\Gamma_{n-2}|$$

with $|\tau\text{-tilt}\,\Gamma_1| = 1$ *and* $|\tau\text{-tilt}\,\Gamma_2| = 3$. *Hence*

$$|\tau\text{-tilt}\,\Gamma_n| = \frac{(3+\sqrt{13})^n - (3-\sqrt{13})^n}{\sqrt{13}\cdot 2^n}.$$

Proof. Applying Theorem 2.5 to Γ_n and Δ_n, we have

$$\begin{aligned} |\tau\text{-tilt}\,\Gamma_n| &= C_0 \cdot |\tau\text{-tilt}(\Gamma_n/\langle e_1\rangle)| + C_1 \cdot |\tau\text{-tilt}(\Gamma_n/\langle e_1 + e_2\rangle)| \\ &= |\tau\text{-tilt}\,\Delta_{n-1}| + |\tau\text{-tilt}\,\Gamma_{n-1}| \end{aligned} \tag{1}$$

and

$$\begin{aligned} |\tau\text{-tilt}\,\Delta_n| &= C_0 \cdot |\tau\text{-tilt}(\Delta_n/\langle e_0\rangle)| + C_1 \cdot |\tau\text{-tilt}(\Delta_n/\langle e_0 + e_1\rangle)| \\ &\quad + C_2 \cdot |\tau\text{-tilt}(\Delta_n/\langle e_0 + e_1 + e_2\rangle)| \\ &= |\tau\text{-tilt}\,\Gamma_n| + |\tau\text{-tilt}\,\Delta_{n-1}| + 2|\tau\text{-tilt}\,\Gamma_{n-1}|. \end{aligned} \tag{2}$$

The formula (1) implies

$$|\tau\text{-tilt}\,\Delta_{n-1}| = |\tau\text{-tilt}\,\Gamma_n| - |\tau\text{-tilt}\,\Gamma_{n-1}|.$$

Applying it to (2), we have

$$|\tau\text{-tilt}\,\Gamma_n| = 3|\tau\text{-tilt}\,\Gamma_{n-1}| + |\tau\text{-tilt}\,\Gamma_{n-2}| \tag{3}$$

This is a linear homogeneous recurrence relation of degree 2 and its characteristic equation is $x^2 - 3x - 1 = 0$. The proof is finished. ■

Let Λ be an algebra. Recall that a module $M \in \operatorname{mod}\Lambda$ is called *tilting* if

- $\operatorname{pd}_\Lambda M \leq 1$;
- $\operatorname{Ext}^1_\Lambda(M, M) = 0$;
- $|M| = |\Lambda|$.

Thus a module $M \in \operatorname{mod}\Lambda$ is tilting if and only if it is τ-tilting and $\operatorname{pd}_\Lambda M \leq 1$, by the Auslander–Reiten formula.

The set of all tilting Λ-modules is denoted by $\operatorname{tilt}\Lambda$. The following result is part of [16, Theorem 2.8]. Here we give another proof.

PROPOSITION 3.2. $|\operatorname{tilt}\Gamma_n| = 2^{n-1}$.

Proof. Note that P_1 is the unique Γ_n-module which has S_1 as a composition factor and its projective dimension is at most 1. By Remark 2.6 and the above argument, $P_1 \oplus N_1$ is a tilting Γ_n-module if and only if N_1 is a tilting $\Gamma_n/\langle e_1\rangle$-module, since $\operatorname{pd}_{\Gamma_n} N_1 = \operatorname{pd}_{\Gamma_n/\langle e_1\rangle} N_1$. Thus

$$|\operatorname{tilt}\Gamma_n| = |\operatorname{tilt}(\Gamma_n/\langle e_1\rangle)| = |\operatorname{tilt}\Delta_{n-1}|.$$

Note that P_0 and S_0 are the only two Δ_n-modules which have S_0 as a composition factor and their projective dimension is at most 1. Similarly, we get

$$|\operatorname{tilt}\Delta_n| = |\operatorname{tilt}(\Delta_n/\langle e_0\rangle)| + |\operatorname{tilt}(\Delta/\langle e_0 + e_1\rangle)| = |\operatorname{tilt}\Gamma_n| + |\operatorname{tilt}\Delta_{n-1}|.$$

Thus $|\operatorname{tilt}\Gamma_n| = 2|\operatorname{tilt}\Gamma_{n-1}|$ with $|\operatorname{tilt}\Gamma_1| = 1$, and so $|\operatorname{tilt}\Gamma_n| = 2^{n-1}$. ∎

As generalizations of simple modules and semisimple modules, bricks and semibricks were introduced and studied in [8, 15]. Let Λ be an algebra. A Λ-module M is called a *brick* if $\operatorname{Hom}_\Lambda(M, M)$ is a K-division algebra, and a *semibrick* is a set consisting of isoclasses of pairwise Hom-orthogonal bricks. Recall from [3] that a semibrick $\mathcal{S}$ is called *left finite* if the smallest torsion class $T(\mathcal{S})$ containing $\mathcal{S}$ is functorially finite. There exists a bijection between $\mathrm{s}\tau\text{-tilt}\,\Lambda$ and the set of left finite semibricks of Λ [3, Theorem 2.3]. Note that every torsion class is functorially finite for a representation-finite algebra. So, for a Nakayama algebra Λ, there exists a bijection between $\mathrm{s}\tau\text{-tilt}\,\Lambda$ and the set $\operatorname{sbrick}\Lambda$ of semibricks of Λ, and hence $|\mathrm{s}\tau\text{-tilt}\,\Lambda| = |\operatorname{sbrick}\Lambda|$. Asai gave a method to calculate the number of semibricks over $KA_n/\operatorname{rad}^r$. In fact, we have the following more general result.

PROPOSITION 3.3. *Let Λ be a Nakayama algebra of type A_n. Then*

(1) $|\mathrm{s}\tau\text{-tilt}\,\Lambda| = 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_n\rangle)| + \sum\limits_{i=2}^{l(I_n)} C_{i-1}|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]}\rangle)|$,

(2) $|\mathrm{s}\tau\text{-tilt}\,\Lambda| = 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_1\rangle)| + \sum\limits_{i=2}^{l(P_1)} C_{i-1}|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_{[1,i]}\rangle)|$.

Proof. (1) For a given brick X of Λ with $\operatorname{top} X = S_i$ and $\operatorname{soc} X = S_j$, we will denote $S_{i,j} := X$.

We define W_0 as the subset of $\operatorname{sbrick}\Lambda$ consisting of the semibricks without S_n as a composition factor. It is clear that $|W_0| = |\operatorname{sbrick}(\Lambda/\langle e_n\rangle)|$.

Let W_i $(i = 1, \dots, l(I_n))$ be the subset of $\operatorname{sbrick}\Lambda$ consisting of the semibricks which contain the brick $S_{n-i+1,n}$.

First, there is a bijection

$$W_1 \mapsto \operatorname{sbrick}(\Lambda/\langle e_n\rangle)$$

defined by $\mathcal{S} \mapsto \mathcal{S} \setminus \{S_{n,n}\}$. So $|W_0| = |\operatorname{sbrick}(\Lambda/\langle e_n\rangle)|$.

Secondly, for $i = 2, 3, \dots, l(I_n)$, there exists a bijection

$$W_1 \mapsto \operatorname{sbrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle) \times \operatorname{sbrick}(\Lambda/\langle 1 - e_{[n-i+2,n-1]}\rangle)$$

defined by

$$\begin{aligned}\mathcal{S} \mapsto (\{S \in \mathcal{S} \mid \operatorname{Supp} S \cap [n-i+1, n] = \emptyset\},&\\ \{S \in \mathcal{S} \mid \operatorname{Supp} S \subset [n-i+2, n-1]\}),&\end{aligned}$$

where $\operatorname{Supp} S$ stands for the support of S. Note that $\operatorname{sbrick}\Lambda = \bigcup_{i=0}^{l(I_n)} W_i$. Thus we obtain

$$\begin{aligned}|\mathrm{s}\tau\text{-tilt}\,\Lambda| = |\operatorname{sbrick}\Lambda| &= \sum_{i=0}^{l(I_n)} |W_i|\\
&= 2|\operatorname{sbrick}(\Lambda/\langle e_n\rangle)|\\
&\quad + \sum_{i=2}^{l(I_n)} |\operatorname{sbrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle)| \cdot |\operatorname{sbrick}(\Lambda/\langle 1 - e_{[n-i+2,n-1]}\rangle)|\\
&= 2|\operatorname{sbrick}(\Lambda/\langle e_n\rangle)| + \sum_{i=2}^{l(I_n)} |\operatorname{sbrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle)| \cdot |\operatorname{sbrick}(KA_{i-2})|\\
&= 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_n\rangle)| + \sum_{i=2}^{l(I_n)} |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]}\rangle)| \cdot |\mathrm{s}\tau\text{-tilt}(KA_{i-2})|\\
&= 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_n\rangle)| + \sum_{i=2}^{l(I_n)} C_{i-1} \cdot |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]}\rangle)|.\end{aligned}$$

(2) Note that there is a bijection between $\mathrm{s}\tau\text{-tilt}\,\Lambda$ and $\mathrm{s}\tau\text{-tilt}\,\Lambda^{\mathrm{op}}$ [1, Theorem 2.14]). Now the assertion follows from (1). ∎

We give the following example to illustrate Proposition 3.3.

EXAMPLE 3.4. Let Λ be the algebra given by the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \to 4$$

with the relation $\alpha\beta = 0$. By Proposition 3.3(1), we have

$$\begin{aligned}|\mathrm{s}\tau\text{-tilt}\, \Lambda| &= 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_4\rangle)| + |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_3+e_4\rangle)| \\ &\quad + 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_2+e_3+e_4\rangle)| \\ &= 2\times 12 + 5 + 2\times 2 = 33.\end{aligned}$$

On the other hand, by Proposition 3.2(2),

$$|\mathrm{s}\tau\text{-tilt}\, \Lambda| = 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_1\rangle)| + |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_1+e_2\rangle)| = 2\times 14 + 5 = 33.$$

The following result gives a formula for $|\mathrm{s}\tau\text{-tilt}\, \Gamma_n|$.

THEOREM 3.5. *We have*

$$|\mathrm{s}\tau\text{-tilt}\, \Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\, \Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\, \Gamma_{n-2}|$$

with $|\mathrm{s}\tau\text{-tilt}\, \Gamma_1| = 2$ *and* $|\mathrm{s}\tau\text{-tilt}\, \Gamma_2| = 12$. *Hence*

$$|\mathrm{s}\tau\text{-tilt}\, \Gamma_n| = \frac{(3+2\sqrt{3})^n - (3-2\sqrt{3})^n}{2\sqrt{3}}.$$

Proof. Applying Proposition 3.3(2) to Γ_n and Δ_n respectively, we have

$$\begin{aligned}|\mathrm{s}\tau\text{-tilt}\, \Gamma_n| &= 2|\mathrm{s}\tau\text{-tilt}(\Gamma_n/\langle e_1\rangle)| + C_1\cdot|\mathrm{s}\tau\text{-tilt}(\Gamma_n/\langle e_1+e_2\rangle)| \\ &= 2|\mathrm{s}\tau\text{-tilt}\, \Delta_{n-1}| + |\mathrm{s}\tau\text{-tilt}\, \Gamma_{n-1}|\end{aligned} \tag{4}$$

and

$$\begin{aligned}|\mathrm{s}\tau\text{-tilt}\, \Delta_n| &= 2|\mathrm{s}\tau\text{-tilt}(\Delta_n/\langle e_0\rangle)| + C_1\cdot|\mathrm{s}\tau\text{-tilt}(\Delta_n/\langle e_0+e_1\rangle)| \\ &\quad + C_2\cdot|\mathrm{s}\tau\text{-tilt}(\Delta_n/\langle e_0+e_1+e_2\rangle)| \\ &= 2|\mathrm{s}\tau\text{-tilt}\, \Gamma_n| + |\mathrm{s}\tau\text{-tilt}\, \Delta_{n-1}| + 2|\mathrm{s}\tau\text{-tilt}\, \Gamma_{n-1}|.\end{aligned}$$

This implies

$$|\mathrm{s}\tau\text{-tilt}\, \Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\, \Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\, \Gamma_{n-2}|. \tag{5}$$

This is a linear homogeneous recurrence relation of degree 2 and its characteristic equation is $x^2 - 6x - 3 = 0$. The proof is finished. ■

Let $\overline{\Gamma}_n$ be the algebra given by the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n$$

with the relations $a_{2k-1}a_{2k} = 0$ for $1 \le k \le n-1$, and let $\overline{\Delta}_n$ be the algebra given by the quiver

$$0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n$$

with the relations $a_{2k-1}a_{2k} = 0$ for $1 \le k \le n-1$. By using the same argument as in Theorem 3.5, we can obtain

$$|\mathrm{s}\tau\text{-tilt}\, \overline{\Delta}_n| = 6|\mathrm{s}\tau\text{-tilt}\, \overline{\Delta}_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\, \overline{\Delta}_{n-2}|.$$

4. The case for Γ'_n. In this section, we will give formulas for $|\tau\text{-tilt}\, \Gamma'_n|$ and $|\mathrm{s}\tau\text{-tilt}\, \Gamma'_n|$.

Let X_n be the set of all support τ-tilting Γ_n-modules which do not have $P_1, \ldots, P_{2n-3}$ as direct summands, and let Y_n be the set of all support τ-tilting Δ_n-modules which do not have $P_0, P_1, \ldots, P_{2n-3}$ as direct summands. Let X'_n be the set of all support τ-tilting $\overline{\Gamma}_n$-modules which do not have $P_1, \ldots, P_{2n-2}$ as direct summands, and let Y'_n be the set of all support τ-tilting $\overline{\Delta}_n$-modules which do not have $P_0, P_1, \ldots, P_{2n-2}$ as direct summands.

We need the following lemma.

Lemma 4.1.

(1) $|X_n| = 3|X_{n-1}| + |X_{n-2}|$ *and* $|Y_n| = 3|Y_{n-1}| + |Y_{n-2}|$.
(2) $|X'_n| = 3|X'_{n-1}| + |X'_{n-2}|$ *and* $|Y'_n| = 3|Y'_{n-1}| + |Y'_{n-2}|$.

Proof. (1) By Lemma 2.4, all support τ-tilting Γ_n-modules which have S_1, S_2 as composition factors must have P_1 as a direct summand. Hence X_n consists of two parts: the first part comes from all support τ-tilting Γ_n-modules which do not have $P_1, \ldots, P_{2n-3}$ as direct summands and do not have S_1 as a composition factor (their number is exactly $|Y_{n-1}|$); the second part comes from all support τ-tilting Γ_n-modules which do not have $P_1, \ldots, P_{2n-3}$ as direct summands and have S_1 as a composition factor but not S_2 (their number is exactly $|X_{n-1}|$). Hence, $|X_n| = |Y_{n-1}| + |X_{n-1}|$. Similarly, we have $|Y_n| = |X_n| + |Y_{n-1}| + 2|X_{n-1}|$. These two equalities imply $|X_n| = 3|X_{n-1}| + |X_{n-2}|$ and $|Y_n| = 3|Y_{n-1}| + |Y_{n-2}|$.

(2) The proof is similar. ∎

The following result gives a formula for $|\tau\text{-tilt}\, \Gamma'_n|$.

Theorem 4.2. *We have*

$$|\tau\text{-tilt}\, \Gamma'_n| = 3|\tau\text{-tilt}\, \Gamma'_{n-1}| + |\tau\text{-tilt}\, \Gamma'_{n-2}|$$

with $|\tau\text{-tilt}\, \Gamma'_1| = 3$ *and* $|\tau\text{-tilt}\, \Gamma'_2| = 11$. *Hence*

$$|\tau\text{-tilt}\, \Gamma'_n| = \frac{(3+\sqrt{13})^n + (3-\sqrt{13})^n}{2^n}.$$

Proof. We claim that every proper support τ-tilting Γ'_n-module M which has S_1, S_2 as composition factors must have a projective Γ'_n-module as a direct summand. Indeed, if M does not have S_{2n} as a composition factor, then it has P_1 as a direct summand by Lemma 2.4. Now, assume that M has $S_i, S_{i+1}, \ldots, S_{2n}, S_1, S_2$ as composition factors, but not S_{i-1}. Then M has P_i as a direct summand by Lemma 2.4.

Now, $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\, \Gamma'_n$ consists of the following two parts:

(i) U_1: the subset of modules which do not have S_2 as a composition factor.

(ii) U_2: the subset of modules which have S_2 as a composition factor, but not S_1.

Since $\overline{\Lambda} := \Gamma'_n/\langle e_2\rangle$ is the quiver

$$3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n \xrightarrow{a_{2n}} 1$$

with the relations $a_{2k-1}a_{2k} = 0$ for $2 \le k \le n$, U_1 is exactly the set of support τ-tilting $\overline{\Lambda}$-modules which do not have $P_3, P_4, \ldots, P_{2n-1}$ as direct summands, and so $|U_1| = |X_n|$. Note that $\overline{\Gamma} := \Gamma'_n/\langle e_1\rangle$ is the quiver

$$2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n$$

with the relations $a_{2k-1}a_{2k} = 0$ for $2 \le k \le n-1$. Thus, the number of support τ-tilting $\overline{\Gamma}$-modules which do not have $P_2, P_4, \ldots, P_{2n-2}$ as direct summands is exactly $|Y'_{n-1}|$. Moreover, the number of support τ-tilting $\overline{\Gamma}$-modules which do not have $P_2, P_4, \ldots, P_{2n-2}$ as direct summands and do not have S_2 as a composition factor is exactly $|X'_{n-1}|$. Therefore, $|U_2| = |Y'_{n-1}| - |X'_{n-1}|$. By Theorem 2.2, we obtain

$$|\tau\text{-tilt}\,\Gamma'_n| = |\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\,\Gamma'_n| = |U_1| + |U_2| = |X_n| + |Y'_{n-1}| - |X'_{n-1}|.$$

Now, the recurrence relation for $|\tau\text{-tilt}\,\Gamma'_n|$ follows from Lemma 4.1. ∎

The following result gives a formula for $|\mathrm{s}\tau\text{-tilt}\,\Gamma'_n|$.

THEOREM 4.3. *We have*

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma'_n| = 6|\mathrm{s}\tau\text{-tilt}\,\Gamma'_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\Gamma'_{n-2}|$$

with $|\mathrm{s}\tau\text{-tilt}\,\Gamma'_1| = 6$ *and* $|\mathrm{s}\tau\text{-tilt}\,\Gamma'_2| = 42$. *Hence*

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma'_n| = (3+2\sqrt{3})^n + (3-2\sqrt{3})^n.$$

Proof. The set $\mathrm{sbrick}\,\Gamma'_n$ of semibricks of Γ'_n consists of five parts:

(i) V_0: the semibricks without S_1 as a composition factor.
(ii) V_1: the semibricks which contain S_1 but not the brick I_2.
(iii) V_2: the semibricks which contain I_1.
(iv) V_3: the semibricks which contain P_1.
(v) V_4: the semibricks which contain I_2.

Obviously, $|V_0| = |\mathrm{sbrick}(\Gamma'_n/\langle e_1\rangle)| = |\mathrm{sbrick}\,\overline{\Delta}_{n-1}|$.

There is a bijection $V_1 \mapsto \mathrm{sbrick}(\Gamma'_n/\langle e_1\rangle)$ defined by $\mathcal{S} \mapsto \mathcal{S} \setminus \{S_1\}$, so

$$|V_1| = |\mathrm{sbrick}(\Gamma'_n/\langle e_1\rangle)| = |\mathrm{sbrick}\,\overline{\Delta}_{n-1}|.$$

Similarly, there are bijections

$$V_2 \mapsto \mathrm{sbrick}(\Gamma'_n/\langle e_1 + e_{2n}\rangle) \quad \text{and} \quad V_3 \mapsto \mathrm{sbrick}(\Gamma'_n/\langle e_1 + e_2\rangle),$$

so

$$|V_2| = |\mathrm{sbrick}(\Gamma'_n/\langle e_1 + e_{2n}\rangle)| = |\mathrm{sbrick}\,\Delta_{n-1}|,$$
$$|V_3| = |\mathrm{sbrick}(\Gamma'_n/\langle e_1 + e_2\rangle)| = |\mathrm{sbrick}\,\Delta^{\mathrm{op}}_{n-1}|.$$

Finally, we can define a bijection

$$V_4 \mapsto \operatorname{sbrick}(\mathit{\Gamma}'_n/\langle e_1 + e_2 + e_{2n}\rangle) \times \operatorname{sbrick}(\mathit{\Gamma}'_n/\langle 1 - e_1\rangle)$$

by $V_4 \ni \mathcal{S} \mapsto (\mathcal{S} \setminus \{S_1, I_2\}, S_1 \cap \mathcal{S})$. Thus

$$|V_4| = |\operatorname{sbrick}(\mathit{\Gamma}'_n/\langle e_1 + e_2 + e_{2n}\rangle)| \cdot |\operatorname{sbrick}(\mathit{\Gamma}'_n/\langle 1 - e_1\rangle)| = 2|\operatorname{sbrick} \mathit{\Gamma}_{n-1}|.$$

Therefore

$$\begin{aligned} |\mathrm{s}\tau\text{-tilt}\, \mathit{\Gamma}'_n| = |\operatorname{sbrick} \mathit{\Gamma}'_n| &= \sum_{i=0}^{4} |V_i| \\ &= 2|\operatorname{sbrick} \overline{\mathit{\Delta}}_{n-1}| + |\operatorname{sbrick} \mathit{\Delta}_{n-1}| + |\operatorname{sbrick} \mathit{\Delta}^{\mathrm{op}}_{n-1}| + 2|\operatorname{sbrick} \mathit{\Gamma}_{n-1}| \\ &= 2|\mathrm{s}\tau\text{-tilt}\, \overline{\mathit{\Delta}}_{n-1}| + |\mathrm{s}\tau\text{-tilt}\, \mathit{\Delta}_{n-1}| + |\mathrm{s}\tau\text{-tilt}\, \mathit{\Delta}^{\mathrm{op}}_{n-1}| + 2|\mathrm{s}\tau\text{-tilt}\, \mathit{\Gamma}_{n-1}| \\ &= 2|\mathrm{s}\tau\text{-tilt}\, \overline{\mathit{\Delta}}_{n-1}| + 2|\mathrm{s}\tau\text{-tilt}\, \mathit{\Delta}_{n-1}| + 2|\mathrm{s}\tau\text{-tilt}\, \mathit{\Gamma}_{n-1}|. \end{aligned}$$

Note that $|\mathrm{s}\tau\text{-tilt}\, \mathit{\Delta}_{n-1}|$ is a linear combination of $|\mathrm{s}\tau\text{-tilt}\, \mathit{\Gamma}_n|$ and $|\mathrm{s}\tau\text{-tilt}\, \mathit{\Gamma}_{n-1}|$ by (4), so $|\mathrm{s}\tau\text{-tilt}\, \mathit{\Delta}_n|$ has the same recurrence relation as $|\mathrm{s}\tau\text{-tilt}\, \mathit{\Gamma}_n|$. In particular, $|\mathrm{s}\tau\text{-tilt}\, \overline{\mathit{\Delta}}_n|$, $|\mathrm{s}\tau\text{-tilt}\, \mathit{\Delta}_n|$, $|\mathrm{s}\tau\text{-tilt}\, \mathit{\Gamma}_n|$ have the same recurrence relations, and so $|\mathrm{s}\tau\text{-tilt}\, \mathit{\Gamma}'_n|$ also has the same recurrence relation. ■

5. Examples. In this section, we list the numbers of (support) τ-tilting modules over $\mathit{\Gamma}_n$ and $\mathit{\Gamma}'_n$ in the following table. The sequence $|\tau\text{-tilt}\, \mathit{\Gamma}_n|$ is listed in the On-line Encyclopedia of Integer Sequences (OEIS) as the sequence A006190 and $|\tau\text{-tilt}\, \mathit{\Gamma}'_n|$ as A006497.

n	1	2	3	4	5	6	7	8
$\lvert\tau\text{-tilt}\, \mathit{\Gamma}_n\rvert$	1	3	10	33	109	360	1189	3927
$\lvert\mathrm{s}\tau\text{-tilt}\, \mathit{\Gamma}_n\rvert$	2	12	78	504	3258	21060	136134	879984
$\lvert\tau\text{-tilt}\, \mathit{\Gamma}'_n\rvert$	3	11	36	119	393	1298	4287	114159
$\lvert\mathrm{s}\tau\text{-tilt}\, \mathit{\Gamma}'_n\rvert$	6	42	270	17464	11286	72954	471582	3048354

Acknowledgements. The authors thank the referee for useful and detailed suggestions.

This work was partially supported by NSFC (Nos. 11971225, 12101320, 12171207) and Jiangsu Provincial Double-Innovation Doctor Program (JSSCBS20210353).

REFERENCES

[1] T. Adachi, O. Iyama and I. Reiten, *τ-tilting theory*, Compos. Math. 50 (2014), 415–452.
[2] T. Adachi, *The classification of τ-tilting modules over Nakayama algebras*, J. Algebra 452 (2016), 227–262.
[3] S. Asai, *Semibricks*, Int. Math. Res. Notices 2020, 4993–5054.

[4] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras*, London Math. Soc. Student Texts 65, Cambridge Univ. Press, Cambridge, 2006.

[5] M. Auslander and S. O. Smalø, *Almost split sequences in subcategories*, J. Algebra 69 (1981), 426–454.

[6] T. Brüstle, L. Hille, C. M. Ringel and G. Röhrle. *The Δ-filtered modules without self-extensions for the Auslander algebra of* $k[T]/\langle T^n \rangle$, Algebras Represent. Theory 2 (1999), 295–312.

[7] S. Fomin and A. Zelevinsky, *Y-systems and generalized associahedra*, Ann. of Math. 158 (2003), 977–1018.

[8] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France 90 (1962), 323–448.

[9] H. Gao and R. Schiffler, *On the number of τ-tilting modules over Nakayama algebras*, SIGMA Symmetry Integrability Geom. Methods Appl. 16 (2020), art. 058, 13 pp.

[10] D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. 274 (1982), 399–443.

[11] O. Iyama and X. Zhang, *Classifying τ-tilting modules over the Auslander algebra of* $K[x]/(x^n)$, J. Math Soc. Japan 72 (2020), 731–764.

[12] N. Kajita, *The number of tilting modules over hereditary algebras and tilting modules over Auslander algebras*, Master Thesis, Grad. School of Math., Nagoya Univ., 2008 (in Japanese).

[13] B. Keller and D. Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. Sér. A 40 (1988), 239–253.

[14] A. Obaid, S. K. Nauman, W. M. Fakieh and C. M. Ringel, *The number of support-tilting modules for a Dynkin algebra*, J. Integer Sequences 18 (2015), no. 10, art. 15.10.6, 24 pp.

[15] C. M. Ringel, *Representations of K-species and bimodules*, J. Algebra 41 (1976), 269–302.

[16] X. Zhang, *Classifying tilting modules over the Auslander algebras of radical square zero Nakayama algebras*, J. Algebra Appl. 21 (2022), no. 2, art. 2250041.

Hanpeng Gao
School of Mathematical Sciences
Anhui University
230601 Hefei, Anhui Province, P.R. China
E-mail: hpgao07@163.com

Zhaoyong Huang (corresponding author)
Department of Mathematics
Nanjing University
210093 Nanjing, Jiangsu Province, P.R. China
E-mail: huangzy@nju.edu.cn

Zongzhen Xie
Department of Mathematics
and Computer Science
School of Biomedical Engineering
and Informatics
Nanjing Medical University
211166 Nanjing, Jiangsu Province, P.R. China
E-mail: zzhx@njmu.edu.cn