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PSEUDO-HOMOTOPIES BETWEEN MAPS ON g-GROWTH HYPERSPACES OF CONTINUA

BY

FÉLIX CAPULÍN, ENRIQUE CASTAÑEDA-ALVARADO, LEONARDO JUÁREZ-VILLA and DAVID MAYA (Toluca)

Abstract. We introduce the concept of g-growth hyperspace: if X is a continuum, then a non-empty subset $\mathcal{H}$ of 2^X is a *g-growth hyperspace of* X provided that if $\mathcal{A}$ is a subcontinuum of 2^X and $\mathcal{A}\cap\mathcal{H}\neq\emptyset$, then $\bigcup\mathcal{A}\in\mathcal{H}$. We study pseudo-homotopies between maps of hyperspaces of continua. As a consequence, we show that pseudo-contractibility and contractibility are equivalent in g-growth hyperspaces.

1. Introduction. R. H. Bing introduced the notions of pseudo-homotopy and pseudo-contractibility. However, W. Kuperberg was the first to give an example to show that these concepts are different from homotopy and contractibility, respectively.

On the other hand, it is well known that the study of hyperspaces can provide information about the topological behavior of the original space and vice versa. In continuum theory, a large number of results on this topic can be found in [14] and [18].

In this paper, we will give general facts about pseudo-homotopies between maps of some hyperspaces of continua. As a consequence, we find that the concepts of pseudo-homotopy and homotopy coincide for maps on some hyperspaces. In particular, these concepts are equivalent for induced maps of g-growth hyperspaces of continua. This implies, among other things, that pseudo-contractibility and contractibility are the same in g-growth hyperspaces. This equivalence was proved in [6] for the hyperspaces $2^X, C_n(X)$ and $C_\infty(X)$ using other techniques. For more about pseudo-homotopies and pseudo-contractibility, the reader is referred to [4, 5, 6, 11, 13, 15, 19, 20].

This paper is organized as follows. After some preliminaries, in Section 3 we study pseudo-homotopies between maps on hyperspaces of continua, and pseudo-homotopies between maps of continua and their relation-

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ship to some induced maps. In Section 4, we generalize some results about contractibility and pseudo-contractibility in hyperspaces of continua. In Section 5, we give relationships between pseudo-homotopically equivalent spaces and some of their hyperspaces. In Section 6, we present results about pseudo-contractibility on products. Finally, in Section 7, we exhibit some classes of maps preserving contractibility of hyperspaces and we formulate some questions about contractibility of hyperspaces.

2. Preliminaries. Throughout this paper a *continuum* means a non-empty compact connected metric space, and a *map* means a continuous function. An *arc* is a homeomorphic image of $I = [0, 1] \subset \mathbb{R}$. If any two points of a space can be joined by an arc, the space is called *arcwise connected*. The symbol $\mathbb{N}$ will denote the set of all positive integers.

Given a continuum X, the following hyperspaces are well known:

$$\begin{aligned} 2^X &= \{A \subseteq X : A \text{ is non-empty and closed}\}, \\ C(X) &= \{A \in 2^X : A \text{ is connected}\}, \\ C_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ components}\} \quad \text{for } n \in \mathbb{N}, \\ C_\infty(X) &= \{A \in 2^X : A \text{ has finitely many components}\}, \\ F_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ points}\} \quad \text{for } n \in \mathbb{N}, \\ F_\infty(X) &= \{A \in 2^X : A \text{ contains finitely many points}\}. \end{aligned}$$

These hyperspaces are topologized with the *Hausdorff metric*. In particular, $F_1(X)$, the hyperspace of singletons of X, is homeomorphic to X. Notice that $C_1(X) = C(X)$, $F_1(X) \subset C(X) \subset C_n(X) \subset C_\infty(X) \subset 2^X$, $F_1(X) \subset F_n(X) \subset F_\infty(X) \subset C_\infty(X) \subset 2^X$, and $F_n(X) \subset C_n(X)$ for each $n \in \mathbb{N}$.

From now on, given a continuum X, a hyperspace of X contained in 2^X and defined using a set of topological properties will be denoted by $H(X)$.

LEMMA 2.1. *Let X, K be continua. The map $e' : 2^X \times 2^K \to 2^{X\times K}$ defined by $e'((A, B)) = A \times B$ is an embedding fulfilling:*

(1) $e'(H(X) \times H(K)) \subset H(X \times K)$ *for* $H(-) \in \{C(-), C_\infty(-), F_\infty(-)\}$.
(2) $e'(C_n(X) \times C_m(K)) \subset C_{nm}(X \times K)$ *for all* $n, m \geq 1$.
(3) $e'(F_n(X) \times F_m(K)) \subset F_{nm}(X \times K)$ *for all* $n, m \geq 1$.
(4) $e'(H(X) \times F_1(K)) \subset H(X \times K)$ *for* $H(-) \in \{C_n(-), C_\infty(-), F_\infty(-)\}$.
(5) $e'(F_1(X) \times H(K)) \subset H(X \times K)$ *for* $H(-) \in \{C_n(-), C_\infty(-), F_\infty(-)\}$.

Proof. Notice that e' is well defined, because if A, B are closed subsets of X and K respectively, then $A \times B$ is a closed subset of $X \times K$. The continuity of e' follows from [17, formula 9, p. 339]. So, to prove that e' is an embedding, it is enough to prove that e' is one-to-one. If $(A, B) \neq (A', B')$, then either $A \neq A'$ or $B \neq B'$. So $A \times B \neq A' \times B'$.

The proofs of (1)–(5) are immediate consequences of the definitions of the product and of each hyperspace. ■

Now, we will introduce a class of hyperspaces contained in 2^X which includes some traditional hyperspaces of a continuum.

DEFINITION 2.2. Let X be a continuum. A nonempty subset $\mathcal{H}$ of 2^X is called a *g-growth hyperspace* of X provided that if $\mathcal{A} \in C(2^X)$ and $\mathcal{A} \cap \mathcal{H} \neq \emptyset$, then $\bigcup \mathcal{A} \in \mathcal{H}$.

Observe that every g-growth hyperspace contains X, and the intersection of an arbitrary family of g-growth hyperspaces is also a g-growth hyperspace. Each growth hyperspace (see [9, p. 271]) is a g-growth hyperspace. By [18, Lemmas 1.48 and 1.49] and [8, Proposition 3.1], 2^X, $C(X)$, $C_n(X)$, and $C_\infty(X)$ are g-growth hyperspaces. Also, each hyperspace in the following list is a g-growth hyperspace:

- $\{A \in 2^X : \text{int}(A) \neq \emptyset\}$;
- $2^X_p = \{A \in 2^X : p \in A\}$, $C(X,p) = \{A \in C(X) : p \in A\}$, $C_n(X,p) = \{A \in C_n(X) : p \in A\}$ for any $p \in X$ and $n \in \mathbb{N}$;
- $\mu^{-1}([t,1])$ for each Whitney map $\mu : 2^X \to [0,1]$ and $t \in [0,1]$;
- $\langle U_1, \dots, U_m \rangle$ (Vietoris set), where $U_1, \dots, U_m$ are subsets of X with $X = \bigcup_{k=1}^m U_k$;
- $\{B \in 2^X : H(X,B) \leq \varepsilon\}$ for each $\varepsilon > 0$.

Next, we present some results concerning g-growth hyperspaces.

PROPOSITION 2.3. *Let X be a continuum, let $\mathcal{H}$ be a g-growth hyperspace of X, let μ be a Whitney map for $C(X)$, and let $t \in [0,1]$. Then $\mu^{-1}(t) \subseteq \mathcal{H}$ if and only if $\mu^{-1}([t,1]) \subseteq \mathcal{H}$.*

Proof. Suppose that $\mu^{-1}(t) \subseteq \mathcal{H}$. Let $A \in \mu^{-1}((t,1])$. From the fact that $\mu|_{C(A)}$ is a Whitney map for $C(A)$, it follows that $(\mu|_{C(A)})^{-1}(t) = \{B \in \mu^{-1}(t) : B \subseteq A\} \in C(2^X)$. Since $\mathcal{H} \cap \{B \in \mu^{-1}(t) : B \subseteq A\} \neq \emptyset$, we find that $A = \bigcup\{B \in \mu^{-1}(t) : B \subseteq A\} \in \mathcal{H}$. The converse implication is immediate. ■

The next result follows from the above proposition and the fact that $F_1(X) = \mu^{-1}(0)$ for each Whitney map μ for $C(X)$.

COROLLARY 2.4. *Let X be a continuum and let $\mathcal{H}$ be a g-growth hyperspace of X. Then $F_1(X) \subseteq \mathcal{H}$ if and only if $C(X) \subseteq \mathcal{H}$.*

THEOREM 2.5. *Each g-growth hyperspace of a continuum is arcwise connected.*

Proof. Let $\mathcal{H}$ be a g-growth hyperspace of a continuum X. It is enough to show that an arc from any element of $\mathcal{H}$ to X is contained in $\mathcal{H}$. Let $G \in \mathcal{H}$. According to [14, proof of Theorem 14.9, p. 113], there exists an order arc

$\alpha : [0,1] \to 2^X$ from G to X. Observe that if $t \in [0,1]$, then $\alpha([0,t]) \in C(2^X)$, $G \in \mathcal{H} \cap \alpha([0,t])$, and $\alpha(t) = \bigcup \alpha([0,t])$. Since $\mathcal{H}$ is a g-growth hyperspace, $\alpha([0,1])$ is contained in $\mathcal{H}$. So, $\mathcal{H}$ is arcwise connected. ∎

A connected topological space Z has *property* (b) provided that for each map f from Z into the unit circle S^1 in the Euclidean plane, there exists a map $h : Z \to \mathbb{R}$ such that $f = \exp \circ h$, where $\exp : \mathbb{R} \to S^1$ is defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$. It is well known that property (b) is a topological property and that if X is a compact metric space, then X has property (b) if and only if each mapping from X to S^1 is homotopic to a constant mapping (see [21, Theorem 6.2, p. 225]).

Recall that a connected space X is *unicoherent* provided that if A and B are connected closed subsets of X such that $X = A \cup B$, then $A \cap B$ is connected. It is known that each connected metric space having property (b) is unicoherent (see [21, Theorem 7.3, p. 227]).

THEOREM 2.6. *Each g-growth hyperspace of a continuum has property* (b).

Proof. Let $\mathcal{H}$ be a g-growth hyperspace of a continuum. Letting $\mathcal{K}_A = \{B \in \mathcal{H} : A \subseteq B\}$ for each $A \in \mathcal{H}$, we can use similar arguments to those in [1, proof of Lemma 13, p. 2004] to prove that $\mathcal{H}$ has property (b). ∎

From this result and [21, Theorem 6.2, p. 225] we infer that each mapping from a g-growth hyperspace of a continuum to S^1 is homotopic to a constant mapping. On the other hand, in Corollary 4.4, we will prove that if a continuum X is pseudo-contractible, then each hyperspace $H(X)$ having property e is pseudo-contractible. So, every mapping from $H(X)$ to a polyhedron is homotopically trivial. In the case where $H(X)$ is a compact pseudo-contractible space, it has trivial shape. It is known that if X is a continuum, the g-growth hyperspaces 2^X and $C(X)$ have trivial shape (see for instance [14, Theorem 19.10, p. 160]). We do not know if every mapping from a g-growth hyperspace of a continuum to a polyhedron is homotopically trivial; in particular, we do not know if every compact g-growth hyperspace has trivial shape. Note that there are hyperspaces that do not have trivial shape, for example $F_2(S^1)$.

COROLLARY 2.7. *Each g-growth hyperspace of a continuum is unicoherent.*

Proof. Let $\mathcal{H}$ be a g-growth hyperspace of a continuum. First, by Proposition 2.5 we find that $\mathcal{H}$ is connected. Then, we apply Theorem 2.6 and [21, Theorem 7.3, p. 227] to conclude that $\mathcal{H}$ is unicoherent. ∎

We end this section with the following question.

QUESTION 2.8. *Is every mapping from a g-growth hyperspace of a continuum to a polyhedron homotopically trivial? In particular, does each compact g-growth hyperspace have trivial shape?*

3. Pseudo-homotopies between maps on hyperspaces. Let X and Y be topological spaces, and let I be the unit interval. Two maps $f,\ g : X \to Y$ are called *homotopic* (written $f \simeq g$) if there exists a map $G : X \times I \to Y$ (called a *homotopy*) satisfying $G(x, 0) = f(x)$ and $G(x, 1) = g(x)$ for each $x \in X$. We say that f and g are *pseudo-homotopic* if there exist a continuum K, points $a, b \in K$ and a map $G : X \times K \to Y$ such that $G(x, a) = f(x)$ and $G(x, b) = g(x)$ for each $x \in X$. The map G is called a *pseudo-homotopy* between f and g with factor space K. We then write $f \simeq_K g$.

REMARK 3.1. If two maps are homotopic, then they are pseudo-homotopic.

The converse of Remark 3.1 is not true, as shown by W. Kuperberg.

REMARK 3.2. If the factor space K is arcwise connected, then $f \simeq_K g$ is equivalent to $f \simeq g$. This fact will be helpful throughout this paper.

In this section, we will give some results concerning pseudo-homotopic maps between hyperspaces.

THEOREM 3.3. *Let X, Y and K be continua, and let $\mathcal{H}$ be a g-growth hyperspace of Y. If $G : X \times K \to \mathcal{H}$ is a map, then the function $\hat{G} : X \times C(K) \to \mathcal{H}$ defined by $\hat{G}(x, B) = \bigcup G(\{x\} \times B)$ satisfies the following conditions:*

(1) *$\hat{G}$ is well defined and continuous.*
(2) *$\hat{G}(x, \{t\}) = G(x, t)$ for each $(x, t) \in X \times K$.*
(3) *If $t \in T \in C(K)$ and $G(x, t) \in \mathcal{H}'$ for some g-growth hyperspace $\mathcal{H}'$ of Y contained in $\mathcal{H}$, then $\hat{G}(x, T) \in \mathcal{H}'$.*

Proof. From the continuity of G together with [18, Lemma 1.48] and the definition of g-growth hyperspace we obtain (1). Condition (2) is immediate from the definition of $\hat{G}$. Condition (3) follows from the definition of g-growth hyperspace. ∎

THEOREM 3.4. *Let X and Y be continua, let $\mathcal{H}$ be a g-growth hyperspace of Y and let $f, g : X \to \mathcal{H}$ be maps. Then f and g are pseudo-homotopic if and only if f and g are homotopic.*

Proof. Assume that f and g are pseudo-homotopic. Then there exist a continuum K, points $a, b \in K$ and a map $G : X \times K \to \mathcal{H}$ such that $G(x, a) = f(x)$ and $G(x, b) = g(x)$ for each $x \in X$. So, by Theorem 3.3, the map $\hat{G} : X \times C(K) \to \mathcal{H}$ satisfies $\hat{G}(x, \{a\}) = f(x)$ and $\hat{G}(x, \{b\}) = g(x)$ for each $x \in X$. Since $C(K)$ is an arcwise connected continuum, by Remark 3.2 it follows that f and g are homotopic. The converse is immediate. ∎

Let X and Y be continua. If $H(X)$ is a hyperspace of X defined through a set $\mathbf{P}$ of topological properties, then the hyperspace $H(Y)$ of Y will be defined by the same set of properties. That is, if there is a set $\mathbf{P}$ of topological properties such that $H(X) = \{A \in 2^X : A \text{ satisfies } \mathbf{P}\}$, then $H(Y) = \{B \in 2^Y : B \text{ satisfies } \mathbf{P}\}$. We will only mention $\mathbf{P}$ if there is risk of confusion.

Let X and Y be continua. Let $H(X)$ and $H(Y)$ be hyperspaces of X and Y respectively. If for every map $f : X \to Y$, the image $f(A) \in H(Y)$ for each $A \in H(X)$ and this assignment is continuous, we say that $H(X)$ *admits an H-induced map*. So, the H-induced map of f is represented by $H(f) : H(X) \to H(Y)$ and defined by $H(f)(A) = f(A)$. By [14, Lemma 13.3, p. 106] the hyperspaces 2^X, $C(X)$, $C_n(X)$, $C_\infty(X)$, $F_n(X)$ and $F_\infty(X)$ admit induced maps. Usually, we use the notation 2^f, $C(f)$, $C_n(f)$, $C_\infty(f)$, $F_n(f)$ and $F_\infty(f)$ to represent the induced maps between the corresponding spaces. Throughout this work, if Z is a continuum, every hyperspace $H(Z)$ admits H-induced maps. We say that the hyperspace $H(X)$ has *property e* if the function $e : H(X) \times K \to H(X \times K)$ given by $e(A, x) = A \times \{x\}$ is an embedding. For example, the hyperspaces 2^X, $C(X)$, $C_n(X)$, $C_\infty(X)$, $F_n(X)$ and $F_\infty(X)$ have property e.

THEOREM 3.5. *Let X, Y and K be continua, let $H(X)$ be a hyperspace having property e and let $G : X \times K \to Y$ be a map. Define $\tilde{G} : H(X) \times K \to H(Y)$ by $\tilde{G}((A, t)) = G(A \times \{t\})$. Then the function $\tilde{G}$ is well defined and continuous.*

Proof. The map $\tilde{G}$ is clearly well defined, and it is continuous because $\tilde{G} = H(G) \circ e$, where e is the embedding from $H(X) \times K$ to $H(X \times K)$ defined above. ∎

Now, under the assumption that every hyperspace involved has property e, we will show that if two maps between continua are pseudo-homotopic, their induced maps are pseudo-homotopic too.

THEOREM 3.6. *Let X and Y be continua, and let $H(X)$ be a hyperspace having property e. Then the H-induced maps of pseudo-homotopic maps are pseudo-homotopic and if $H(Y)$ is a g-growth hyperspace, then the H-induced maps are homotopic.*

Proof. Let $f, g : X \to Y$ be pseudo-homotopic. Then there exist a continuum K, points $a, b \in K$ and a map $G : X \times K \to Y$ such that $G(x, a) = f(x)$ and $G(x, b) = g(x)$ for each $x \in X$. Theorem 3.5 guarantees that there exists a map $\tilde{G}$ which satisfies $\tilde{G}(S, a) = G(S \times \{a\}) = f(S) = H(f)(S)$ and $\tilde{G}(S, b) = G(S \times \{b\}) = g(S) = H(g)(S)$. Then $\tilde{G}$ is a pseudo-homotopy between $H(f)$ and $H(g)$ with factor space K.

If $H(Y)$ is a g-growth hyperspace, we apply Theorem 3.4 to deduce that $H(f)$ and $H(g)$ are homotopic. ∎

Given a topological space Z, we denote by id_Z the identity map of Z. The following result is easy to prove.

THEOREM 3.7. *Let X, Y and Z be continua. If $f : X \to Y$ and $g : Y \to Z$ are maps, and $H(X)$ is a hyperspace of X, then:*

(1) $H(g \circ f) = H(g) \circ H(f)$.
(2) $H(\mathrm{id}_X) = \mathrm{id}_{H(X)}$.

Let X be a continuum and let $G(X)$ and $H(X)$ be two hyperspaces of 2^X such that $G(X) \subseteq H(X)$. A *quotient hyperspace* $Q(X)$ for X is the quotient space $H(X)/G(X)$ obtained by identifying $G(X)$ to a point. We say that a quotient hyperspace $Q(X)$ is a *g-growth quotient hyperspace* for X if $H(X)$ and $G(X)$ are g-growth hyperspaces. Let $\rho_X : H(X) \to Q(X)$ and $\rho_Y : H(Y) \to Q(Y)$ be the quotient maps. The *q-induced map* of a map $f : X \to Y$ between continua is the map $Q(f) : Q(X) \to Q(Y)$ satisfying $Q(f) \circ \rho_X = \rho_Y \circ H(f)$. For instance, if $n, m \in \mathbb{N}$ with $m \leq n$, then $F^n_m(X)$, $HS_n(X)$ and $C^n_m(X)$ denote the quotient spaces $F_n(X)/F_m(X)$, $C_n(X)/F_n(X)$ and $C_n(X)/C_m(X)$, respectively. In these cases we can define q-induced maps. Throughout this work, if X is a continuum, every quotient hyperspace $Q(X)$ admits q-induced maps. We say that the quotient hyperspace $Q(X) = H(X)/G(X)$ *has property* e if $H(X)$ and $G(X)$ have property e.

PROPOSITION 3.8. *Let X, Y, Z be continua and let $Q(X)$ be a quotient hyperspace for X. If $f : X \to Y$ and $g : Y \to Z$ are maps, then:*

(1) $Q(g \circ f) = Q(g) \circ Q(f)$.
(2) $Q(\mathrm{id}_X) = \mathrm{id}_{Q(X)}$.

Proof. For the first part, notice that $Q(g) \circ Q(f) \circ \rho_X = Q(g) \circ \rho_Y \circ H(f) = \rho_Z \circ H(g) \circ H(f)$. On the other hand, $Q(g \circ f) \circ \rho_X = \rho_Z \circ H(g \circ f) = \rho_Z \circ H(g) \circ H(f)$. So, $Q(g \circ f) = Q(g) \circ Q(f)$. For the second part, $Q(\mathrm{id}_X) \circ \rho_X = \rho_X \circ H(\mathrm{id}_X) = \rho_X \circ \mathrm{id}_{H(X)} = \rho_X = \mathrm{id}_{Q(X)} \circ \rho_X$. ∎

THEOREM 3.9. *Let X and Y be continua, and let $Q(X)$ be a quotient hyperspace for X having property e. The q-induced maps of pseudo-homotopic maps are pseudo-homotopic and if $Q(Y)$ is a g-growth quotient hyperspace for Y, then the q-induced maps of pseudo-homotopic maps are homotopic.*

Proof. Let $f, g : X \to Y$ be pseudo-homotopic. Applying similar arguments to the proof of Theorem 3.6, we can see that if K is a continuum, $a, b \in K$ and $M : X \times K \to Y$ is a map satisfying $M(x, a) = f(x)$ and $M(x, b) = g(x)$ for each $x \in X$, then the map $\tilde{M} : H(X) \times K \to H(Y)$ defined by $\tilde{M}(A, t) = M(A \times \{t\})$ satisfies $\tilde{M}(A, a) = H(f)(A)$ and $\tilde{M}(A, b) = H(g)(A)$ for each $A \in H(X)$. Since $G(X)$ has property e, we have $\tilde{M}(G(X) \times K) \subseteq G(Y)$. Thus, the Transgression Lemma [12, Theorem 3.2, p. 123] ensures that there exists a map $L : Q(X) \times K \to Q(Y)$ such that $L(\rho_X(A), t) = \rho_Y(\tilde{M}(A, t))$. Observe that $L(\rho_X(A), a) = \rho_Y(\tilde{M}(A, a)) =$

$\rho_Y(H(f)(A)) = Q(f)(A)$, and $L(\rho_Y(A), b) = Q(g)(A)$ for each $A \in H(X)$. So, $Q(f)$ and $Q(g)$ are pseudo-homotopic.

Assume that $H(Y)$ and $G(Y)$ are g-growth hyperspaces of Y. The function $S : H(X)\times C(K) \to H(Y)$ defined by $S(A, T) = \bigcup \tilde{M}(\{A\}\times T)$ is well defined and continuous, with $S(A, \{a\}) = H(f)(A)$ and $S(A, \{b\}) = H(g)(A)$ for each $A \in H(X)$, and $S(B, \{a\}) = \tilde{M}(B, a) = H(f)(B) = f(B) \in G(Y)$, and $S(B, \{b\}) = \tilde{M}(B, b) = g(B) \in G(Y)$ for each $B \in G(X)$. Now, let $l : [0, 1] \to C(K)$ be a map such that $l(0) = \{a\}$, $l(1) = \{b\}$ and $l(s) \cap \{a, b\} \neq \emptyset$ for each $s \in [0, 1]$. Hence, if $(B, s) \in G(X) \times [0, 1]$, then $\tilde{M}(\{B\} \times l(s)) \in C(2^X)$ and $\tilde{M}(\{B\} \times l(s)) \cap G(Y) \neq \emptyset$. Since $G(Y)$ is a g-growth hyperspace for Y, we have $S(G(X) \times l([0, 1])) \subseteq G(Y)$. Hence, by the Transgression Lemma, there exists a map $R : Q(X) \times [0, 1] \to Q(Y)$ such that $R(\rho_X(A), s) = \rho_Y(S(A, l(s)))$. Observe that $R(\rho_X(A), 0) = \rho_Y(S(A, l(0))) = \rho_Y(S(A, a)) = \rho_Y(H(f)(A)) = Q(f)(\rho_X(A))$ and $R(\rho_X(A), 1) = \rho_Y(S(A, l(1))) = \rho_Y(S(A, b)) = \rho_Y(H(g)(A)) = Q(g)(\rho_X(A))$. We conclude that $Q(f)$ and $Q(g)$ are homotopic. ∎

4. Pseudo-contractibility in hyperspaces. We recall that a topological space X is said to be (*pseudo-*)*contractible* if its identity map is (pseudo-)homotopic to a constant map in X. Notice that every contractible space is pseudo-contractible. A general problem is to determine classes of spaces where both concepts coincide. The following result gives a relevant example; it is a consequence of Theorem 3.4, and it is one of the main results of this paper. Compare with [6, Theorem 3.2 and Corollary 3.5, p. 363].

THEOREM 4.1. *Let X be a continuum and let $\mathcal{H}$ be a g-growth hyperspace of X. Then $\mathcal{H}$ is pseudo-contractible if and only if $\mathcal{H}$ is contractible.*

A subspace Y of X is said to be (*pseudo-*)*contractible in X* if the inclusion map from Y into X is (pseudo-)homotopic to a constant map in X. Observe that each subspace Z of a (pseudo-)contractible subspace Y of a space X is (pseudo-)contractible in X.

The following result extends [16, Lemma 3.1, p. 25].

THEOREM 4.2. *Let X be a continuum and let $\mathcal{H}$ be a g-growth hyperspace of X containing $F_1(X)$. Then the following statements are equivalent:*

(1) 2^X *is contractible.*
(2) $F_1(X)$ *is contractible in* 2^X.
(3) $\mathcal{H}$ *is contractible.*
(4) $F_1(X)$ *is contractible in* $\mathcal{H}$.
(5) $\mathcal{H}$ *is pseudo-contractible.*
(6) $F_1(X)$ *is pseudo-contractible in* $\mathcal{H}$.

Proof. By [16, Lemma 3.1, p. 25] conditions (1) and (2) are equivalent. The equivalence between (3) and (5) is guaranteed by Theorem 4.1. Conditions (4) and (6) follow from (3) and (5), respectively.

Now, assume $\mathcal{H}$ is contractible. Since $F_1(X)$ is a subspace of the contractible subspace $\mathcal{H}$ of 2^X, $F_1(X)$ is contractible in 2^X. So, (3) implies (2).

Now, suppose that 2^X is contractible. Then there exists a map $L : 2^X \times [0,1] \to 2^X$ satisfying $L(A,0) = A$ and $L(A,1) = X$ for each $A \in 2^X$. Define $M : \mathcal{H} \times [0,1] \to \mathcal{H}$ by $M(A,t) = \bigcup L(\{A\} \times [0,t])$ for each $(A,t) \in \mathcal{H} \times [0,1]$. In order to prove that M is well defined, let $(A,t) \in \mathcal{H} \times [0,1]$. The continuity of L implies that $L(\{A\} \times [0,t]) \in C(2^X)$. Since $A \in L(\{A\} \times [0,t]) \cap \mathcal{H}$ and $\mathcal{H}$ is a g-growth hyperspace of X, we get $M(A,t) \in \mathcal{H}$. The continuity of M follows from that of L and [18, Lemma 1.49]. Finally, notice that $M(A,0) = A$ and $M(A,1) = X$ for each $A \in \mathcal{H}$. In conclusion, $\mathcal{H}$ is contractible. So, (1) implies (3).

Finally, assume that $F_1(X)$ is pseudo-contractible in $\mathcal{H}$. Let $i : F_1(X) \to \mathcal{H}$ be the inclusion map. Our assumption guarantees that i is pseudo-homotopic to a constant map. Hence, in light of Theorem 3.4, i is homotopic to a constant map. Then $F_1(X)$ is contractible in $\mathcal{H}$ (cf. [6, Theorem 3.3, p. 363]). So, (6) implies (4). ■

Notice that the condition $F_1(X) \subset \mathcal{H}$ for a g-growth hyperspace $\mathcal{H}$ is necessary for the contractibility of $\mathcal{H}$ to imply the contractibility of 2^X. First, in [2, Theorem 6.3] it is proved that if X is a continuum and $n \in \mathbb{N}$, then the space $C_n(X,K) = \{A \in C_n(X) : K \subset A\}$ is contractible for each $K \in 2^X$. As a consequence, if X is a continuum and $p \in X$, then the hyperspace $C(X,p)$ is contractible. On the other hand, $C(X,p)$ is a g-growth hyperspace of X such that $F_1(X) \not\subset C(X,p)$. Finally, taking a continuum X such that 2^X is not contractible (see [14, p. 158] for the existence of such a continuum), we see that contractibility of a g-growth hyperspace does not imply the contractibilty of 2^X. Another example of this fact is the following.

Example 4.3. *There exists a dendroid X such that $C(X)$ is not contractible but $C(X) \setminus \{\{p\}\}$ is a contractible g-growth hyperspace of X for certain $p \in X$.*

Proof. Let X be a dendroid such that there exist proper subcontinua Y and W of X and $p \in X$ satisfying:

(1) $X = Y \cup W$;
(2) $Y \cap W = \{p\}$;
(3) X does not have the property of Kelley at p (see [14, p. 167] for the definition);
(4) Y and W have the property of Kelley;
(5) $\{p\}$ is an R^3-set of X (see [14, Definition 24.12]).

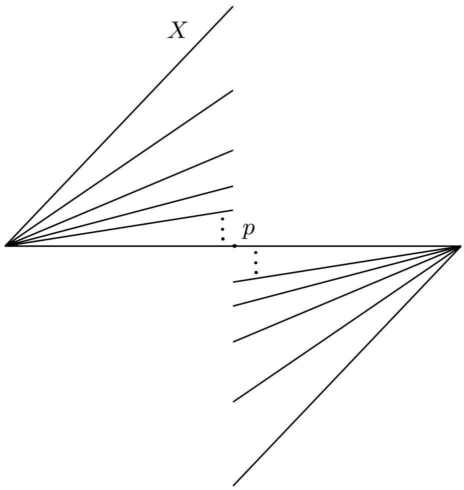

Fig. 1. Dendroid with an R^3-continuum

One example is pictured in Figure 1.

Fix a Whitney map $\mu : C(X) \to [0,1]$ such that $\mu(Y) = \mu(W) = 1/2$. Define $F : C(Y) \times [0,1/2] \to C(Y)$ and $G : C(W) \times [0,1/2] \to C(W)$ by

$$F(A,t) = \bigcup\{B \in C(Y) : \mu(B) = t, B \cap A \neq \emptyset\},$$
$$G(A,t) = \bigcup\{B \in C(W) : \mu(B) = t, B \cap A \neq \emptyset\}.$$

Because Y and W have property of Kelley, the functions F and G are continuous. Set $\mathcal{B} = \{B \in C(X) \setminus \{\{p\}\} : B \cap Y \neq \emptyset, B \cap W \neq \emptyset\}$. Define $L : (C(X) \setminus \{\{p\}\}) \times [0,1] \to C(X) \setminus \{\{p\}\}$ by

$$L(A,t) = \begin{cases} F(A,t) & \text{if } (A,t) \in C(Y) \times [0,1/2], \\ Y \cup G(\{p\}, t-1/2) & \text{if } (A,t) \in C(Y) \times [1/2,1], \\ G(A,t) & \text{if } (A,t) \in C(W) \times [0,1/2], \\ W \cup F(\{p\}, t-1/2) & \text{if } (A,t) \in C(W) \times [1/2,1], \\ F\big(A \cap Y, \frac{\mu(A\cap Y)}{\mu(A)} t\big) \cup G\big(A \cap W, \frac{\mu(A\cap W)}{\mu(A)} t\big) & \text{if } (A,t) \in \mathcal{B} \times [0,1]. \end{cases}$$

Then $L(A,0) = A$ and $L(A,1) = X$ for every $A \in C(X) \setminus \{\{p\}\}$. ■

Similar results to the one below were proved in [19, Corollary, p. 44], and [6, Corollary 3.7] using different tools. Here, they are corollaries of the previous results.

THEOREM 4.4. *If X is a pseudo-contractible continuum and $H(X)$ is a hyperspace having property e, then $H(X)$ is pseudo-contractible, and if $H(X)$ is a g-growth hyperspace for X, then $H(X)$ is contractible.*

Proof. Since X is a pseudo-contractible continuum, id_X is pseudo-homotopic to a constant map in X. By Theorem 3.6, the H-induced map $H(\mathrm{id}_X)$ is pseudo-homotopic to a constant map in $H(X)$. Thus, by Theorem 3.7, $\mathrm{id}_{H(X)}$ is pseudo-homotopic to a constant map, so $H(X)$ is pseudo-contractible.

If $H(X)$ is a g-growth hyperspace of X, then Theorem 4.1 shows that $H(X)$ is contractible. ∎

Recall that a closed subset Z of a space X is a *retract* of X if there exists a map $r : X \to Z$ such that $r|_Z = \mathrm{id}_Z$. The map r is called a *retraction.* We say that Z is a *pseudo-deformation retract* of X if there exists a retraction $r : X \to Z$ and a continuum K such that $r \simeq_K \mathrm{id}_X$. We say that Z is a *strong pseudo-deformation retract* of X if there exist a retraction $r : X \to Z$, a continuum K, points $a, b \in K$ and a map $G : X \times K \to X$ such that $G(x, a) = x$, $G(x, b) = r(x)$ for each $x \in X$, and $G(z, t) = z$ for each $z \in Z$ and each $t \in K$. The map G is called a *strong pseudo-deformation retraction.*

PROPOSITION 4.5. *Let X be a continuum, let Y be a subcontinuum of X and let $\mathcal{H}$ be a g-growth hyperspace of X containing $F_1(X)$ such that $F_1(X)$ is a retract of $\mathcal{H}$. If Y is pseudo-contractible in X, then Y is contractible in X.*

Proof. Consider the following maps: the inclusion map $i : Y \to X$, the embedding $j : X \to \mathcal{H}$ defined by $j(x) = \{x\}$, a retraction $r : \mathcal{H} \to F_1(X)$ and finally the homeomorphism $h : F_1(X) \to X$ defined by $h(\{x\}) = x$. By hypothesis, i is pseudo-homotopic to a constant map in X. Since j is continuous, $j \circ i$ is pseudo-homotopic to a constant map from Y into $\mathcal{H}$. Theorem 3.4 implies that $j \circ i$ is homotopic to a constant map. Hence, $i = h \circ r \circ j \circ i$ is homotopic to a constant map. ∎

The following results show some cases where pseudo-contractibility and contractibility are equivalent, and we will prove some cases where (pseudo-)contractibility of a space implies (pseudo-)contractibility of other ones.

The following proposition generalizes [6, Proposition 3.1].

PROPOSITION 4.6. *Let X be a continuum and let $\mathcal{H}$ be a g-growth hyperspace of X containing $F_1(X)$ such that $F_1(X)$ is a retract of $\mathcal{H}$. Then X is pseudo-contractible if and only if X is contractible.*

Proof. If X is pseudo-contractible, Corollary 4.4 shows that $\mathcal{H}$ is contractible. Since retractions preserve contractibility, $F_1(X)$ is contractible. So, X is contractible. Another proof uses Proposition 4.5 with $Y = X$. The converse is trivial. ∎

The following result is proved in [6, Theorem 3.8].

PROPOSITION 4.7. *Let X be a continuum. Then $F_1(X)$ is pseudo-contractible in $F_\infty(X)$ if and only if $F_\infty(X)$ is pseudo-contractible.*

THEOREM 4.8. *Let X be a continuum and let $H(X)$ be a hyperspace having property e such that $F_1(X) \subset H(X) \subset F_\infty(X)$. If $F_1(X)$ is a retract of $F_\infty(X)$, then the following statements are equivalent:*

(1) *X is (pseudo-)contractible.*
(2) *$H(X)$ is (pseudo-)contractible.*
(3) *$F_\infty(X)$ is (pseudo-)contractible.*

Proof. We prove the pseudo-contractibility version. Corollary 4.4 shows that (1) implies (2). Now, assume (2). Since $F_1(X)$ is contained in the pseudo-contractible subspace $H(X)$ of $F_\infty(X)$, $F_1(X)$ is pseudo-contractible in $F_\infty(X)$. By Proposition 4.7 the hyperspace $F_\infty(X)$ is pseudo-contractible. So, (2) implies (3). Finally, if $F_\infty(X)$ is pseudo-contractible, then our assumption together with [5, Theorem 19] implies that $F_1(X)$ is pseudo-contractible and so is X. ∎

THEOREM 4.9. *Let X be a continuum, let $\mathcal{H}$ be a g-growth hyperspace of X containing $F_1(X)$ and let $G(X)$ be a hyperspace of X such that $F_1(X) \subseteq G(X) \subseteq \mathcal{H} \cap F_\infty(X)$. If $G(X)$ is a retract of $\mathcal{H}$, then the following statement are equivalent:*

(1) *2^X is contractible.*
(2) *$\mathcal{H}$ is contractible.*
(3) *$G(X)$ is contractible.*
(4) *$F_\infty(X)$ is contractible.*

Proof. In light of Theorem 4.2, conditions (1) and (2) are equivalent. Now, assume that $\mathcal{H}$ is contractible. Since each retract of a contractible space is contractible, we conclude that $G(X)$ is contractible. Now if we suppose (3), then $F_1(X)$ is contractible in $G(X)$, and since $G(X) \subset F_\infty(X)$, we find that $F_1(X)$ is contractible in $F_\infty(X)$. Hence, by [14, 78.49], $F_\infty(X)$ is contractible. Finally, if $F_\infty(X)$ is contractible, then $F_1(X)$ is contractible in $F_\infty(X)$. Since $F_\infty(X) \subset 2^X$, and $F_1(X)$ is contractible in 2^X, by [14, Theorem 20.1], 2^X is contractible. ∎

We know by Corollary 4.4 that if X is pseudo-contractible continuum then each hyperspace $H(X)$ having property e is pseudo-contractible. The following result gives a condition ensuring contractibility of $H(X)$ and $F_\infty(X)$. On the other hand, in Example 4.13, we will show a continuum X such that $F_n(X)$ and $F_\infty(X)$ are pseudo-contractible but $F_n(X)$ and $F_\infty(X)$ are not contractible.

COROLLARY 4.10. *Let X be a continuum, let $\mathcal{H}$ be a g-growth hyperspace of X containing $F_1(X)$ and let $H(X)$ be a hyperspace such that $F_1(X) \subseteq H(X) \subseteq \mathcal{H} \cap F_\infty(X)$. If X pseudo-contractible and $H(X)$ is a retract of $\mathcal{H}$, then $H(X)$ and $F_\infty(X)$ are contractible.*

Corollary 4.11. *Let X be a continuum, let $\mathcal{H}$ be a g-growth hyperspace of X containing $F_1(X)$ and let $H(X)$ be a hyperspace such that $F_1(X) \subseteq H(X) \subseteq \mathcal{H} \cap F_\infty(X)$. If $H(X)$ is a retract of $\mathcal{H}$, then the following statements are equivalent:*

(1) *$F_\infty(X)$ is pseudo-contractible.*
(2) *$F_\infty(X)$ is contractible.*
(3) *$H(X)$ is pseudo-contractible.*
(4) *$H(X)$ is contractible.*

Proof. Observe that (2) implies (1), and (3) follows from (4). In light of Theorem 4.9, conditions (2) and (4) are equivalent. Notice that (1) and (3) each imply that $F_1(X)$ is pseudo-contractible in 2^X and so, by Theorem 4.2 the hyperspace $\mathcal{H}$ is contractible. Therefore, from this and Theorem 4.9 it follows that (1) implies (2) and (3) implies (4). ∎

Theorem 4.12. *Let X be a continuum and let $H(X)$ be a hyperspace having property e such that $F_1(X) \subseteq H(X) \subseteq F_\infty(X)$. If $\mathcal{H}$ is a g-growth hyperspace of X containing $F_1(X)$ and $F_1(X)$ is a retract of $\mathcal{H}$, then the following statement are equivalent:*

(1) *X is contractible.*
(2) *X is pseudo-contractible.*
(3) *2^X is contractible.*
(4) *$\mathcal{H}$ is contractible.*
(5) *$H(X)$ is contractible.*
(6) *$H(X)$ is pseudo-contractible.*

Proof. Observe that (1)$\Rightarrow$(2)$\Rightarrow$(6)$\Rightarrow$(3)$\Leftrightarrow$(4), and (1)$\Rightarrow$(5)$\Rightarrow$(3)$\Leftrightarrow$(4). Finally, if we assume (4), then since retractions preserve contractibility, we see that X is contractible. This proves (4)$\Rightarrow$(1). ∎

The following example shows that the concepts of pseudo-contractibility and contractibility are not equivalent in some hyperspaces, even if the hyperspaces contain $F_1(X)$. To see this, we will consider the hyperspaces $F_n(X)$ and $F_\infty(X)$ for Kuperberg's continuum X. Note that these hyperspaces are not g-growth hyperspaces.

Example 4.13. First, let us describe Kuperberg's continuum X. Let $\mathbb{C}$ be the complex plane and let $X_0 = \left\{\frac{t+2}{t+1}e^{it} : t \in [0,\infty)\right\}$ be the spiral approaching the unit circle S^1. Let $X = X_0 \cup \{x : |x| \leq 1\} \subset \mathbb{C}$. Note that X is not pathwise connected. It is well known that X is pseudo-contractible. On the other hand, by Corollary 4.4 the hyperspaces $F_n(X)$ and $F_\infty(X)$ are pseudo-contractible but not contractible. Indeed, if we suppose that $F_n(X)$ and $F_\infty(X)$ are contractible, then both $F_n(X)$ and $F_\infty(X)$ are pathwise connected, and by [7, Proposition 1.2] and [10, Lemma 2.3] the space X is pathwise connected, a contradiction.

The above example negatively answers the following question.

QUESTION 4.14 ([6, Question 7]). *Let $H(X) \in \{F_\infty(X), F_n(X)\}$ $(n \in \mathbb{N})$. Does pseudo-contractibility of $H(X)$ imply contractibility of $H(X)$ for any continuum X?*

We ask the following question.

QUESTION 4.15. *Does there exist a continuum X such that X is pseudo-contractible, noncontractible and $F_1(X)$ is a retract of $F_\infty(X)$?*

Recall that a space Z is said to be (*pseudo-*)*contractible with respect to X* provided that each map from Z into X is (pseudo-)homotopic to a constant map in X. Note that if $Y \subseteq X$ and Y is (pseudo-)contractible with respect to X, then Y is (pseudo-)contractible in X.

THEOREM 4.16. *Let X, Y be continua and let $\mathcal{H}$ be a g-growth hyperspace of Y. Then X is pseudo-contractible with respect to $\mathcal{H}$ if and only if X is contractible with respect to $\mathcal{H}$.*

Proof. Assume that X is pseudo-contractible with respect to $\mathcal{H}$. Let $f : X \to \mathcal{H}$ be a map. Our assumption guarantees that f is pseudo-homotopic to a constant map. Hence, in light of Theorem 3.4, f is homotopic to a constant map. Thus X is contractible with respect to $\mathcal{H}$. The converse is immediate. ∎

The last theorems in this section give a relationship between the pseudo-contractibility of a continuum and the pseudo-contractibility of its quotient hyperspaces having property e.

COROLLARY 4.17. *If X is a pseudo-contractible continuum, then each quotient hyperspace $Q(X)$ having property e is pseudo-contractible, and if $Q(X)$ is a g-growth quotient hyperspace of X, then it is contractible.*

Proof. Our assumption implies that the map id_X is pseudo-homotopic to a constant map. By Theorem 3.9, the q-induced map $Q(\mathrm{id}_X)$ is pseudo-homotopic to a constant map. Finally, since $Q(\mathrm{id}_X) = \mathrm{id}_{Q(X)}$, we conclude that $Q(X)$ is pseudo-contractible. If $Q(X)$ is a g-growth quotient hyperspace, then by Theorem 3.9 the quotient $Q(X)$ is contractible. ∎

In particular, $F_n^m(X)$ and $HS_n(X)$ are pseudo-contractible, and $C_n^m(X)$ is contractible if X is pseudo-contractible.

COROLLARY 4.18. *Let X be a continuum and let $H(X)$ be a hyperspace having property e such that $F_1(X) \subset H(X) \subset F_\infty(X)$. If $F_1(X)$ is a retract of $F_\infty(X)$ and $H(X)$ is pseudo-contractible, then each quotient hyperspace $Q(X)$ having property e is pseudo-contractible and each g-growth quotient hyperspace of X is contractible.*

Proof. Since $H(X)$ is pseudo-contractible, Theorem 4.8 shows that X is pseudo-contractible. Thus, the result follows from Corollary 4.17. ∎

THEOREM 4.19. *Let X and Y be spaces such that Y is a closed subspace of X. If Y is a strong pseudo-deformation retract of X, then the quotient space X/Y is pseudo-contractible.*

Proof. By hypothesis, there exist a continuum K, points $a, b \in K$ and a map $G : X \times K \to X$ such that $G(x, a) = x$, $G(x, b) = r(x)$ and $G(y, t) = y$ for each $x \in X$, each $y \in Y$ and each $t \in K$. Let $q : X \to X/Y$ be the quotient map. Let $G' : X/Y \times K \to X/Y$ be given by $G'(\mathbf{x}, t) = Y$ if $\mathbf{x} = Y$ and $G'(\mathbf{x}, t) = q(G(q^{-1}(\mathbf{x}), t))$ if $\mathbf{x} \neq Y$. Observe that G' is continuous and $G'(\mathbf{x}, a) = \mathbf{x}$ and $G'(\mathbf{x}, b) = Y$. So, X/Y is pseudo-contractible. ∎

COROLLARY 4.20. *Let X be a continuum and let $G(X)$ and $H(X)$ be hyperspaces of X. If $G(X)$ is a strong pseudo-deformation retract of $H(X)$, then the quotient $Q(X)$ is pseudo-contractible.*

We end this section with the following questions.

QUESTION 4.21. *For what kind of continua (pseudo-)contractibility of a quotient hyperspace $Q(X) = H(X)/G(X)$ implies (pseudo-)contractibility of $H(X)$?*

QUESTION 4.22. *Does there exist a continuum X such that $Q(X) = H(X)/G(X)$ is pseudo-contractible but not contractible? In other words, does pseudo-contractibility of $Q(X)$ imply its contractibility?*

5. Pseudo-homotopically equivalent hyperspaces. In this part, we will study the concept of pseudo-homotopy equivalence applied to hyperspaces of continua. As a result, we show that $H(-)$ defines a pseudo-homotopic functor.

Let X and Y be topological spaces. We say that Y is *semi-pseudo-homotopically equivalent to* X, written $Y \approx_P^{SE} X$, if there exist a continuum C and maps $g : Y \to X$ and $f : X \to Y$ such that $f \circ g \simeq_C \mathrm{id}_Y$. We say that Y is *semi-homotopically equivalent* to X (written $Y \approx^{SE} X$) if $f \circ g \simeq \mathrm{id}_Y$. On the other hand, X and Y are *pseudo-homotopically equivalent* (or have the *same pseudo-homotopy type*), written $X \approx_P^E Y$, if there exist continua C, D, and maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq_C \mathrm{id}_Y$ and $g \circ f \simeq_D \mathrm{id}_X$. If $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$, then Y is *homotopically equivalent to* X, written $Y \approx^E X$.

THEOREM 5.1. *Let X and Y be continua and let $\mathcal{H}$ be a g-growth hyperspace of Y. If $\mathcal{H} \approx_P^{SE} X$, then $\mathcal{H} \approx^{SE} X$.*

Proof. From our assumption, there exist maps $g : \mathcal{H} \to X$ and $f : X \to \mathcal{H}$, and a continuum C such that $f \circ g \simeq_C \mathrm{id}_{\mathcal{H}}$. By Theorem 3.4 we have $f \circ g \simeq \mathrm{id}_{\mathcal{H}}$. This proves that $\mathcal{H}$ is semi-homotopically equivalent to X. ∎

From Theorem 5.2 until Corollary 5.15 we assume that each $H(X)$ has property e.

THEOREM 5.2. *Let X, Y be continua. If $Y \approx_P^{SE} X$ and $H(X)$ is a hyperspace of X, then $H(Y) \approx_P^{SE} H(X)$ and if $H(Y)$ and $H(X)$ are g-growth hyperspaces, then $H(Y) \approx^{SE} H(X)$.*

Proof. Suppose that $Y \approx_P^{SE} X$. Then there exist maps $g : Y \to X$ and $f : X \to Y$ and a continuum C such that $f \circ g \simeq_C \mathrm{id}_Y$.

Since $f \circ g \simeq_C \mathrm{id}_Y$, by Theorem 3.6 we obtain $H(f \circ g) \simeq_C H(\mathrm{id}_Y)$. From Theorem 3.7, it follows that $H(f) \circ H(g) \simeq_C \mathrm{id}_{H(Y)}$. Therefore, $H(Y) \approx_P^{SE} H(X)$. If $H(Y)$ and $H(X)$ are g-growth hyperspaces, by Theorem 3.6 we conclude $H(Y) \approx^{SE} H(X)$. ■

COROLLARY 5.3. *Let X and Y be continua. If $Y \approx_P^{SE} X$ and $Q(X)$ is a quotient hyperspace of X, then $Q(Y) \approx_P^{SE} Q(X)$, and if $Q(Y)$ and $Q(X)$ are g-growth quotient hyperspaces, then $Q(Y) \approx^{SE} Q(X)$.*

COROLLARY 5.4. *Let X and Y be continua. If $Y \approx_P^{E} X$ and $H(X)$ is a hyperspace of X, then $H(Y) \approx_P^{E} H(X)$ and if $H(Y)$ and $H(X)$ are g-growth hyperspaces, then $H(Y) \approx^{E} H(X)$.*

The converse of Corollary 5.4 does not hold. Taking $X = I$ and $Y = S^1$ we have $2^Y \approx^E 2^X$ and $C(X) \approx^E C(Y)$ because they are homeomorphic. However, Y is not pseudo-homotopically equivalent to X.

COROLLARY 5.5. *Let X and Y be continua. If $Y \approx_P^{E} X$ and $Q(X)$ is a quotient hyperspace of X, then $Q(Y) \approx_P^{E} Q(X)$, and if $Q(Y)$ and $Q(X)$ are g-growth quotient hyperspaces, then $Q(Y) \approx^{E} Q(X)$.*

PROBLEM 5.6. (1) *Let $\mathcal{H}(X)$ be a g-growth hyperspace of a continuum X. Characterize the continua Y such that if $\mathcal{H}(Y) \approx_P^{SE} \mathcal{H}(X)$, then $Y \approx_P^{SE} X$.*

(2) *Let $G(X)$ be a hyperspace of a continuum X. Characterize the continua Y such that if $G(Y) \approx_P^{SE} G(X)$, then $Y \approx_P^{SE} X$.*

PROBLEM 5.7. (1) *Let $\mathcal{H}(X)$ be a g-growth hyperspace of a continuum X. Characterize the continua Y such that if $\mathcal{H}(Y) \approx_P^{E} \mathcal{H}(X)$, then $Y \approx_P^{E} X$.*

(2) *Let $G(X)$ be a hyperspace of a continuum X. Characterize the continua Y such that if $G(Y) \approx_P^{E} G(X)$, then $Y \approx_P^{E} X$.*

PROBLEM 5.8. (1) *Let $\mathcal{H}(X)$ be a g-growth hyperspace of a continuum X. Characterize the continua X such that for each continuum Y fulfilling $\mathcal{H}(Y) \approx_P^{SE} \mathcal{H}(X)$, we have $Y \approx_P^{SE} X$.*

(2) *Let $G(X)$ be a hyperspace of a continuum X. Characterize the continua X such that for each continuum Y fulfilling $G(Y) \approx_P^{SE} G(X)$, we have $Y \approx_P^{SE} X$.*

PROBLEM 5.9. (1) *Let $\mathcal{H}(X)$ be a g-growth hyperspace of a continuum X. Characterize the continua X such that for each continuum Y fulfilling $\mathcal{H}(Y) \approx_P^E \mathcal{H}(X)$, we have $Y \approx_P^E X$.*

(2) *Let $G(X)$ be a hyperspace of a continuum X. Characterize the continua X such that for each continuum Y fulfilling $G(Y) \approx_P^E G(X)$, we have $Y \approx_P^E X$.*

COROLLARY 5.10. *Let X and Y be continua such that $Y \approx_P^{SE} X$ and let $H(X)$ be a hyperspace of X. If $H(X)$ is (pseudo-)contractible, then $H(Y)$ is (pseudo-)contractible and if $H(X)$ and $H(Y)$ are g-growth hyperspaces and $H(X)$ is contractible, then $H(Y)$ is contractible.*

Proof. Since $Y \approx_P^{SE} X$, by Theorem 5.2 we obtain $H(Y) \approx_P^{SE} H(X)$. Since $H(X)$ is (pseudo-)contractible, so is $H(Y)$. ∎

COROLLARY 5.11. *Let X and Y be continua and let $\mathcal{H}$ be a g-growth hyperspace of X. If X is pseudo-contractible and $Y \simeq_P^{SE} \mathcal{H}$, then Y is contractible.*

Proof. In light of Corollary 4.4, $\mathcal{H}$ is contractible. So, Y must be contractible. ∎

COROLLARY 5.12. *Let X be continuum and let $\mathcal{H}$ be a g-growth hyperspace of X such that $X \simeq_P^{SE} \mathcal{H}$. Then X is pseudo-contractible if and only if X is contractible.*

Proof. Each contractible continuum is pseudo-contractible. On the other hand, if X is pseudo-contractible, then applying Corollary 5.11 we find that X is contractible. ∎

COROLLARY 5.13. *Let X and Y be continua and let $Q(X)$ be a quotient hyperspace of X. If $Y \approx_P^{SE} X$ and $Q(X)$ is pseudo-contractible, then $Q(Y)$ is pseudo-contractible.*

COROLLARY 5.14. *Let X and Y be continua such that $Y \approx_P^E X$ and let $H(X)$ be a hyperspace of a continuum X. Then $H(X)$ is (pseudo-)contractible if and only if $H(Y)$ is (pseudo-)contractible, and if $H(X)$ and $H(Y)$ are g-growth hyperspaces, then pseudo-contractibility of one hyperspace implies contractibility of the other.*

COROLLARY 5.15. *Let X and Y be continua such that $Y \approx_P^E X$ and let $Q(X)$ be a quotient hyperspace of X. Then $Q(Y)$ is pseudo-contractible if and only if $Q(X)$ is pseudo-contractible.*

The following continuum is an example of Corollary 5.14 for some specific hyperspaces.

EXAMPLE 5.16. Let M be the Möbius strip. We know that M is not pseudo-contractible and it is homotopically equivalent to S^1. The following are true:

(1) 2^M is contractible.
(2) $C(M)$ is contractible.
(3) $C_n(M)$ is contractible, for each $n \in \mathbb{N}$,
(4) $C_\infty(M)$ is contractible
(5) $F_n(M)$ is not pseudo-contractible, for each $n \in \mathbb{N}$,
(6) $F_\infty(M)$ is contractible.

6. Pseudo-contractibility of hyperspaces of products and products of hyperspaces. Concerning products we have the following results.

THEOREM 6.1. *Let X and Y be topological spaces. If $X \approx_P^E Y$, then $X \times Z$ has the same pseudo-homotopy type as $Y \times Z$ for each topological space Z.*

Proof. By hypothesis, there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq_C \mathrm{id}_Y$ and $g \circ f \simeq_K \mathrm{id}_X$. Let G be a pseudo-homotopy between $g \circ f$ and id_X. We consider the maps $\hat{f} : X \times Z \to Y \times Z$ and $\hat{g} : Y \times Z \to X \times Z$ given by $\hat{f}(x, z) = (f(x), z)$ and $\hat{g}(y, z) = (g(y), z)$ respectively.

We will just show that $\hat{g} \circ \hat{f} \simeq_K \mathrm{id}_{X\times Z}$; the assertion $\hat{f} \circ \hat{g} \simeq_C \mathrm{id}_{Y\times Z}$ is proved in a similar way. Consider $\hat{G} : (X \times Z) \times K \to X \times Z$ defined as $\hat{G}((x, z), k) = (G(x, k), z)$. So, $\hat{G}((x, z), a) = (G(x, a), z) = ((g \circ f)(x), z) = (\hat{g} \circ \hat{f})(x, z)$. On the other hand, $\hat{G}((x, z), b) = (G(x, b), z) = (x, z)$. Therefore, $X \times Z$ and $Y \times Z$ have the same pseudo-homotopy type. ∎

It is known that if X is pseudo-contractible, then X has the same pseudo-homotopy type as a point (see [5, Theorem 40]).

COROLLARY 6.2. *Let X be a topological space. If X is pseudo-contractible, then $X \times Y$ has the same pseudo-homotopy type as Y for each topological space Y.*

Proof. If X is pseudo-contractible then $X \approx_P^E \{x_0\}$. Therefore, $X\times Y \approx_P^E \{x_0\} \times Y$. Since $\{x_0\} \times Y \approx Y$ for each Y, we see that $X \times Y$ has the same pseudo-homotopy type as Y. ∎

The next corollary is a consequence of Corollaries 5.14 and 6.2.

COROLLARY 6.3. *Let X and Y be continua, and let $H(X)$ be a hyperspace of X having property e. If Y is pseudo-contractible, then the following statements hold:*

(1) *$H(X)$ is pseudo-contractible if and only if $H(X\times Y)$ is pseudo-contractible.*
(2) *If $H(X)$ is a g-growth hyperspace, then $H(X)$ is contractible if and only if $H(X \times Y)$ is contractible.*
(3) *If X is pseudo-contractible, then $H(X \times Y)$ is pseudo-contractible.*

(4) *If X is pseudo-contractible and $H(X)$ is a g-growth hyperspace, then $H(X \times Y)$ is contractible.*

LEMMA 6.4. *Let X and Y be continua such that $Y \subset X$ and let $H(X)$ be a hyperspace of X. If Y is a retract of X, then $H(Y)$ is a retract of $H(X)$.*

Proof. If r is a retraction from X to Y and $H(r)$ is the induced map, then $H(r)$ is a retraction from $H(X)$ to $H(Y)$, because $H(r)(K) \in H(Y)$ for every $K \in H(X)$ and $H(r)(K) = r(K) = K = H(\mathrm{id}_Y)(K)$ for every $K \in H(Y)$. ■

COROLLARY 6.5. *Let X be a continuum, let $H(X)$ be a hyperspace having property e, and let Y be a subcontinuum of X such that Y is a retract of X. If $H(X)$ is pseudo-contractible, then $H(Y)$ is pseudo-contractible, and if $H(X)$ is a g-growth hyperspace, then $H(Y)$ is contractible.*

The next result follows from Theorem 3.6 and Lemma 6.4.

PROPOSITION 6.6. *Let X and Y be continua such that $Y \subset X$ and let $H(X)$ be a hyperspace of X having property e. If Y is a pseudo-deformation retract of X, then $H(Y)$ is a pseudo-deformation retract of $H(X)$, and if $H(X)$ is a g-growth hyperspace, then $H(Y)$ is a deformation retract of $H(X)$.*

PROPOSITION 6.7. *Let X be a topological space and let $Y \subset X$. If Y is a pseudo-deformation retract of X, then $Y \approx_P^E X$.*

Proof. Let r be a retraction from X onto Y. If $G : X \times K \to Y$ is a pseudo-homotopy between the maps id_X and r, then the map G induces pseudo-homotopies between $i \circ r$ and id_X and between $r \circ i$ and id_Y, where the map i is the inclusion map from Y into X. ■

We get the following three results as a consequence of Proposition 6.6, Proposition 6.7 and Corollary 6.5.

COROLLARY 6.8. *Let X and Y be continua such that $Y \subset X$. If Y is a pseudo-deformation retract of X and $H(X)$ is a hyperspace having property e, then $H(X)$ is pseudo-contractible if and only if $H(Y)$ is pseudo-contractible and if $H(X)$ is a g-growth hyperspace of X, then $H(X)$ is contractible if and only if $H(Y)$ is contractible.*

COROLLARY 6.9. *Let X be a continuum, let $\mathcal{H}$ be a g-growth hyperspace of X and let $Y \subset \mathcal{H}$. If Y is a pseudo-deformation retract of $\mathcal{H}$, then Y is contractible if and only if Y is pseudo-contractible.*

PROPOSITION 6.10. *Let X and Y be continua and let $H(X)$ be a hyperspace having property e. If $H(X \times Y)$ is pseudo-contractible, then $H(X)$ and $H(Y)$ are pseudo-contractible.*

Proof. Let $x_0 \in X$ and $y_0 \in Y$. Since $R_X : X \times Y \to X \times \{y_0\}$ and $R_Y : X \times Y \to \{x_0\} \times Y$ defined by $R_X(x, y) = (x, y_0)$ and $R_Y(x, y) = (x_0, y)$

are retractions, we find that $H(X \times \{y_0\})$ and $H(\{x_0\} \times Y)$ are retracts of $H(X \times Y)$. Thus, $H(X \times \{y_0\})$ and $H(\{x_0\} \times Y)$ are pseudo-contractible. Since $H(X \times \{y_0\})$ and $H(\{x_0\} \times Y)$ are homeomorphic to $H(X)$ and $H(Y)$, respectively, $H(X)$ and $H(Y)$ are pseudo-contractible. ■

Corollary 6.11. *Let X and Y be continua and let $H(X)$ be a hyperspace of X having property e. If $H(X \times Y)$ is pseudo-contractible, then $H(X) \times H(Y)$ is pseudo-contractible.*

Proposition 6.12. *Let X and Y be continua and let $\mathcal{H}(X)$ be a g-growth hyperspace such that $F_1(X) \subset \mathcal{H}(X)$. Then $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are contractible if and only if $\mathcal{H}(X \times Y)$ is contractible.*

Proof. If $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are contractible, then by Theorem 4.2 we infer that 2^X and 2^Y are contractible. So, $2^X \times 2^Y$ is contractible. By Lemma 2.1, $2^X \times 2^Y$ is embedded in $2^{X \times Y}$. Moreover, $e'(2^X \times 2^Y)$ is a contractible continuum in $2^{X \times Y}$ containing $F_1(X \times Y)$. Then $F_1(X \times Y)$ is contractible in $2^{X \times Y}$. Thus, [16, Lemma 3.1, p. 25] implies that $2^{X \times Y}$ is contractible. Again by Theorem 4.2, $\mathcal{H}(X \times Y)$ is contractible. The converse implication is a consequence of Proposition 6.10. ■

Using [6, Theorem 3.8] and similar ideas to those used in the proofs of Propositions 6.10 and 6.12, we can prove the following result.

Proposition 6.13. *Let X and Y be continua. Then the hyperspaces $F_\infty(X)$ and $F_\infty(Y)$ are pseudo-contractible if and only if the hyperspace $F_\infty(X \times Y)$ is pseudo-contractible.*

The following example shows the (pseudo-)contractibility of some hyperspaces of a continuum may hold by product theorems, even if the space itself is not pseudo-contractible.

Example 6.14. Let $T = S^1 \times S^1$ be the torus. It is well known that T is not contractible, but the following are true:

(1) 2^T is contractible.
(2) $C(T)$ is contractible.
(3) For each $n \in \mathbb{N}$, $C_n(T)$ is contractible.
(4) $C_\infty(T)$ is contractible.
(5) For each $n \in \mathbb{N}$, $F_n(T)$ is not pseudo-contractible.
(6) $F_\infty(T)$ is pseudo-contractible.

We ask the following questions.

Question 6.15. *Let X and Y be continua and let $H(X)$ be a hyperspace having property e of X such that it is not a g-growth hyperspace. If $H(X)$ and $H(Y)$ are pseudo-contractible, is $H(X \times Y)$ pseudo-contractible?*

In particular, we have the following question.

QUESTION 6.16. *If X and Y are continua for which $F_n(X)$ and $F_n(Y)$ are pseudo-contractible, is the hyperspace $F_n(X \times Y)$ pseudo-contractible?*

A partial answer of this question is given in the third part of Corollary 6.3.

7. Maps, pseudo-contractibility and questions. In this section, we exhibit some classes of maps that preserve or do not preserve contractibility of hyperspaces. First, we recall the following definitions. Let X and Y be continua and let $f : X \to Y$ be an onto map. We say that f is:

- *monotone* if for every subcontinuum B of Y, $f^{-1}(B)$ is a continuum;
- *open* if for every open set U of X, $f(U)$ is open in Y;
- *light* if $f^{-1}(y)$ is totally disconnected for each $y \in Y$;
- *weakly confluent* if for each subcontinuum Q of Y there is a component of $f^{-1}(Q)$ which is mapped by f onto Q;
- *confluent* if for each subcontinuum Q of Y, each component of $f^{-1}(Q)$ is mapped by f onto Q.

It is well known that each monotone map is confluent, each open map is confluent and every confluent map is weakly confluent.

The following result is related to the goal of this section.

THEOREM 7.1 ([18, Theorem 16.39]). *If X is a continuum such that 2^X (or $C(X)$) is contractible and Y is an open image of X, then 2^Y and $C(Y)$ are contractible.*

The above result can be extended to g-growth hyperspaces using Theorem 4.2.

COROLLARY 7.2. *If X is a continuum such that 2^X is contractible and Y is an open image of X, then any g-growth hyperspace $\mathcal{H}$ of Y containing $F_1(Y)$ is contractible.*

COROLLARY 7.3. *Let X be a pseudo-contractible continuum. If $f : X \to Y$ is an onto open map, then any g-growth hyperspace $\mathcal{H}$ of Y containing $F_1(Y)$ is contractible.*

The following problem was stated in [18, Problem 16.40, p. 560].

PROBLEM 7.4. *If X is a continuum whose hyperspaces are contractible, what kinds of images of X have contractible hyperspaces? For example, open images do by Theorem 7.1.*

EXAMPLE 7.5. There exist continua X and Y and a monotone map from X to Y such that X is contractible, therefore every hyperspace $H(X) \subset 2^X$ having property e is contractible, Y is not contractible and every g-growth hyperspace of Y containing $F_1(Y)$ having property e is non-contractible.

Notice that the dendroid X pictured in Figure 2 is contractible, and applying similar arguments to the proof of Theorem 4.4, we conclude that every hyperspace $H(X)$ having property e is contractible. Now, let Y be the quotient continuum obtained by identifying the arc ab to a point. Observe that the quotient map is monotone. Since Y has an R^3-set, by [3, Corollary 3.8, p. 318] the hyperspace 2^Y is not contractible. Hence each g-growth hyperspace of Y containing $F_1(Y)$ is non-contractible by Theorem 4.2. On the other hand, every hyperspace $H(Y) \subset 2^Y$ having property e and containing $F_1(Y)$ is non-pseudo-contractible, because if $H(Y)$ is pseudo-contractible, then $F_1(Y)$ is pseudo-contractible in $H(Y)$ and so is in 2^Y. Hence, by Theorem 4.2 we infer that 2^Y is contractible, a contradiction.

This example shows that monotone maps between non-degenerate continua may not preserve (pseudo-)contractibility of some hyperspaces.

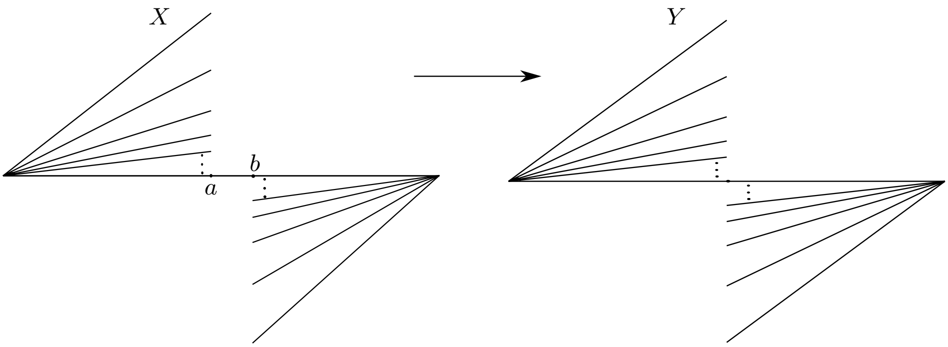

Fig. 2. Monotone map from X to Y

We formulate the following questions.

QUESTION 7.6. *Do open maps preserve pseudo-contractibility?*

QUESTION 7.7. *Under what conditions is there a pseudo-contractible continuum Z and an open map f from Z whose image is or is not pseudo-contractible?*

We have a partial answer to this question.

PROPOSITION 7.8. *Let X be a pseudo-contractible continuum and let $\mathcal{H}$ be a g-growth hyperspace of Y containing $F_1(Y)$. If Y is an open image of X and $F_1(Y)$ is a retract of $\mathcal{H}$, then Y is contractible.*

Proof. Since X is pseudo-contractible, 2^X is contractible. By Corollary 7.2 the hyperspace $\mathcal{H}$ is contractible, so Y is contractible. ∎

In a similar way, we can prove the following corollaries.

COROLLARY 7.9. *Let X be a continuum and let $H(X)$ be a hyperspace of X containing $F_1(X)$. If $H(X)$ is pseudo-contractible and Y is a continuum* *that is a continuous image of X under an open map, then every g-growth hyperspace $\mathcal{H}$ containing $F_1(Y)$ is contractible.*

Proof. Since $H(X)$ is pseudo-contractible, $F_1(X)$ is pseudo-contractible in 2^X, and therefore 2^X is contractible. The result follows from Corollary 7.2. ∎

COROLLARY 7.10. *Let X be a continuum and let $H(X)$ be a hyperspace of X containing $F_1(X)$. If $H(X)$ is pseudo-contractible, Y is a continuum that is a continuous image of X under an open map and $F_1(Y)$ is a retract of a g-growth hyperspace $\mathcal{H}$ of Y, then Y is contractible.*

EXAMPLE 7.11. There exist continua X and Y, and a map $f : X \to Y$, such that:

(1) The map f is light open and weakly confluent.
(2) Every growth hyperspace of X is contractible and the hyperspaces $F_n(X)$ and $F_\infty(X)$ are not contractible.
(3) Every g-growth hyperspace of Y is contractible and $F_\infty(Y)$ is contractible but $F_n(Y)$ is not pseudo-contractible for each $n \in \mathbb{N}$.
(4) The continua X and Y are not (pseudo-)contractible.

Let $\mathbb{C}$ be the complex plane and let $X_0 = \left\{\frac{t+2}{t+1}e^{it} : t \in [0,\infty)\right\}$ be the spiral approaching S^1. Let $X = X_0 \cup Y \subset \mathbb{C}$ where $Y = S^1$ and let $f : X \to Y$ be the projection map. The continua X and Y satisfy conditions (1)–(4). Moreover, we can see that the condition that $F_1(Y)$ be a retract of 2^Y in Corollary 7.10 is necessary.

QUESTION 7.12. *Let X be a continuum and let $H(X)$ be a hyperspace such that $F_1(X) \subset H(X)$ and $H(X)$ is pseudo-contractible. If Y is a continuum that is a continuous image of X under an open map, is $H(Y)$ (pseudo-) contractible?*

EXAMPLE 7.13. Monotone maps between non-degenerate continua may not preserve non-pseudo-contractibility.

Consider Kuperberg's pseudo-contractible continuum X and $Y = X \cup S$, where S is a copy of S^1 that meets X in one point. Notice that X is the quotient space of Y by identifying the subcontinuum S to a point. The space Y is not pseudo-contractible and the quotient map from Y onto X is monotone.

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Félix Capulín, Enrique Castañeda-Alvarado, Leonardo Juárez-Villa, David Maya
Facultad de Ciencias, Departamento de Matemáticas
Universidad Autónoma del Estado de México
Instituto Literario No. 100
Colonia Centro
C.P. 50000, Toluca, Estado de México, México
E-mail: fcapulin@gmail.com, fcp@uaemex.mx
eca@uaemex.mx
juvile06@gmail.com
dmayae@outlook.com, dmayae@uaemex.mx