COLLOQUIUM MATHEMATICUM

Online First version

SUBSYMMETRIC BASES HAVE THE FACTORIZATION PROPERTY

BY

RICHARD LECHNER (Linz)

Abstract. Let $(e_j)_{j=1}^{\infty}$ denote a Schauder basis for a Banach space X, and let $(e_j^*)_{j=1}^{\infty}$ denote the biorthogonal functionals. We say that $(e_j^*)_{j=1}^{\infty}$ has the factorization property if the identity operator I_{X^*} on X^* factors through every bounded operator $T\colon X^* \to X^*$ with a δ-large diagonal, i.e., $\inf_j |\langle Te_j^*, e_j\rangle| \geq \delta > 0$. We show that if $(e_j^*)_{j=1}^{\infty}$ is subsymmetric and non-ℓ^1-splicing (there is no disjointly supported ℓ^1-sequence in X^*), then $(e_j^*)_{j=1}^{\infty}$ has the factorization property, even when X^* is non-separable. This property is stable under ℓ^p-direct sums of such Banach spaces for all $1 \leq p \leq \infty$, i.e., the standard basis $(e_{n,j}^*)_{n,j=1}^{\infty}$ of $\ell^p(X^*)$ also has the factorization property.

Moreover, we find a condition $(\star)$ for unconditional bases $(e_j)_{j=1}^{n}$ of finite-dimensional Banach spaces X_n, which is expressed in terms of the quantities $\|e_1 + \cdots + e_n\|$ and $\|e_1^* + \cdots + e_n^*\|$, under which any operator $T\colon X_n \to X_n$ with large diagonal can be inverted when restricted to $X_\sigma = [e_j : j \in \sigma]$ for a "large" set $\sigma \subset \{1, \ldots, n\}$ (restricted invertibility). In their seminal works [Israel J. Math. 1987, London Math. Soc. Lecture Note Ser. 1989], J. Bourgain and L. Tzafriri proved restricted invertibility results for unconditional bases satisfying a lower r-estimate and for subsymmetric bases which satisfy certain conditions in terms of Boyd indices. Using condition $(\star)$, we are able to prove restricted invertibility results with reasonably large sets σ which apply to a wider range of bases.

1. Introduction. An infinite-dimensional Banach space X is called *indecomposable* if whenever X is the direct sum of two closed subspaces Y and Z of X, then either Y or Z has to be finite-dimensional. In 1971, J. Lindenstrauss asked the important question whether indecomposable Banach spaces exist [25]. In 1993, W. T. Gowers and B. Maurey gave an affirmative answer to that question [13]. Complementary to the above question, J. Lindenstrauss also asked in [25] to identify the primary Banach spaces, and whether the classical Banach spaces are primary. A Banach space X is called *primary* if whenever X is the direct sum of two complemented spaces Y and Z, either Y or Z is isomorphic to X. Much progress on this question has been made since then and many classical Banach spaces were identified to be primary.

2020 *Mathematics Subject Classification*: Primary 46B25; Secondary 46B26, 47A68, 46B07, 15A09.

Key words and phrases: factorization property, subsymmetric, non-separable, local theory, restricted invertibility, primarity.

Received 1 October 2021; revised 20 January 2022.

Published online 12 May 2022.

DOI: 10.4064/cm8678-1-2022

Given bounded operators $S, T\colon X \to X$, we say that S *factors through* T if there exist bounded operators $A, B\colon X \to X$ such that $S = ATB$. To prove that a Banach space X is primary commonly consists of the following two main steps:

- ▷ showing that X has the *primary factorization property*, i.e., for every bounded operator $T\colon X \to X$, the identity operator I_X on X factors either through T or through $I_X - T$;
- ▷ using Pełczyński's decomposition method to show that for every bounded projection $P\colon X \to X$, either $P(X)$ or $(I_X - P)(X)$ is isomorphic to X.

(Note that X having the primary factorization property only tells us that either $P(X)$ or $(I_X - P)(X)$ contains a complemented subspace that it isomorphic to X.) A closely related concept to the primary factorization property is the factorization property: If $(e_j)_{j=1}^\infty$ is a Schauder basis for X, we say that $(e_j)_{j=1}^\infty$ has the *factorization property* if I_X factors through every bounded linear operator $T\colon X \to X$ which has a large diagonal (with respect to $(e_j)_{j=1}^\infty$). An operator T has *large diagonal* (with respect to $(e_j)_{j=1}^\infty$) if $\inf_j |\langle Te_j, e_j^*\rangle| > 0$, where $(e_j^*)_{j=1}^\infty$ denotes the dual functionals to $(e_j)_{j=1}^\infty$. The factorization property was first implicitly studied by A. D. Andrew in [2] and was then later revisited in [14] and the subsequent works [20, 15, 16, 18, 17, 22, 23]. Formally, the factorization property was conceived in [22] and further investigated in [23]. Although closely related, the factorization property and the primary factorization property are distinct:

- ▷ The James space is primary [8], but its boundedly complete basis does not have the factorization property [22, Proposition 2.5].
- ▷ The standard unit vector basis of the Tsirelson space has the factorization property [21], but the Tsirelson space does not have the primary factorization property (this follows from Theorem IV.c.1, Theorem VII.b.2 and Proposition I.9(3) in [11]).

For general information on the James space and the Tsirelson space, we refer to [12] and [11].

On the other hand, in many classical Banach spaces X with a Schauder basis $(e_j)_{j=1}^\infty$, the factorization property of $(e_j)_{j=1}^\infty$ often implies the primary factorization property of X. In particular, by viewing a given operator on X through the lens of a suitable subsequence, it is not hard to see that if $(e_j)_{j=1}^\infty$ is subsymmetric and has the factorization property, then X has the primary factorization property. In [10] (see also [26, Proposition 3.b.8]), P. G. Casazza and B. L. Lin showed that Banach spaces X with a subsymmetric Schauder basis $(e_j)_{j=1}^\infty$ have the primary factorization property. Barring the case $X = \ell^1$, a simple modification to their proof also shows that $(e_j)_{j=1}^\infty$ has the factorization property.

On the other hand, let us now consider the dual X^* of X together with its weak* Schauder basis $(e_j^*)_{j=1}^\infty$. We say that $(e_j^*)_{j=1}^\infty$ is *non-ℓ^1-splicing* if there exists no disjointly supported ℓ^1-sequence in X^*. If X^* is non-separable, it is in general not known whether X^* has the primary factorization property. However, if $(e_j^*)_{j=1}^\infty$ is subsymmetric and non-ℓ^1-splicing, it was recently established that X^* has the primary factorization property [19, Theorem 1.1].

In contrast to the separable case, one cannot simply modify the proof of [19, Theorem 1.1] to obtain the stronger result that $(e_j^*)_{j=1}^\infty$ has the factorization property. Hence, the question whether $(e_j^*)_{j=1}^\infty$ has the factorization property remained open. By further developing the techniques from [19], the first main result Theorem 3.1 of the present paper answers this question affirmatively: non-ℓ^1-splicing and subsymmetric weak* Schauder bases have the factorization property. Moreover, in Theorem 3.2 we are able to demonstrate that the factorization property of $(e_j^*)_{j=1}^\infty$ transfers over to the standard basis $(e_{n,j}^*)_{n,j=1}^\infty$ of the direct sum $\ell^p(X^*)$, $1 \leq p \leq \infty$, i.e., $(e_{n,j}^*)_{n,j=1}^\infty$ has the factorization property whenever $(e_j^*)_{j=1}^\infty$ is subsymmetric and non-ℓ^1-splicing.

As mentioned above, dealing with factorization problems in non-separable Banach spaces is particularly difficult. Our approach is to exploit that our weak* Schauder basis is non-ℓ^1-splicing and to work in the space directly. Another approach to deal with non-separable Banach spaces, nowadays known as Bourgain's localization method, was introduced by J. Bourgain [4] in 1983: If a Banach space X is isomorphic to $\ell^p((X_n)_{n=1}^\infty)$ for some $1 \leq p \leq \infty$, where X_n denote the finite-dimensional building blocks of X, then for a given operator $T\colon X \to X$:

- ▷ T will first be diagonalized with respect to X_n, i.e., $T = (T_n)_{n=1}^\infty$ and $T_n\colon X_n \to X_n$,
- ▷ then the finite-dimensional quantitative factorization problems are solved in X_n,
- ▷ and finally their solutions are combined to solve the factorization problem in X.

Thus, Bourgain's localization method strongly connects infinite-dimensional factorization problems with finite-dimensional quantitative factorization problems, which themselves are independently important classical problems in Banach space theory. In particular, we would like to highlight the seminal paper [5] by J. Bourgain and L. Tzafriri, in which they proved the following remarkable restricted invertibility result (see [5, Theorem 6.1]): Given an operator having large diagonal with respect to an unconditional basis $(e_j)_{j=1}^n$ of an n-dimensional Banach space X which satisfies a lower r-estimate for some $1 < r < \infty$, the operator can be inverted when restricted to a "large" subspace. The lower r-estimate measures how far $(e_j)_{j=1}^n$ is "away from ℓ^1". By introducing a new measure of "being away from ℓ^1", we are able to extend the restricted invertibility result to a wider class of bases $(e_j)_{j=1}^n$ (see Theorem 3.3).

2. Preliminaries. Let X and Y denote Banach spaces and suppose there exists a bilinear map $\langle\cdot,\cdot\rangle\colon X\times Y\to\mathbb{R}$ such that

(B1) whenever $x\in X$ and $\langle x,y\rangle=0$ for all $y\in Y$, then $x=0$;
(B2) whenever $y\in Y$ and $\langle x,y\rangle=0$ for all $x\in X$, then $y=0$;
(B3) there exists a constant $C_d>0$ such that $|\langle x,y\rangle|\le C_d\|x\|_X\|y\|_Y$ for all $x\in X$, $y\in Y$.

Then we denote by $\sigma(X,Y)$ the locally convex topology on X generated by the collection of seminorms $\{x\mapsto|\langle x,y\rangle| : y\in Y\}$. Let the normalized sequences $(e_j)_{j=1}^\infty$ in X and $(f_j)_{j=1}^\infty$ in Y be such that

(B4) $\langle e_j,f_j\rangle=1$ and $\langle e_j,f_k\rangle=0$, for all $j\neq k$;
(B5) every $x\in X$ has the unique representation $x=\sum_{j=1}^\infty\langle x,f_j\rangle e_j$, where the series converges in the $\sigma(X,Y)$-topology.

If (B1)–(B5) are satisfied, we say that $((e_j,f_j))_{j=1}^\infty$ is a *topological basis* for the dual pair $(X,Y,\langle\cdot,\cdot\rangle)$ with constant C_d. Suppose $(z_j)_{j=1}^\infty$ is a normalized Schauder basis for the Banach space Z, let Z^* denote the dual of Z and let $(z_j^*)_{j=1}^\infty$ denote the biorthogonal functionals to $(z_j)_{j=1}^\infty$. Then $((z_j^*,z_j))_{j=1}^\infty$ is a topological basis for $(Z^*,Z,\langle\cdot,\cdot\rangle)$ with constant 1, where $\langle\cdot,\cdot\rangle\colon Z^*\times Z\to\mathbb{R}$ denotes the natural pairing given by $\langle z^*,z\rangle=z^*(z)$. The topological basis $(z_j^*)_{j=1}^\infty$ is called a *weak* Schauder basis* for Z^*. For more information on weak* Schauder bases we refer to [26, Section 1.b] and [28, Chapter 1, §13].

We say that the sequence $(e_j)_{j=1}^\infty$ is C_u*-unconditional* if for all scalar sequences $(a_j)_{j=1}^\infty$ and $(\gamma_j)_{j=1}^\infty$ we have

$$\Big\|\sum_{j=1}^\infty\gamma_j a_j e_j\Big\|_X\le C_u\sup_k|\gamma_k|\Big\|\sum_{j=1}^\infty a_j e_j\Big\|_X, \tag{2.1}$$

where we demand the above series converge in the $\sigma(X,Y)$-topology. Moreover, we say that $(e_j)_{j=1}^\infty$ is C_s*-spreading* if $(e_j)_{j=1}^\infty$ is C_s-equivalent to each of its increasing subsequences, i.e.,

$$\frac{1}{C_s}\Big\|\sum_{j=1}^\infty a_j e_{n_j}\Big\|_X\le\Big\|\sum_{j=1}^\infty a_j e_j\Big\|_X\le C_s\Big\|\sum_{j=1}^\infty a_j e_{n_j}\Big\|_X \tag{2.2}$$

for all increasing sequences $(n_j)_{j=1}^\infty$ in $\mathbb{N}$. Again, we demand that the above series converge in the $\sigma(X,Y)$-topology. If (2.1) and (2.2) are both satisfied, we say that $(e_j)_{j=1}^\infty$ is (C_u,C_s)*-subsymmetric*. If $(e_j)_{j=1}^\infty$ is C-unconditional or C-spreading or (C,C)-subsymmetric for some C, we say that $(e_j)_{j=1}^\infty$ is *unconditional* or *spreading* or *subsymmetric*. We also say that $((e_j,f_j))_{j=1}^\infty$ is (C_u-)unconditional, (C_s-)spreading or ((C_u,C_s)-)subsymmetric if $(e_j)_{j=1}^\infty$

is (C_u-)unconditional, (C_s-)spreading or ((C_u, C_s)-)subsymmetric. For more information on unconditional or subsymmetric bases we refer to [26, 28].

Let $((e_j, f_j))_{j=1}^\infty$ be a topological basis for the dual pair $(X, Y, \langle\cdot,\cdot\rangle)$, and let I_X denote the identity operator on X. We say that:

- ▷ I_X *C-factors through* a bounded linear operator $T\colon X \to X$ if there exist operators $A, B\colon X \to X$ such that $I_X = ATB$ and $\|A\| \cdot \|B\| \leq C$; if I_X C-factors through a bounded linear operator $T\colon X \to X$ for some $C > 0$, then we simply say I_X *factors through* T;
- ▷ I_X *almost C-factors through* a bounded linear operator $T\colon X \to X$ if I_X $(C + \eta)$-factors through T for all $\eta > 0$;
- ▷ a bounded linear operator $T\colon X \to X$ has *δ-large diagonal* (*with respect to* $((e_j, f_j))_{j=1}^\infty$) if $\delta := \inf_j |\langle Te_j, f_j\rangle| > 0$;
- ▷ a bounded linear operator $T\colon X \to X$ has *large diagonal* (*with respect to* $((e_j, f_j))_{j=1}^\infty$) if T has δ-large diagonal for some $\delta > 0$;
- ▷ given a function $K\colon (0, \infty) \to (0, \infty)$, we say that $((e_j, f_j))_{j=1}^\infty$ has the *K-factorization property*, if for every $\delta > 0$ and every bounded linear operator $T\colon X \to X$ with δ-large diagonal, I_X almost $K(\delta)$-factors through T;
- ▷ $((e_j, f_j))_{j=1}^\infty$ has the *uniform factorization property* if $((e_j, f_j))_{j=1}^\infty$ has the K-factorization property for some $K\colon (0, \infty) \to (0, \infty)$;
- ▷ $((e_j, f_j))_{j=1}^\infty$ has the *factorization property* if I_X factors through every bounded linear operator $T\colon X \to X$ which has large diagonal with respect to $((e_j, f_j))_{j=1}^\infty$;
- ▷ X has the *primary factorization property* if for every bounded linear operator $T\colon X \to X$, either I_X factors through T or through $I_X - T$;
- ▷ X is *primary* if for every bounded projection $Q\colon X \to X$ either $Q(X)$ or $(I_X - Q)(X)$ is isomorphic to X.

Sometimes, we will say that $(e_j)_{j=1}^\infty$ has the (K-)factorization property if the dual pair is $(X, X^*, \langle\cdot,\cdot\rangle)$ or $(X^*, X, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ denotes the natural pairing. The notion of large diagonal first appeared implicitly in [2] and was later formally introduced in [14]. The factorization property was conceived in [22] and further investigated in [23].

Finally, we will define non-ℓ^1-splicing bases, which were introduced in [19]. We would like to mention that prior to formally introducing the notion "non-ℓ^1-splicing", the underlying properties of such bases were exploited, e.g., in [16, 4] and can be traced back all the way to Lindenstrauss' proof showing that ℓ^∞ is prime [24] (see also [26, Theorem 2.a.7]).

We assume now that $(e_j)_{j=1}^\infty$ is unconditional and define for each $\Lambda \subset \mathbb{N}$ the bounded projection $P_\Lambda\colon X \to X$ by

$$P_\Lambda\Big(\sum_{j=1}^\infty a_j e_j\Big) = \sum_{j\in\Lambda} a_j e_j, \tag{2.3}$$

where the above series converge in the $\sigma(X,Y)$-topology. We then say that the unconditional sequence $(e_j)_{j=1}^\infty$ in X is *non-ℓ^1-splicing* if for every infinite set $\mathcal{I} \subset \mathbb{N}$ and every $\theta > 0$ we can find a sequence $(\Lambda_j)_{j=1}^\infty$ of pairwise disjoint and infinite subsets of $\mathcal{I}$ such that for all sequences $(x_j)_{j=1}^\infty \subset X$ with $\|x_j\|_X \leq 1$ for $j \in \mathbb{N}$ there exists a sequence of scalars $(a_j)_{j=1}^\infty \in \ell^1$ with $\|(a_j)_{j=1}^\infty\|_{\ell^1} = 1$ such that

$$\Big\| \sum_{j=1}^\infty a_j P_{\Lambda_j} x_j \Big\|_X \leq \theta. \tag{2.4}$$

Otherwise, we say that $(e_j)_{j=1}^\infty$ is *ℓ^1-splicing*. If $(e_j)_{j=1}^\infty$ is subsymmetric, J. L. Ansorena [3, Theorem 4] characterized ℓ^1-splicing bases in the following way: $(e_j)_{j=1}^\infty$ is ℓ^1-splicing if and only if there is a disjointly supported sequence $(x_j)_{j=1}^\infty$ in X which is equivalent to the standard unit vector system in ℓ^1. (Although this characterization is stated for weak* Schauder bases, examining the proof reveals that it only uses (2.1) and (2.2) and therefore easily carries over to topological bases.) Examples of non-ℓ^1-splicing weak* Schauder bases are provided in [19] and [3].

Now, let $(e_j)_{j=1}^n$ be a basis for a finite-dimensional Banach space X. Given $1 < r < \infty$, we say that $(e_j)_{j=1}^n$ *satisfies a lower r-estimate with constant $c_r > 0$* if

$$\Big\|\sum_{j=1}^n a_j e_j\Big\|_X \geq c_r \Big(\sum_{j=1}^n |a_j|^r\Big)^{1/r}$$

for all scalars $(a_j)_{j=1}^n$.

For a set $\mathcal{A}$, the cardinality of $\mathcal{A}$ is denoted by $|\mathcal{A}|$. Given a sequence $(x_j)_{j=1}^\infty$ of vectors in a Banach space X and $\mathcal{A} \subset \mathbb{N}$, $[x_j : j \in \mathcal{A}]$ denotes the norm-closure of $\operatorname{span}\{x_j : j \in \mathcal{A}\}$.

3. Results.

This section is divided into two parts: Section 3.1 is devoted to the qualitative infinite-dimensional factorization results Theorem 3.1 and Theorem 3.2, and in Section 3.2 we present our finite-dimensional quantitative factorization results Theorem 3.3 and Corollary 3.5.

3.1. Qualitative factorization results.

P. G. Casazza and B. L. Lin showed in [10] (see also [26, Proposition 3.b.8]) that a Banach space with a subsymmetric Schauder basis has the primary factorization property. Their result was extended in [19, Theorem 1.1] to non-ℓ^1-splicing subsymmetric weak* Schauder bases. As was mentioned in the introduction, it is was not clear how to prove that all non-ℓ^1-splicing subsymmetric weak* Schauder bases have the factorization property. We will now describe the challenges in more detail.

In non-separable spaces, our approach forces us to construct block bases $(b_j)_{j=1}^\infty$ which consist of at least two vectors of (e_j) (see Lemma 4.1) and are equivalent to $(e_j)_{j=1}^\infty$. In general, the large diagonal of a given operator T does not get transferred onto any block basis $(b_j)_{j=1}^\infty$ consisting of more than one vector of $(e_j)_{j=1}^\infty$. However, keeping the diagonal large with respect to the block basis is crucial for us to invert the operator on the block basis. In [2], A. D. Andrew was able to transfer the large diagonal onto the block basis by semiprobabilistically modulating the coefficients of his block basis and increasing the size of the supports $\mathcal{B}_j := \operatorname{supp}(b_j)$, i.e., $\lim_{j\to\infty} |\mathcal{B}_j| = \infty$. This principle of supports growing in size while randomly modulating coefficients was successfully exploited in subsequent works on large diagonals [14, 20, 15, 16, 18, 17, 22, 23].

Unfortunately, in the subsymmetric case, growing support size poses a significant problem: by the nature of our construction, all the coefficients of $(b_j)_{j=1}^\infty$ have absolute value 1, and hence, by [1, Proposition 4], $(b_j)_{j=1}^\infty$ is in general not equivalent to $(e_j)_{j=1}^\infty$. The only way forward to ensure that $(b_j)_{j=1}^\infty$ is equivalent to $(e_j)_{j=1}^\infty$ appears to be [1, Proposition 3], which is keeping the size of our supports bounded: $\sup_j |\mathcal{B}_j| < \infty$. The drawback is that it limits the probabilistic choice of coefficients for $(b_j)_{j=1}^\infty$, making it much more difficult to preserve the large diagonal of T with respect to $(b_j)_{j=1}^\infty$. We compensate for the limited choice in coefficients by additionally randomizing the position of each $\mathcal{B}_j$ and thereby achieve our goal of preserving the large diagonal:

THEOREM 3.1. *Let $((e_j),(f_j))_{j=1}^\infty$ denote a topological basis for the dual pair $(X, Y, \langle\cdot,\cdot\rangle)$ and assume that $(e_j)_{j=1}^\infty$ is (C_u, C_s)-subsymmetric as well as non-ℓ^1-splicing. Then $((e_j),(f_j))_{j=1}^\infty$ has the K-factorization property with $K(\delta) = 2C_u^5C_s^3/\delta$, $\delta > 0$.*

The proof of Theorem 3.1 is given in Section 5.1.

In [9], P. G. Casazza, C. A. Kottman and B. L. Lin showed that the spaces $\ell^p(X)$, $1 < p < \infty$, and $c_0(X)$ are primary whenever X has a (sub)symmetric Schauder basis and is not isomorphic to ℓ^1. Samuel [27] proved that the spaces $\ell^p(\ell^q)$, $1 \le p, q \le \infty$, are primary. Capon [7] showed that $\ell^1(X)$ and $\ell^\infty(X)$ are primary whenever X has a (sub)symmetric Schauder basis. In [19], the author extended the above results to Banach spaces X with a non-ℓ^1-splicing subsymmetric weak* Schauder basis. To be precise, [19, Theorem 1.2] asserts that the Banach spaces $\ell^p(X)$, $1 \le p \le \infty$, have the primary factorization property and are therefore primary. For the same reasons described above Theorem 3.1, it also remained open whether the standard basis of $\ell^p(X)$ has the factorization property. Theorem 3.2 below gives an affirmative answer whenever the subsymmetric basis is non-ℓ^1-splicing. We would like to point out that mixing of different Banach spaces in the direct sum is allowed.

THEOREM 3.2. *For each $n \in \mathbb{N}$, let $((e_{n,j}), (f_{n,j}))_{j=1}^{\infty}$ denote a topological basis for the dual pair $(X_n, Y_n, \langle\cdot,\cdot\rangle_n)$ with constant C_d (uniformly in n). For all $n \in \mathbb{N}$, suppose that $(e_{n,j})_{j=1}^{\infty}$ is (C_u, C_s)-subsymmetric (uniformly in n) and non-ℓ^1-splicing. Then for all $1 \le p \le \infty$, $((e_{n,j}), (f_{n,j}))_{n,j=1}^{\infty}$ has the uniform factorization property in $\ell^p((X_n)_{n=1}^{\infty})$.*

We refer to Section 5.2 for the proof of Theorem 3.2.

3.2. Finite-dimensional quantitative factorization results. In [5, Theorem 6.1], J. Bourgain and L. Tzafriri obtained the following restricted invertibility result for operators acting on n-dimensional Banach spaces with an unconditional basis $(e_j)_{j=1}^{n}$ which satisfies a lower r-estimate for some $1 < r < \infty$: For any $0 < \varepsilon < 1$, there exists a subset σ with $|\sigma| \ge n^{1-\varepsilon}$ on which a given operator is invertible when restricted to the subspace $[e_j : j \in \sigma]$. Analyzing their proof, we recognize that we can relax the condition that $(e_j)_{j=1}^{\infty}$ has to satisfy a lower r-estimate (see Theorem 3.3 and Remark 3.4). We would like to point out the closely related recent works [18, 17], in which the dependence on the dimension for quantitative factorization results in one- and two-parameter Hardy and BMO spaces was improved from super-exponential estimates to polynomial estimates.

In this subsection, $(e_j)_{j=1}^{\infty}$ denotes a normalized basis for the Banach space X and $(e_j^*)_{j=1}^{\infty}$ denotes the biorthogonal functionals to $(e_j)_{j=1}^{\infty}$. We define the function $\tau\colon \mathbb{N} \to [0,\infty)$ by putting

$$\tau(n) = \max \min\Big(\max_{1\le j\le n} \Big\| \sum_{\substack{i=1\\ i\ne j}}^{n} \varepsilon_{ij} e_i \Big\|_X, \max_{1\le i\le n} \Big\| \sum_{\substack{j=1\\ j\ne i}}^{n} \varepsilon_{ij} e_j^* \Big\|_{X^*} \Big) \tag{3.1}$$

where the maximum is taken over all $\varepsilon_{ij} \in \{\pm 1\}$ and $1 \le i, j \le n$. Note that if $(e_j)_{j=1}^{\infty}$ is C_u-unconditional, we have the estimates

$$C_u^{-1} \le \tau(n) \le C_u \min\Big(\Big\| \sum_{i=1}^{n} e_i \Big\|_X, \Big\| \sum_{j=1}^{n} e_j^* \Big\|_{X^*} \Big), \qquad n \ge 2. \tag{3.2}$$

We are now ready to state our first result on restricted invertibility.

THEOREM 3.3. *Let $(e_j)_{j=1}^{\infty}$ be a normalized C_u-unconditional basis for a Banach space X, let $(e_j^*)_{j=1}^{\infty}$ denote the biorthogonal functionals to $(e_j)_{j=1}^{\infty}$ and put $X_n = [e_j : 1 \le j \le n]$, $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $\delta, \Gamma, \eta > 0$ be such that*

$$\frac{\delta \min(1,\eta)}{4\Gamma n} \le \tau(n) \le \frac{\delta \min(1,\eta)}{2^{10}\Gamma} \cdot \frac{[16 + \min(\eta, (1+\eta)^{-1})n]^2}{n}. \tag{3.3}$$

Let $T\colon X_n \to X_n$ denote an operator satisfying

$$\|T\| \le \Gamma \quad \text{and} \quad |\langle T e_j, e_j^* \rangle| \ge \delta, \quad 1 \le j \le n, \tag{3.4}$$

and let $D\colon X_n \to X_n$ *denote the diagonal operator of* T, *i.e.,*

$$De_i = \langle Te_i, e_i^* \rangle e_i, \qquad 1 \le i \le n.$$

For each $\sigma \subset \{1,\dots,n\}$ *define the restriction operators* $R_\sigma\colon X_n \to X_n$ *by* $R_\sigma(\sum_{i=1}^n a_i e_i) = \sum_{i\in\sigma} a_i e_i$. *Then there exists a subset* $\sigma \subset \{1,\dots,n\}$ *with*

$$|\sigma| \ge \sqrt{\frac{\delta \min(1,\eta)}{16\Gamma}} \cdot \sqrt{\frac{n}{\tau(n)}} \tag{3.5}$$

such that the operator $R_\sigma D^{-1} T R_\sigma$ *is invertible and satisfies*

$$\|(R_\sigma D^{-1} T R_\sigma)^{-1}\| \le 1 + \eta. \tag{3.6}$$

Moreover, if we define $X_\sigma = [e_j : j \in \sigma]$, *there exist operators* $E\colon X_\sigma \to X_n$ *and* $P\colon X_n \to X_\sigma$ *with* $\|E\|\|P\| \le C_u^2 \frac{1+\eta}{\delta}$ *such that* $I_{X_\sigma} = PTE$.

Theorem 3.3 will be proved in Section 5.3.

We will now relate Theorem 3.3 to [5, Theorem 6.1].

Remark 3.4. Let $1 < r, s < \infty$ with $1/r + 1/s = 1$ and suppose that $(e_j)_{j=1}^n$ satisfies a lower r-estimate with constant c_r. Then one can easily verify that $(e_j^*)_{j=1}^n$ satisfies an upper s-estimate with constant $1/c_r$, i.e.,

$$\Big\|\sum_{j=1}^n a_j e_j^*\Big\| \le \frac{1}{c_r}\Big(\sum_{j=1}^n |a_j|^s\Big)^{1/s}.$$

In particular, we obtain the estimate $\tau(n) \le \frac{1}{c_r}(n-1)^{1/s}$. Thus, if we choose

$$n \ge \left(\frac{2^{10}\Gamma}{c_r \delta \min(1,\eta)\min(\eta,(1+\eta)^{-1})}\right)^{1/r},$$

Theorem 3.3 yields a subset $\sigma \subset \{1,\dots,n\}$ with

$$|\sigma| \ge \sqrt{\frac{c_r \delta \min(1,\eta)}{16\Gamma}} \cdot n^{1/(2r)}.$$

In [6, Corollary 4.4], Bourgain and Tzafriri obtained linear estimates for the size of the set σ whenever the subsymmetric basis $(e_j)_{j=1}^\infty$ satisfies certain conditions in terms of Boyd indices. Using Theorem 3.3, we can get rid of these restrictions entirely and obtain reasonable quantitative estimates for all subsymmetric bases.

Corollary 3.5. *Let* $(e_j)_{j=1}^\infty$ *be a normalized* (C_u, C_s)*-subsymmetric basis for a Banach space* X, *let* $(e_j^*)_{j=1}^\infty$ *denote the biorthogonal functionals to* $(e_j)_{j=1}^\infty$ *and put* $X_n = [e_j : 1 \le j \le n]$, $n \in \mathbb{N}$. *Let* $n \in \mathbb{N}$ *and* $\delta, \Gamma, \eta > 0$ *be such that*

$$n \ge 1 + \frac{C_u \delta \min(1,\eta)}{4\Gamma} \quad \text{and} \quad n \ge \frac{2^{11} C_s^3}{\delta^2 \min(1,\eta^2)\min(\eta^4,(1+\eta)^{-4})}. \tag{3.7}$$

Let $T\colon X_n \to X_n$ denote an operator satisfying

$$\|T\| \le \Gamma \quad \text{and} \quad |\langle Te_j, e_j^*\rangle| \ge \delta, \quad 1 \le j \le n. \tag{3.8}$$

As in Theorem 3.3, $D\colon X_n \to X_n$ denotes the diagonal operator of T and $R_\sigma\colon X_n \to X_n$, $\sigma \subset \{1,\dots,n\}$ the restriction operators. Then there exists a subset $\sigma \subset \{1,\dots,n\}$ with

$$|\sigma| \ge 4 \cdot (2C_u^3 C_s^3)^{-1/4} \cdot \sqrt{\frac{\delta \min(1,\eta)}{\Gamma}} \cdot n^{1/4} \tag{3.9}$$

such that the operator $R_\sigma D^{-1} T R_\sigma$ is invertible and satisfies

$$\|(R_\sigma D^{-1} T R_\sigma)^{-1}\| \le 1 + \eta. \tag{3.10}$$

Thus, for $k = |\sigma|$, there exist operators $E\colon X_k \to X_n$ and $P\colon X_n \to X_k$ with $\|E\|\,\|P\| \le C_u^2 C_s^2 (1+\eta)/\delta$ such that $I_{X_k} = PTE$.

The proof of Corollary 3.5 can be found in Section 5.3.

4. Tools. Here, we provide the mathematical tools that will be used in Sections 5.1 and 5.2. They comprise subspace annihilation results (see Lemmata 4.1 and 4.4) and techniques by which we preserve the large diagonal of an operator under blocking of the basis (see Lemma 4.2 and Proposition 4.3). Moreover, we provide estimates for basic factorization operators that are obtained by blocking a subsymmetric basis (see Proposition 4.5).

LEMMA 4.1. *Let $\mathcal{I} \subset \mathbb{N}$ denote an infinite set, let $n \in \mathbb{N}$, $L \in 2\cdot\mathbb{N}$ and $\eta > 0$. Then for all bounded sequences $(x_j)_{j=1}^\infty$ in X and $(y_j)_{j=1}^\infty$ in Y and every $n \in \mathbb{N}$ there exists an infinite set $\Lambda \subset \mathcal{I}$ such that*

$$\sup\Big\{\Big|\Big\langle \sum_{k\in\mathcal{B}} \varepsilon_k x_k, y_j\Big\rangle\Big| : \mathcal{B} \subset \Lambda,\ |\mathcal{B}| = L,\ \varepsilon \in \mathcal{E}(\mathcal{B}),\ 1 \le j \le n\Big\} \le \eta,$$

$$\sup\Big\{\Big|\Big\langle x_j, \sum_{k\in\mathcal{B}} \varepsilon_k y_k\Big\rangle\Big| : \mathcal{B} \subset \Lambda,\ |\mathcal{B}| = L,\ \varepsilon \in \mathcal{E}(\mathcal{B}),\ 1 \le j \le n\Big\} \le \eta,$$

where

$$\mathcal{E}(\mathcal{B}) = \Big\{(\varepsilon_k) \in \{\pm 1\}^{\mathcal{B}} : \sum_{k\in\mathcal{B}} \varepsilon_k = 0\Big\}. \tag{4.1}$$

Proof. The proof is a straightforward adaptation of the argument given for [19, Lemma 3.1]. For the sake of completeness, we give a short proof.

First, we put

$$\mathcal{G}_k^1 = \{i \in \mathcal{I} : (k-1)\eta/L < \langle x_i, y_1\rangle \le k\eta/L\}$$

and note that by (B3), $\bigcup_{k\in\mathbb{Z}} \mathcal{G}_k^1 = \mathcal{I}$ and that there are only finitely many non-empty $\mathcal{G}_k^1$, $k \in \mathbb{Z}$. Since $\mathcal{I}$ is infinite, there exists at least one $k_1 \in \mathbb{Z}$ such that $\mathcal{G}_{k_1}^1$ is also infinite. Next, we define

$$\mathcal{G}_k^2 = \{i \in \mathcal{G}_{k_1}^1 : (k-1)\eta/L < \langle x_i, y_2\rangle \le k\eta/L\}$$

and repeat the previous step to obtain a $k_2 \in \mathbb{Z}$ such that $\mathcal{G}^2_{k_2}$ is infinite. After n steps, we obtain an infinite set $\mathcal{G} = \mathcal{G}^n_{k_n}$ such that

$$\sup\{|\langle x_{i_0}, y_j\rangle - \langle x_{i_1}, y_j\rangle| : i_0, i_1 \in \mathcal{G},\, 1 \le j \le n\} \le 2\eta/L. \tag{4.2}$$

We will now repeat the above process but with the roles of x and y reversed and with $\mathcal{G}$ instead of $\mathcal{I}$. To illustrate this, we now put $\mathcal{H}^1_k = \{i \in \mathcal{G} : (k-1)\eta/L < \langle x_1, y_i\rangle \le k\eta/L\}$. With the same reasoning as above, we can find $l_1 \in \mathbb{Z}$ such that $\mathcal{H}^1_{l_1}$ is infinite. Iterating this procedure and stopping after n steps yields an infinite set $\Lambda = \mathcal{H}^n_{l_n} \subset \mathcal{G}$ such that

$$\sup\{|\langle x_j, y_{i_0}\rangle - \langle x_j, y_{i_1}\rangle| : i_0, i_1 \in \Lambda,\, 1 \le j \le n\} \le 2\eta/L. \tag{4.3}$$

Note that for each $\mathcal{B} \subset \Lambda$ with $|\mathcal{B}| = L$ and each $(\varepsilon_k) \in \mathcal{E}(\mathcal{B})$, there are exactly $L/2$ elements $k \in \mathcal{B}$ such that $\varepsilon_k = 1$ and $L/2$ elements $k \in \mathcal{B}$ such that $\varepsilon_k = -1$. This observation together with (4.2) and (4.3) proves the assertion. ∎

For each $L \in 2\cdot\mathbb{N}$ and $\mathcal{A} \subset \mathbb{N}$ with $L \le |\mathcal{A}| < \infty$, we define a finite set $\Omega^{\mathcal{A}}_L$ by

$$\Omega^{\mathcal{A}}_L = \{(\mathcal{B}, (\varepsilon_k)) : \mathcal{B} \subset \mathcal{A},\, |\mathcal{B}| = L,\, (\varepsilon_k) \in \mathcal{E}(\mathcal{B})\}. \tag{4.4}$$

We denote by $\mathbb{E}^{\mathcal{A}}_L$ the average over all elements in $\Omega^{\mathcal{A}}_L$. For each $(\mathcal{B}, (\varepsilon_k)) \in \Omega^{\mathcal{A}}_L$, we define

$$b^{(\varepsilon_k)}_{\mathcal{B}} = \sum_{k\in\mathcal{B}} \varepsilon_k e_k \quad\text{and}\quad d^{(\varepsilon_k)}_{\mathcal{B}} = \sum_{k\in\mathcal{B}} \varepsilon_k f_k. \tag{4.5}$$

Finally, we define $\nu\colon \mathbb{N} \to [0,\infty)$ by

(4.6)
$$\nu(n) = \sup\Big\{\min\Big(\max_{l\in\mathcal{A}}\Big\|\sum_{k\in\mathcal{A}\setminus\{l\}} e_k\Big\|_X, \max_{k\in\mathcal{A}}\Big\|\sum_{l\in\mathcal{A}\setminus\{k\}} f_l\Big\|_Y\Big) : \mathcal{A} \subset \mathbb{N},\, |\mathcal{A}| = n\Big\}.$$

The following lemma uses a randomization technique from [21]. In contrast to the randomization over all possible choices of signs in [21], in Lemma 4.2 we average over all signs in the smaller set $\mathcal{E}(\mathcal{B})$. Selecting signs limited to $\mathcal{E}(\mathcal{B})$ is essential in non-separable spaces, as it allows us to use Lemma 4.1 to diagonalize large parts of a given operator. The drawback of averaging over $\mathcal{E}(\mathcal{B})$ is that it introduces a negative bias when averaging $\varepsilon_k\varepsilon_l$, $k \ne l$ (see (4.8)). Eventually, this bias is controlled in terms of ν.

LEMMA 4.2. *Let $((e_j),(f_j))_{j=1}^\infty$ be a topological basis for a dual pair $(X, Y, \langle\cdot,\cdot\rangle)$. Let $T\colon X \to X$ denote a bounded linear operator such that $\delta := \inf_j \langle Te_j, f_j\rangle > 0$. Let $L \in 2\cdot\mathbb{N}$, $N \in \mathbb{N}$ with $N \ge L$ and pick any $\mathcal{A} \subset \mathbb{N}$ with $|\mathcal{A}| = N$. Then*

$$\mathbb{E}^{\mathcal{A}}_L \langle Tb^{(\varepsilon_k)}_{\mathcal{B}}, d^{(\varepsilon_k)}_{\mathcal{B}}\rangle \ge \Big[\delta - C_d \frac{\|T\|}{N-1}\nu(N)\Big] \cdot L.$$

Proof. Let $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{B}| = L$ be fixed and note that by (4.5), (4.4) and (4.1),

$$\frac{1}{\binom{L}{L/2}} \sum_{(\varepsilon_k)\in\mathcal{E}(\mathcal{B})} \langle Tb_{\mathcal{B}}^{(\varepsilon_k)}, d_{\mathcal{B}}^{(\varepsilon_k)}\rangle = \sum_{k\in\mathcal{B}} \langle Te_k, f_k\rangle + \sum_{k\neq l\in\mathcal{B}} \frac{1}{\binom{L}{L/2}} \sum_{(\varepsilon_k)\in\mathcal{E}(\mathcal{B})} \varepsilon_k\varepsilon_l\langle Te_k, f_l\rangle. \tag{4.7}$$

A straightforward calculation shows that

$$\frac{1}{\binom{L}{L/2}} \sum_{(\varepsilon_k)\in\mathcal{E}(\mathcal{B})} \varepsilon_k\varepsilon_l = \frac{-1}{L-1}, \qquad k \neq l. \tag{4.8}$$

Combining (4.7) with (4.8) yields

$$\begin{aligned}\frac{1}{\binom{L}{L/2}} \sum_{(\varepsilon_k)\in\mathcal{E}(\mathcal{B})} \langle Tb_{\mathcal{B}}^{(\varepsilon_k)}, d_{\mathcal{B}}^{(\varepsilon_k)}\rangle &= \sum_{k\in\mathcal{B}} \langle Te_k, f_k\rangle - \frac{1}{L-1} \sum_{k\neq l\in\mathcal{B}} \langle Te_k, f_l\rangle \\ &= A_1 - \frac{1}{L-1} A_2.\end{aligned} \tag{4.9}$$

We will now separately average A_1 and A_2 over all $\binom{N}{L}$ possible selections $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{B}| = L$. First, averaging A_1 yields

$$\begin{aligned}\frac{1}{\binom{N}{L}} \sum_{\substack{\mathcal{B}\subset\mathcal{A}\\|\mathcal{B}|=L}} A_1 &= \frac{1}{\binom{N}{L}} \sum_{\substack{\mathcal{B}\subset\mathcal{A}\\|\mathcal{B}|=L}} \sum_{k\in\mathcal{B}} \langle Te_k, f_k\rangle \\ &= \sum_{k\in\mathcal{A}} \langle Te_k, f_k\rangle \frac{|\{\mathcal{B}\subset A : \mathcal{B}\ni k,\ |\mathcal{B}|=L\}|}{\binom{N}{L}} = \frac{\binom{N-1}{L-1}}{\binom{N}{L}} \sum_{k\in\mathcal{A}} \langle Te_k, f_k\rangle\end{aligned}$$

and we record

$$\frac{1}{\binom{N}{L}} \sum_{\substack{\mathcal{B}\subset\mathcal{A}\\|\mathcal{B}|=L}} A_1 = \frac{L}{N} \sum_{k\in\mathcal{A}} \langle Te_k, f_k\rangle. \tag{4.10}$$

Secondly, averaging A_2 gives us

$$\begin{aligned}\frac{1}{\binom{N}{L}} \sum_{\substack{\mathcal{B}\subset\mathcal{A}\\|\mathcal{B}|=L}} A_2 &= \frac{1}{\binom{N}{L}} \sum_{\substack{\mathcal{B}\subset\mathcal{A}\\|\mathcal{B}|=L}} \sum_{k\neq l\in\mathcal{B}} \langle Te_k, f_l\rangle \\ &= \sum_{k\neq l\in\mathcal{A}} \langle Te_k, f_l\rangle \frac{|\{\mathcal{B}\subset A : \mathcal{B}\ni k, l,\ |\mathcal{B}|=L\}|}{\binom{N}{L}} \\ &= \frac{\binom{N-2}{L-2}}{\binom{N}{L}} \sum_{k\neq l\in\mathcal{A}} \langle Te_k, f_l\rangle,\end{aligned}$$

from which it immediately follows that

$$\frac{1}{\binom{N}{L}} \sum_{\substack{\mathcal{B}\subset\mathcal{A}\\|\mathcal{B}|=L}} A_2 = \frac{L(L-1)}{N(N-1)} \sum_{k\neq l\in\mathcal{A}} \langle Te_k, f_l\rangle. \tag{4.11}$$

Combining (4.9)–(4.11) yields

$$\begin{aligned}\mathbb{E}_L^{\mathcal{A}} \langle Tb_{\mathcal{B}}^{(\varepsilon_k)}, d_{\mathcal{B}}^{(\varepsilon_k)}\rangle &= \frac{L}{N}\sum_{k\in\mathcal{A}}\langle Te_k, f_k\rangle - \frac{L}{N(N-1)}\sum_{k\neq l\in\mathcal{A}}\langle Te_k, f_l\rangle\\ &= \frac{L}{N}B_1 - \frac{L}{N(N-1)}B_2.\end{aligned} \tag{4.12}$$

By definition of δ and $\mathcal{A}$, we obtain

$$B_1 = \sum_{k\in\mathcal{A}} \langle Te_k, f_k\rangle \geq \delta N. \tag{4.13}$$

Now we will estimate B_2 in two different ways, each exploiting the linearity of $\langle\cdot,\cdot\rangle$ in the respective component. Exploiting the linearity in the second component of the bilinear form and using (B3) yields

$$\begin{aligned}B_2 = \sum_{k\neq l\in\mathcal{A}} \langle Te_k, f_l\rangle &= \sum_{k\in\mathcal{A}} \Big\langle Te_k, \sum_{l\in\mathcal{A}\setminus\{k\}} f_l\Big\rangle \leq C_d\|T\| \sum_{k\in\mathcal{A}} \Big\|\sum_{l\in\mathcal{A}\setminus\{k\}} f_l\Big\|\\ &\leq C_d\|T\|N \max_{k\in\mathcal{A}} \Big\|\sum_{l\in\mathcal{A}\setminus\{k\}} f_l\Big\| = C_d\|T\|N \max_{k\in\mathcal{A}} \Big\|\sum_{l\in\mathcal{A}\setminus\{k\}} f_l\Big\|_Y.\end{aligned}$$

Using the exact same steps as above but exploiting the linearity in the first component of the bilinear form yields

$$B_2 \leq C_d\|T\|N \max_{l\in\mathcal{A}} \Big\|\sum_{k\in\mathcal{A}\setminus\{l\}} e_k\Big\|_X.$$

Thus, we have the estimate

$$B_2 \leq C_d\|T\|N\nu(N). \tag{4.14}$$

Combining (4.12)–(4.14) concludes the proof. ■

For $n\in\mathbb{N}$ we define the functions $\lambda,\mu\colon\mathbb{N}\to[0,\infty)$ by

$$\lambda(n) = \Big\|\sum_{j=1}^n e_j\Big\|_X \quad\text{and}\quad \mu(n) = \Big\|\sum_{j=1}^n f_j\Big\|_Y. \tag{4.15}$$

If $(e_j)_{j=1}^\infty$ is C_s-spreading, then by (4.6) and (2.2), we obtain

$$\nu(n) \leq C_s \min(\lambda(n-1), \mu(n-1)), \qquad n\in\mathbb{N}. \tag{4.16}$$

We note that keeping track of the constants in [26, proof of Propositions 3.a.6 and 3.a.4] yields

$$\lambda(n)\mu(n) \leq 2C_uC_sn. \tag{4.17}$$

We are now ready to prove Proposition 4.3 by specializing Lemma 4.2 to the case where $(e_j)_{j=1}^\infty$ is subsymmetric.

PROPOSITION 4.3. *Let $((e_j),(f_j))_{j=1}^\infty$ be a topological basis for a dual pair $(X,Y,\langle\cdot,\cdot\rangle)$ and suppose that $(e_j)_{j=1}^\infty$ is (C_u,C_s)-subsymmetric. Let $T\colon X\to X$ denote a bounded linear operator such that $\delta := \inf_j \langle Te_j, f_j\rangle > 0$. For $L\in 2\cdot\mathbb{N}$, $0<\kappa<1$ define*

$$N = \max\Big(L, 1 + \Big\lceil \frac{2C_d^2C_uC_s^3\|T\|^2}{\kappa^2\delta^2}\Big\rceil\Big). \tag{4.18}$$

Then for each $\mathcal{A}\subset\mathbb{N}$ with $|\mathcal{A}|=N$,

$$\mathbb{E}_L^{\mathcal{A}}\,\langle Tb_{\mathcal{B}}^{(\varepsilon_k)}, d_{\mathcal{B}}^{(\varepsilon_k)}\rangle \ge (1-\kappa)\delta L. \tag{4.19}$$

In particular, there exists a set $\mathcal{B}\subset\mathcal{A}$ with $|\mathcal{B}|=L$ and a choice of signs $(\varepsilon_k)\in\mathcal{E}(\mathcal{B})$ such that

$$\Big\langle T\sum_{k\in\mathcal{B}}\varepsilon_k e_k, \sum_{k\in\mathcal{B}}\varepsilon_k f_k\Big\rangle \ge (1-\kappa)\delta L. \tag{4.20}$$

Proof. Lemma 4.2 yields

$$\mathbb{E}_L^{\mathcal{A}}\,\langle Tb_{\mathcal{B}}^{(\varepsilon_k)}, d_{\mathcal{B}}^{(\varepsilon_k)}\rangle \ge \Big[\delta - C_d\frac{\|T\|}{N-1}\nu(N)\Big]\cdot L.$$

Using (4.16) and (4.17) gives

$$\begin{aligned}\nu(N) &\le C_s\min(\lambda(N-1),\mu(N-1)) \le C_s\sqrt{\lambda(N-1)\mu(N-1)}\\ &\le \sqrt{2C_uC_s^3(N-1)}.\end{aligned}$$

Thus far, we have proved

$$\mathbb{E}_L^{\mathcal{A}}\,\langle Tb_{\mathcal{B}}^{(\varepsilon_k)}, d_{\mathcal{B}}^{(\varepsilon_k)}\rangle \ge \Big[\delta - C_d\frac{\|T\|}{\sqrt{N-1}}\cdot\sqrt{2C_uC_s^3}\Big]\cdot L.$$

The latter inequality together with (4.18) implies (4.19). Finally, (4.20) directly follows from (4.19) and the definition of $\mathbb{E}_L^{\mathcal{A}}$. ∎

We will now essentially restate the result [19, Lemma 3.2] (which is phrased for weak* Schauder bases) for our dual system.

LEMMA 4.4. *Let $((e_j),(f_j))_{j=1}^\infty$ be a topological basis for a dual pair $(X,Y,\langle\cdot,\cdot\rangle)$ and assume that $(e_j)_{j=1}^\infty$ is an unconditional, non-ℓ^1-splicing sequence. Let $\mathcal{I}\subset\mathbb{N}$ denote an infinite set, $\eta>0$ and $y_1,\dots,y_n\in Y$ and let $T\colon X\to X$ denote a bounded linear operator. Then there exists an infinite set $\Lambda\subset\mathcal{I}$ such that*

$$\sup_{\|x\|_X\le 1}|\langle TP_\Lambda x, y_j\rangle| \le \eta, \qquad 1\le j\le n.$$

Proof. For $n=1$ the proof of [19, Lemma 3.2] applies and is therefore omitted.

We now use induction. In the first step, we apply the lemma with $n = 1$ to y_1 and obtain an infinite set $\Lambda_1 \subset \mathcal{I}$ such that $\sup_{\|x\|_X \le 1} |\langle TP_\Lambda x, y_1\rangle| \le \eta/C_u$. Next, we apply the lemma with $n = 1$ to $\mathcal{I} = \Lambda_1$ and y_2 and obtain an infinite set $\Lambda_2 \subset \Lambda_1$ such that $\sup_{\|x\|_X \le 1} |\langle TP_{\Lambda_2} x, y_2\rangle| \le \eta/C_u$. Continuing in this manner and stopping after n steps yields infinite sets $\Lambda_1 \supset \cdots \supset \Lambda_n$ such that

$$\sup_{\|x\|_X \le 1} |\langle TP_{\Lambda_j} x, y_j\rangle| \le \eta/C_u, \qquad 1 \le j \le n.$$

Define $\Lambda = \Lambda_n$ and observe that for $x \in X$ and $1 \le j \le n$, unconditionality yields

$$|\langle TP_\Lambda x, y_j\rangle| = |\langle TP_{\Lambda_j} P_\Lambda x, y_j\rangle| \le \eta\|P_\Lambda x\|/C_u \le \eta\|x\|$$

as claimed. ∎

The following proposition gives estimates for our basic factorization operators B, Q, defined below.

Proposition 4.5. *Let $((e_j), (f_j))_{j=1}^\infty$ be a topological basis for a dual pair $(X, Y, \langle\cdot,\cdot\rangle)$ and assume that $(e_j)_{j=1}^\infty$ is a (C_u, C_s)-subsymmetric sequence. Let $L \in \mathbb{N}$ and let $(\mathcal{B}_j)$ denote a sequence of sets $\mathcal{B}_j \subset \mathbb{N}$ with $|\mathcal{B}_j| = L$, $\mathcal{B}_j < \mathcal{B}_{j+1}$, $j \in \mathbb{N}$, and let $(\varepsilon_k) \in \{\pm 1\}^{\mathbb{N}}$. Define*

$$b_j = \sum_{k \in \mathcal{B}_j} \varepsilon_k e_k \quad \text{and} \quad d_j = \sum_{k \in \mathcal{B}_j} \varepsilon_k f_k, \tag{4.21}$$

and define operators $B, Q\colon X \to X$ by

$$Bx = \sum_{j=1}^\infty \langle x, f_j\rangle b_j \quad \text{and} \quad Qx = \sum_{j=1}^\infty \langle x, d_j\rangle e_j, \qquad x \in X. \tag{4.22}$$

The operators are well defined, $QB = L \cdot I_X$, and

$$\|B\|, \|Q\| \le C_u C_s \cdot L. \tag{4.23}$$

Proof. Repeating Step 2 in the proof given for [19, Theorem 1.1] essentially gives the result. Since the proof is short, we include it here for the sake of completeness.

First we pick any (n_j^l) such that $\{n_j^l : 1 \le l \le L\} = \mathcal{B}_j$, $j \in \mathbb{N}$, and note that the sequence $(n_j^l)_j$ is increasing for each $1 \le l \le L$. Now, let $x = \sum_{j=1}^\infty a_j e_j$ be a series that converges in the $\sigma(X, Y)$-topology. Hence, by (2.1) and (2.2), $Bx = \sum_{l=1}^L \sum_{j=1}^\infty \varepsilon_{n_j} a_j e_{n_j^l}$ is well defined and satisfies (4.23) as claimed. Similarly, since $Qx = \sum_{l=1}^L \sum_{j=1}^\infty \varepsilon_{n_j^l} a_{n_j^l} e_j$, the operator Q is well defined and satisfies (4.23). The identity $QB = L \cdot I_X$ promptly follows from (B4). ∎

5. Proofs. In this section we will present the proofs of our infinite-dimensional factorization results: Theorem 3.1 (see Section 5.1) and Theorem 3.2 (see Section 5.2), as well as our restricted invertibility results: Theorem 3.3 and Corollary 3.5 (see Section 5.3).

5.1. Banach spaces with a subsymmetric basis. Here, we will prove our first main result, Theorem 3.1. Our strategy is to use Lemmata 4.1 and 4.4 to diagonalize the operator with a block basis $(b_j)_{j=1}^\infty$ of $(e_j)_{j=1}^\infty$, while using Proposition 4.3 to transfer the large diagonal of the operator with respect to $(e_j)_{j=1}^\infty$ over to the block basis $(b_j)_{j=1}^\infty$.

Proof of Theorem 3.1. Let $T\colon X \to X$ be a bounded linear operator which has δ-large diagonal with respect to $(e_j)_{j=1}^\infty$. Given $\eta > 0$, we want to show that I_X K-factors through T with $K(\delta) = 2C_u^5C_s^3/\delta + \eta$, $\delta > 0$. First, we define the multiplier $M\colon X \to X$ by $Me_j = \operatorname{sign}(\langle Te_j, f_j\rangle)e_j$ and note that by (2.1), M is a well defined operator satisfying $\|M\| \le C_u$. Thus, if we define $\widetilde{T} = TM$, then

$$\inf_j \langle \widetilde{T}e_j, f_j\rangle = \inf_j |\langle Te_j, f_j\rangle| = \delta > 0. \tag{5.1}$$

Throughout this proof, we will use the following constants:

$$\begin{aligned} &\kappa = \frac{1}{2 + 4C_u^5C_s^3/\eta\delta}, && \eta_i = \frac{2^{-i-1}}{C_d i}\kappa\delta, \quad i \in \mathbb{N}, \\ &L = 2, && N = \max\Bigl(2, 1 + \Bigl\lceil \frac{2C_uC_s^3\|T\|^2}{\kappa^2\delta^2}\Bigr\rceil\Bigr). \end{aligned} \tag{5.2}$$

Step 1: *Construction of the block basis.* In each step of the subsequent construction, we will employ Lemma 4.1, Lemma 4.4 and Proposition 4.3 to $\widetilde{T}$ and with the constants κ, L and N as defined in (5.2). Other parameters will be explicitly specified when appropriate.

For our initial step, we use Proposition 4.3 with $\mathcal{A} = \mathcal{A}_1 = \{1, \dots, N\}$ to obtain a set $\mathcal{B}_1 \subset \mathcal{A}_1$ with $|\mathcal{B}_1| = 2$ and signs $(\varepsilon_k) \in \mathcal{E}(\mathcal{B}_1)$ such that

$$\langle \widetilde{T}b_1, d_1\rangle \ge 2(1 - \kappa)\delta, \tag{5.3}$$

where we defined

$$b_1 = \sum_{k\in\mathcal{B}_1} \varepsilon_k e_k \quad \text{and} \quad d_1 = \sum_{k\in\mathcal{B}_1} \varepsilon_k f_k. \tag{5.4}$$

This completes the initial step of our construction.

Assume that we have already chosen pairwise disjoint sets $\mathcal{B}_1 < \dots < \mathcal{B}_{i-1}$ with $|\mathcal{B}_j| = 2$, $1 \le j \le i - 1$, that we have selected signs $(\varepsilon_k) \in \mathcal{E}(\mathcal{B}_j)$, $1 \le j \le i - 1$, and that we have defined

$$b_j = \sum_{k\in\mathcal{B}_j} \varepsilon_k e_k \quad \text{and} \quad d_j = \sum_{k\in\mathcal{B}_j} \varepsilon_k f_k, \qquad 1 \le j \le i - 1. \tag{5.5}$$

We will now construct b_i and d_i. First, we apply Lemma 4.4 with

$$\mathcal{I} = \{k \in \Lambda^1_{i-1} : k > \max \mathcal{B}_{i-1}\}, \qquad \eta = \eta_i/C_u, \qquad y_j = d_j, \quad 1 \le j \le i-1,$$

to obtain an infinite set $\Lambda^0_i \subset \mathcal{I}$ such that

$$\sup_{\|x\|_X \le 1} |\langle \widetilde{T} P_{\Lambda^0_i} x, d_j\rangle| \le \eta_i/C_u \tag{5.6}$$

for all $1 \le j \le i-1$. Next, using Lemma 4.1 with $\mathcal{I} = \Lambda^0_i$, $\eta = \eta_i$, $x_j = Tb_j$, $y_j = f_j$, $1 \le j \le i-1$, yields an infinite set $\Lambda^1_i \subset \Lambda^0_i$ such that

$$\sup\Big\{\Big|\Big\langle \widetilde{T} b_j, \sum_{k\in\mathcal{B}} \varepsilon_k f_k\Big\rangle\Big| : \mathcal{B} \subset \Lambda^1_i,\, |\mathcal{B}| = 2,\, \varepsilon \in \mathcal{E}(\mathcal{B}),\, 1 \le j \le i-1\Big\} \le \eta_i. \tag{5.7}$$

Now we select any $\mathcal{A}_i \subset \Lambda^1_i$ with $|\mathcal{A}_i| = N$. Applying Proposition 4.3 to $\mathcal{A} = \mathcal{A}_i$ gives a set $\mathcal{B}_i \subset \mathcal{A}_i$ with $|\mathcal{B}_i| = 2$ and $(\varepsilon_k) \in \mathcal{E}(\mathcal{B}_i)$ such that

$$\langle \widetilde{T} b_i, d_i\rangle \ge 2(1-\kappa)\delta, \tag{5.8}$$

where we put

$$b_i = \sum_{k\in\mathcal{B}_i} \varepsilon_k e_k \quad \text{and} \quad d_i = \sum_{k\in\mathcal{B}_i} \varepsilon_k f_k. \tag{5.9}$$

This concludes the inductive construction of our block basis.

To summarize, we proved the following estimates. From (5.7) and (5.8) we obtain

$$|\langle \widetilde{T} b_j, d_i\rangle| \le \eta_i \quad \text{and} \quad \langle \widetilde{T} b_i, d_i\rangle \ge 2(1-\kappa)\delta, \qquad i,j \in \mathbb{N},\, j \le i-1, \tag{5.10}$$

and since $\mathcal{B}_i \subset \Lambda^1_i \subset \Lambda^0_i \subset \Lambda^1_{i-1}$ with $|\mathcal{B}_i| = 2$, (5.6) yields

$$\Big|\Big\langle \widetilde{T} \sum_{j=i+1}^{\infty} a_j b_j, d_i\Big\rangle\Big| \le \frac{\eta_i}{C_u}\Big\|\sum_{j=i+1}^{\infty} a_j b_j\Big\| \tag{5.11}$$

for all $i,j \in \mathbb{N}$ with $1 \le j \le i-1$ and all $(a_j)_{j=i+1}^{\infty}$ such that the tail $\sum_{j=i+1}^{\infty} a_j b_j$ converges in the $\sigma(X,Y)$-topology.

Step 2: *Factorization.* Let B, Q be defined as in (4.22) and put $Z = B(X)$. Proposition 4.5 yields

$$2 \cdot I_X = QB \quad \text{and} \quad \|B\|, \|Q\| \le 2C_uC_s. \tag{5.12}$$

Next, we define $P\colon X \to Z$ by

$$Px = \sum_{j=1}^{\infty} \frac{\langle x, d_j\rangle}{\langle \widetilde{T} b_j, d_j\rangle} b_j, \qquad x \in X, \tag{5.13}$$

and observe that by the large diagonal of $\widetilde{T}$ (see (5.1)), the unconditionality of $(e_j)_{j=1}^\infty$ (see (2.1)) and Proposition 4.5 we obtain

$$\|Px\|_X \leq \frac{C_u^2 C_s}{(1-\kappa)\delta}\|x\|_X, \quad x \in X. \tag{5.14}$$

Let $z = \sum_{j=1}^\infty a_j b_j \in Z$ and note that

$$P\widetilde{T}z - z = \sum_{i=1}^\infty \sum_{j=1}^{i-1} a_j \frac{\langle \widetilde{T}b_j, d_i\rangle}{\langle \widetilde{T}b_i, d_i\rangle} b_i + \sum_{i=1}^\infty \frac{\langle \widetilde{T}\sum_{j=i+1}^\infty a_j b_j, d_i\rangle}{\langle \widetilde{T}b_i, d_i\rangle} b_i. \tag{5.15}$$

Combining (5.10), (5.11) and (5.2) with $2|a_j| = |\langle z, b_j\rangle| \leq 2C_d\|z\|_X$ to estimate (5.15) yields

$$\|P\widetilde{T}z - z\|_X \leq \frac{\kappa}{(1-\kappa)}\|z\|_X, \quad z \in Z. \tag{5.16}$$

Define $J\colon Z \to X$ by $Jz = z$ and note that by (5.16), $P\widetilde{T}J\colon Z \to Z$ is invertible. Hence, if we define $V = (P\widetilde{T}J)^{-1}P$, then by (5.14) and (5.16) we obtain

$$I_Z = V\widetilde{T}J \quad \text{and} \quad \|V\| \leq \frac{C_u^2 C_s}{(1-2\kappa)\delta}. \tag{5.17}$$

Combining (5.12) with (5.17) and recalling that $\widetilde{T} = TM$ yields

$$I_X = QI_Z B/2 = QV\widetilde{T}JB/2 = QVTMJB/2 = ETF, \tag{5.18}$$

where we put $E = QV$ and $F = MJB/2$. By (5.12), (5.17), (5.2) and since $\|M\| \leq C_u$, we obtain

$$\|E\| \cdot \|F\| \leq \frac{2C_u^5 C_s^3}{(1-2\kappa)\delta} \leq \frac{2C_u^5 C_s^3}{\delta} + \eta,$$

which together with (5.18) concludes the proof. ∎

5.2. Direct sums of Banach spaces with a subsymmetric basis. In this subsection, we will provide a proof for Theorem 3.2. In each coordinate $n \in \mathbb{N}$, we will construct a block basis $(b_{n,j})_{j=1}^\infty$ of $(e_{n,j})_{j=1}^\infty$ such that $(b_{n,j})_{n,j=1}^\infty$ diagonalizes a given operator $T\colon \ell^p((X_n)_{n=1}^\infty) \to \ell^p((X_n)_{n=1}^\infty)$ and also preserves the large diagonal. To achieve this goal, we will need vector-valued versions of Lemma 4.1, Lemma 4.4 and Proposition 4.3.

Some of the tools needed in the proof of Theorem 3.2 have already been anticipated in [19]: [19, Lemma 5.1] is a vector-valued version of Lemma 4.1 and [19, Lemma 5.2] of Lemma 4.4 (both are formulated for weak* bases, but reviewing their proofs reveals that the lemmata hold for our topological bases as well). The two-parameter version of Proposition 4.3 can be obtained by coordinatewise application of Proposition 4.3, i.e., replacing $(e_j)_{j=1}^\infty$ and $(f_j)_{j=1}^\infty$ with $(e_{n,j})_{j=1}^\infty$ and $(f_{n,j})_{j=1}^\infty$ for fixed n.

Proof of Theorem 3.2. Heavily exploiting that $(e_j)_{j=1}^{\infty}$ is non-ℓ^1-splicing, it was possible to construct the block basis $(b_j)_{j=1}^{\infty}$ in the proof of [19, Theorem 1.2] with the same two basic steps as in the proof of [19, Theorem 1.1]. The ith step of the construction reads as follows:

(P1) annihilating the previously constructed vectors Tb_j, $1 \leq j \leq i-1$, by choosing b_i in $[e_k : k \in \mathcal{A}_i]$;
(P2) annihilating the vectors that will be constructed in later steps by preselecting infinitely many suitable coordinates $\mathcal{A}_{i+1}$.

For the proof of Theorem 3.2, we can use the same scheme as described above, i.e., we replace the two basic construction steps (P1) and (P2) by the following three steps described in the ith inductive step in the proof of Theorem 3.1:

(F1) using Lemma 4.4 to preselect infinitely many coordinates Λ_i^0 so that vectors in $TP_{\Lambda_i^0}(X)$ annihilate the previously constructed d_j, $1 \leq j \leq i-1$;
(F2) preselecting another infinite set $\Lambda_i^1 \subset \Lambda_i^0$ by utilizing Lemma 4.1, so that we can choose among many disjointly supported candidates in $[e_k : k \in \Lambda_i^1]$ for the current block basis element d_i, all of which annihilate the previously constructed vectors $\widetilde{T}b_j$, $1 \leq j \leq i-1$;
(F3) using Proposition 4.3 to find among those candidates some block basis elements b_i and d_i so that the diagonal is kept large, i.e., $\langle \widetilde{T}b_i, d_i\rangle \geq 2(1-\kappa)\delta$.

Finally, we note that Step 3 in the proof of [19, Theorem 1.2] reduces to the following case: we have $\mathcal{I} = \mathbb{N}$, $\mathcal{J}_i = \mathcal{K}_i = \mathbb{N}$ and $H = \widetilde{T}$. Also, that reduced case is essentially Step 2 in the proof of Theorem 3.1. Rerunning those arguments with the described modifications concludes the proof. ∎

5.3. Finite-dimensional quantitative factorization results. We first prove Theorem 3.3 and then conclude this work with the proof of Corollary 3.5.

In this section $(e_j)_{j=1}^{\infty}$ denotes a normalized basis for the Banach space X and $(e_j^*)_{j=1}^{\infty}$ denotes the biorthogonal functionals to $(e_j)_{j=1}^{\infty}$. Recall that in (3.1), we defined the function $\tau\colon \mathbb{N} \to [0,\infty)$ by

$$\tau(n) = \max\min\Big(\max_{1\leq j\leq n}\Big\|\sum_{\substack{i=1\\ i\neq j}}^{n} \varepsilon_{ij}e_i\Big\|_X, \max_{1\leq i\leq n}\Big\|\sum_{\substack{j=1\\ j\neq i}}^{n} \varepsilon_{ij}e_j^*\Big\|_{X^*}\Big),$$

where the maximum is taken over all $\varepsilon_{ij} \in \{\pm 1\}$, and $1 \leq i, j \leq n$.

Mainly, the proof of Theorem 3.3 follows the method used by Bourgain and Tzafriri for [5, Theorem 6.1], but instead of exploiting the lower r-

estimate which allows them to extract a large submatrix with small entries, we use an upper estimate involving τ.

Proof of Theorem 3.3. We begin by defining the constants

$$\kappa_1 = \min(1,\eta)/4, \quad \kappa_2 = \min(\eta,(1+\eta)^{-1})/4, \quad \alpha = \sqrt{\frac{\delta\kappa_1}{\Gamma}}\cdot\frac{1}{\sqrt{n\tau(n)}}, \tag{5.19}$$

and note that $\alpha \le 1$ by the lower estimate in (3.3). Now let $(\xi_i)_{i=1}^n$ denote independent random variables over the probability space $(\Omega,\Sigma,\mathbb{P})$ taking values in $\{0,1\}$ with $\mathbb{E}\,\xi_i = \alpha$, where $\mathbb{E}$ denotes conditional expectation. For each $\omega\in\Omega$ we define the operators

$$A(\omega) = R(\omega)D^{-1}TR(\omega) \quad\text{and}\quad R(\omega)e_i = \xi_i(\omega)e_i,\ 1\le i\le n, \tag{5.20}$$

and note that

$$A\Big(\sum_{i=1}^n a_i e_i\Big) = \sum_{i,j=1}^n a_i\xi_i\xi_j\frac{\langle Te_i,e_j^*\rangle}{\langle Te_j,e_j^*\rangle}e_j. \tag{5.21}$$

Let $x=\sum_{i=1}^n a_ie_i$ and observe that by (5.21),

$$\|(A-R)x\| \le \sum_{i\ne j}|a_i|\xi_i\xi_j\frac{|\langle Te_i,e_j^*\rangle|}{|\langle Te_j,e_j^*\rangle|} \le \frac{\|x\|}{\delta}\sum_{i\ne j}\xi_i\xi_j|\langle Te_i,e_j^*\rangle|.$$

Taking the supremum over all $\|x\|\le 1$ and taking the expectation yields

$$\mathbb{E}\,\|A-R\| \le \frac{1}{\delta}\sum_{i\ne j}\mathbb{E}\,\xi_i\xi_j|\langle Te_i,e_j^*\rangle| = \frac{\alpha^2}{\delta}\sum_{i\ne j}|\langle Te_i,e_j^*\rangle|. \tag{5.22}$$

Defining $\varepsilon_{ij}=\operatorname{sign}(\langle Te_i,e_j^*\rangle)$ we deduce from (5.22) that

$$\mathbb{E}\,\|A-R\| \le \frac{\alpha^2}{\delta}\sum_i\Big\langle Te_i,\sum_{j:j\ne i}\varepsilon_{ij}e_j^*\Big\rangle \le \alpha^2\frac{\Gamma}{\delta}n\max_i\Big\|\sum_{j:j\ne i}\varepsilon_{ij}e_j^*\Big\|. \tag{5.23}$$

Similarly, we obtain

$$\mathbb{E}\,\|A-R\| \le \alpha^2\frac{\Gamma}{\delta}n\max_j\Big\|\sum_{i:i\ne j}\varepsilon_{ij}e_i\Big\|. \tag{5.24}$$

Combining (5.23), (5.24), (3.1) and (5.19) yields

$$\mathbb{E}\,\|A-R\| \le \alpha^2\frac{\Gamma}{\delta}n\tau(n) = \kappa_1. \tag{5.25}$$

Now define

$$\Omega' = \Big\{\omega\in\Omega : \Big|\sum_{i=1}^n\xi_i(\omega)-\alpha n\Big| \le \alpha n/2\Big\}$$

and observe that by (5.19) and (3.3),

$$\begin{aligned}\mathbb{P}(\Omega'^c) &= \mathbb{P}\Big(\Big\{\omega \in \Omega : \Big|\sum_{i=1}^n \xi_i - \alpha n\Big| > \alpha n/2\Big\}\Big)\\ &\leq \frac{4}{\alpha^2 n^2}\,\mathbb{E}\Big(\sum_{i=1}^n \xi_i - \alpha n\Big)^2 = \frac{4(1-\alpha)}{\alpha n} \leq \kappa_2.\end{aligned}$$

Thus, combining this measure estimate with (5.25), we can find an $\omega_0 \in \Omega'$ such that

$$\|A(\omega_0) - R(\omega_0)\| \leq \kappa_1/(1-\kappa_2). \tag{5.26}$$

Since $\kappa_1/(1-\kappa_2) < 1$, $A(\omega_0)$ is invertible, hence by (5.19) we obtain

$$\|A^{-1}(\omega_0)\| \leq \frac{1-\kappa_2}{1-\kappa_1-\kappa_2} \leq 1+\eta. \tag{5.27}$$

We put $\sigma = \{1 \leq i \leq n : \xi_i(\omega_0) = 1\}$ and note that since $\omega_0 \in \Omega'$, we have $|\sigma| \geq \alpha n/2$; appealing to (5.19) shows (3.5). Since $R(\omega_0) = R_\sigma$, (3.6) follows from (5.27) and the definition of A (see (5.20)).

We conclude the proof by defining $E\colon X_\sigma \to X_n$ by $Ex = x = R_\sigma x$ and $P\colon X_n \to X_\sigma$ by $P = (R_\sigma D^{-1} T R_\sigma)^{-1} R_\sigma D^{-1}$ and using $\|D^{-1}\| \leq C_u/\delta$ together with (3.6). ∎

To conclude, we apply Theorem 3.3 to a subsymmetric basis and thereby prove Corollary 3.5.

Proof of Corollary 3.5. Recalling (4.6) and observing that if $(e_j)_{j=1}^\infty$ is C_u-unconditional, we have

$$\tau(n) \leq C_u \nu(n), \qquad n \in \mathbb{N}. \tag{5.28}$$

Moreover, if $(e_j)_{j=1}^\infty$ is (C_u, C_s)-subsymmetric, then by (5.28) and (4.16) we obtain

$$\tau(n) \leq C_u C_s \min(\lambda(n-1), \mu(n-1)), \qquad n \in \mathbb{N}. \tag{5.29}$$

Thus, by (4.17), we also have

$$\tau(n) \leq \sqrt{2C_u^3 C_s^3 (n-1)}, \qquad n \in \mathbb{N}. \tag{5.30}$$

Using (5.30), (3.7) and the inequality $\tau(n) \geq 1/C_u$, $n \geq 2$, we obtain (3.3); thus, Theorem 3.3 yields (3.9) and (3.10). Noting that X_k is C_s-isomorphic to X_σ, we obtain the estimate $\|E\|\,\|P\| \leq C_u^2 C_s^2 (1+\eta)/\delta$. ∎

Acknowledgements. Supported by the Austrian Science Foundation (FWF) Pr.Nr. P32728.

REFERENCES

[1] Z. Altshuler, P. G. Casazza and B. L. Lin, *On symmetric basic sequences in Lorentz sequence spaces*, Israel J. Math. 15 (1973), 140–155.

[2] A. D. Andrew, *Perturbations of Schauder bases in the spaces* $C(K)$ *and* L^p, $p > 1$, Studia Math. 65 (1979), 287–298.
[3] J. L. Ansorena, *Primarity of direct sums of Orlicz spaces and Marcinkiewicz spaces*, Banach J. Math. Anal. 14 (2020), 950–969.
[4] J. Bourgain, *On the primarity of* H^∞*-spaces*, Israel J. Math. 45 (1983), 329–336.
[5] J. Bourgain and L. Tzafriri, *Invertibility of "large" submatrices with applications to the geometry of Banach spaces and harmonic analysis*, Israel J. Math. 57 (1987), 137–224.
[6] J. Bourgain and L. Tzafriri, *Restricted invertibility of matrices and applications*, in: Analysis at Urbana (Urbana, IL, 1986–1987), Vol. II, London Math. Soc. Lecture Note Ser. 138, Cambridge Univ. Press, Cambridge, 1989, 61–107.
[7] M. Capon, *Primarité de certains espaces de Banach*, Proc. London Math. Soc. (3) 45 (1982), 113–130.
[8] P. G. Casazza, *James' quasi-reflexive space is primary*, Israel J. Math. 26 (1977), 294–305.
[9] P. G. Casazza, C. A. Kottman and B. L. Lin, *On some classes of primary Banach spaces*, Canad. J. Math. 29 (1977), 856–873.
[10] P. G. Casazza and B. L. Lin, *Projections on Banach spaces with symmetric bases*, Studia Math. 52 (1974), 189–193.
[11] P. G. Casazza and T. J. Shura, *Tsirelson's Space*, Lecture Notes in Math. 1363, Springer, Berlin, 1989.
[12] H. Fetter and B. Gamboa de Buen, *The James Forest*, London Math. Soc. Lecture Note Ser. 236, Cambridge Univ. Press, Cambridge, 1997.
[13] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 6 (1993), 851–874.
[14] N. J. Laustsen, R. Lechner and P. F. X. Müller, *Factorization of the identity through operators with large diagonal*, J. Funct. Anal. 275 (2018), 3169–3207.
[15] R. Lechner, *Factorization in mixed norm Hardy and BMO spaces*, Studia Math. 242 (2018), 231–265.
[16] R. Lechner, *Factorization in* SL^∞, Israel J. Math. 226 (2018), 957–991.
[17] R. Lechner, *Dimension dependence of factorization problems: biparameter Hardy spaces*, Proc. Amer. Math. Soc. 147 (2019), 1639–1652.
[18] R. Lechner, *Dimension dependence of factorization problems: Hardy spaces and* SL^∞_n, Israel J. Math. 232 (2019), 677–693.
[19] R. Lechner, *Subsymmetric weak* Schauder bases and factorization of the identity*, Studia Math. 248 (2019), 295–319.
[20] R. Lechner, *Direct sums of finite dimensional* SL^∞_N *spaces*, Houston J. Math. 46 (2020), 705–745.
[21] R. Lechner, P. Motakis, P. F. X. Müller and Th. Schlumprecht, *The factorization property of Tsirelson and Schreier spaces*, in preparation.
[22] R. Lechner, P. Motakis, P. F. X. Müller and Th. Schlumprecht, *Strategically reproducible bases and the factorization property*, Israel J. Math. 238 (2020), 13–60.
[23] R. Lechner, P. Motakis, P. F. X. Müller and Th. Schlumprecht, *The factorisation property of* $\ell^\infty(X_k)$, Math. Proc. Cambridge Philos. Soc. 171 (2021), 421–448.
[24] J. Lindenstrauss, *On complemented subspaces of* m, Israel J. Math. 5 (1967), 153–156.
[25] J. Lindenstrauss, *Decomposition of Banach spaces*, Indiana Univ. Math. J. 20 (1971), 917–919.
[26] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. I, Sequence Spaces*, Ergeb. Math. Grenzgeb. 92, Springer, Berlin, 1977.
[27] C. Samuel, *Primarité des produits d'espaces de suites*, Colloq. Math. 39 (1978), 123–132.

[28] I. Singer, *Bases in Banach Spaces. I*, Grundlehren Math. Wiss. 154, Springer, New York, 1970.

Richard Lechner
Institute of Analysis
Johannes Kepler University Linz
Altenberger Straße 69
A-4040 Linz, Austria
E-mail: Richard.Lechner@jku.at