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On smooth Weyl sums over biquadrates and Waring's problem

by

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1. Introduction. Our focus in this memoir lies on the moments of quartic smooth Weyl sums

$$g(\alpha;P,R)=\sum_{x\in\mathcal{A}(P,R)}e(\alpha x^4),$$

where $e(z)=e^{2\pi iz}$ and $\mathcal{A}(P,R)$ denotes the set of numbers $n\in[1,P]$ all of whose prime divisors are at most R. In this paper, we refer to the number Δ_t as an *admissible exponent* for the positive real number t if there exists a positive number η such that, whenever $1\le R\le P^\eta$, one has

$$\int_0^1 |g(\alpha;P,R)|^t\,\mathrm{d}\alpha \ll P^{t-4+\Delta_t}.$$

Recent work [5, Theorem 1.3] of the authors shows that $\Delta_{10}=0.1991466$ is an admissible exponent. It is implicit in work of Vaughan [10, Lemma 5.2], moreover, that the exponent $\Delta_{12}=0$ is admissible. Hitherto, the sharpest upper bounds available for admissible exponents Δ_t in the range $10\le t\le 12$ stem from linear interpolation, via Hölder's inequality, between the 10th and 12th moments. Our principal goal in this paper is to derive estimates going beyond this classical convexity barrier. In particular, we seek to establish the existence of a number t_0, with $t_0<12$, having the property that the exponent $\Delta_{t_0}=0$ is admissible. It transpires that the existence of such a number t_0 has attractive consequences for additive problems involving biquadrates.

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A complete description of our new admissible exponents would be cumbersome to report at this stage, so we defer a full account to §§4 and 5. An indication of the kind of results available is provided in the following theorem.

THEOREM 1.1. *The exponents Δ_t presented in Table 1 are all admissible.*

The exponents in Table 1 are all rounded up in the final decimal place presented. A more precise determination of the number t_0 to which we alluded above is given in our second theorem.

Table 1. Admissible exponents for $10 \le t \le 12$

t	10.00	11.00	11.50	11.75	11.96	12.00
Δ_t	0.1991466	0.0806719	0.0323341	0.0128731	0.0000000	0.0000000

THEOREM 1.2. *Whenever $t \ge 11.95597$, the exponent $\Delta_t = 0$ is admissible. Thus, there exists a positive number η such that, when $1 \le R \le P^\eta$, one has*

$$\int_0^1 |g(\alpha; P, R)|^t \, \mathrm{d}\alpha \ll P^{t-4}. \tag{1.1}$$

The upper bound (1.1) presented in Theorem 1.2 is the first established in which a moment of a biquadratic smooth Weyl sum beyond the 4th but smaller than the 12th has the conjectured order of magnitude. As experts will recognise, such a mean value offers the prospect of establishing results of Waring type concerning sums of 12 or more smooth biquadrates. In this context, we shall refer to a positive integer n as being *R-smooth* when all of its prime divisors are no larger than R. We record the following consequence of Theorem 1.2.

COROLLARY 1.3. *There exists a positive number κ having the property that every sufficiently large integer n satisfying $n \equiv r \pmod{16}$, with $1 \le r \le 12$, can be written as the sum of 12 biquadrates of $(\log n)^\kappa$-smooth integers.*

Recall that for all integers m one has $m^4 \in \{0, 1\} \pmod{16}$, so whenever n is the sum of 12 biquadrates, it follows that $n \equiv r \pmod{16}$ for some integer r with $0 \le r \le 12$. Moreover, the integer $31 \cdot 16^s$ ($s \ge 0$) is never the sum of 12 biquadrates. The condition on r in Corollary 1.3 is therefore implied by local solubility considerations.

An earlier conclusion of Harcos [8] delivers a conclusion similar to that of Corollary 1.3 for sums of 17 biquadrates, though with smoothness parameter $(\log n)^\kappa$ replaced by $\exp(c(\log n \log\log n)^{1/2})$ for a suitable positive constant c. By adapting the treatment of [3, §5], concerning Waring's problem for cubes of smooth numbers, to the present setting, it would be routine using Theorem 1.2 to establish a version of Corollary 1.3 for sums of 12 biquadrates of $\exp(c(\log n \log\log n)^{1/2})$-smooth integers. The reduction of the smoothness parameter to $(\log n)^\kappa$ is made possible by recent work of Drappeau and Shao [7]. Once equipped with the estimate (1.1), the details of the proof of Corollary 1.3 are a routine, though not especially brief, modification of the argument of [7]. Since this is hardly the main point of the present memoir, we eschew any account of the proof of Corollary 1.3, leaving the reader to follow the pedestrian walkway already provided in [7].

A second application of Theorem 1.2 concerns the solubility of pairs of diagonal quartic equations of the shape

$$a_1x_1^4 + \cdots + a_sx_s^4 = 0,$$
$$b_1x_1^4 + \cdots + b_sx_s^4 = 0,$$

wherein $a_i, b_i \in \mathbb{Z}$ are fixed with $(a_i, b_i) \neq (0,0)$ for $1 \leq i \leq s$. Suppose that $s \geq 22$, and that in any diagonal quartic form lying in the pencil of the two forms defining these equations, there are at least 12 variables having non-zero coefficients. The authors show in [6] that, provided this system has non-singular real and p-adic solutions for each prime number p, it has $\mathcal{N}(P) \gg P^{s-8}$ integral solutions $\mathbf{x}$ with $|x_i| \leq P$ $(1 \leq i \leq s)$. This conclusion improves on an earlier one [4] of the authors in which the condition on the pencil insists that at least $s - 7$ variables have non-zero coefficients. An important ingredient in the proof of this new result is an optimal upper bound of the shape (1.1) for some $t < 12$, as provided by Theorem 1.2.

We establish Theorems 1.1 and 1.2 by applying estimates for the mean values

$$U_s(P,R) = \int_0^1 |g(\alpha; P, R)|^s \, d\alpha, \tag{1.2}$$

with various values of $s \in [4, 12]$. Seminal work of Vaughan [10, Theorem 4.3] derives useful admissible exponents when $s \in \{6, 8, 10\}$ and shows also, implicitly, that the exponent $\Delta_{12} = 0$ is admissible. Following some refinement in these exponents in subsequent work of Vaughan [11, Theorem 1.3], the second author introduced a new approach [13] in which moments of fractional order can be estimated in a manner more efficient than mere application of Hölder's inequality to interpolate between admissible exponents available for even values of s. This tool was fully exploited in work [2, Theorem 2 and p. 393] of the authors. Despite recent progress on the 10th moment (see [5, Theorem 1.3]), the sharpest upper bounds hitherto available for admissible exponents Δ_s in the range $10 \leq s \leq 12$ stem from linear interpolation, via Hölder's inequality, between 10th and 12th moments.

A pedestrian application of the iterative method of [13] would seek to break the classical convexity barrier, between 10th and 12th moments, by applying an 8th moment of an auxiliary exponential sum of the shape

$$\widetilde{F}_1(\alpha)=\sum_{\substack{u\in\mathcal{A}(P^\phi R,R)\\ u>P^\phi}}\sum_{\substack{z_1,z_2\in\mathcal{A}(P,R)\\ z_1\equiv z_2 \pmod{u^4}\\ z_1\neq z_2}} e(\alpha u^{-4}(z_1^4-z_2^4)). \tag{1.3}$$

Here, the parameter ϕ is chosen appropriately in the range $0\le\phi\le 1/4$. It transpires that this approach bounds the mean value $U_s(P,R)$ defined in (1.2) in terms of corresponding bounds for $U_{s-2}(P,R)$ and $U_t(P,R)$, wherein t is a parameter to be chosen with $\frac{8}{7}(s-2)\le t\le\frac{4}{3}(s-2)$. This, it turns out, is too inefficient to be useful. What makes the exponential sum awkward to handle is the constraint that z_1 and z_2 both be smooth. Drawing inspiration from an argument presented in [14, §§1, 3], in this paper we estimate the auxiliary integral

$$\int_0^1 \widetilde{F}_1(\alpha)|g(\alpha;P^{1-\phi},R)|^{s-2}\,\mathrm{d}\alpha$$

in terms of the mediating mean value

$$\int_0^1 |\widetilde{F}_1(\alpha)^2 g(\alpha;P^{1-\phi},R)^8|\,\mathrm{d}\alpha. \tag{1.4}$$

By orthogonality, this mean value counts the number of solutions of an underlying Diophantine equation. This observation permits us the expedient step of discarding the constraint that z_1 and z_2 be smooth in the exponential sum $\widetilde{F}_1(\alpha)$ defined in (1.3), and in this way a useful bound may be derived. One may then introduce the array of tools developed by previous scholars to handle classical analogues of $\widetilde{F}_1(\alpha)$.

The approach outlined above succeeds in bounding $U_s(P,R)$ in terms of $U_t(P,R)$ and the mean value (1.4), with $t=2s-12$. Since $\widetilde{F}_1(\alpha)$ may be thought of roughly as having the weight of two smooth Weyl sums, the mean value (1.4) behaves approximately as a 12th moment. Yet, with the smoothness constraint discarded, we are able to obtain an optimal upper bound for this mean value. It is the latter that permits our efficient application of ideas from the machinery associated with breaking classical convexity. A careful analysis of these ideas would show, in fact, that admissible exponents Δ_{12-u} exist satisfying $\Delta_{12-u}\ll u^\beta$, for small values of u, wherein

$$\beta>\frac{\log(38/15)}{\log 2}=1.341\ldots.$$

Since the approach of Δ_{12-u} towards 0 as $u\to 0$ is more rapid than linear decay, we may apply a Weyl-type estimate to establish the existence of a positive number u_0 for which $\Delta_{12-u_0} = 0$ is admissible. Appeal to the Keil–Zhao device, just as in [14, §6], delivers the sharpest conclusions presently available concerning upper bounds for permissible values of u_0.

This memoir is organised as follows. We begin in §2 by deriving the auxiliary mean value estimate associated with (1.4). Then, in §3, we employ this mean value within the infrastructure permeating the theory of breaking convexity so as to derive the mean value estimates required in deriving new admissible exponents. The iterative relations delivering these new exponents are derived in §4, and numerical values follow. These results establish all of the admissible exponents asserted in Theorem 1.1 save for the claim that $\Delta_{11.96} = 0$. In §5 we discuss the Keil–Zhao device and its implications for mean values of biquadratic smooth Weyl sums. The final assertion of Theorem 1.1 follows, as does its more precise analogue recorded in Theorem 1.2.

In this paper, we adopt the convention that whenever ε, P or R appear in a statement, either implicitly or explicitly, then for each $\varepsilon > 0$, there exists a positive number $\eta = \eta(\varepsilon)$ such that the statement holds whenever $R \le P^\eta$ and P is sufficiently large in terms of ε and η. Implicit constants in Vinogradov's notation $\ll$ and $\gg$ will depend at most on ε and η. Since our iterative methods involve only a finite number of statements (depending at most on ε), there is no danger of losing control of implicit constants. Finally, we write $\|\theta\| = \min_{y\in\mathbb{Z}} |\theta - y|$.

2. The auxiliary mean value estimate. We begin by recalling some upper bounds on admissible exponents.

LEMMA 2.1. *The exponents*

$$\Delta_8 = 0.594193, \qquad \Delta_{10} = 0.1991466, \qquad \Delta_{12} = 0$$

are admissible.

Proof. The conclusion concerning Δ_8 follows from [2, Theorem 2] and the discussion surrounding the table of exponents on [2, p. 393]. The assertion concerning Δ_{10} is established in [5, Theorem 1.3]. Finally, the validity of the admissible exponent $\Delta_{12} = 0$ is a consequence of [10, Lemma 5.2]. ∎

In advance of the introduction of the mean value estimate central to our subsequent deliberations, we must introduce some notation. Let ϕ be a real number with $0 \le \phi \le 1/4$, and write

$$M = P^\phi, \quad H = PM^{-4} \quad \text{and} \quad Q = PM^{-1}. \tag{2.1}$$

We define the difference polynomial

$$\Psi(z,h,m) = m^{-4}((z + hm^4)^4 - (z - hm^4)^4) = 8hz(z^2 + h^2m^8),$$

and then introduce the exponential sum having argument $\Psi(z,h,m)$, namely

$$F_1(\alpha)=\sum_{1\le h\le H}\sum_{M<m\le MR}\sum_{1\le z\le 2P} e(8\alpha hz(z^2+h^2m^8)). \tag{2.2}$$

It is convenient to work with the exponential sum $g_\flat(\alpha)=g(\alpha;2Q,R)$.

The remainder of this section is devoted to the estimation of the auxiliary mean value

$$T=\int_0^1 |F_1(\alpha)^2 g_\flat(\alpha)^8|\,\mathrm{d}\alpha. \tag{2.3}$$

Lemma 2.2. *One has*

$$T\ll P^\varepsilon(PHM)^2Q^4(1+(PM^{-6})^{1/10}).$$

Proof. We follow closely certain aspects of the argument of the proof of [5, Lemma 2.4] associated with the corresponding analysis therein of the mean value defined by [5, equation (2.2)]. In this way, writing $\mathcal{B}(l)$ for the set of all integers z with $1\le z\pm l\le 4P$ and $z\equiv l \pmod 2$, we find by applying Cauchy's inequality that

$$|F_1(\alpha)|^2\ll P^{1+\varepsilon}H^2M^2+P^\varepsilon HM(D(\alpha)E(\alpha))^{1/2}, \tag{2.4}$$

in which

$$D(\alpha)=\sum_{1\le h\le H}\sum_{1\le l\le 2P}\Big|\sum_{z\in\mathcal{B}(l)} e(6\alpha hlz^2)\Big|^2,$$

$$E(\alpha)=\sum_{1\le h\le H}\sum_{1\le l\le 2P}\Big|\sum_{M<m\le MR} e(8\alpha lh^3m^8)\Big|^2.$$

The trivial estimate $E(\alpha)\ll PH(MR)^2$ combines with (2.3) and (2.4) to give

$$T\ll P^{1+\varepsilon}H^2M^2T_1+P^\varepsilon(PH^3M^4)^{1/2}T_2, \tag{2.5}$$

where

$$T_1=\int_0^1|g_\flat(\alpha)|^8\,\mathrm{d}\alpha\quad\text{and}\quad T_2=\int_0^1 D(\alpha)^{1/2}|g_\flat(\alpha)|^8\,\mathrm{d}\alpha. \tag{2.6}$$

We estimate T_2 via the Hardy–Littlewood method. Given integers a and q with $0\le a\le q\le P$ and $(a,q)=1$, let $\mathfrak{P}(q,a)$ denote the set of all $\alpha\in[0,1)$ with $|q\alpha-a|\le PQ^{-4}$, and let $\mathfrak{P}$ denote the union of these intervals. Note that this union is disjoint. Define the function $\Phi:[0,1)\to[0,1]$ by putting

$$\Phi(\alpha)=(q+Q^4|q\alpha-a|)^{-1}$$

when $\alpha\in\mathfrak{P}(q,a)\subseteq\mathfrak{P}$, and put $\Phi(\alpha)=0$ when $\alpha\notin\mathfrak{P}$. Having introduced essentially the same notation here as that employed in the proof of

[5, Lemma 2.4], we find that when $\alpha \in [0,1)$, the proof of [10, Lemma 3.1] shows that

$$D(\alpha) \ll P^{2+\varepsilon} H + P^{3+\varepsilon} H \Phi(\alpha).$$

Write

$$T_3 = \int_{\mathfrak{P}} \Phi(\alpha)^{1/2} |g_\flat(\alpha)|^8 \, \mathrm{d}\alpha.$$

Then we deduce from (2.5) and (2.6) that

$$\begin{aligned} T &\ll P^{1+\varepsilon} H^2 M^2 T_1 + P^\varepsilon (P^3 H^4 M^4)^{1/2} T_1 + P^\varepsilon (P^4 H^4 M^4)^{1/2} T_3 \\ &\ll P^{3/2+\varepsilon} H^2 M^2 T_1 + P^{2+\varepsilon} H^2 M^2 T_3. \end{aligned} \tag{2.7}$$

We see from Lemma 2.1 that there is an admissible exponent Δ_8 smaller than 3/5, and thus $T_1 \ll Q^{23/5}$. Consequently, it follows from (2.1) that

$$P^{-1/2} T_1 \ll P^{-1/2} Q^{23/5} = Q^4 (PM^{-6})^{1/10}. \tag{2.8}$$

Meanwhile, an application of Schwarz's inequality reveals that

$$T_3 \le \Big(\int_{\mathfrak{P}} \Phi(\alpha) |g_\flat(\alpha)|^4 \, \mathrm{d}\alpha \Big)^{1/2} \Big(\int_0^1 |g_\flat(\alpha)|^{12} \, \mathrm{d}\alpha \Big)^{1/2}.$$

It follows from [1, Lemma 2] that

$$\int_{\mathfrak{P}} \Phi(\alpha) |g_\flat(\alpha)|^4 \, \mathrm{d}\alpha \ll Q^{\varepsilon-4} (PQ^2 + Q^4) \ll Q^\varepsilon.$$

On the other hand, by Lemma 2.1 we have

$$\int_0^1 |g_\flat(\alpha)|^{12} \, \mathrm{d}\alpha \ll Q^8.$$

We thus conclude that

$$T_3 \ll (Q^\varepsilon)^{1/2} (Q^8)^{1/2} \ll Q^{4+\varepsilon}.$$

On substituting this estimate together with (2.8) into (2.7), we infer that

$$T \ll P^\varepsilon (PHM)^2 Q^4 (PM^{-6})^{1/10} + P^\varepsilon (PHM)^2 Q^4.$$

This completes the proof of Lemma 2.2. ∎

3. Mean values associated with breaking convexity. The auxiliary mean value T defined in (2.3) captures the essentials of what is needed in our application of the second author's work [13] on breaking convexity, but not the details. We must therefore expend further effort in order that the intricacies of our full argument be accommodated. We begin with some additional notation. We define the modified set of smooth numbers $\mathcal{B}(L,\pi,R)$ for prime numbers π by putting

$$\mathcal{B}(L,\pi,R) = \{ n \in \mathcal{A}(L\pi, R) : n > L,\ \pi \mid n, \text{ and } \pi' \mid n \text{ implies that } \pi' \ge \pi \}.$$

In this definition we use π' to denote a prime number. We note that this definition corrects the analogous definition in the preamble to [14, (3.1)]. Recalling the notation (2.1), we put

$$\widetilde{F}_{d,e}(\alpha;\pi)=\sum_{u\in\mathcal{B}(M/d,\pi,R)}\sum_{\substack{x,y\in\mathcal{A}(P/(de),R)\\ (x,u)=(y,u)=1\\ x\equiv y\ (\mathrm{mod}\ u^4)\\ y<x}} e(\alpha u^{-4}(x^4-y^4)), \tag{3.1}$$

$$F_{d,e}(\alpha)=\sum_{1\le z\le 2P/(de)}\sum_{1\le h\le Hd^3/e}\sum_{M/d<u\le MR/d} e(8\alpha hz(z^2+h^2u^8)) \tag{3.2}$$

and

$$\widetilde{f}(\alpha;P,M,R)=\max_{m>M}\Big|\sum_{x\in\mathcal{A}(P/m,R)} e(\alpha x^4)\Big|. \tag{3.3}$$

We note that $F_{d,e}(\alpha)=0$ when $e>Hd^3$. Finally, we put

$$\Upsilon_{d,e,\pi}(P,R;\phi)=\int_0^1 |\widetilde{F}_{d,e}(\alpha;\pi)^2\widetilde{f}(\alpha;P/(de),M/d,\pi)^8|\,\mathrm{d}\alpha. \tag{3.4}$$

Our initial step is to bound $\Upsilon_{d,e,\pi}(P,R;\phi)$ in terms of a similar mean value in which $F_{d,e}(\alpha)$ is substituted for $\widetilde{F}_{d,e}(\alpha;\pi)$.

LEMMA 3.1. *When $\pi\le R$, one has*

$$\Upsilon_{d,e,\pi}(P,R;\phi)\ll P^{\varepsilon}\int_0^1 |F_{d,e}(\alpha)^2 g(\alpha;2Q/e,R)^8|\,\mathrm{d}\alpha.$$

Proof. As in a similar treatment offered during the proof of [14, Lemma 3.1], the maximal property of the sum $\widetilde{f}(\alpha;P/(de),M/d,\pi)$ is readily eliminated by application of a standard argument employing the Dirichlet kernel. Define

$$\mathcal{D}_K(\theta)=\sum_{|m|\le K^4} e(m\theta)\quad\text{and}\quad \mathcal{D}_K^*(\theta)=\min\{2K^4+1,\|\theta\|^{-1}\}.$$

Then for $K\ge 1$ one has the familiar estimate

$$\int_0^1 \mathcal{D}_K^*(\theta)\,\mathrm{d}\theta\ll\log(2K). \tag{3.5}$$

Recalling (2.1) once more, we see that whenever $m>M$, one has

$$\sum_{x\in\mathcal{A}(P/m,R)} e(\alpha x^4)=\int_0^1 g(\alpha+\theta;Q,R)\mathcal{D}_{P/m}(\theta)\,\mathrm{d}\theta.$$

When $m > M$, we have $\mathcal{D}_{P/m}(\theta) \ll \mathcal{D}^*_{P/m}(\theta) \le \mathcal{D}^*_Q(\theta)$, and so it follows from (3.3) that

$$\widetilde{f}(\alpha; P/(de), M/d, \pi) \ll \int_0^1 |g(\alpha+\theta; Q/e, \pi)|\mathcal{D}^*_Q(\theta)\,\mathrm{d}\theta. \tag{3.6}$$

We substitute eight copies of (3.6) into (3.4), deducing that

$$\Upsilon_{d,e,\pi}(P,R;\phi) \ll \int_0^1 \int_{[0,1)^8} |\widetilde{F}_{d,e}(\alpha;\pi)|^2 \Big(\prod_{i=1}^8 |g(\alpha+\theta_i; Q/e,\pi)|\mathcal{D}^*_Q(\theta_i)\Big)\,\mathrm{d}\boldsymbol{\theta}\,\mathrm{d}\alpha.$$

We next put

$$\Xi_{d,e,\pi}(\theta) = \int_0^1 |\widetilde{F}_{d,e}(\alpha;\pi)^2 g(\alpha+\theta; Q/e,\pi)^8|\,\mathrm{d}\alpha. \tag{3.7}$$

Then by applying the elementary bound $|z_1\cdots z_8| \le |z_1|^8+\cdots+|z_8|^8$, and invoking symmetry, we discern via (3.5) that

$$\begin{aligned} \Upsilon_{d,e,\pi}(P,R;\phi) &\ll \Big(\int_0^1 \Xi_{d,e,\pi}(\theta_1)\mathcal{D}^*_Q(\theta_1)\,\mathrm{d}\theta_1\Big)\prod_{i=2}^8\int_0^1 \mathcal{D}^*_Q(\theta_i)\,\mathrm{d}\theta_i \\ &\ll Q^\varepsilon \int_0^1 \Xi_{d,e,\pi}(\theta)\mathcal{D}^*_Q(\theta)\,\mathrm{d}\theta. \end{aligned} \tag{3.8}$$

We relate $\Xi_{d,e,\pi}(\theta)$ to the number of integral solutions of the equation

$$u_1^{-4}(x_1^4-y_1^4) - u_2^{-4}(x_2^4-y_2^4) = \sum_{j=1}^4 (w_{2j-1}^4 - w_{2j}^4), \tag{3.9}$$

wherein, for $i=1$ and 2, one has the constraints

$$u_i \in \mathcal{B}(M/d,\pi,R), \qquad x_i, y_i \in \mathcal{A}(P/(de),R),$$
$$(x_i,u_i) = (y_i,u_i) = 1, \qquad x_i \equiv y_i \pmod{u_i^4}, \qquad y_i < x_i,$$

and in addition $w_j \in \mathcal{A}(Q/e,\pi)$ $(1 \le j \le 8)$. Indeed, by orthogonality, it follows from (3.1) and (3.7) that $\Xi_{d,e,\pi}(\theta)$ counts the number of these solutions, with each solution counted with weight

$$e\Big(-\theta\sum_{j=1}^4 (w_{2j-1}^4 - w_{2j}^4)\Big).$$

Since this weight is unimodular, we find that $|\Xi_{d,e,\pi}(\theta)|$ is bounded above by the corresponding unweighted count of solutions, and hence by the number of integral solutions of the equation (3.9) with the constraints, for $i = 1$ and 2,

$$M/d < u_i \le MR/d, \quad 1 \le y_i < x_i \le P/(de), \quad x_i \equiv y_i \pmod{u_i^4},$$

and in addition $w_j \in \mathcal{A}(Q/e, R)$ $(1 \le j \le 8)$.

Next we substitute $z_i = x_i + y_i$ and $h_i = (x_i - y_i)u_i^{-4}$ $(i = 1, 2)$. Then we see from the conditions on x_i and y_i that $1 \le h_i \le (P/(de))(M/d)^{-4}$ $(i = 1, 2)$. Moreover, one has

$$2x_i = z_i + h_i u_i^4 \quad \text{and} \quad 2y_i = z_i - h_i u_i^4 \quad (i = 1, 2).$$

Then since

$$u^{-4}((z + hu^4)^4 - (z - hu^4)^4) = 8hz(z^2 + h^2u^8),$$

we deduce via (2.1) that $|\Xi_{d,e,\pi}(\theta)|$ is bounded above by the number of integral solutions of the equation

$$8h_1z_1(z_1^2 + h_1^2u_1^8) - 8h_2z_2(z_2^2 + h_2^2u_2^8) = \sum_{j=1}^{4}(w_{2j-1}^4 - w_{2j}^4),$$

in which, for $i = 1$ and 2, one has

$$M/d < u_i \le MR/d, \quad 1 \le z_i \le 2P/(de), \quad 1 \le h_i \le Hd^3/e,$$

and in addition $w_j \in \mathcal{A}(2Q/e, R)$ $(1 \le j \le 8)$.

We may now recall (3.2) and invoke orthogonality to obtain the upper bound

$$|\Xi_{d,e,\pi}(\theta)| \le \int_0^1 |F_{d,e}(\alpha)^2 g(\alpha; 2Q/e, R)^8| \, \mathrm{d}\alpha.$$

By substituting this upper bound into (3.8) and recalling (3.5), we thus conclude that

$$\begin{aligned} \Upsilon_{d,e,\pi}(P, R; \phi) &\ll Q^{\varepsilon} \Big(\int_0^1 \mathcal{D}_Q^*(\theta) \, \mathrm{d}\theta \Big) \int_0^1 |F_{d,e}(\alpha)^2 g(\alpha; 2Q/e, R)^8| \, \mathrm{d}\alpha \\ &\ll Q^{2\varepsilon} \int_0^1 |F_{d,e}(\alpha)^2 g(\alpha; 2Q/e, R)^8| \, \mathrm{d}\alpha. \end{aligned}$$

This completes the proof of Lemma 3.1. ∎

By applying Lemma 3.1, we relate $\Upsilon_{d,e,\pi}(P, R; \phi)$ to the mean value T defined in (2.3), and bounded in Lemma 2.2.

Lemma 3.2. *Suppose that*

$$\pi \le R, \quad 1 \le d \le M, \quad 1 \le e \le \min\{Q, Hd^3\}, \quad 1/6 \le \phi \le 1/4.$$

Then

$$\Upsilon_{d,e,\pi}(P, R; \phi) \ll P^{\varepsilon}(PHM)^2 Q^4 d^{5/2} e^{-8}.$$

Proof. On recalling (2.2) and (3.2), we find from Lemma 2.2 that

$$\int_0^1 |F_{1,1}(\alpha)^2 g(\alpha;2Q,R)^8|\,\mathrm{d}\alpha \ll P^\varepsilon (PHM)^2Q^4(1+(PM^{-6})^{1/10}).$$

We apply this estimate with $P/(de)$ in place of P, and with M/d in place of M. In alignment with (2.1), we then also have Hd^3/e in place of H, and Q/e in place of Q. The hypotheses of the lemma concerning e and ϕ ensure that

$$(M/d)^4(P/(de))^{-1} = e/(Hd^3) \le 1,$$

whence $(M/d)^4 \le P/(de)$, as well as

$$(P/(de))(M/d)^{-6} = (PM^{-6})d^5e^{-1} \le d^5.$$

Hence we obtain the bound

$$\int_0^1 |F_{d,e}(\alpha)^2 g(\alpha;2Q/e,R)^8|\,\mathrm{d}\alpha \ll P^\varepsilon \left(\frac{P}{de}\cdot\frac{Hd^3}{e}\cdot\frac{M}{d}\right)^2\left(\frac{Q}{e}\right)^4(1+(d^5)^{1/10}).$$

This bound applied in concert with Lemma 3.1 delivers the conclusion of the lemma. ∎

Finally, we recall an estimate for the mean value

$$\widetilde{U}_s(P,M,R) = \int_0^1 \widetilde{f}(\alpha;P,M,R)^s\,\mathrm{d}\alpha. \tag{3.10}$$

LEMMA 3.3. *Suppose that $s>1$ and that Δ_s is an admissible exponent. Then whenever $P>M$ and $R>2$, one has $\widetilde{U}_s(P,M,R) \ll_s (P/M)^{s-4+\Delta_s+\varepsilon}$.*

Proof. This is immediate from [13, Lemma 3.2], on noting the definition of an admissible exponent Δ_s used within this paper. ∎

4. New admissible exponents for $s>10$. The mean value estimates of §§2 and 3 may be converted into admissible exponents by utilising the machinery of [13, §§2–4]. In this context, we write

$$\Omega_{d,e,\pi}(P,R;\phi) = \int_0^1 |\widetilde{F}_{d,e}(\alpha;\pi)\widetilde{f}(\alpha;P/(de),M/d,\pi)^{s-2}|\,\mathrm{d}\alpha, \tag{4.1}$$

and then define

$$\mathcal{U}_s(P,R) = \sum_{1\le d\le D}\sum_{\pi\le R}\sum_{1\le e\le Q} d^{2-s/2}e^{s/2-1}\Omega_{d,e,\pi}(P,R;\phi). \tag{4.2}$$

The key lemma for our present deliberations is the following.

Lemma 4.1. *Suppose that $s>4$ and $0<\phi\le 1/4$. Suppose also that Δ_s and Δ_{s-2} are admissible exponents, and put $\mu_t=t-4+\Delta_t$ $(t=s-2,s)$. Then whenever $1\le D\le P^{1/4}$, one has*

$$U_s(P,R)\ll P^{\mu_s+\varepsilon}D^{s/2-\mu_s}+MP^{1+\mu_{s-2}+\varepsilon}+P^{\frac{s-3}{s-2}\mu_s+\varepsilon}V_s(P,R),$$

where

$$V_s(P,R)=(PM^{s-2}Q^{\mu_{s-2}}+M^{s-3}\mathcal{U}_s(P,R))^{1/(s-2)}.$$

Proof. On noting the definition of an admissible exponent, the stated conclusion is immediate on substituting the conclusion of [13, Lemma 3.3] into that of [13, Lemma 2.3]. ∎

We may now announce our new admissible exponents.

Lemma 4.2. *Let u be a real number with $0\le u\le 2$. Suppose that the exponents Δ_{10-u} and Δ_{12-2u} are both admissible and satisfy*

$$2\Delta_{10-u}-4/5\le \Delta_{12-2u}\le 2\Delta_{10-u}. \tag{4.3}$$

Put

$$\Delta^*_{12-u}=\frac{3\Delta_{12-2u}}{8-2\Delta_{10-u}+\Delta_{12-2u}}.$$

Then whenever $\Delta_{12-u}>\Delta^*_{12-u}$, the exponent Δ_{12-u} is admissible.

Proof. We initiate our discussion by estimating the mean value $\Omega_{d,e,\pi}(P,R;\phi)$. Here and throughout the proof, we set $s=12-u$. Suppose that

$$d\le M,\qquad e\le Q,\qquad \pi\le R,\qquad 1/6\le\phi\le 1/4.$$

Then on recalling (3.4) and (3.10), an application of Schwarz's inequality to (4.1) reveals that

$$\Omega_{d,e,\pi}(P,R;\phi)\le (\Upsilon_{d,e,\pi}(P,R;\phi))^{1/2}\big(\widetilde{U}_{2s-12}(P/(de),M/d,\pi)\big)^{1/2}. \tag{4.4}$$

Observe that since $0\le u\le 2$ and $\Delta_{12-2u}\ge 0$, we have $2u-\Delta_{12-2u}\le 4$. Then since $2s-12=12-2u$, we deduce from Lemmata 3.2 and 3.3 that when $e\le Hd^3$, one has

$$\begin{aligned}\Omega_{d,e,\pi}(P,R;\phi)&\ll P^{\varepsilon}((PHM)^2Q^4d^{5/2}e^{-8})^{1/2}((Q/e)^{8-2u+\Delta_{12-2u}})^{1/2}\\ &\ll P^{1+\varepsilon}HMQ^{6-u+\frac12\Delta_{12-2u}}d^{5/4}e^{-6}.\end{aligned} \tag{4.5}$$

When $e>Hd^3$ instead, it follows from (3.2) that $F_{d,e}(\alpha)=0$, and hence we deduce from Lemma 3.1 that $\Upsilon_{d,e,\pi}(P,R;\phi)=0$. In such circumstances we infer from (4.4) that $\Omega_{d,e,\pi}(P,R;\phi)=0$.

Provided that we make a choice of D with $D\le M$, it therefore follows by substituting (4.5) into (4.2) that

$$\mathcal{U}_s(P,R)\ll P^{1+\varepsilon}HMQ^{6-u+\frac12\Delta_{12-2u}}\Sigma_0,$$

where

$$\Sigma_0=\sum_{1\le d\le D}\sum_{\pi\le R}\sum_{1\le e\le \min\{Q,Hd^3\}} d^{13/4-s/2}e^{-1}.$$

Thus, on recalling our convention concerning ε and R, we deduce that

$$\mathcal{U}_s(P,R)\ll P^{1+\varepsilon}HMQ^{6-u+\frac12\Delta_{12-2u}}.$$

In the notation of Lemma 4.1, we thus obtain the bound

$$V_s(P,R)^{s-2}\ll P^{\varepsilon}M^{s-3}(\Psi_1+\Psi_2),$$

where

$$\Psi_1=PMQ^{6-u+\Delta_{10-u}}\quad\text{and}\quad \Psi_2=PHMQ^{6-u+\frac12\Delta_{12-2u}}.$$

By reference to (2.1), the equation $\Psi_1=\Psi_2$ implicitly determines a linear equation for ϕ, namely

$$1+\phi+(6-u+\Delta_{10-u})(1-\phi)=2-3\phi+(6-u+\tfrac12\Delta_{12-2u})(1-\phi).$$

This equation has the solution $\phi=\phi_0$, where

$$\phi_0=\frac{1+\frac12\Delta_{12-2u}-\Delta_{10-u}}{4+\frac12\Delta_{12-2u}-\Delta_{10-u}}.$$

Observe that the hypothesis (4.3) ensures that $\phi_0\le 1/4$, and also that

$$6\phi_0-1=\frac{\frac52\big(\Delta_{12-2u}+\frac45-2\Delta_{10-u}\big)}{\frac{18}{5}+\frac12\big(\Delta_{12-2u}+\frac45-2\Delta_{10-u}\big)}\ge 0,$$

whence $\phi_0\ge 1/6$. This justifies our earlier assumption that $1/6\le\phi\le 1/4$. We define the exponent μ_s via the relation

$$\mu_s=\mu_{s-2}(1-\phi_0)+1+(s-2)\phi_0,$$

and then put $\Delta_s^*=\mu_s+4-s$. Thus we have

$$\begin{aligned}\Delta_s^*=\Delta_{s-2}(1-\phi_0)+4\phi_0-1&=\frac{3\Delta_{10-u}+\frac32\Delta_{12-2u}-3\Delta_{10-u}}{4+\frac12\Delta_{12-2u}-\Delta_{10-u}}\\&=\frac{3\Delta_{12-2u}}{8+\Delta_{12-2u}-2\Delta_{10-u}}.\end{aligned}$$

Put $D=P^{\omega}$, where ω is any sufficiently small, but fixed, positive number. Then we may follow the discussion of [13, §4] so as to confirm via Lemma 4.1 that whenever Δ_{12-2u} and Δ_{10-u} are admissible exponents, then one has the upper bound

$$U_s(P,R)\ll P^{\mu_s+\varepsilon},$$

whence $\Delta_s=\Delta_{12-u}$ is also an admissible exponent whenever $\Delta_{12-u}>\Delta_{12-u}^*$. This completes the proof of Lemma 4.2. ∎

Note that in view of Lemma 2.1, it follows by applying linear interpolation via Hölder's inequality that when $0 \le u \le 2$, one has

$$\Delta_{12-2u} \le \Delta_{10-u} \le 2\Delta_{10-u} \le \Delta_{12-2u} + \Delta_8 \le \Delta_{12-2u} + \tfrac{3}{5}. \tag{4.6}$$

Hence the hypothesis (4.3) will always be satisfied in the applications to come.

The conclusion of Lemma 4.2 permits the bulk of Theorem 1.1 to be established. Since exponents Δ_s admissible throughout the interval $10 \le s \le 12$ may be of use in future applications, we provide explicit formulae.

THEOREM 4.3. *Suppose that* $0 \le t \le 1$. *Then the exponent*

$$\Delta_{10+t} = 0.1991466 - 0.1184747t$$

is admissible. In particular, the exponent $\Delta_{11} = 0.0806719$ *is admissible.*

Proof. Working within the environment (4.6), put

$$\Delta_{11}^* = \frac{3\Delta_{10}}{8 - 2\Delta_9 + \Delta_{10}}.$$

Then, by applying Lemma 4.2 with $u = 1$, we find that when Δ_9 and Δ_{10} are admissible exponents, then the exponent Δ_{11} is admissible whenever $\Delta_{11} > \Delta_{11}^*$. By linear interpolation using Schwarz's inequality, we may assume that $\Delta_9 = \frac{1}{2}(\Delta_8 + \Delta_{10})$ is admissible, and thus

$$\Delta_{11}^* \le \frac{3\Delta_{10}}{8 - \Delta_8}. \tag{4.7}$$

But in view of Lemma 2.1, we may suppose that $\Delta_8 = 0.594193$ and $\Delta_{10} = 0.1991466$. Thus we find from (4.7) that $\Delta_{11}^* \le 0.080671803$, and the final conclusion of the theorem follows.

By linear interpolation using Hölder's inequality, it follows from this admissible exponent Δ_{11} that when $0 \le t \le 1$, the exponent

$$\Delta_{10+t} = (1-t)\Delta_{10} + t\Delta_{11}$$

is admissible. The first conclusion of the theorem therefore follows with a modicum of computation. ∎

It might be thought that for values of t with $0 < t < 1$, a more direct application of Lemma 4.2 would yield admissible exponents superior to those obtained in Theorem 4.3 via linear interpolation. However, working within the environment (4.6), put

$$\Delta_{10+t}^* = \frac{3\Delta_{8+2t}}{8 - 2\Delta_{8+t} + \Delta_{8+2t}} \qquad (0 \le t \le 1).$$

Then an application of Lemma 4.2 with $u = 2 - t$ shows that the exponent Δ_{10+t} is admissible whenever $\Delta_{10+t} > \Delta_{10+t}^*$. Here, by linear interpolation using Hölder's inequality, we may suppose that

$$\Delta_{8+2t} \le (1-t)\Delta_8 + t\Delta_{10} \quad \text{and} \quad 2\Delta_{8+t} \le \Delta_8 + \Delta_{8+2t}.$$

Thus we deduce that

$$\Delta^*_{10+t} \le \frac{3\Delta_8 - 3t(\Delta_8 - \Delta_{10})}{8-\Delta_8} < 0.2407002 - 0.1600283t.$$

This estimate is inferior to that of Theorem 4.3 in all cases save $t = 1$, in which situation it matches the conclusion of the theorem.

THEOREM 4.4. *Suppose that* $0 \le t \le 1/2$. *Then the exponent*

$$\Delta_{11+t} = \frac{0.0806719 - 0.0959852t}{1 + 0.0213477t}$$

is admissible. In particular, the exponent $\Delta_{11.5} = 0.0323341$ *is admissible.*

Proof. Working within the environment (4.6), put

$$\Delta^*_{11.5} = \frac{3\Delta_{11}}{8 + \Delta_{11} - 2\Delta_{9.5}}.$$

By applying Lemma 4.2 with $u = 1/2$, we find that when $\Delta_{9.5}$ and Δ_{11} are admissible exponents, then so too is $\Delta_{11.5}$ whenever $\Delta_{11.5} > \Delta^*_{11.5}$. By linear interpolation using Schwarz's inequality, we have $\Delta_{9.5} \le \frac{1}{4}(\Delta_8 + 3\Delta_{10})$, and thus

$$\Delta^*_{11.5} \le \frac{6\Delta_{11}}{16 + 2\Delta_{11} - \Delta_8 - 3\Delta_{10}}.$$

On making use of the admissible exponents $\Delta_8 = 0.594193$, $\Delta_{10} = 0.1991466$ and $\Delta_{11} = 0.0806719$ available from Lemma 2.1 and Theorem 4.3, we thus see that the exponent $\Delta_{11.5} = 0.0323341$ is admissible.

Put

$$\Delta^*_{11+t} = \frac{3\Delta_{10+2t}}{8 + \Delta_{10+2t} - 2\Delta_{9+t}} \qquad (0 \le t \le 1/2).$$

Then, more generally, by applying Lemma 4.2 with $u = 1-t$, we find that the exponent Δ_{11+t} is admissible whenever $\Delta_{11+t} > \Delta^*_{11+t}$. Applying linear interpolation as before, we find that

$$\Delta^*_{11+t} \le \frac{(3-6t)\Delta_{10} + 6t\Delta_{11}}{8 + (1-2t)\Delta_{10} + 2t\Delta_{11} - \Delta_8(1-t) - \Delta_{10}(1+t)} = \frac{3\Delta_{10} - 6t(\Delta_{10} - \Delta_{11})}{8 - \Delta_8 + (\Delta_8 - 3\Delta_{10} + 2\Delta_{11})t}. \tag{4.8}$$

We may suppose that $\Delta_8 = 0.594193$, $\Delta_{10} = 0.1991466$ and $\Delta_{11} = 0.0806719$, and thus

$$\frac{3\Delta_{10}}{8-\Delta_8} < 0.0806719, \qquad \frac{6(\Delta_{10} - \Delta_{11})}{8-\Delta_8} > 0.0959852$$

and

$$\frac{\varDelta_8 - 3\varDelta_{10} + 2\varDelta_{11}}{8 - \varDelta_8} > 0.0213477.$$

Thus we deduce that the upper bound for $\varDelta_{11+t}$ claimed in the theorem does indeed follow from (4.8). ∎

THEOREM 4.5. *Suppose that* $0 \le t \le 1/4$. *Then the exponent*

$$\varDelta_{11.5+t} = \frac{0.0323341 - 0.0769435t}{1 + 0.0693668t + 0.0022534t^2}$$

is admissible. In particular, the exponent $\varDelta_{11.75} = 0.0128731$ *is admissible.*

Proof. Working within the environment (4.6), put

$$\varDelta^*_{11.75} = \frac{3\varDelta_{11.5}}{8 + \varDelta_{11.5} - 2\varDelta_{9.75}}.$$

Then, by applying Lemma 4.2 with $u = 1/4$, we find that if $\varDelta_{9.75}$ and $\varDelta_{11.5}$ are admissible exponents, then so too is $\varDelta_{11.75}$ whenever $\varDelta_{11.75} > \varDelta^*_{11.75}$. By linear interpolation, we have $\varDelta_{9.75} \le \frac{1}{8}(\varDelta_8 + 7\varDelta_{10})$, and thus

$$\varDelta^*_{11.75} \le \frac{12\varDelta_{11.5}}{32 + 4\varDelta_{11.5} - \varDelta_8 - 7\varDelta_{10}}.$$

On making use of the admissible exponents $\varDelta_8 = 0.594193$, $\varDelta_{10} = 0.1991466$ and $\varDelta_{11.5} = 0.0323341$ made available by Lemma 2.1 and Theorem 4.4, we see that $\varDelta_{11.75} \le 0.0128731$.

Put

$$\varDelta^*_{11.5+t} = \frac{3\varDelta_{11+2t}}{8 + \varDelta_{11+2t} - 2\varDelta_{9.5+t}} \qquad (0 \le t \le 1/4).$$

Then, more generally, by applying Lemma 4.2 with $u = 1/2 - t$, we find that the exponent $\varDelta_{11.5+t}$ is admissible whenever $\varDelta_{11.5+t} > \varDelta^*_{11.5+t}$. By linear interpolation, we have $\varDelta_{9.5+t} \le \frac{1}{4}((1-2t)\varDelta_8 + (3+2t)\varDelta_{10})$. By substituting this estimate together with that supplied by Theorem 4.4 for $\varDelta_{11+2t}$, we obtain the first conclusion of the theorem following a modicum of computation. ∎

5. The Keil–Zhao device.

We take a simple approach to the application of the Keil–Zhao device (see [15, equation (3.10)] and [9, p. 608]). This permits estimates more or less half the strength of a corresponding minor arc estimate for a classical Weyl sum, though applied to smooth Weyl sums. A careful application of the method enables us to apply major arc estimates in a manner that avoids any consideration of smooth Weyl sums on minor arcs.

THEOREM 5.1. *Suppose that $s \geq 8$ and that the exponent Δ_s is admissible, and satisfies $\Delta_s < 1/8$. Suppose also that $u > s + 16\Delta_s > 10$. Then*

$$\int_0^1 |g(\alpha; P, R)|^u \, d\alpha \ll P^{u-4}. \tag{5.1}$$

In particular, the exponent $\Delta_w = 0$ is admissible for $w \geq u$.

Proof. We assume the hypotheses of the statement, and define

$$\delta = \tfrac{1}{2}(u - s - 16\Delta_s). \tag{5.2}$$

Then $\delta > 0$, and since $s + 16\Delta_s > 10$, it follows that $u > 10 + 2\delta$. It is convenient throughout to abbreviate $g(\alpha; P, R)$ simply to $g(\alpha)$. Also, put

$$I = \int_0^1 |g(\alpha)|^u \, d\alpha.$$

We establish the bound (5.1) by means of the Hardy–Littlewood method. Define the set $\mathfrak{M}$ of major arcs to be the union of the intervals

$$\mathfrak{M}(q, a) = \left\{\alpha \in [0, 1) : |q\alpha - a| \leq \tfrac{1}{8} P^{-3}\right\},$$

with $0 \leq a \leq q \leq \frac{1}{8}P$ and $(a, q) = 1$, and then put $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$. Finally, write

$$G(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^4),$$

and observe that the methods of [12, Chapter 4] (compare [10, proof of Lemma 5.1]) establish that

$$\int_{\mathfrak{M}} |G(\alpha)|^{5+\delta} \, d\alpha \ll_\delta P^{1+\delta}. \tag{5.3}$$

We introduce auxiliary sets of major and minor arcs in order to transform our mean value into one correctly configured for the application of the Keil–Zhao device. Let $\mathfrak{n}$ denote the set of real numbers $\alpha \in [0, 1)$ satisfying

$$|g(\alpha)| \leq 2P^{15/16}. \tag{5.4}$$

Then, when T is a real number with $T \geq 1$, denote by $\mathfrak{N}(T)$ the set of real numbers $\alpha \in [0, 1)$ for which

$$T < |g(\alpha)| \leq 2T. \tag{5.5}$$

Thus, on writing

$$\mathfrak{N} = \bigcup_{\substack{j=0 \\ 2^j \leq P^{1/16}}}^{\infty} \mathfrak{N}(2^{-j} P),$$

we see that $\mathfrak{N} \cup \mathfrak{n} = [0,1)$. It follows that

$$I \le I_0 + \sum_{\substack{j=0 \\ 2^j \le P^{1/16}}}^{\infty} I_1(2^{-j}P), \tag{5.6}$$

where

$$I_0 = \int_{\mathfrak{n}} |g(\alpha)|^u \,\mathrm{d}\alpha \quad \text{and} \quad I_1(T) = \int_{\mathfrak{N}(T)} |g(\alpha)|^u \,\mathrm{d}\alpha.$$

The analysis of I_0 is direct. In view of (5.2) and the bound (5.4), together with the definition of an admissible exponent, one sees that

$$I_0 \le \Big(\sup_{\alpha \in \mathfrak{n}} |g(\alpha)| \Big)^{16\Delta_s + 2\delta} \int_0^1 |g(\alpha)|^s \,\mathrm{d}\alpha \ll (P^{15/16})^{16\Delta_s + 2\delta} P^{s-4+\Delta_s}.$$

On recalling (5.2) once again, we therefore discern that

$$I_0 = o(P^{u-4}). \tag{5.7}$$

Consider next any value of T with $P^{15/16} \le T \le P$. We deduce from (5.2) and (5.5) that

$$I_1(T) \ll T^{16\Delta_s - 2} \int_{\mathfrak{N}(T)} |g(\alpha)|^{u-16\Delta_s+2} \,\mathrm{d}\alpha = T^{16\Delta_s-2} K(T), \tag{5.8}$$

where

$$K(T) = \int_{\mathfrak{N}(T)} |g(\alpha)|^{s+2\delta+2} \,\mathrm{d}\alpha.$$

By Cauchy's inequality, one has

$$K(T) = \sum_{x,y \in \mathcal{A}(P,R)} \int_{\mathfrak{N}(T)} |g(\alpha)|^{s+2\delta} e(\alpha(x^4 - y^4)) \,\mathrm{d}\alpha \le P K^*(T)^{1/2}, \tag{5.9}$$

where

$$\begin{aligned} K^*(T) &= \sum_{1 \le x,y \le P} \Big| \int_{\mathfrak{N}(T)} |g(\alpha)|^{s+2\delta} e(\alpha(x^4 - y^4)) \,\mathrm{d}\alpha \Big|^2 \\ &= \sum_{1 \le x,y \le P} \int_{\mathfrak{N}(T)} \int_{\mathfrak{N}(T)} |g(\alpha) g(\beta)|^{s+2\delta} e((\alpha - \beta)(x^4 - y^4)) \,\mathrm{d}\alpha \,\mathrm{d}\beta \\ &= \int_{\mathfrak{N}(T)} \int_{\mathfrak{N}(T)} |g(\alpha) g(\beta)|^{s+2\delta} |G(\alpha - \beta)|^2 \,\mathrm{d}\alpha \,\mathrm{d}\beta. \end{aligned}$$

Since $[0,1) = \mathfrak{M} \cup \mathfrak{m}$, it follows that

$$K^*(T) \ll K^*(T; \mathfrak{M}) + K^*(T; \mathfrak{m}),$$

where, for $\mathfrak{B} \subseteq [0,1)$, we write

$$K^*(T;\mathfrak{B}) = \int\limits_{\mathfrak{N}(T)} \int\limits_{\substack{\mathfrak{N}(T) \\ \alpha-\beta\in\mathfrak{B}}} |g(\alpha)g(\beta)|^{s+2\delta}|G(\alpha-\beta)|^2\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{5.10}$$

By applying Weyl's inequality (see [12, Lemma 2.4]), one obtains the bound

$$\sup_{\alpha-\beta\in\mathfrak{m}} |G(\alpha-\beta)| \ll P^{7/8+\varepsilon}.$$

Thus, invoking symmetry and the trivial estimate (5.5) for $|g(\alpha)|$ and $|g(\beta)|$, one arrives at the estimate

$$K^*(T;\mathfrak{m}) \ll (P^{7/8+\varepsilon})^2(T^{2\delta})^2\Big(\int_0^1 |g(\alpha)|^s\,\mathrm{d}\alpha\Big)^2 \ll (P^{7/8+\varepsilon}T^{2\delta})^2(P^{s-4+\Delta_s})^2.$$

On recalling (5.2), we deduce that

$$\begin{aligned}(T^{16\Delta_s-2})^2P^2K^*(T;\mathfrak{m}) &\ll P^\varepsilon(P^{15/8-2\delta-15\Delta_s}T^{16\Delta_s-2+2\delta})^2(P^{u-4})^2\\ &= P^\varepsilon\Big(\frac{P^{15/16}}{T}\Big)^{4-32\Delta_s}\Big(\frac{T}{P}\Big)^{4\delta}(P^{u-4})^2.\end{aligned}$$

Then since, by hypothesis, one has $P^{15/16} \le T \le P$ and $\Delta_s < 1/8$, we obtain

$$(T^{16\Delta_s-2})^2P^2K^*(T;\mathfrak{m}) \ll (T/P)^{2\delta}(P^{u-4})^2. \tag{5.11}$$

Next, as $T < |g(\alpha)| \le 2T$ when $\alpha \in \mathfrak{N}(T)$, we find from (5.2) and (5.10) that when $T \ge P^{15/16}$, one has

$$\begin{aligned}K^*(T;\mathfrak{M}) &\ll (T^{2-16\Delta_s})^2 \int\limits_{\mathfrak{N}(T)} \int\limits_{\substack{\mathfrak{N}(T) \\ \alpha-\beta\in\mathfrak{M}}} |g(\alpha)g(\beta)|^{s+2\delta+16\Delta_s-2}|G(\alpha-\beta)|^2\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &= (T^{2-16\Delta_s})^2\Omega_0,\end{aligned} \tag{5.12}$$

where

$$\Omega_0 = \int\limits_{\mathfrak{N}(T)} \int\limits_{\substack{\mathfrak{N}(T) \\ \alpha-\beta\in\mathfrak{M}}} |g(\alpha)g(\beta)|^{u-2}|G(\alpha-\beta)|^2\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

An application of Hölder's inequality shows that

$$\Omega_0^{5+\delta} \le \Omega_1\Omega_2\Omega_3^{3+\delta} \sup_{(\alpha,\beta)\in\mathfrak{N}(T)^2} |g(\alpha)g(\beta)|^{u-10-2\delta}, \tag{5.13}$$

where we have written

$$\Omega_1 = \int\limits_{\substack{\mathfrak{N}(T)\ \mathfrak{N}(T)\\ \alpha-\beta\in\mathfrak{M}}}\!\!\int |G(\alpha-\beta)|^{5+\delta}|g(\alpha)|^u \,\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{5.14}$$

$$\Omega_2 = \int\limits_{\substack{\mathfrak{N}(T)\ \mathfrak{N}(T)\\ \alpha-\beta\in\mathfrak{M}}}\!\!\int |G(\alpha-\beta)|^{5+\delta}|g(\beta)|^u \,\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{5.15}$$

$$\Omega_3 = \int\limits_{\mathfrak{N}(T)}\int\limits_{\mathfrak{N}(T)} |g(\alpha)g(\beta)|^u \,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{5.16}$$

By a change of variable, we find from (5.14) and (5.3) that

$$\Omega_1 \le \Bigl(\int_{\mathfrak{M}} |G(\theta)|^{5+\delta}\,\mathrm{d}\theta\Bigr)\Bigl(\int_0^1 |g(\alpha)|^u\,\mathrm{d}\alpha\Bigr) \ll P^{1+\delta}I.$$

A symmetrical argument bounds the mean value Ω_2 defined in (5.15), and thus

$$\Omega_1\Omega_2 \ll (P^{1+\delta}I)^2. \tag{5.17}$$

On the other hand, it is immediate from (5.16) that

$$\Omega_3 \le \Bigl(\int_0^1 |g(\alpha)|^u\,\mathrm{d}\alpha\Bigr)^2 = I^2. \tag{5.18}$$

On substituting (5.17) and (5.18) within (5.13), and noting (5.5), we conclude thus far that

$$\Omega_0^{5+\delta} \ll (P^{1+\delta}I)^2(I^2)^{3+\delta}T^{2(u-10-2\delta)} \ll P^{2+2\delta}I^{8+2\delta}T^{2(u-10-2\delta)}.$$

We now find from (5.12) that

$$(T^{16\Delta_s-2})^2P^2K^*(T;\mathfrak{M}) \ll (I^{8+2\delta}P^{12+4\delta}T^{2(u-10-2\delta)})^{1/(5+\delta)}.$$

Combining this estimate with (5.11), and substituting into (5.9) and thence into (5.8), we discern that

$$I_1(T) \ll (T/P)^\delta P^{u-4} + (I^{4+\delta}P^{6+2\delta}T^{u-10-2\delta})^{1/(5+\delta)},$$

so that, in view of our earlier observation that $u > 10+2\delta$, we obtain the relation

$$\sum_{\substack{j=0\\ 2^j\le P^{1/16}}}^{\infty} I_1(2^{-j}P) \ll P^{u-4} + (I^{4+\delta}P^{u-4})^{1/(5+\delta)}.$$

Referring back to (5.6) and (5.7), we arrive at the upper bound

$$I \ll P^{u-4} + (I^{4+\delta}P^{u-4})^{1/(5+\delta)},$$

whence $I \ll P^{u-4}$. This completes the proof of Theorem 5.1. ∎

COROLLARY 5.2. *Provided that $u \geq 11.95597$, one has*

$$\int_0^1 |g(\alpha; P, R)|^u \, d\alpha \ll P^{u-4}. \tag{5.19}$$

In particular, the exponent $\Delta_u = 0$ is admissible.

Proof. We apply Theorem 5.1 with $s = 11.75$ and the admissible exponent $\Delta_{11.75} = 0.0128731$ supplied by Theorem 4.5. We thus deduce that whenever

$$u > 11.75 + 16\Delta_{11.75} = 11.9559696,$$

then the desired conclusion (5.19) holds. This establishes that $\Delta_u = 0$ is admissible, completing the proof of the corollary. ■

This corollary implies and is more or less equivalent to Theorem 1.2. We performed extensive numerical computations in order to determine the optimal choice for s in Theorem 5.1 in order that the value of u, with the exponent $\Delta_u = 0$, be minimised. It transpires that this optimal value is equal to 11.75. We should remark that it is not altogether surprising that the optimal value occurs at a value of s of the shape $s = 12 - 2^{-j}$ for some non-negative integer j, because at each such value, it follows from Lemma 4.2 and the kind of arguments underlying Theorems 4.3 to 4.5 that there is a jump in the derivative of Δ_s with respect to s. Here, we are thinking of Δ_s as representing the least permissible admissible exponent as a function of s.

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