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ON DENSE SUBSETS IN SPACES OF METRICS

BY

YOSHITO ISHIKI (Tsukuba and Saitama)

Abstract. In spaces of metrics, we investigate topological distributions of the doubling property, uniform disconnectedness, and uniform perfectness, which are quasi-symmetrically invariant properties appearing in the David–Semmes theorem. We show that the set of all doubling metrics and the set of all uniformly disconnected metrics are dense in spaces of metrics on finite-dimensional and zero-dimensional compact metrizable spaces, respectively. Conversely, this denseness implies the finite-dimensionality, zero-dimensionality, and compactness of metrizable spaces. We also determine the topological distribution of the set of all uniformly perfect metrics in the space of metrics on the Cantor set.

1. Introduction.

For a metrizable topological space X, we denote by $\mathrm{M}(X)$ the set of all metrics on X that generate the topology of X. Define a metric $\mathcal{D}_X : \mathrm{M}(X)^2 \to [0,\infty]$ by

$$\mathcal{D}_X(d,e) = \sup_{x,y\in X} |d(x,y) - e(x,y)|.$$

In [9], the author introduced the notion of a transmissible property which unifies geometric properties defined by finite subsets of metric spaces, and proved that for every non-discrete metrizable space X, the set of all metrics in $\mathrm{M}(X)$ not satisfying a transmissible property with a singular transmissible parameter is dense G_δ in $\mathrm{M}(X)$. Since the doubling property and uniform disconnectedness are transmissible properties with singular parameters, the set of all non-doubling metrics and the set of all non-uniformly-disconnected metrics are dense G_δ in spaces of metrics (see [9]). In contrast to [9], in this paper, we investigate topological distributions of the doubling property, uniform disconnectedness, and uniform perfectness in spaces of metrics.

Niemytzki and Tychonoff [12] proved that a metrizable space X is compact if and only if all metrics in $\mathrm{M}(X)$ are complete, and the author [10] proved an ultrametric analogue of their result (see [10, Corollary 1.3], see also [5, Proposition 4.10]). Nomizu and Ozeki [13] proved that a second count-

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able differentiable manifold is compact if and only if all Riemannian metrics on the manifold are complete. As a development of these theorems, in the present paper, we characterize the compactness and the finite-dimensionality or 0-dimensionality of a metrizable space by the denseness of the set of all doubling metrics or all uniformly disconnected metrics in the space of metrics.

In this paper, a topological space is said to be *finite-dimensional* (resp. *0-dimensional*) if its covering dimension is finite (resp. 0). The definition and basic properties of the covering dimension can be seen in [1].

For a metric space (X, d) and for a subset A of X, we denote by $\delta_d(A)$ the diameter of A, and we define $\alpha_d(A) = \inf\{d(x, y) \mid x \neq y,\ x, y \in A\}$. A metric space (X, d) is said to be *doubling* if there exist $\beta \in (0, \infty)$ and $C \in [1, \infty)$ such that for every finite subset A of X we have

$$\mathrm{Card}(A) \leq C \cdot (\delta_d(A)/\alpha_d(A))^{\beta},$$

where Card stands for cardinality. Let X be a topological space. A subset S is said to be F_σ (resp. G_δ) if S is the union of countably many closed subsets of X (resp. the intersection of countably many open subsets of X).

THEOREM 1.1. *Let X be a metrizable topological space. Then X is finite-dimensional and compact if and only if the set of all doubling metrics in* $\mathrm{M}(X)$ *is dense F_σ in* $(\mathrm{M}(X), \mathcal{D}_X)$.

A metric space is said to be *uniformly disconnected* if there exists $\delta \in (0, 1)$ such that for every non-constant finite sequence $\{z_i\}_{i=1}^N$ in X we have

$$\delta d(z_1, z_N) \leq \max_{1 \leq i \leq N} d(z_i, z_{i+1}).$$

This notion was introduced in [2] in a different but equivalent way. Note that a metric space is uniformly disconnected if and only if it is bi-Lipschitz embeddable into an ultrametric space (see [2, Proposition 15.7]). Similarly to Theorem 1.1, we obtain:

THEOREM 1.2. *Let X be a metrizable space. Then X is 0-dimensional and compact if and only if the set of all uniformly disconnected metrics in* $\mathrm{M}(X)$ *is dense F_σ in* $(\mathrm{M}(X), \mathcal{D}_X)$.

Let X be a set. A metric d on X is said to be an *ultrametric* if for all $x, y, z \in X$ the metric d satisfies the so-called *strong triangle inequality*:

$$d(x, y) \leq d(x, z) \vee d(z, y),$$

where $\vee$ stands for maximum.

We say that a set S is a *range set* if S is a subset of $[0, \infty)$ and $0 \in S$. For a range set S, we say that a metric d on X is *S-valued* if $d(X^2)$ is contained in S. For a range set S, and for a topological space X, we denote by $\mathrm{UM}(X, S)$ the set of all S-valued ultrametrics on X that generate the topology of X. For a topological space X, and for a range set S, we define a function $\mathcal{UD}_X^S : \mathrm{UM}(X,S)^2 \to [0,\infty]$ by letting $\mathcal{UD}_X^S(d,e)$ be the infimum of $\epsilon \in S \sqcup \{\infty\}$ such that for all $x, y \in X$ we have

$$d(x,y) \le e(x,y) \vee \epsilon \quad \text{and} \quad e(x,y) \le d(x,y) \vee \epsilon.$$

The function $\mathcal{UD}_X^S$ is an ultrametric on $\mathrm{UM}(X,S)$ valued in $\mathrm{CL}(S) \sqcup \{\infty\}$, where $\mathrm{CL}(S)$ is the closure of S in $[0,\infty)$.

In [10], the author investigated the topological distributions of metrics not satisfying a transmissible property in $(\mathrm{UM}(X,S), \mathcal{UD}_X^S)$.

We say that a range set S has *countable coinitiality* if there exists a strictly decreasing sequence $\{r_i\}_{i\in\mathbb{Z}_{\ge 0}}$ in S converging to 0 as $i \to \infty$. As an ultrametric analogue of Theorem 1.1, we obtain:

THEOREM 1.3. *Let S be a range set with countable coinitiality. Let X be an ultrametrizable space. Then X is compact if and only if the set of all doubling metrics in* $\mathrm{UM}(X,S)$ *is dense F_σ in* $(\mathrm{UM}(X,S), \mathcal{UD}_X^S)$.

REMARK 1.1. There are some results on relations between topological properties of metrizable spaces and properties of spaces of ultrametrics. Let X be an ultrametrizable space. Dovgoshey–Shcherbak [5] proved that X is compact if and only if all $d \in \mathrm{UM}(X,[0,\infty))$ are totally bounded, and X is separable if and only if for all $d \in \mathrm{UM}(X,[0,\infty))$ we have $\mathrm{Card}(\{d(x,y) \mid x,y \in X\}) \le \aleph_0$.

Let $c \in (0,1)$. A metric space (X,d) is said to be *c-uniformly perfect* if for every $x \in X$ and every $r \in (0, \delta_d(X))$, there exists $y \in X$ with

$$cr \le d(x,y) \le r.$$

A metric space is said to be *uniformly perfect* if it is c-uniformly perfect for some $c \in (0,1)$. We next investigate topological distributions of the set of all uniformly perfect metrics and its complement in the space of metrics on the Cantor set. In this paper, Γ denotes the (middle-third) Cantor set.

THEOREM 1.4. *The following statements hold true:*

(1) *The set of all uniformly perfect metrics in* $\mathrm{M}(\Gamma)$ *is dense F_σ in* $(\mathrm{M}(\Gamma), \mathcal{D}_\Gamma)$.
(2) *The set of all non-uniformly perfect metrics in* $\mathrm{M}(\Gamma)$ *is a dense G_δ set in* $(\mathrm{M}(\Gamma), \mathcal{D}_\Gamma)$.

We say that a range set S is *exponential* if there exist $a \in (0,1)$ and $M \in [1,\infty)$ such that for every $n \in \mathbb{Z}_{\ge 0}$ we have

$$[M^{-1}a^n, Ma^n] \cap S \ne \emptyset.$$

As an ultrametric analogue of Theorem 1.4, we obtain:

Theorem 1.5. *Let S be a range set.*

(1) *The set S is exponential if and only if the set of all uniformly perfect metrics in* $\mathrm{UM}(\Gamma, S)$ *is dense F_σ in* $(\mathrm{UM}(\Gamma, S), \mathcal{UD}_\Gamma^S)$.
(2) *The set of all non-uniformly perfect metrics in* $\mathrm{UM}(\Gamma, S)$ *is dense G_δ in* $(\mathrm{UM}(\Gamma, S), \mathcal{UD}_\Gamma^S)$.

Let (X, d) and (Y, e) be metric spaces. A homeomorphism $f : X \to Y$ is said to be *quasi-symmetric* if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for all $x, y, z \in X$ and $t \in [0, \infty)$,

$$d(x, y) \leq td(x, z) \quad \text{implies} \quad e(f(x), f(y)) \leq \eta(t)e(f(x), f(z)).$$

For example, all bi-Lipschitz homeomorphisms are quasi-symmetric. Note that the doubling property, uniform disconnectedness, and uniform perfectness are invariant under quasi-symmetric maps. David and Semmes [2] proved that if a compact metric space is doubling, uniformly disconnected, and uniformly perfect, then it is quasi-symmetrically equivalent to the Cantor set Γ equipped with the Euclidean metric [2, Proposition 15.11]. To simplify our description, the symbols $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ stand for the doubling property, uniform disconnectedness, and uniform perfectness, respectively. Before stating our results, for the sake of simplicity, we introduce the following notions:

Definition 1.1. If a metric space (X, d) satisfies a property $\mathcal{P}$, then we write $T_{\mathcal{P}}(X, d) = 1$; otherwise, $T_{\mathcal{P}}(X, d) = 0$. For a triple $(u_1, u_2, u_3) \in \{0, 1\}^3$, we say that a metric space (X, d) is *of type* (u_1, u_2, u_3) if $T_{\mathcal{P}_k}(X, d) = u_k$ for all $k \in \{1, 2, 3\}$.

A topological space is said to be a *Cantor space* if it is homeomorphic to the Cantor set Γ. For a metric space (X, d), we denote by $\mathcal{G}(X, d)$ the *conformal gauge* of (X, d) defined as the quasi-symmetric equivalence class of (X, d), which is a basic concept in conformal dimension theory (see, e.g., [11]). For each $(u, v, w) \in \{0, 1\}^3$, we define

$$\mathcal{M}(u, v, w) = \{\mathcal{G}(X, d) \mid (X, d) \text{ is a Cantor space of type } (u, v, w)\}.$$

The David–Semmes theorem mentioned above states that $\mathcal{M}(1, 1, 1)$ is a singleton. In contrast to this, the author [8] proved that for every $(u, v, w) \in \{0, 1\}^3$ except $(1, 1, 1)$, we have $\mathrm{Card}(\mathcal{M}(u, v, w)) = 2^{\aleph_0}$ (see [8, Theorem 2]). As a development of this result, we investigate topological distributions of metrics of type (u, v, w) for all $(u, v, w) \in \{0, 1\}^3$.

For every $(u, v, w) \in \{0, 1\}^3$, we put

$$\mathrm{E}(u, v, w) = \{d \in \mathrm{M}(\Gamma) \mid (\Gamma, d) \text{ is of type } (u, v, w)\}.$$

Let X be a topological space. A subset M of X is said to be $F_{\sigma\delta}$ (resp. $G_{\delta\sigma}$) if M is the intersection of countably many F_σ subsets of X (resp. the union of countably many G_δ subsets of X).

THEOREM 1.6. *The following three statements hold true:*
(1) *The set* $\mathrm{E}(1,1,1)$ *is a dense* F_σ *subset of* $(\mathrm{M}(\Gamma), \mathcal{D}_\Gamma)$.
(2) *The set* $\mathrm{E}(0,0,0)$ *is a dense* G_δ *subset of* $(\mathrm{M}(\Gamma), \mathcal{D}_\Gamma)$.
(3) *For every* $(u,v,w) \in \{0,1\}^3$ *except* $(1,1,1)$ *and* $(0,0,0)$, *the set* $\mathrm{E}(u,v,w)$ *is a dense* $G_{\delta\sigma}$ *and* $F_{\sigma\delta}$ *subset of* $(\mathrm{M}(\Gamma), \mathcal{D}_\Gamma)$.

2. The doubling property. The following is known as the McShane–Whitney extension theorem.

THEOREM 2.1. *Let* $l \in (0,\infty)$. *Let* (X,d) *be a metric space, and let* A *be a subset of* X. *Then, for every* l*-Lipschitz map* $f : A \to \mathbb{R}$, *there exists an* l*-Lipschitz map* $F : X \to \mathbb{R}$ *such that* $F|_A = f$. *Moreover, every* l*-Lipschitz map* f *from* $(A, d|_{A^2})$ *into* $\mathbb{R}^n$ *with the* ℓ^∞*-norm can be extended to an* l*-Lipschitz map from* (X,d) *into* $\mathbb{R}^n$ *with the* ℓ^∞*-norm.*

THEOREM 2.2. *Let* X *be a metrizable compact finite-dimensional space. Let* Q *be the set of all metrics* $d \in \mathrm{M}(X)$ *for which* (X,d) *is isometrically embeddable into a Euclidean space equipped with the* ℓ^∞*-norm. Then the set* Q *is dense in* $(\mathrm{M}(X), \mathcal{D}_X)$.

Proof. In this proof, let L_N denote the metric induced from the ℓ^∞-norm on $\mathbb{R}^N$ for all $N \in \mathbb{Z}_{\geq 1}$. Put $n = \dim(X)$. Take arbitrary $d \in \mathrm{M}(X)$ and $\epsilon \in (0,\infty)$. Since X is compact, there exists a finite sequence $P = \{p_i\}_{i=1}^m$ in X such that $\alpha_d(P) \geq \epsilon$ and $X = \bigcup_{i=1}^m U(p_i, \epsilon)$. By the Kuratowski embedding theorem (see [7]), there exists an isometric embedding from $(P, d|_{P^2})$ into $(\mathbb{R}^m, L_m)$. Thus, by Theorem 2.1, there exists a 1-Lipschitz map $F : X \to \mathbb{R}^m$ such that $F|_P$ is an isometry. By [1, Theorem 9.6], there exists a topological embedding $I : X \to \mathbb{R}^{2n+1}$. Since $\mathbb{R}^{2n+1}$ is homeomorphic to its open ball with radius $\epsilon/2$, we may assume that $\delta_{L_{2n+1}}(I(X)) < \epsilon$. Define $D \in \mathrm{M}(X)$ by

$$D(x,y) = L_m(F(x), F(y)) \vee L_{2n+1}(I(x), I(y)).$$

Then the map $E : (X, D) \to (\mathbb{R}^{m+2n+1}, L_{m+2n+1})$ defined by $E(x) = (F(x), I(x))$ is an isometric embedding. Since $F|_P$ is an isometric embedding, and since $\alpha_d(P) \geq \epsilon$, we have $D(p_i, p_j) = d(p_i, p_j)$. Since F is 1-Lipschitz, for every $x \in X$, the inequality $d(x, p_i) < \epsilon$ implies $D(x, p_i) < \epsilon$. For all $x, y \in X$, take p_i, p_j with $d(x, p_i) < \epsilon$ and $d(y, p_j) < \epsilon$. Then

$$|D(x,y) - d(x,y)| \leq d(x,p_i) + d(y,p_j) + D(x,p_i) + D(y,p_j) < 4\epsilon.$$

Thus, we conclude that Q is dense in $(\mathrm{M}(X), \mathcal{D}_X)$. ∎

The following is the Hausdorff metric extension theorem [6]:

THEOREM 2.3. *Let* X *be a metrizable space, and* A *a closed subset of* X. *Then for every* $d \in \mathrm{M}(A)$, *there exists* $D \in \mathrm{M}(X)$ *with* $D|_{A^2} = d$.

By [9, Corollary 4.4 and Proposition 4.9], we obtain:

LEMMA 2.4. *For a metrizable space X, the set of all doubling metrics in* $\mathrm{M}(X)$ *is* F_σ *in* $(\mathrm{M}(X), \mathcal{D}_X)$.

Proof of Theorem 1.1. Let T be the set of all doubling metrics in $\mathrm{M}(X)$. Assume first that X is a metrizable compact finite-dimensional space. Since all metric subspaces of Euclidean spaces are doubling, T contains the set Q stated in Theorem 2.2. Then, by Theorem 2.2 and Lemma 2.4, the set T is dense F_σ in $\mathrm{M}(X)$.

We next prove the converse. If there exists a doubling metric in $\mathrm{M}(X)$, then, by the Assouad embedding theorem (see [7, Theorem 12.1]), X is finite-dimensional. For the sake of contradiction, suppose that X is not compact. Then there exists a countable closed discrete subspace F of X. Let e be the metric on F such that $e(x, y) = 1$ whenever $x \neq y$. By Theorem 2.3, there exists $D \in \mathrm{M}(X)$ with $D|_{F^2} = e$. Let U be the open ball centered at D with radius $1/2$ in $(\mathrm{M}(X), \mathcal{D}_X)$. Take $d \in U$. Then, for every finite subset A of F, we have $\delta_D(A) - 1/2 \leq \delta_d(A)$ and $\alpha_d(A) \leq \alpha_D(A) + 1/2$. Since $1 \leq \alpha_D(A)$ and $1 \leq \delta_D(A)$, we have $\delta_D(A)/2 \leq \delta_d(A)$ and $\alpha_d(A) \leq 2\alpha_D(A)$. Since D is not doubling, for every $C \in [1, \infty)$ and every $\beta \in (0, \infty)$, there exists a finite subset B of F with $4C \cdot (\delta_D(B)/\alpha_D(B))^\beta < \mathrm{Card}(B)$, and hence $C \cdot (\delta_d(B)/\alpha_d(B))^\beta < \mathrm{Card}(B)$, so d is not doubling. Thus, the open set U consists of non-doubling metrics, and hence the set T is not dense in $\mathrm{M}(X)$. This finishes the proof. ∎

3. Amalgamation lemmas.

We provide new amalgamation lemmas for metrics and ultrametrics.

PROPOSITION 3.1. *Let I be a set. Let (X, d) be a metric space. Let $\{B_i\}_{i \in I}$ be a covering of X consisting of mutually disjoint clopen subsets, and let $\{p_i\}_{i \in I}$ be points with $p_i \in B_i$. Let $\{e_i\}_{i \in I}$ be a set of metrics such that $e_i \in \mathrm{M}(B_i)$. Define a function $D : X^2 \to [0, \infty)$ by*

$$D(x, y) = \begin{cases} e_i(x, y) & \text{if } x, y \in B_i, \\ e_i(x, p_i) + d(p_i, p_j) + e_j(p_j, y) & \text{if } x \in B_i \text{ and } y \in B_j. \end{cases}$$

Then $D \in \mathrm{M}(X)$ and $D|_{B_i^2} = e_i$ for all $i \in I$. Moreover, if for every $i \in I$ we have $\delta_d(B_i) \leq \epsilon$ and $\delta_{e_i}(B_i) \leq \epsilon$, then $\mathcal{D}_X(D, d) \leq 4\epsilon$.

Proof. We first show that D satisfies the triangle inequality. Take distinct $i, j, k \in I$, and take $x, y, z \in X$. If $x, y \in B_i$ and $z \in B_j$, we have $D(x, y) = e_i(x, y) \leq e_i(x, p_i) + e_i(p_i, y) \leq D(x, z) + D(z, y)$. If $x \in B_i$ and $y, z \in B_j$, we have

$$\begin{aligned} D(x, y) &= e_i(x, p_i) + d(p_i, p_j) + e_j(p_j, y) \\ &\leq e_i(x, p_i) + d(p_i, p_j) + e_j(p_j, z) + e_i(z, y) = D(x, z) + D(z, y). \end{aligned}$$

If $x \in B_i$, $y \in B_j$ and $z \in B_k$, we have

$$\begin{aligned} D(x,y) &= e_i(x,p_i) + d(p_i,p_j) + e_j(p_j,y) \\ &\le e_i(x,p_i) + d(p_i,p_k) + d(p_k,p_j) + e_j(p_j,z) + e_i(z,y) \\ &\le D(x,z) + D(z,y). \end{aligned}$$

Since i, j, k and x, y, z are arbitrary, we conclude that D satisfies the triangle inequality.

Since $\{B_i\}_{i\in I}$ is a disjoint family of clopen subsets, X is homeomorphic to the disjoint union space induced from $\{B_i\}_{i\in I}$. Thus, $D \in \mathrm{M}(X)$.

We next prove the last part. Take $x, y \in X$. If $x, y \in B_i$, then, by the assumption, we have $|D(x,y) - d(x,y)| \le 2\epsilon$. If $x \in B_i$ and $y \in B_j$ for some $i, j \in I$ with $i \ne j$, then

$$\begin{aligned} |D(x,y) - d(x,y)| &\le D(x,p_i) + D(p_j,y) + d(x,p_i) + d(y,p_j) \\ &\le \delta_d(B_i) + \delta_{e_i}(B_i) + \delta_d(B_j) + \delta_{e_j}(B_j) \le 4\epsilon. \end{aligned}$$

This completes the proof. ∎

By replacing "+" with "$\vee$" in the proof of Proposition 3.1, we obtain the following proposition:

Proposition 3.2. *Let I be a set. Let S be a range set. Let (X,d) be an S-valued ultrametric space. Let $\{B_i\}_{i\in I}$ be a covering of X consisting of mutually disjoint clopen subsets, and let $\{p_i\}_{i\in I}$ be points with $p_i \in B_i$. Let $\{e_i\}_{i\in I}$ be a set of ultrametrics with $e_i \in \mathrm{UM}(B_i, S)$. Define a function $D : X^2 \to [0,\infty)$ by*

$$D(x,y) = \begin{cases} e_i(x,y) & \text{if } x, y \in B_i, \\ e_i(x,p_i) \vee d(p_i,p_j) \vee e_j(p_j,y) & \text{if } x \in B_i \text{ and } y \in B_j. \end{cases}$$

Then $D \in \mathrm{UM}(X,S)$ and $D|_{B_i^2} = e_i$ for all $i \in I$. Moreover, if for every $i \in I$ we have $\delta_d(B_i) \le \epsilon$ and $\delta_{e_i}(B_i) \le \epsilon$, then $\mathcal{UD}_X^S(D,d) \le \epsilon$.

Remark 3.1. Let X be a metrizable space, let A be a closed subset of X, and let $\{B_i\}_{i\in I}$ be a covering of X by mutually disjoint clopen subsets. The Hausdorff metric extension theorem (Theorem 2.3) states that a metric defined on the squared set A^2 can be extended to a metric defined on X^2 (see also Theorem 3.7). On the other hand, Proposition 3.1 states that a metric defined on $\coprod_{i\in I} B_i^2$ can be extended to a metric on X^2. Note that $\coprod_{i\in I} B_i^2$ is not a squared subset of X^2 in general. Dovgoshey–Martio–Vuorinen [4] found a necessary and sufficient condition under which a weight of a weighted graph can be extended to a pseudometric on the vertex set of the graph. Dovgoshey–Petrov [3] proved an ultrametric version of it. Propositions 3.1 and 3.2 can be considered as a generalization of Dovgoshey–Martio–Vuorinen's and Dovgoshey–Petrov's results. We also remark that Proposition 3.2 is a generalization of Dovgoshey–Shcherbak's construction of ultrametrics (see [5, (4.11) in the proof of Theorem 4.7]).

By the definition of uniform perfectness, we obtain:

LEMMA 3.3. *A metric space (X, d) is uniformly perfect if and only if there exist $c \in (0, 1)$ and $\delta \in (0, \infty)$ such that for every $x \in X$ and every $r \in (0, \delta)$, there exists $y \in X$ with $cr \leq d(x, y) \leq r$.*

Recall that $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ stand for the doubling property, uniform disconnectedness, and uniform perfectness, respectively.

LEMMA 3.4. *Let (X, d) be a metric space. Let $\{B_i\}_{i=1}^n$ be a covering of X consisting of mutually disjoint clopen subsets, and let $\{p_i\}_{i=1}^n$ be points with $p_i \in B_i$. Let $\{e_i\}_{i=1}^n$ be a set of metrics such that $e_i \in \mathrm{M}(B_i)$. Let D be the metric constructed in Proposition 3.1 from d and $\{e_i\}_{i=1}^n$. Then for every $k \in \{1, 2, 3\}$, the following hold true:*

(1) *If each e_i satisfies $\mathcal{P}_k$, then so does (X, D).*
(2) *If some e_i does not satisfy $\mathcal{P}_k$, then neither does (X, D).*

Proof. We first prove (1).

If $k = 1$, take a finite subset A of X, and put $A_i = A \cap B_i$ for all $i \in \{1, \ldots, n\}$. Since each S_i is doubling, for each $i \in \{1, \ldots, n\}$ there exist $C_i \in (0, \infty)$ and β_i such that $\mathrm{Card}(A_i) \leq C_i(\delta_{e_i}(A_i)/\alpha_{e_i}(A_i))^{\beta_i}$. Put $C = \max_{1 \leq i \leq n} C_i$ and $\beta = \max_{1 \leq i \leq n} \beta_i$; then $\mathrm{Card}(A) \leq nC(\delta_D(A)/\alpha_D(A))^{\beta}$. This proves (1) for $k = 1$.

We next deal with $k = 2$. By assumption and [2, Proposition 15.7], for each $i \in \{1, \ldots, n\}$ there exist an ultrametric $w_i \in \mathrm{M}(B_i)$ and a bi-Lipschitz map $f_i : (B_i, e_i) \to (B_i, w_i)$. By applying Proposition 3.2 to $\{w_i\}_{i=1}^n$, we obtain an ultrametric $R \in \mathrm{M}(X)$. Define $f : (X, D) \to (X, R)$ by $f(x) = f_i(x)$ if $x \in B_i$. Then f is bi-Lipschitz, and so (X, D) is uniformly disconnected.

We next handle the case of $k = 3$. Assume that all B_i are c-uniformly perfect for some $c \in (0, 1)$. Put $m = \min_{1 \leq i \leq n} \delta_{e_i}(B_i)$. Since each B_i has at least two elements, we have $m > 0$. Put $C = (1/2) \min \{c, m/\delta_D(X)\}$. Then (X, D) is C-uniformly perfect.

We now prove the statement (2). Since the doubling property and uniform disconnectedness are hereditary to all metric subspaces, statements (2) for $k = 1, 2$ are true. We now treat the case of $k = 3$. Take $c \in (0, 1)$. Put $\delta = \min \{d(p_i, p_j) \mid i \neq j\}$. We may assume that e_1 is not uniformly perfect. Then, by Lemma 3.3, there exist $x \in B_1$ and $r \in (0, \delta)$ such that for all $y \in B_1$ we have $e_1(x, y) < cr$ or $r < e_1(x, y)$. By the definition of D, for all $y \in X$ we have $D(x, y) < cr$ or $r < D(x, y)$. Thus, D is not uniformly perfect. ∎

By [9, Corollary 4.4 and Proposition 4.11], we obtain:

Lemma 3.5. *For a metrizable space X, the set of all uniformly disconnected metrics in* $\mathrm{M}(X)$ *is* F_σ *in* $(\mathrm{M}(X), \mathcal{D}_X)$.

Proof of Theorem 1.2. Assume that X is compact and 0-dimensional. Take arbitrary $d \in \mathrm{M}(X)$ and $\epsilon \in (0, \infty)$. Since X is compact and 0-dimensional, there exists a covering $\{B_i\}_{i=1}^n$ of X by mutually disjoint clopen subsets with $\delta_d(B_i) \leq \epsilon$. Since each B_i is 0-dimensional, there exists a uniformly disconnected metric $e_i \in \mathrm{M}(B_i)$ with $\delta_{e_i}(B_i) \leq \epsilon$. Applying Proposition 3.1 to d and $\{e_i\}_{i=1}^n$, we obtain $D \in \mathrm{M}(X)$ with $\mathcal{D}_X(D, d) \leq 4\epsilon$. By Lemma 3.4, the metric D is uniformly disconnected. Thus, by Lemma 3.5, the set of all uniformly disconnected metrics is dense F_σ in $\mathrm{M}(X)$.

We prove the converse. If there exists a uniformly disconnected metric in $\mathrm{M}(X)$, then X is 0-dimensional. For the sake of contradiction, suppose that X is not compact. Then there exists a countable closed discrete subset F of X. Identify F with $\mathbb{Z}$, and let e be the relative Euclidean metric on $\mathbb{Z}$ $(= F)$. By Theorem 2.3, there exists $D \in \mathrm{M}(X)$ such that $D|_{F^2} = e$. Let U be the open ball centered at D with radius $1/2$ in $(\mathrm{M}(X), \mathcal{D}_X)$ Take $d \in U$. Since e is not uniformly disconnected, for every $\delta \in (0, 1)$ there exists a non-constant finite sequence $\{z_i\}_{i=1}^N$ in F with

$$\max_{1 \leq i \leq N} D(z_i, z_{i+1}) < 4\delta D(z_1, z_N).$$

Since $\mathcal{D}_X(d, D) < 1/2$, and since $1 \leq D(z_1, z_Z)$ and $1 \leq \max_{1 \leq i \leq N} D(z_i, z_{i+1})$, we have $D(z_1, z_N) \leq 2d(z_1, z_N)$ and

$$\max_{1 \leq i \leq N} d(z_i, z_{i+1}) \leq 2 \max_{1 \leq i \leq N} D(z_i, z_{i+1}) < \max_{1 \leq i \leq N} d(z_i, z_{i+1}) < \delta d(z_1, z_N).$$

This implies that d is not uniformly disconnected. Thus, the open subset U consists of non-uniformly-disconnected metrics, and hence the set of all uniformly disconnected metrics is not dense in $\mathrm{M}(X)$. This finishes the proof. ∎

Similarly to Lemma 3.4, we obtain:

Lemma 3.6. *Let S be a range set. Let (X, d) be an S-valued ultrametric space. Let $\{B_i\}_{i=1}^n$ be a covering of X consisting of mutually disjoint clopen subsets, and let $\{p_i\}_{i=1}^n$ be points with $p_i \in B_i$. Let $\{e_i\}_{i=1}^n$ be metrics such that $e_i \in \mathrm{UM}(B_i, S)$. Let D be the metric constructed in Proposition* 3.2 *from d and $\{e_i\}_{i=1}^n$. Then for $k \in \{1, 3\}$, the following hold true:*

(1) *If each e_i satisfies $\mathcal{P}_k$, then so does (X, D).*
(2) *If some e_i does not satisfy $\mathcal{P}_k$, then neither does (X, D).*

The author [10] proved the extension theorem on ultrametrics, which is an ultrametric analogue of Theorem 2.3 (see [10, Theorem 1.2]).

Theorem 3.7. *Let S be a range set. Let X be a topological space with $\mathrm{UM}(X, S) \neq \emptyset$, and let A be a closed subset of X. Then for every $d \in \mathrm{UM}(A, S)$ there exists $D \in \mathrm{UM}(X, S)$ with $D|_{A^2} = d$.*

By [10, Corollary 6.4 and Proposition 6.8], we obtain:

LEMMA 3.8. *Let S be a range set. For a topological space X, the set of all doubling metrics in* $\mathrm{UM}(X,S)$ *is* F_σ *in* $\mathrm{UM}(X,S)$.

Proof of Theorem 1.3. By Lemma 3.8, the set of all doubling metrics in $\mathrm{UM}(X,S)$ is F_σ in $\mathrm{UM}(X,S)$. Assume first that X is compact. By [10, Proposition 2.12], we have $\mathrm{UM}(X,S) \neq \emptyset$. Take arbitrary $d \in \mathrm{UM}(X,S)$ and $\epsilon \in (0,\infty)$. Since X is compact and 0-dimensional, there exists a disjoint covering $\{B_i\}_{i=1}^n$ by clopen subsets with $\delta_d(B_i) \leq \epsilon$. Then there exists a doubling metric $e_i \in \mathrm{UM}(B_i,S)$ with $\delta_{e_i}(B_i) \leq \epsilon$. Applying Proposition 3.2 to d and $\{e_i\}_{i=1}^n$, we obtain $D \in \mathrm{UM}(X,S)$ with $\mathcal{UD}_X^S(D,d) \leq \epsilon$. By Lemma 3.6, the metric D is doubling. Thus, the set of all doubling metrics in $\mathrm{UM}(X,S)$ is dense in $\mathrm{UM}(X,S)$.

We next prove the converse. Similarly to the proof of Theorem 1.1, by using Theorem 3.7, we conclude that if the set of all doubling metrics in $\mathrm{UM}(X,S)$ is dense in $\mathrm{UM}(X,S)$, then X is compact. ∎

4. Uniform perfectness. Fix a countable dense subset P of Γ. For $c \in (0,1)$, let $K(c)$ denote the set of all $d \in \mathrm{M}(\Gamma)$ such that for every $r \in (0,1)\cap\mathbb{Q}$, and for every $x \in P$, there exists $y \in P$ satisfying $cr \leq d(x,y) \leq r$. Let K denote the set of all uniformly perfect metrics in $\mathrm{M}(\Gamma)$.

LEMMA 4.1. *The set K is an* F_σ *subset of* $\mathrm{M}(\Gamma)$.

Proof. By the definitions, we have $K = \bigcup_{c\in(0,1)\cap\mathbb{Q}} K(c)$. For every $c \in (0,1)\cap\mathbb{Q}$, we prove that $\mathrm{CL}(K(c))$ is contained in $K(c/4)$, where CL is the closure operator of $\mathrm{M}(\Gamma)$. Take $d \in \mathrm{CL}(K(c))$, and take a sequence $\{d_n\}_{n\in\mathbb{Z}_{\geq 0}}$ in $K(c)$ such that $d_n \to d$ as $n \to \infty$ in $(\mathrm{M}(\Gamma), \mathcal{D}_\Gamma)$. Then for every $n \in \mathbb{Z}_{\geq 0}$, every $r \in (0,1)\cap\mathbb{Q}$, and every $x \in P$, there exists $y(n,r,x) \in P$ such that $cr/2 \leq d_n(x, y(n,r,x)) \leq r/2$. Since $(0,1)\cap\mathbb{Q}$ and P are countable, and since Γ is compact, we can apply Cantor's diagonal argument to $y(n,r,x)$, and hence obtain a strictly increasing map $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ such that for every $r \in (0,1)\cap\mathbb{Q}$ and every $x \in P$, the sequence $\{y(\phi(n),r,x)\}_{n\in\mathbb{Z}_{\geq 0}}$ converges to a point in X, say $z(r,x)$. Take $p(r,x) \in P$ with $d(p(r,x), z(r,x)) \leq cr/4$. Letting $n \to \infty$, we have $cr/2 \leq d(x, z(r,x)) \leq r/2$, and hence $cr/4 \leq d(x, p(r,x)) \leq r$. Thus, $d \in K(c/4)$, and hence $\mathrm{CL}(K(c))$ is contained in $K(c/4)$. By this observation, we conclude that $K = \bigcup_{c\in(0,1)\cap\mathbb{Q}} \mathrm{CL}(K(c))$. Thus K is F_σ in $\mathrm{M}(\Gamma)$. ∎

Similarly to Lemma 4.1, we obtain:

LEMMA 4.2. *Let S be a range set. The set* $K \cap \mathrm{UM}(\Gamma,S)$ *is* F_σ *in* $\mathrm{UM}(\Gamma,S)$.

Proof of Theorem 1.4. By Lemma 4.1, it suffices to show that K is dense in $\mathrm{M}(\Gamma)$. Take arbitrary $d \in \mathrm{M}(\Gamma)$ and $\epsilon \in (0, \infty)$. Since Γ is 0-dimensional and compact, there exists a covering $\{B_i\}_{i=1}^{n}$ by mutually disjoint clopen non-empty subsets with $\delta_d(B_i) \leq \epsilon$. Note that each B_i is a Cantor space (see [14, Corollary 30.4]). Identify B_i and Γ, and let $e_i \in \mathrm{M}(B_i)$ be the metric identified with $\epsilon \cdot E$, where E is the relative Euclidean metric on Γ. Then each e_i is uniformly perfect and satisfies $\delta_{e_i}(B_i) \leq \epsilon$. Applying Proposition 3.1 to d and $\{e_i\}_{i=1}^{n}$, we obtain $D \in \mathrm{M}(\Gamma)$ with $\mathcal{D}_\Gamma(d, D) \leq 4\epsilon$. Lemma 3.4 implies that D is uniformly perfect. Thus K is dense in $\mathrm{M}(\Gamma)$. This finishes the proof. ∎

LEMMA 4.3. *Let S be an exponential range set. Then there exists a strictly decreasing sequence $\{s(n)\}_{n \in \mathbb{Z}_{\geq 0}}$ in S with the property that there exist $a \in (0,1)$ and $M \in [1, \infty)$ such that $M^{-1}a^n \leq s(n) \leq Ma^n$ for every $n \in \mathbb{Z}_{\geq 0}$.*

Proof. By the assumption on S, there exist $b \in (0,1)$ and $M \in [1, \infty)$ such that $[M^{-1}b^n, Mb^n] \cap S \neq \emptyset$ for every $n \in \mathbb{Z}_{\geq 0}$. Put $p = -\log M / \log b$ and $a = b^{2p+1}$. Then $[M^{-1}a^n, Ma^n] \cap S \neq \emptyset$ and $Ma^{n+1} < M^{-1}a^n$ for all $n \in \mathbb{Z}_{\geq 0}$. This leads to the lemma. ∎

A sequence $s : \mathbb{Z}_{\geq 0} \to (0, \infty)$ said to be *shrinking* if it is a strictly decreasing sequence in $(0, \infty)$ convergent to 0. For a shrinking sequence $s : \mathbb{Z}_{\geq 0} \to (0, \infty)$ and $m \in \mathbb{Z}_{\geq 0}$, define $s^{\{m\}} : \mathbb{Z}_{\geq 0} \to (0, \infty)$ by $s^{\{m\}}(n) = s(m + n)$. Then $s^{\{m\}}$ is shrinking.

Let 2^ω be the set of all maps from $\mathbb{Z}_{\geq 0}$ into $\{0, 1\}$. Define a valuation $v : 2^\omega \times 2^\omega \to [0, \infty]$ by $v(x, y) = \min \{ n \in \mathbb{Z}_{\geq 0} \mid x(n) \neq y(n) \}$ if $x \neq y$; otherwise $v(x, y) = \infty$. Let s be a shrinking sequence. Put $s(\infty) = 0$. Define a metric d_s on 2^ω by $d_s(x, y) = s(v(x, y))$. Then for every shrinking sequence $s : \mathbb{Z}_{\geq 0} \to (0, \infty)$ and every $m \in \mathbb{Z}_{\geq 0}$, the metric space $(2^\omega, d_{s^{\{m\}}})$ is a Cantor space. Note that for every $x \in 2^\omega$ and every $n \in \mathbb{Z}_{\geq 0}$, there exists $d_s(x, y) = s(n)$. The metric space $(2^\omega, d_s)$ is called a *sequentially metrized Cantor space* in the author's paper [8], and the author investigated the doubling property, uniform disconnectedness, and uniform perfectness of such spaces. The following lemma is essentially contained in [8, Lemma 6.4].

LEMMA 4.4. *Let S be a range set. Let $s : \mathbb{Z}_{\geq 0} \to S$ be a shrinking sequence in S. If there exist $a \in (0, 1)$ and $M \in [1, \infty)$ such that $M^{-1}a^n \leq s(n) \leq Ma^n$, then for every $m \in \mathbb{Z}_{\geq 0}$, the metric space $(2^\omega, d_{s^{\{m\}}})$ is uniformly perfect.*

Proof. It suffices to handle the case of $m = 0$. Put $c = M^{-2}a$. Take arbitrary $x \in 2^\omega$. For every $r \in (0, \infty)$, take $n \in \mathbb{Z}_{\geq 0}$ such that $s(n+1) < r \leq s(n)$. Take $y \in 2^\omega$ with $d(x, y) = s(n+1)$. Then $cr \leq d(x, y) \leq r$. This implies that $(2^\omega, d_s)$ is uniformly perfect. ∎

Proof of Theorem 1.5. Similarly to Theorem 1.3, statement (2) follows from Proposition 3.2 and Lemmas 3.6 and 4.2.

We now prove (1). Assume first that S is exponential. Then by Lemma 4.3, there exist $a \in (0,1)$ and $M \in [1,\infty)$ and a strictly decreasing sequence $\{s(n)\}_{n\in\mathbb{Z}_{\geq 0}}$ in S such that $M^{-1}a^n \leq s(n) \leq Ma^n$. Take arbitrary $d \in \mathrm{UM}(\Gamma, S)$ and $\epsilon \in (0,\infty)$. Then there exists a covering $\{B_i\}_{i=1}^N$ of Γ consisting of mutually disjoint non-empty clopen subsets of with $\delta_d(B_i) \leq \epsilon$. Note that each B_i is a Cantor space. For a sufficiently large $m \in \mathbb{Z}_{\geq 0}$, the space $(2^\omega, d_{s\{m\}})$ satisfies $\delta_{d_{s\{m\}}}(2^\omega) \leq \epsilon$. Identify B_i and 2^ω, and let $e_i \in \mathrm{UM}(B_i, S)$ be the metric identified with $d_{s\{m\}}$. By applying Proposition 3.2 to $\{e_i\}_{i=1}^N$ and d, we get an S-valued ultrametric D in $\mathrm{UM}(\Gamma, S)$ with $\mathcal{UD}_X^S(D,d) \leq \epsilon$. By Lemmas 3.4 and 4.4, the metric D is uniformly perfect. This leads to the first part of statement (1). Assume next that S is not exponential. We now prove that no $d \in \mathrm{UM}(\Gamma, S)$ is uniformly perfect. Take arbitrary $c \in (0,1)$. Since S is not exponential, there exists $n \in \mathbb{Z}_{\geq 0}$ with $[c^{n+1}, c^{n-1}] \cap S = \emptyset$. Take any $x \in \Gamma$. Put $r = c^{n-1}$. Then every $y \in \Gamma$ satisfies $d(x,y) \leq cr$ or $r \leq d(x,y)$. Thus, no $d \in \mathrm{UM}(\Gamma, S)$ is uniformly perfect. This finishes the proof. ∎

Before proving Theorem 1.6, we remark that each $\mathrm{E}(u,v,w)$ contains a metric with arbitrarily small diameter. This follows from the facts that $\mathrm{E}(u,v,w)$ is not empty (see [8, Theorem 1.2]), and that for every $(u,v,w) \in \{0,1\}^3$ and every $\epsilon \in (0,\infty)$, if $d \in \mathrm{E}(u,v,w)$, then $\epsilon d \in \mathrm{E}(u,v,w)$.

Proof of Theorem 1.6. Fix $(u,v,w) \in \{0,1\}^3$. We now prove that $\mathrm{E}(u,v,w)$ is dense in $\mathrm{M}(\Gamma)$. Take arbitrary $d \in \mathrm{M}(\Gamma)$ and $\epsilon \in (0,\infty)$. As Γ is compact and 0-dimensional, there exists a covering $\{B_i\}_{i=1}^n$ of Γ consisting of mutually disjoint non-empty clopen subsets with $\delta_d(B_i) \leq \epsilon$. Since the set $\mathrm{E}(u,v,w)$ contains a metric with arbitrarily small diameter, there exists $e_i \in \mathrm{M}(B_i)$ of type (u,v,w) with $\delta_{e_i}(B_i) \leq \epsilon$. Applying Proposition 3.1 to d and $\{e_i\}_{i=1}^n$, we obtain $D \in \mathrm{M}(\Gamma)$ with $\mathcal{D}_\Gamma(D,d) \leq 4\epsilon$. By Lemma 3.4, the metric D is of type (u,v,w). Thus, $\mathrm{E}(u,v,w)$ is dense in $\mathrm{M}(\Gamma)$. For each $k \in \{1,2,3\}$, let $W(k,1)$ denote the set of all metrics satisfying property $\mathcal{P}_k$ in $\mathrm{M}(\Gamma)$, and put $W(k,0) = \mathrm{M}(\Gamma) \setminus W(k,1)$. By Lemmas 2.4, 3.5, and 4.1, for all $k \in \{1,2,3\}$, the sets $W(k,1)$ and $W(k,0)$ are F_σ and G_δ in $\mathrm{M}(\Gamma)$, respectively. Thus, for all $(u,v,w) \in \{0,1\}^3$, we have

$$\mathrm{E}(u,v,w) = W(1,u) \cap W(2,v) \cap W(3,w). \tag{4.1}$$

By (4.1), the sets $\mathrm{E}(1,1,1)$ and $\mathrm{E}(0,0,0)$ are F_σ and G_δ, respectively. If $(u,v,w) \in \{0,1\}^3$ is neither $(0,0,0)$ nor $(1,1,1)$, then by (4.1), the set $\mathrm{E}(u,v,w)$ is the intersection of an F_σ set and a G_δ set in $\mathrm{M}(\Gamma)$. Since $\mathrm{M}(\Gamma)$ is a metrizable space, the set $\mathrm{E}(u,v,w)$ is $F_{\sigma\delta}$ and $G_{\delta\sigma}$ in $\mathrm{M}(\Gamma)$. This finishes the proof. ∎

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Yoshito Ishiki
Graduate School of Pure and Applied Sciences
University of Tsukuba
Tennodai 1-1-1, Tsukuba, Ibaraki, 305-8571, Japan

Current address:
Photonics Control Technology Team
RIKEN Center for Advanced Photonics
2-1 Hirasawa, Wako, Saitama 351-0198, Japan
E-mail: yoshito.ishiki@riken.jp