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AN IMPROVEMENT OF THE PINNED DISTANCE SET PROBLEM IN EVEN DIMENSIONS

BY

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Abstract. We study the pinned distance set $\Delta_x(E) = \{|x-y| : y \in E\}$ in even dimensions. We utilize the orthogonal projection method as in [X. Du et al., Math. Ann. 380 (2021)] to show that for any compact subset $E \subset \mathbb{R}^d$, where d is an even integer and $d/2 < \dim_{\mathcal{H}}(E) < d/2+1$, there exists a point $x \in E$ such that

$$\dim_{\mathcal{H}}(\Delta_x(E)) \geq \min\left\{\tfrac{2d}{d+1}\dim_{\mathcal{H}}(E) - \tfrac{d-1}{d+1}\left(d + \tfrac{d-2}{2(d-1)}\right), 1\right\}.$$

This improves the partial result of A. Iosevich and B. Liu [Trans. Amer. Math. Soc. 371 (2019)].

1. Introduction

1.1. Falconer distance problem. The Falconer distance set problem is famous in geometric measure theory and harmonic analysis. It was first introduced by Falconer [8] in 1985. Given a compact subset $E \subset \mathbb{R}^d$, we could define the *distance set* of E by

$$\Delta(E) = \{|x-y| : x, y \in E\}.$$

Let $\dim_{\mathcal{H}} E$ denote the Hausdorff dimension of the set E and $|E|$ denote the Lebesgue measure of E. Falconer [8] proved that if $d \geq 2$ and $E \subset \mathbb{R}^d$ is a Borel set with $\dim_{\mathcal{H}} E > \frac{d+1}{2}$, then $|\Delta(E)| > 0$. Moreover, when $\frac{d}{2} < \dim_{\mathcal{H}} E \leq \frac{d+1}{2}$, one has

$$\dim_{\mathcal{H}}(\Delta(E)) \geq \dim_{\mathcal{H}} E - \tfrac{d}{2} + \tfrac{1}{2}. \tag{1.1}$$

He also showed that for $d \geq 2$ and $0 < s \leq d$, there exists a compact set $E \subset \mathbb{R}^d$ with

$$\dim_{\mathcal{H}} E = s \quad \text{and} \quad \dim_{\mathcal{H}}(\Delta(E)) \leq \tfrac{2}{d}\dim_{\mathcal{H}} E.$$

This means that there exists a compact set E with

$$0 < \dim_{\mathcal{H}} E < \tfrac{d}{2} \quad \text{but} \quad \dim_{\mathcal{H}} \Delta(E) < 1.$$

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Falconer's distance conjecture can be described as follows.

Conjecture 1.1 (Strong version). If $d \geq 2$ and $E \subset \mathbb{R}^d$ is a Borel set with $\dim_{\mathcal{H}} E > d/2$, then $|\Delta(E)| > 0$.

A weaker conjecture on dimensional level is

Conjecture 1.2 (Weak version). If $d \geq 2$ and $E \subset \mathbb{R}^d$ is a Borel set with $\dim_{\mathcal{H}} E > d/2$, then $\dim_{\mathcal{H}}(\Delta(E)) = 1$.

In [17], Mattila developed a powerful analytic method to treat the Falconer distance set problem. This method builds a bridge between the decay of the Fourier transform of a fractal measure on the sphere and the Falconer distance set. More precisely, it can be stated as follows.

Theorem A. *Let $s \in (0, d)$, $d \geq 2$ and $E \subset \mathbb{R}^d$ be a compact set with $\dim_{\mathcal{H}} E > s$. Suppose $0 < t \leq s$. Assume that there exists a probability measure μ supported in E such that*

$$\int_{\mathbb{S}^{d-1}} |\widehat{\mu}(r\theta)|^2 \, d\sigma(\theta) \lesssim r^{-t}, \qquad \forall r > 0. \tag{1.2}$$

(1) *If $t \geq d - s$, then $|\Delta(E)| > 0$.*
(2) *If $t < d - s$, then $\dim_{\mathcal{H}} \Delta(E) \geq s + t - d + 1$.*

In [17] Mattila established that if μ is an α-dimensional measure, then

$$\int_{\mathbb{S}^{d-1}} |\widehat{\mu}(r\theta)|^2 \, d\sigma(\theta) \lesssim r^{-\frac{2}{3}\alpha - \frac{1}{6}(d-5)}, \qquad \forall r > 0. \tag{1.3}$$

Through Theorem A, we can deduce that if $\dim_{\mathcal{H}} E > \frac{d+1}{2}$ then $|\Delta(E)| > 0$ and if $\frac{d}{2} < \dim_{\mathcal{H}} E \leq \frac{d+1}{2}$ then

$$\dim_{\mathcal{H}}(\Delta(E)) \geq \tfrac{5}{3} \dim_{\mathcal{H}} E - \tfrac{5}{6} d + \tfrac{1}{6},$$

which was slightly weaker than Falconer's result (1.1). However, based on Mattila's approach, Wolff [22] improved Falconer's result in dimension 2, and then the general dimension d case was considered by Erdoğan [7]. Let $\mathcal{M}(\mathbb{R}^d)$ denote the set of all finite Borel measures on $\mathbb{R}^d$.

Theorem B ([7, 10, 22]). *For all $\frac{d-2}{2} \leq \alpha < d$ $(d \geq 2)$, if $\mu \in \mathcal{M}(\mathbb{R}^d)$ with μ being supported in $B(0,1)$ and an α-dimensional measure, then*

$$\int_{\mathbb{S}^{d-1}} |\widehat{\mu}(r\theta)|^2 \, d\sigma(\theta) \lesssim r^{-\frac{d+2\alpha-2}{4}}, \qquad \forall r > 1. \tag{1.4}$$

Combining Theorem A with Theorem B, one could deduce that for $d \geq 2$, when the compact set E satisfies $\dim_{\mathcal{H}} E > \frac{d}{2} + \frac{1}{3}$, then $|\Delta(E)| > 0$, and if $\dim_{\mathcal{H}} E > \frac{d}{2}$, then

$$\dim_{\mathcal{H}}(\Delta(E)) \geq \min\left\{\tfrac{3}{2} \dim_{\mathcal{H}} E - \tfrac{3}{4} d + \tfrac{1}{2}, 1\right\}. \tag{1.5}$$

Using Mattila's approach one finds that when $d = 2$, the exponent $\frac{4}{3}$ is the best possible, while for $d = 3$, the best possible is $\frac{5}{3}$. Recently, by using polynomial partitioning and refined Strichartz estimates, Du et al. [5] proved that for $d = 3$, if the compact set E satisfies $\dim_{\mathcal{H}} E > 1.8$, then $|\Delta(E)| > 0$. Combining this with Theorem A(2), one can deduce that if $\dim_{\mathcal{H}}(E) > \frac{3}{2}$, then

$$\dim_{\mathcal{H}} \Delta(E) \geq \min\left\{\tfrac{5}{3}\dim_{\mathcal{H}} E - 2, 1\right\}.$$

Du and Zhang [6] gave a sharp estimate for Schrödinger maximal function which implied that when $d \geq 4$, if the compact set E satisfies

$$\dim_{\mathcal{H}} E > \tfrac{d}{2} + \tfrac{1}{4} + \tfrac{1}{8d-4},$$

then $|\Delta(E)| > 0$. Similarly, if $\dim_{\mathcal{H}}(E) > \frac{d}{2}$, then

$$\dim_{\mathcal{H}} \Delta(E) \geq \min\left\{\tfrac{2d-1}{d}\dim_{\mathcal{H}} E - d + 1, 1\right\}.$$

Besides Mattila's approach, other methods (with a heavier geometric flavor) have been tried as well for Falconer's problem in the plane. Relying on the earlier work of Katz and Tao [12], Bourgain [1] proved that if $E \subset \mathbb{R}^2$ satisfies $\dim_{\mathcal{H}} E \geq 1$, then

$$\dim_{\mathcal{H}}(\Delta(E)) > \tfrac{1}{2} + \delta,$$

where $\delta > 0$ is a universal constant. Keleti and Shmerkin [14] established an effective framework which implied that if E is a compact set with $1 < \dim_{\mathcal{H}} E < \frac{4}{3}$, then

$$\dim_{\mathcal{H}}(\Delta(E)) \geq \tfrac{37}{54} = 0.6851851\ldots.$$

We remark that Keleti and Shmerkin's result is better than Wolff's result when the Hausdorff dimension of E is close to 1. However, when the Hausdorff dimension of E is close to $\frac{4}{3}$, Keleti and Shmerkin's result is worse. Recently, Shmerkin [21] improved Keleti and Shmerkin's result. He proved that if E is a compact set with $1 < \dim_{\mathcal{H}} E < 4/3$, then

$$\dim_{\mathcal{H}}(\Delta(E)) \geq \tfrac{40}{57} = 0.702\ldots.$$

1.2. Pinned distance problem. A stronger version of Falconer's distance problem is the pinned distance problem. Given $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, the *pinned distance set* is defined by

$$\Delta_x(E) = \{|x - y| : y \in E\}.$$

Since $\Delta_x(E) \subset \Delta(E)$, when $x \in E$, the condition $|\Delta_x(E)| > 0$ implies $|\Delta(E)| > 0$ and $\dim_{\mathcal{H}}(\Delta(E)) \geq \dim_{\mathcal{H}}(\Delta_x(E))$.

The pinned distance problem was first studied by Peres and Schlag [20].

THEOREM C. *Given a Borel set $E \subset \mathbb{R}^d$, $d \geq 2$, we have*

$$\dim_{\mathcal{H}}(\{x \in \mathbb{R}^d : |\varDelta_x(E)| = 0\}) \leq d + 1 - \dim_{\mathcal{H}} E, \tag{1.6}$$

$$\dim_{\mathcal{H}}(\{x \in \mathbb{R}^d : \dim_{\mathcal{H}}(\varDelta_x(E)) < t\}) \leq d + t - \max\{\dim_{\mathcal{H}} E, 1\}, \tag{1.7}$$

with $0 < t \leq 1$.

By a simple computation, (1.6) implies that if $\dim_{\mathcal{H}} E > \frac{d+1}{2}$, there exists $x \in E$ such that $|\varDelta_x(E)| > 0$. It follows from (1.7) that if $\dim_{\mathcal{H}} E > \frac{d}{2}$, then there is $x \in E$ such that

$$\dim_{\mathcal{H}}(\varDelta_x(E)) \geq \min\{2\dim_{\mathcal{H}} E - d - \epsilon, 1\}, \quad \forall \epsilon > 0.$$

By using the local smoothing estimates of Fourier integral operators, Iosevich and Liu [11] partially improved Peres–Schlag's result as follows.

THEOREM D. *Given a Borel set $E \subset \mathbb{R}^d$, $d \geq 2$, we have*

$$\dim_{\mathcal{H}}(\{x \in \mathbb{R}^d : |\varDelta_x(E)| = 0\}) \leq \tfrac{d(d+1)}{d-1} - \tfrac{d+1}{d-1}\dim_{\mathcal{H}} E, \tag{1.8}$$

$$\dim_{\mathcal{H}}(\{x \in \mathbb{R}^d : \dim_{\mathcal{H}}(\varDelta_x(E)) < t\}) \leq d + 1 + \tfrac{d+1}{d-1}t - \tfrac{d+1}{d-1}\dim_{\mathcal{H}} E. \tag{1.9}$$

Inequality (1.9) shows that if $\dim_{\mathcal{H}} E > \frac{d}{2}$, then there exists $x \in E$ such that

$$\dim_{\mathcal{H}}(\varDelta_x(E)) \geq \min\left\{\tfrac{2d}{d+1}\dim_{\mathcal{H}} E - d + 1 - \epsilon, 1\right\}, \quad \forall \epsilon > 0.$$

Recently, Liu [15] gave an effective framework to the pinned distance set problem by establishing an L^2-identity. Under this new framework, he proved that if $E \subset \mathbb{R}^d$ ($d \geq 2$) is a compact set with

$$\dim_{\mathcal{H}} E > \tfrac{d}{2} + \tfrac{1}{3},$$

then there is $x \in E$ such that $|\varDelta_x(E)| > 0$.

Returning to the planar case, based on Liu's framework, refined decoupling and Orponen's results [19] about radial projections, Guth et al. [9] proved that if $E \subset \mathbb{R}^2$ is a compact set with $\dim_{\mathcal{H}} E > 5/4$, then $|\varDelta_x(E)| > 0$. Based on their work, Liu [16] improved (1.9) in $d = 2$.

THEOREM E (Liu, [16]). *Given any compact set $E \subset \mathbb{R}^2$, $\dim_{\mathcal{H}} E > 1$, and $t \in (0,1)$, we have*

$$\dim_{\mathcal{H}}(\{x \in \mathbb{R}^2 : \dim_{\mathcal{H}}(\varDelta_x(E)) < t\}) \leq \max\{2 + 3t - 3\dim_{\mathcal{H}} E, 2 - \dim_{\mathcal{H}} E\}.$$

In particular, for any compact set $E \subset \mathbb{R}^2$, $\dim_{\mathcal{H}} E > 1$, there exists $x \in E$ such that

$$\dim_{\mathcal{H}}(\varDelta_x E) \geq \min\left\{\tfrac{4}{3}\dim_{\mathcal{H}} E - \tfrac{2}{3} - \epsilon, 1\right\}, \quad \forall \epsilon > 0. \tag{1.10}$$

For general dimensions d, using the orthogonal projection [13] (also [18]) and the approach in [9], Du et al. [4] proved that if $d \geq 4$ is an even integer and $E \subset \mathbb{R}^d$ is a compact set with

$$\dim_{\mathcal{H}} E > \tfrac{d}{2} + \tfrac{1}{4},$$

then there exists $x \in E$ such that $|\Delta_x(E)| > 0$. In this paper, we will use the orthogonal projection method as in [4] to establish a lower bound of the Hausdorff dimension of the pinned distance set which improves (1.9) in even dimensions. In the plane case, Shmerkin [21] deduced that if $E \subset \mathbb{R}^2$, $\dim_{\mathcal{H}} E > 1$, then there exists $x \in E$ such that

$$\dim_{\mathcal{H}}(\Delta_x(E)) > \tfrac{29}{42} = 0.690\ldots. \tag{1.11}$$

Finally, we give a summary of the best known results.

To ensure $|\Delta(E)| > 0$, one needs

- $d = 2$, $\dim_{\mathcal{H}} E > \frac{5}{4}$ ([9]);
- $d \geq 3$, odd, $\dim_{\mathcal{H}} E > \frac{d}{2} + \frac{1}{4} + \frac{1}{8d-1}$ ([5], [6]);
- $d \geq 4$, even, $\dim_{\mathcal{H}} E > \frac{d}{2} + \frac{1}{4}$ ([4]).

Suppose $\dim_{\mathcal{H}} E > \frac{d}{2}$. Then

- if $d=2$, then $\dim_{\mathcal{H}}(\Delta(E)) \geq \max\left\{\min\left\{\frac{4}{3}\dim_{\mathcal{H}} E - \frac{2}{3}, 1\right\}, \frac{40}{57}\right\}$ ([16], [21]);
- if $d \geq 3$, then $\dim_{\mathcal{H}}(\Delta(E)) \geq \min\left\{\frac{2d-1}{d}\dim_{\mathcal{H}} E - d + 1, 1\right\}$ ([5], [6]).

To ensure $|\Delta_x(E)| > 0$ for some $x \in E$, one needs

- $d = 2$, $\dim_{\mathcal{H}} E > \frac{5}{4}$ ([9]);
- $d \geq 3$, odd, $\dim_{\mathcal{H}} E > \frac{d}{2} + \frac{1}{4} + \frac{1}{8d-1}$ ([5], [6]);
- $d \geq 4$, even, $\dim_{\mathcal{H}} E > \frac{d}{2} + \frac{1}{4}$ ([4]).

Suppose $\dim_{\mathcal{H}} E > \frac{d}{2}$. Then for any $\epsilon > 0$,

- if $d = 2$, then $\dim_{\mathcal{H}}(\Delta_x(E)) \geq \max\left\{\min\left\{\frac{4}{3}\dim_{\mathcal{H}} E - \frac{2}{3} - \epsilon, 1\right\}, \frac{29}{42}\right\}$ ([16], [21]).

1.3. Main result. Now, we give the main result of this paper.

THEOREM 1.1. *Let $d \geq 4$ be an even integer. Suppose $E \subset \mathbb{R}^d$ is a compact set with $\frac{d}{2} < \dim_{\mathcal{H}}(E) < \frac{d}{2} + 1$. Let $\tau \in (0,1)$. Then*

$$\begin{aligned} &\dim_{\mathcal{H}}(\{x \in \mathbb{R}^d : \dim_{\mathcal{H}}(\Delta_x(E)) < \tau\}) \\ &\qquad \leq \max\left\{d + \tfrac{d-2}{2(d-1)} + \tfrac{d+1}{d-1}\tau - \tfrac{d+1}{d-1}\dim_{\mathcal{H}}(E), d - \dim_{\mathcal{H}}(E)\right\}. \end{aligned} \tag{1.12}$$

Moreover, there is $x \in E$ such that

$$\dim_{\mathcal{H}}(\Delta_x(E)) \geq \min\left\{\tfrac{2d}{d+1}\dim_{\mathcal{H}}(E) - \tfrac{d-1}{d+1}\left(d + \tfrac{d-2}{2(d-1)}\right) - \epsilon, 1\right\}, \quad \forall \epsilon > 0. \tag{1.13}$$

REMARK 1.2. The restriction of even dimensions stems from the fact that the dimension of a subspace is $\frac{d}{2} + 1$ which should be an integer in the proof of Lemma 4.2. Compared with the result in [4], we consider (1.13) in the whole range $\frac{d}{2} < \dim_{\mathcal{H}} E \leq \frac{d}{2} + 1$. Our result (1.12) improves (1.9) in even dimensions $d \geq 4$.

The structure of the paper is as follows. In Section 2, we give some notations and main lemmas. In Section 3, we prove Theorem 1.1. The proof of Proposition 2.3, which is a variation of Propositions 2.1 and 2.2 in [4], will be given in Section 4.

We write

- $A \lesssim B$ if $A \leq CB$ for a constant C which only depends on some unimportant fixed variables such as d, α and sometimes ϵ;
- $A \sim B$ if $A \lesssim B$ and $B \lesssim A$;
- $A \lessapprox B$ if $A \leq C_\epsilon R^\epsilon B$ for any $\epsilon > 0$, $R > 1$;
- $A \lesssim_\epsilon B$ if $A \leq C_\epsilon B$;
- $\widehat{\mu}$ for the Fourier transform of measure μ;
- $\mathrm{RapDec}(R)$ for those quantities that are bounded by a huge negative power of R, i.e. $\mathrm{RapDec}(R) \leq C_N R^{-N}$ for arbitrary $N > 0$;
- σ_t for the normalized surface measure on $t\mathbb{S}^{d-1}$ $(d \geq 2)$;
- $d_*^x(\mu)$ for the push-forward measure of μ defined by

$$\int h(t)\, d_*^x(\mu)(t) := \int h(|x-y|)\, d\mu(y), \tag{1.14}$$

where h is a continuous function in $\mathbb{R}$.

If we slightly abuse notation, regarding μ as its density, we find that

$$d_*^x(\mu)(t) = c_d \cdot \mu * \sigma_t(x), \tag{1.15}$$

where $c_d = |\mathbb{S}^{d-1}|$. An orthogonal projection of a set E is usually written as $\pi(E)$, and $\pi_*(\mu)$ denotes the push-forward measure induced by orthogonal projection.

2. Some key lemmas. In this section, we will give two key lemmas.

Let $E \subset \mathbb{R}^d$ be a compact set. For any $\alpha < \dim_{\mathcal{H}}(E)$, by Frostman's Lemma, there is a probability measure μ_E such that

$$\mu_E(B(x,r)) \lesssim r^s, \qquad \forall x \in \mathbb{R}^d,\ r > 0. \tag{2.1}$$

We say that μ_E is an *α-Frostman measure* if it satisfies (2.1). Given $s > 0$, we define the *s-energy integral* of μ_E by

$$I_s(\mu_E) := \iint |x-y|^{-s}\, d\mu_E(x)\, d\mu_E(y). \tag{2.2}$$

It follows from [17, Theorem 3.10] that

$$I_s(\mu_E) = \gamma(d,s) \int |\widehat{\mu_E}(\xi)|^2 |\xi|^{s-d}\, d\xi, \qquad 0 < s < d, \tag{2.3}$$

where $\gamma(d,s)$ is a constant depending on d and s. If μ_E is an α-Frostman measure, then for any $0 < s < \alpha$, we have

$$I_s(\mu_E) < \infty. \tag{2.4}$$

Now, we recall the following criterion to determine the dimension of pinned distance sets.

Lemma 2.1 ([16]). *Given a compact set $E \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$ and a probability measure μ_E on E. Suppose there exist $\tau \in (0,1]$, $K \in \mathbb{Z}_+$, $\beta > 0$ such that*

$$\mu_E(\{y : |y-x| \in D_k\}) < 2^{-k\beta} \quad \textit{for any } D_k = \bigcup_{j=1}^{M} I_j,$$

where $k > K$, $M \leq 2^{k\tau}$ are arbitrary integers and each I_j is an interval of length $\approx 2^{-k}$. Then

$$\dim_{\mathcal{H}}(\Delta_x(E)) \geq \tau. \tag{2.5}$$

Next, we recall a famous lemma in probability theory.

Lemma 2.2 (Borel–Cantelli Lemma, [3]). *Let μ be a Borel probability measure supported on F and $\{F_{(k)}\}_{k=1}^{\infty}$ be a sequence of μ-measurable subsets of F. If $\sum_{k=1}^{\infty} \mu(F_{(k)}) < \infty$, then*

$$\mu\Big(\overline{\lim_{k\to\infty}}\, F_{(k)}\Big) = 0. \tag{2.6}$$

By the same trick as in [16], we need to modify slightly Propositions 2.1 and 2.2 in [4] to get a quantitative version, Proposition 2.3 below. To do this, we first give some notations.

Let $d \geq 4$ be an even integer, and k be a large integer. Let $\alpha_E, \alpha_F > 0$. Suppose E, F are compact subsets of the unit ball $B(0,1) \subset \mathbb{R}^d$ with

$$\mathcal{H}^{\alpha_E}(E) > 0, \quad \mathcal{H}^{\alpha_F}(F) > 0, \quad \operatorname{dist}(E,F) \gtrsim 1.$$

Let μ_E be an α_E-Frostman measure and μ_F be an α_F-Frostman measure. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be a non-negative function satisfying $\int_{\mathbb{R}^d} \phi(x)\, dx = 1$, and $\phi \geq 1$ on $B(0,1/2)$. Write $\phi_{2^{-k}}(\cdot) = 2^{dk}\phi(2^k \cdot)$. We denote

$$\mu_E^{2^{-k}} = \mu_E * \phi_{2^{-k}}. \tag{2.7}$$

Since (see [18, p. 80])

$$\widehat{\pi_*(\mu_E^{2^{-k}})}(\xi) = \widehat{\mu_E^{2^{-k}}}(\xi), \qquad \xi \in P,$$

where π_* is an orthogonal projection mapping $\mathbb{R}^d$ onto P which is a subspace of $\mathbb{R}^d$ and the dimension of P is bigger than E, one has $I_{s_E}(\pi_*(\mu_E^{2^{-k}})) < \infty$. Precisely, we find a subspace P with dimension $d/2+1 > \dim E$ and we define

$$I_{s_E}(\pi_*(\mu_E^{2^{-k}})) := \int_P |\widehat{\pi_*(\mu_E^{2^{-k}})}(\xi)|\, |\xi|^{s_E-d}\, d\xi < \int_{\mathbb{R}^d} |\widehat{\mu_E^{2^{-k}}}(\xi)|\, |\xi|^{s_E-d}\, d\xi < \infty.$$

Write $\widetilde{\mu}_E = \pi_*(\mu_E)$.

Finally, we state a key proposition which plays an important role in the proof of Theorem 1.1; we will prove it in Section 4.

PROPOSITION 2.3. *Suppose that the orthogonal projection* π *mapping* $\mathbb{R}^d$ *onto a* $\left(\frac{d}{2}+1\right)$*-dimensional subspace* $P \subset \mathbb{R}^d$ *satisfies* $\mathrm{dist}(\pi(E), \pi(F)) \gtrsim 1$. *For all* $d/2 < s_E < \alpha_E$ *and all* $d - s_E < s_F < \min\{\alpha_F, d/2\}$, *assume that*

$$I_{s_E}(\pi_*(\mu_E)) + I_{s_F}(\pi_*(\mu_F)) < \infty. \tag{2.8}$$

Then there exists a constant $c = c(s_E, s_F) > 0$ *such that for any* $1 \gg \delta > 0$, $2^k \gg R_0(k) \gg 1$, *one can decompose*

$$\mu_E^{2^{-k}} = \mu_{E,\,\mathrm{good}}^{2^{-k}} + \mu_{E,\,\mathrm{bad}}^{2^{-k}}, \tag{2.9}$$

where $\mu_{E,\,\mathrm{good}}^{2^{-k}}$, $\mu_{E,\,\mathrm{bad}}^{2^{-k}}$ *are complex-valued Schwartz functions such that*

(1) *there exists* $F' \subset F$ *such that* $\mu_F(F') \geq 1 - R_0(k)^{-c\delta}$, *and for each* $x \in F'$,

$$\int_{\mathbb{R}} |d_*^x(\mu_E^{2^{-k}})(t) - d_*^x(\mu_{E,\,\mathrm{good}}^{2^{-k}})(t)|\, dt \lesssim R_0(k)^{-c\delta}; \tag{2.10}$$

(2) *there exists a constant* $C(\delta) > 0$ *such that*

$$\begin{aligned} &\iint \left| d_*^x(\mu_{E,\,\mathrm{good}}^{2^{-k}})(t) \right|^2 dt\, d\mu_F(x) \\ &\lesssim C(\delta) R_0(k)^{O(1)} \int |\widehat{\mu_E}(\xi)|^2\, |\xi|^{-\frac{d}{2(d+1)} - \frac{(d-1)\alpha_F}{d+1} + O(\delta)}\, d\xi + \mathrm{RapDec}(R_0(k)). \end{aligned} \tag{2.11}$$

Here all implicit constants may depend on $\mathrm{dist}(E, F)$, E, F, *and the implicit constant in* (2.1), *but are independent of* k; *and* $\mathrm{RapDec}(R)$ *denotes quantities that are bounded by a huge negative power of* R, *i.e.* $\mathrm{RapDec}(R) \leq C_N R^{-N}$ *for arbitrary* $N > 0$.

3. Proof of Theorem 1.1. In this section, we will utilize Proposition 2.3 to prove Theorem 1.1.

We argue by contradiction. Let $F \subset \mathbb{R}^d$ be any compact set with positive α_F-dimensional Hausdorff measure, where

$$\alpha_F > \lambda := \max\left\{d + \tfrac{d-2}{2(d-1)} + \tfrac{d+1}{d-1}\tau - \tfrac{d+1}{d-1}\dim_{\mathcal{H}}(E), d - \dim_{\mathcal{H}}(E)\right\}. \tag{3.1}$$

Without loss of generality, we may assume that E and F are contained in the unit ball. Let $\frac{d}{2} < \alpha_E < \dim_{\mathcal{H}}(E)$ satisfy

$$\alpha_F > \lambda_{\alpha_E} := \max\left\{d + \tfrac{d-2}{2(d-1)} + \tfrac{d+1}{d-1}\tau - \tfrac{d+1}{d-1}\alpha_E, d - \alpha_E\right\}. \tag{3.2}$$

By Frostman's Lemma [17], there exists an α_E-Frostman measure μ_E supported on E and an α_F-Frostman measure μ_F supported on F.

STEP 1: *First reduction.* To prove Theorem 1.1, it suffices to show that there exists $F^* \subset F$ with $\mu_F(F^*) > 0$ such that for μ_F-a.e. $x \in F^*$,

$$\dim_{\mathcal{H}}(\varDelta_x(E)) \geq \tau. \tag{3.3}$$

Furthermore, we only need to show that the above result for E and F also holds for the subsets of E and F. In fact, for any subsets $E_1 \subset E$ and

$F_1 \subset F$ with $\mu_E(E_1) > 0$ and $\mu_F(F_1) > 0$, we define

$$\mu_{E_1} = \frac{\mu_E|_{E_1}}{\mu_E(E_1)} \quad \text{and} \quad \mu_{F_1} = \frac{\mu_F|_{F_1}}{\mu_F(F_1)}, \tag{3.4}$$

where $\mu|_E$ denotes the restriction of μ to E. If we can prove that for μ_{F_1}-a.e. $x \in F_1$,

$$\dim_{\mathcal{H}}(\varDelta_x(E_1)) \geq \tau, \tag{3.5}$$

then setting $F^* = F_1$, it is easy to see that $\mu_F(F^*) > 0$, and for μ_{F_1}-a.e. $x \in F^*$,

$$\dim_{\mathcal{H}}(\varDelta_x(E)) \geq \dim_{\mathcal{H}}(\varDelta_x(E_1)) \geq \tau. \tag{3.6}$$

Thus, we only need to show (3.5).

STEP 2: *Second reduction.* We use Lemmas 2.1 and 2.2 to get further reduction.

Denote

$$\mathcal{D}_k^\tau = \Big\{ \bigcup_{j=1}^{M} I_j : M \leq 2^{k\tau},\ I_j \text{ is an open interval, } |I_j| \approx 2^{-k},\ j = 1, \ldots, M \Big\}.$$

Let $\beta > 0$ be a small number to be specified later. We define

$$F^1_{(k)} := \big\{ x \in F_1 : \mu_{E_1}(\{y : |y - x| \in D_k\}) \geq 2^{-k\beta} \text{ for some } D_k \in \mathcal{D}_k^\tau \big\}. \tag{3.7}$$

By Lemma 2.1, the Borel–Cantelli Lemma and the same argument as in [16], proving (3.5) can be reduced to showing

$$\sum_{k=1}^{\infty} \mu_{F_1}(F^1_{(k)}) < \infty. \tag{3.8}$$

STEP 3: *Modification of the sets E and F.* To be able to apply Proposition 2.3, we need to modify the sets E and F by using orthogonal projection.

It is well known that for all $\frac{d}{2} < s_E < \alpha_E$ and all $d - s_E < s_F < \min\big\{\alpha_F, \frac{d}{2}\big\}$, one has

$$I_{s_E}(\mu_E) + I_{s_F}(\mu_F) < \infty. \tag{3.9}$$

By the results about orthogonal projections in [18], we can find a $(d/2+1)$-dimensional subspace $P \subset \mathbb{R}^d$ and the orthogonal projection π which maps $\mathbb{R}^d$ onto P such that

$$I_{s_E}(\pi_*(\mu_E)) < \infty \quad \text{and} \quad I_{s_F}(\pi_*(\mu_F)) < \infty$$

for all $\frac{d}{2} < s_E < \alpha_E$ and all $d - s_E < s_F < \min\big\{\alpha_F, \frac{d}{2}\big\}$. It is easy to see that there exist $\widetilde{E} \subset \pi(E)$ and $\widetilde{F} \subset \pi(F)$ such that

$$\operatorname{dist}(\widetilde{E}, \widetilde{F}) \gtrsim 1, \qquad \pi_*(\mu_E)(\widetilde{E}) > 0, \qquad \pi_*(\mu_F)(\widetilde{F}) > 0.$$

Let

$$\widetilde{\mu}_E(\widetilde{X}) := \frac{\pi_*(\mu_E)|_{\widetilde{E}}(\widetilde{X})}{\pi_*(\mu_E)(\widetilde{E})} = \frac{\pi_*(\mu_E)(\widetilde{X} \cap \widetilde{E})}{\pi_*(\mu_E)(\widetilde{E})}, \quad \widetilde{X} \subset P,$$

$$\widetilde{\mu}_F(\widetilde{X}) := \frac{\pi_*(\mu_F)|_{\widetilde{F}}(\widetilde{X})}{\pi_*(\mu_F)(\widetilde{F})} = \frac{\pi_*(\mu_F)(\widetilde{X} \cap \widetilde{F})}{\pi_*(\mu_F)(\widetilde{F})}, \quad \widetilde{X} \subset P.$$

We also define

$$E_1 := \pi^{-1}(\widetilde{E}) \cap E \quad \text{and} \quad F_1 := \pi^{-1}(\widetilde{F}) \cap F.$$

We know that

$$\mu_E(E_1) = \mu_E(\pi^{-1}(\widetilde{E})) = \pi_*(\mu_E)(\widetilde{E}) > 0,$$

which means that $\mathcal{H}^{\alpha_E}(E_1) > 0$. Similarly, we can obtain $\mathcal{H}^{\alpha_F}(F_1) > 0$. Let

$$\mu_{E_1}(X) = \frac{\mu_E|_{E_1}(X)}{\mu_E(E_1)}, \quad X \subset E_1,$$

$$\mu_{F_1}(X) = \frac{\mu_F|_{F_1}(X)}{\mu_F(F_1)}, \quad X \subset F_1.$$

From the constructions, we deduce that the above sets E_1, F_1 and the projection π satisfy

$$\operatorname{dist}(E_1, F_1) \gtrsim 1 \quad \text{and} \quad \operatorname{dist}(\pi(E_1), \pi(F_1)) \gtrsim 1.$$

Meanwhile, we get an α_E-Frostman measure μ_{E_1} supported on E_1 and an α_F-Frostman measure μ_{F_1} supported on F_1 such that

$$I_{s_E}(\mu_{E_1}) < \infty \quad \text{and} \quad I_{s_F}(\mu_{F_1}) < \infty.$$

In addition,

$$I_{s_E}((\pi)_*(\mu_{E_1})) < \infty \quad \text{and} \quad I_{s_F}((\pi)_*(\mu_{F_1})) < \infty,$$

where s_E satisfies $\frac{d}{2} < s_E < \alpha_E$ and s_F satisfies $d - s_E < s_F < \min\left\{\alpha_F, \frac{d}{2}\right\}$.

Therefore, the sets E_1, F_1 and the measures μ_{E_1}, μ_{F_1} satisfy the assumptions of Proposition 2.3.

Step 4: *Proof of* (3.8). We now utilize Proposition 2.3 to show (3.8).

For simplicity, we write E, F for E_1, F_1. It suffices to prove

$$\sum_{k=1}^{\infty} \mu_F(F_{(k)}) < \infty, \tag{3.10}$$

with

$$F_{(k)} = \left\{ x \in F : \exists D_k \in \mathcal{D}_k^{\tau} \ \mu_E(\{y : |y - x| \in D_k\}) \geq 2^{-k\beta} \right\}.$$

Let $R_0(k) = 2^{10k\beta/(c\delta)}$, where c is the constant in (2.10), and $0 < \beta \ll \delta \ll 1$ will be specified later. With this choice we have $\operatorname{RapDec}(R_0(k)) = \operatorname{RapDec}(2^k)$ and $R_0(k)^{-c\delta} \ll 2^{-k\beta}$.

By Proposition 2.3, we can find $F' \subset F$ with

$$\mu_F(F') \geq 1 - R_0(k)^{-c\delta}.$$

Let $F'_{(k)} = F_{(k)} \cap F'$; then

$$\mu_F(F_{(k)} \setminus F'_{(k)}) < R_0(k)^{-c\delta} \ll 2^{-k\beta}. \tag{3.11}$$

Combining this with the basic fact that

$$\mu_F(F_{(k)}) = \mu_F(F_k \setminus F'_{(k)}) + \mu_F(F'_{(k)}),$$

we can reduce the proof of (3.10) to showing

$$\sum_{k=1}^{\infty} \mu_F(F'_{(k)}) < \infty. \tag{3.12}$$

This will follow from

$$\mu_F(F'_{(k)}) \lesssim_{\delta,\beta} 2^{-k\beta}. \tag{3.13}$$

Recall that for $x \in F'_{(k)}$ there exists $D_k \in \mathcal{D}_k^{\tau}$ such that

$$\mu_E(\{y : |x-y| \in D_k\}) \geq 2^{-k\beta}.$$

Proof of (3.13). For any $D_k \in \mathcal{D}_k^{\tau}$ and $x \in F'_{(k)}$, we obtain

$$\mu_E(\{y : |y-x| \in D_k\}) \lesssim \int_{\widetilde{D}_k} d_*^x(\mu_E^{2^{-k}})(t)\,dt, \tag{3.14}$$

where $\widetilde{D}_k$ denotes the 2^{-k}-neighborhood of D_k. In fact,

$$\begin{aligned}
\int_{\widetilde{D}_k} d_*^x(\mu_E^{2^{-k}})(t)\,dt &= \int_{\widetilde{D}_k}\int_{\mathbb{S}^{d-1}(x,t)} \mu_E^{2^{-k}}(z)\,dz\,dt = \int_{|x-z|\in\widetilde{D}_k} \mu_E^{2^{-k}}(z)\,dz \\
&= 2^{dk} \int_{|x-z|\in\widetilde{D}_k} \int \phi(2^k(z-y))\,d\mu_E(y)\,dz \\
&\geq 2^{dk} \int \int_{|u|\in\widetilde{D}_k,\,|x-y-u|\leq 2^{-k-1}} du\,d\mu_E(y) \\
&\gtrsim 2^{dk} \int_{|y-x|\in D_k} \int_{|u|\in B(|x-y|,2^{-k-1})} du\,d\mu_E(y) \\
&\gtrsim \int_{|y-x|\in D_k} d\mu_E(y) = \mu_E(\{y : |y-x| \in D_k\}).
\end{aligned}$$

From the definition of $F_{(k)}$ and (3.14), we have

$$2^{-k\beta}\mu_F(F'_{(k)}) \le \int_{F'_{(k)}} \Big(\sup_{D_k\in\mathcal{D}_k^\tau} \int_{\widetilde{D}_k} |d_*^x(\mu_E^{2^{-k}})(t)|\,dt\Big)\,d\mu_F(x) \tag{3.15}$$
$$\le \int_{F'_{(k)}} \Big(\sup_{D_k\in\mathcal{D}_k^\tau} \int_{\widetilde{D}_k} |d_*^x(\mu_{E,\,\mathrm{good}}^{2^{-k}})(t)|\,dt\Big)\,d\mu_F(x)$$
$$+ \int_{F'_{(k)}} \Big(\int |d_*^x(\mu_E^{2^{-k}})(t) - d_*^x(\mu_{E,\,\mathrm{good}}^{2^{-k}})(t)|\,dt\Big)\,d\mu_F(x)$$
$$:= I_1 + I_2.$$

By (2.10), we have

$$I_2 \lesssim R_0(k)^{-c\delta} \ll 2^{-k\beta}\mu_F(F'_{(k)}). \tag{3.16}$$

For I_1, by Hölder's inequality and (2.11), we have

$$I_1 \le \int_{F'_{(k)}} \sup_{D_k\in\mathcal{D}_k^\tau} \Big(|\widetilde{D}_k| \int |d_*^x(\mu_{E,\,\mathrm{good}}^{2^{-k}})(t)|^2\,dt\Big)^{1/2}\,d\mu_F(x)$$
$$\le \sup_{D_k\in\mathcal{D}_k^\tau}(|\widetilde{D}_k|^{1/2}) \int_{F'_{(k)}} \Big(\int |d_*^x(\mu_{E,\,\mathrm{good}}^{2^{-k}})(t)|^2\,dt\Big)^{1/2}\,d\mu_F(x)$$
$$\le 2^{-\frac{k(1-\tau)}{2}} \int_{F'_{(k)}} \Big(\int |d_*^x(\mu_{E,\,\mathrm{good}}^{2^{-k}})(t)|^2\,dt\Big)^{1/2}\,d\mu_F(x)$$
$$\le 2^{-\frac{k(1-\tau)}{2}} \mu_F(F'_{(k)})^{1/2}\Big(\int_{F'_{(k)}} \int |d_*^x(\mu_{E,\,\mathrm{good}}^{2^{-k}})(t)|^2\,dt\,d\mu_F(x)\Big)^{1/2}$$
$$\le 2^{-\frac{k(1-\tau)}{2}} \mu_F(F'_{(k)})^{1/2}\Big(C(\delta)R_0(k)^{O(1)} \int |\widehat{\mu_E^{2^{-k}}}(\xi)|^2|\xi|^{-\frac{d}{2(d+1)}-\frac{(d-1)\alpha_F}{d+1}+O(\delta)}\,d\xi$$
$$+ \mathrm{RapDec}(R_0(k))\Big)^{1/2}$$
$$\le_{\delta,\beta} 2^{-\frac{k(1-\tau)}{2}} 2^{\frac{O(1)k\beta}{2\delta}} \mu_F(F'_{(k)})^{1/2}\Big(\int |\widehat{\mu_E^{2^{-k}}}(\xi)|^2|\xi|^{-\frac{d}{2(d+1)}-\frac{(d-1)\alpha_F}{d+1}+O(\delta)}\,d\xi$$
$$+ \mathrm{RapDec}(R_0(k))\Big)^{1/2}.$$

Plugging this and (3.16) into (3.15), we obtain

$$\mu_F(F'_{(k)})$$
$$\lesssim_{\delta,\beta} 2^{-k(1-\tau-O(\beta/\delta)-2\beta)} \int |\widehat{\mu_E}(\xi)|^2|\widehat{\phi}(2^{-k}\xi)|^2|\xi|^{-\frac{d}{2(d+1)}-\frac{(d-1)\alpha_F}{d+1}+O(\delta)}\,d\xi$$
$$+ \mathrm{RapDec}(R_0(k))$$

$$\lesssim_{\delta,\beta} 2^{-k(1-\tau-O(\beta/\delta)-2\beta)} \int\limits_{|\xi|\le 2^{k/(1-\delta)}} |\widehat{\mu_E}(\xi)|^2|\xi|^{-\frac{d}{2(d+1)}-\frac{(d-1)\alpha_F}{d+1}+O(\delta)}\,d\xi$$
$$+ \mathrm{RapDec}(R_0(k)).$$

Since $\tau < 1$, we may choose $0 < \beta \ll \delta \ll 1$ such that

$$1 - \tau - O(\beta/\delta) - 3\beta > 0,$$

which implies

$$\begin{aligned} 2^{-k(1-\tau-O(\beta/\delta)-2\beta)} &\lesssim 2^{-k\beta}|\xi|^{-(1-\delta)(1-\tau-O(\beta/\delta)-3\beta)} \\ &= 2^{-k\beta}|\xi|^{-1+\tau+O(\beta/\delta+\delta+\beta)} \end{aligned}$$

in the domain $|\xi| \le 2^{k/(1-\delta)}$. Hence,

$$\mu_F(F'_{(k)}) \lesssim_{\delta,\beta} 2^{-k\beta} \int\limits_{|\xi|\le 2^{k/(1-\delta)}} |\widehat{\mu_E}(\xi)|^2|\xi|^{-1+\tau-\frac{d}{2(d+1)}-\frac{(d-1)\alpha_F}{d+1}+O(\beta/\delta+\delta+\beta)}\,d\xi$$
$$+ \mathrm{RapDec}(R_0(k)).$$

Since $\alpha_F > d + \frac{d-2}{2(d-1)} + \frac{d+1}{d-1}\tau - \frac{d+1}{d-1}\alpha_E$, we may choose s_E very close to α_E and $0 < \beta \ll \delta \ll 1$ such that

$$-1 + \tau - \tfrac{d}{2(d+1)} - \tfrac{(d-1)\alpha_F}{d+1} + O(\beta/\delta + \delta + \beta) < -d + s_E,$$

which guarantees that the energy integral

$$\int |\widehat{\mu_E}(\xi)|^2|\xi|^{-1+\tau-\frac{d}{2(d+1)}-\frac{(d-1)\alpha_F}{d+1}+O(\beta/\delta+\delta+\beta)}\,d\xi$$

is finite, and so (3.13) follows, which implies Theorem 1.1.

4. Proof of Proposition 2.3. In this section, we will prove Proposition 2.3 by a similar argument to [4, 9].

4.1. Constructions. Let $R_0(k)$ ($\ll 2^k$) be a large number to be determined later, and let $R_j(k) = 2^j R_0(k)$ with $j \in \mathbb{Z}^+$ and $\delta > 0$ be a small constant to be chosen later.

In frequency space, we use rectangular blocks τ to cover the annulus $R_{j-1}(k) \le |\omega| \le R_j(k)$, where the dimensions of τ are approximate $R_j(k)^{1/2} \times \cdots \times R_j(k)^{1/2} \times R_j(k)$, with the long direction of each block τ being the radial direction. Choose a partition of unity subordinate to this cover such that

$$1 = \psi_0 + \sum_{j=1}^{\infty}\sum_{\tau} \psi_{j,\tau}. \tag{4.1}$$

In physical space, for each (j,τ), cover the unit ball by tubes T with dimension of T approximately $R_j^{-1/2+\delta}(k) \times \cdots \times R_j^{-1/2+\delta}(k) \times 2$, with the long axis parallel to the long axis of τ. The covering has uniformly bounded overlap. $\mathbb{T}_{j,\tau}$ denotes the collection of all these tubes. Let η_T be a partition of unity such that

$$1 = \sum_{T\in\mathbb{T}_{j,\tau}} \eta_T(x), \qquad \forall x \in B(0,2). \tag{4.2}$$

Let $\mathbb{T}_j = \bigcup_\tau \mathbb{T}_{j,\tau}$ and $\mathbb{T} = \bigcup_{j\geq 1} \mathbb{T}_j$. For each $T \in \mathbb{T}_{j,\tau}$, we define

$$M_T f := \eta_T(\psi_{j,\tau}\widehat{f})^\vee,$$

and $M_0 f := (\psi_0 \widehat{f})^\vee$.

Let $c(s_E, s_F) > 0$ be a large constant to be determined later, and let $4T$ denote the concentric tube of four times the radius. A tube $T \in \mathbb{T}_{j,\tau}$ is called *bad* if

$$\mu_F(4T) \geq R_j(k)^{-d/4+c(s_E,s_F)\delta}. \tag{4.3}$$

A tube $T \in \mathbb{T}$ is called *good* if it is not bad, and we define

$$\mu^{2^{-k}}_{E,\,\mathrm{good}} := M_0\mu_E^{2^{-k}} + \sum_{T\in\mathbb{T},\,\mathrm{good}} M_T\mu_E^{2^{-k}}. \tag{4.4}$$

As in [9, proof of Lemma 5.2], we can prove that $\mu^{2^{-k}}_{E,\,\mathrm{good}}$ is essentially supported in the $R_0(k)$-neighborhood of E with a rapidly decaying tail away from it.

It remains, to show that $\mu_E^{2^{-k}}$ and $\mu^{2^{-k}}_{E,\,\mathrm{good}}$ satisfy the estimates (2.10) and (2.11).

4.2. Proof of (2.10)**.** We first recall a key result of [9]. For any point $x \in F$, define

$$\mathrm{Bad}_j(x) = \bigcup_{\substack{T\in\mathbb{T}_j:\, x\in 2T,\\ T \text{ is bad}}} 2T, \qquad \forall j \geq 1.$$

Let

$$\mathrm{Bad}_j = \{(x_1,x_2) \in E\times F : \text{there is a bad } T \in \mathbb{T}_j \text{ such that } 2T \text{ contains } x_1 \text{ and } x_2\}.$$

Then, for any point x in F,

$$\|d_*^x(\mu^{2^{-k}}_{E,\,\mathrm{good}}) - d_*^x(\mu_E^{2^{-k}})\|_{L^1} \lesssim \sum_{j\geq 1} \mu_E(\mathrm{Bad}_j(x)) + \mathrm{RapDec}(R_0(k)). \tag{4.5}$$

Next we will use the result of [19] to show the bad part is negligible. We define a radial projection map $P_y : \mathbb{R}^d \setminus \{y\} \to \mathbb{S}^{d-1}$ by

$$P_y(x) = \frac{x-y}{|x-y|}.$$

Proposition 4.1 ([19]). *Let μ, ν be measures on two compact sets with disjoint supports, such that $I_s(\mu) < \infty$, $I_u(\nu) < \infty$ for some $u > d-1$,*

$2(d-1)-u<s<d-1$. *Then there is* $p=p(s,u)>1$ *such that* $P_y\nu$ *is absolutely continuous with a density in* $L^p(\mathbb{S}^{d-1})$ *for* μ*-almost all* y. *Moreover,*

$$\int \|P_y\nu\|_{L^p}^p \, d\mu(y) < \infty.$$

By Proposition 4.1, we can prove the following lemma.

LEMMA 4.2. *For each* $\frac{d}{2}<s_E<\alpha_E<\dim_{\mathcal{H}}(E)$ *and all* $d-s_E<s_F<\min\left\{\alpha_F,\frac{d}{2}\right\}$, *there is a constant* $c(s_E,s_F)>0$ *(that appears in the definition of bad tubes) such that for each* $j\geq 1$,

$$\mu_E^{2^{-k}}\times\mu_F(\mathrm{Bad}_j)\lesssim R_j(k)^{-2d\delta}.$$

Proof. Recall that $\widetilde{\mu}_E^{2^{-k}}$ and $\widetilde{\mu}_F$ are measures on a $(d/2+1)$-dimensional subspace and that $I_{s_E}(\widetilde{\mu}_E^{2^{-k}})<\infty$ and $I_{s_F}(\widetilde{\mu}_F)<\infty$. Therefore, applying Proposition 4.1 with $\widetilde{\mu}_E^{2^{-k}}$, $\widetilde{\mu}_F$, we get

$$\int \|P_y\widetilde{\mu}_F\|_{L^p}^p \, d\widetilde{\mu}_E^{2^{-k}}(y) < \infty.$$

By the definition of Bad_j, it suffices to consider tubes $T\in\mathbb{T}_j$ that intersect E and F. Hence, the projected tube $\pi(T)\subset P$ also looks like a tube, with side length ~ 1 in the long direction, and $\sim R_j(k)^{-1/2+\delta}$ in the rest of the directions. Therefore, $\mathbb{T}_j$ gives rise to a collection $\widetilde{\mathbb{T}}_j$ that contains tubes in P of dimensions roughly $1\times R_j(k)^{-1/2+\delta}\times\cdots\times R_j(k)^{-1/2+\delta}$.

One can similarly define a tube $\widetilde{T}\in\widetilde{\mathbb{T}}_j$ to be bad if

$$\widetilde{\mu}_F(4\widetilde{T})\geq R_j(k)^{-d/4+c(s_E,s_F)\delta}.$$

It is easy to see that the badness of a tube is preserved under the projection. Indeed, if $T\in\mathbb{T}_j$ is bad, then

$$\widetilde{\mu}_F(4\pi(T))\geq\mu_F(4T)\geq R_j(k)^{-d/4+c(s_E,s_F)\delta}.$$

Define

$$\widetilde{\mathrm{Bad}}_j:=\{(x_1,x_2)\in P^2: \text{there is a bad } \widetilde{T}\in\widetilde{\mathbb{T}}_j \text{ such that } 2\widetilde{T} \text{ contains } x_1 \text{ and } x_2\}.$$

Then one has

$$\mu_E^{2^{-k}}\times\mu_F(\mathrm{Bad}_j)\leq\widetilde{\mu}_E^{2^{-k}}\times\widetilde{\mu}_F(\widetilde{\mathrm{Bad}}_j)=\int\widetilde{\mu}_F(\widetilde{\mathrm{Bad}}_j(y))\,d\widetilde{\mu}_E^{2^{-k}}(y),$$

where

$$\widetilde{\mathrm{Bad}}_j(y):=\bigcup_{\substack{\widetilde{T}\in\widetilde{\mathbb{T}}_j:\, y\in 2\widetilde{T},\\ \widetilde{T}\text{ is bad}}} 2\widetilde{T}. \tag{4.6}$$

Let $\widetilde{T}\in\widetilde{\mathbb{T}}_j$ be a bad tube and $y\in 2\widetilde{T}\bigcap\pi(E_1)$. Let $A(\widetilde{T})$ be the cap of the sphere $\mathbb{S}^{d/2}$ whose center corresponds to the direction of the long axis of $\widetilde{T}$ and with radius $\sim R_j(k)^{-1/2+\delta}$. Since $d(\pi(E),\pi(F)) \geq 1$, one has $P_y(4\widetilde{T} \cap \pi(F)) \subset A(\widetilde{T})$, hence

$$P_y\widetilde{\mu}_F(A(\widetilde{T})) \geq \widetilde{\mu}_F(4\widetilde{T}) \geq R_j(k)^{-d/4+c(s_E,s_F)\delta}.$$

Thus, $P_y(\widetilde{\mathrm{Bad}}_j(y))$ can be covered by caps $A(\widetilde{T})$ of radius $\sim R_j(k)^{-d/4+\delta}$ each of which satisfies (4.6). By the Vitali covering lemma, there exists a pairwise disjoint family of the $A(\widetilde{T})$'s such that the $5A(\widetilde{T})$'s cover $P_y(\widetilde{\mathrm{Bad}}_j(y))$. Hence, the total number of disjoint $A(\widetilde{T})$'s in the covering is bounded by $R_j(k)^{d/4-c(s_E,s_F)\delta}$, which implies

$$|P_y(\widetilde{\mathrm{Bad}}_j(y))| \lesssim R_j(k)^{d/4-c(s_E,s_F)\delta} \cdot R_j(k)^{-1/2+\delta} = R_j(k)^{-(c(s_E,s_F)-d/2)\delta},$$

where $|\cdot|$ denotes the surface measure on $\mathbb{S}^{d/2}$. Therefore, by Hölder's inequality and by choosing $c(s_E,s_F)$ sufficiently large, one has

$$\begin{aligned}\mu_E^{2^{-k}} \times \mu_F(\mathrm{Bad}_j) &\leq \int \widetilde{\mu}_F(\widetilde{\mathrm{Bad}}_j(y))\, d\widetilde{\mu}_E^{2^{-k}}(y) \leq \int \Big(\int_{P_y(\widetilde{\mathrm{Bad}}_j(y))} P_y\widetilde{\mu}_F \Big)\, d\widetilde{\mu}_E^{2^{-k}}(y) \\ &\leq \sup_y |P_y(\widetilde{\mathrm{Bad}}_j(y))|^{1-1/p} \int \|P_y\widetilde{\mu}_F\|_{L^p}\, d\widetilde{\mu}_E^{2^{-k}} \lesssim R_j(k)^{-2d\delta}. \ \blacksquare\end{aligned}$$

Proof of (2.10). Since

$$\mu_E^{2^{-k}} \times \mu_F(\mathrm{Bad}_j) = \int \mu_E^{2^{-k}}(\mathrm{Bad}_j(x))\, d\mu_F(x),$$

Lemma 4.2 ensures that there exists $B_j \subset F$ with $\mu_F(B_j) \leq R_j(k)^{-(d/2)\delta}$ and for all $x \in F\backslash B_j$,

$$\mu_E(\mathrm{Bad}_j(x)) \lesssim R_j(k)^{-(3d/2)\delta}.$$

Let $F' = F\backslash \bigcup_{j\geq 1} B_j$ and choose $R_0(k)$ sufficiently large (depending on δ and α_E, α_F). One obviously has $\mu_F(F') \geq 1 - R_0(k)^{-c(s_E,s_F)\delta}$, and by (4.5) for each $x \in F'$, we have

$$\|d_*^x(\mu_{E,\mathrm{good}}^{2^{-k}}) - d_*^x(\mu_E^{2^{-k}})\|_{L^1} \lesssim R_0(k)^{-(d/2)\delta}. \ \blacksquare$$

4.3. Proof of (2.11)**.** The proof of (2.11) is based on [15]. We first recall the L^2-identity.

PROPOSITION 4.3 ([15]). *For any Schwartz function f on $\mathbb{R}^d$, $d \geq 2$, and any $x \in \mathbb{R}^d$,*

$$\int_0^\infty |f * \sigma_t(x)|^2 t^{d-1}\, dt = \int_0^\infty |f * \widehat{\sigma}_r(\xi)|^2 r^{d-1}\, dr.$$

An important ingredient in our proof is refined decoupling, which is proved by the l^2 decoupling in [2].

Suppose that $S \subset \mathbb{R}^d$ is a compact and strictly convex C^2 hypersurface with Gaussian curvature ~ 1. For any $\epsilon > 0$, choose $0 < \delta \ll \epsilon$. Let $\mathcal{N}_1(RS)$

be the 1-neighborhood of RS. Partition RS into blocks θ with scale $R^{1/2}\times\cdots\times R^{1/2}\times 1$. For each θ, let $\mathbb{T}_\theta$ denote all the tubes with dimensions $R^{-1/2+\delta}\times\cdots\times R^{-1/2+\delta}\times 1$ where the long axis is perpendicular to θ, and let $\mathbb{T}=\bigcup_\theta \mathbb{T}_\theta$. Each $T\in\mathbb{T}$ belongs to $\mathbb{T}_\theta$ for a single θ, and we let $\theta(T)$ denote this θ. We say that f is *microlocalized* to $(T,\theta(T))$ if f is essentially supported in $2T$ and $\widehat{f}$ is essentially supported in $2\theta(T)$.

PROPOSITION 4.4 ([2]). *Let $2\le p\le \frac{2(d+1)}{d-1}$. For any $\epsilon>0$, suppose there exists $0<\delta\ll\epsilon$ satisfying the following. Let $\mathbb{W}\subset\mathbb{T}$ and suppose that each $T\in\mathbb{W}$ lies in the unit ball. Let $W=|\mathbb{W}|$. Suppose that $f=\sum_{T\in\mathbb{W}} f_T$, where f_T is microlocalized to $(T,\theta(T))$. Suppose that $\|f_T\|_{L^p}$ is $\sim$ constant for each $T\in\mathbb{W}$. Let Y be a union of $R^{-1/2}$-cubes in the unit ball each of which intersects at most M tubes $T\in\mathbb{W}$. Then*

$$\|f\|_{L^p(Y)}\lessapprox\Big(\frac{M}{W}\Big)^{1/2-1/p}\Big(\sum_{T\in\mathbb{W}}\|f_T\|_{L^p}^2\Big)^{1/2}. \tag{4.7}$$

Next, we prove a key lemma.

LEMMA 4.5. *For any $s_F>0$, $r>0$, and δ sufficiently small depending on s_F, ϵ, we have*

$$\begin{aligned}\int_F |\mu_{E,\,\mathrm{good}}^{2^{-k}}*\widehat{\sigma}_r(x)|^2\,d\mu_F(x) \\ \le C(R_0(k))r^{-\frac{d}{2(d+1)}-\frac{(d-1)}{d+1}s_F+\epsilon}r^{-(d-1)}\int|\widehat{\mu_E^{2^{-k}}}(\xi)|\psi_r(\xi)\,d\xi+\mathrm{RapDec}(r),\end{aligned}\tag{4.8}$$

where ψ_r is a weight function which is $\sim$1 on the annulus $r-1\le|\xi|\le r+1$ and decays off it. To be precise, we could take

$$\psi_r(\xi)=\big(1+\big|r-|\xi|\big|\big)^{-100}.$$

Proof of (2.11). By Proposition 4.3 and Lemma 4.5,

$$\begin{aligned}&\int_F\|d_*^x(\mu_{E,\,\mathrm{good}}^{2^{-k}})\|_{L^2}^2\,d\mu_F(x)\\ &\simeq\int_0^\infty\int_F|\mu_{E,\,\mathrm{good}}^{2^{-k}}*\sigma_t(x)|^2\,d\mu_F(x)t^{d-1}\,dt\\ &\simeq\int_0^\infty\int_F|\mu_{E,\,\mathrm{good}}^{2^{-k}}*\widehat{\sigma}_r(x)|^2\,d\mu_F(x)r^{d-1}\,dr\\ &\lesssim_{R_0(k)}\int_0^\infty\int_{\mathbb{R}^d}r^{-\frac{d}{2(d+1)}-\frac{(d-1)\alpha_F}{d+1}+\epsilon}r^{-(d-1)}\psi_r(\xi)|\widehat{\mu_E}(\xi)|^2|\widehat{\phi}(2^{-k}\xi)|^2\,d\xi\,dr\\ &\lesssim\int_{\mathbb{R}^d}|\xi|^{-\frac{d}{2(d+1)}-\frac{(d-1)\alpha_F}{d+1}+\epsilon}|\widehat{\mu_E}(\xi)|^2\,d\xi,\end{aligned}$$

as desired. ∎

Proof of Lemma 4.5. CASE 1: $r > 10R_0(k)$. By definition,

$$\mu_{E,\text{good}}^{2^{-k}} * \widehat{\sigma}_r(x) = \sum_{j:\, R_j(k)\sim r} \sum_{T\in\mathbb{T}_{j,\tau}:\, T\text{ good}} M_T\mu_E^{2^{-k}} * \widehat{\sigma}_r(x) + \text{RapDec}(r).$$

Let η_1 be a bump function adapted to the unit ball and define

$$f_T = \eta_1(M_T\mu_E^{2^{-k}} * \widehat{\sigma}_r). \tag{4.9}$$

Using the dyadic pigeonhole argument, we obtain

$$\begin{aligned}
&\int |\mu_{E,\text{good}}^{2^{-k}} * \widehat{\sigma}_r(x)|^2\, d\mu_F(x) \qquad (4.10)\\
&\lesssim \int \Big|\eta_1(x) \sum_{j:\, R_j(k)\sim r} \sum_{T\in\mathbb{T}_{j,\tau}:\, T\text{ good}} M_T\mu_E^{2^{-k}} * \widehat{\sigma}_r(x)\Big|^2 d\mu_F(x) + \text{RapDec}(r)\\
&\lesssim \Big[\sum_{k=1}^{N_1} \Big(\int |f_k(x)|^2\, d\mu_F(x)\Big)^{1/2}\Big]^2 + \text{RapDec}(r)\\
&\lesssim N_1^2 \int |f_\lambda(x)|^2\, d\mu_F(x) + \text{RapDec}(r)\\
&\lesssim (\log r)^2 \int |f_\lambda(x)|^2\, d\mu_F(x) + \text{RapDec}(r)
\end{aligned}$$

where

$$\begin{aligned}
f_k &= \sum_{T\in\mathbb{W}_k} f_T,\\
\mathbb{W}_k &= \bigcup_{j:\, R_j(k)\sim r} \bigcup_{\tau} \{T\in\mathbb{T}_{j,\tau} : T \text{ good},\ \|f_T\|_{L^p} \sim 2^{-k}\|f\|_{\max}\},\\
\|f\|_{\max} &= \max_{T\in\mathbb{T}} \|f_T\|_{L^p}, \qquad N_1 \sim \log r,
\end{aligned}$$

and

$$\int |f_\lambda(x)|^2\, d\mu_F(x) = \max_{1\le k\le N_1} \int |f_k(x)|^2\, d\mu_F(x). \tag{4.11}$$

It suffices to estimate the term

$$\int |f_\lambda(x)|^2\, d\mu_F(x). \tag{4.12}$$

To do this, we divide the unit ball into $r^{-1/2}$-cubes q. Let $Y_l = \bigcup_{q\in\mathcal{Q}_l} q$, where

$$\mathcal{Q}_l := \{q : q \text{ intersects } {\sim}2^l \text{ tubes } T \in \mathbb{W}_\lambda\},$$

where λ is chosen according to (4.11). Then, we make another dyadic pigeonholing argument to get

$$\begin{aligned}
\int |f_\lambda(x)|^2\, d\mu_F(x) &= \sum_{l=1}^{N_2} \int_{Y_l} |f_\lambda(x)|^2\, d\mu_F(x) \le N_2 \int_{Y_M} |f_\lambda(x)|^2\, d\mu_F(x) \qquad (4.13)\\
&\lesssim (\log r) \int_{Y_M} |f_\lambda(x)|^2\, d\mu_F(x),
\end{aligned}$$

where

$$\int_{Y_M} |f_\lambda(x)|^2 \, d\mu_F(x) = \max_{1\le l\le N_2} \int_{Y_l} |f_\lambda(x)|^2 \, d\mu_F(x)$$

and $2^{N_2} \sim r^{O(1)}$. On the other hand, by the same argument as in [9], we deduce that the $r^{-1/2}$-neighborhood of Y_M has measure

$$\mu_F(N_{r^{-1/2}}(Y_M)) \lesssim \frac{|\mathbb{W}_\lambda| r^{-d/4+c(\alpha_E,\alpha_F)\delta}}{M}. \tag{4.14}$$

Let $p = \frac{2(d+1)}{d-1}$. Then, by Hölder's inequality, refined decoupling inequality (Proposition 4.4), (4.14) and

$$\|\mu_F * \eta_{1/r}\|_{L^\infty} \lesssim r^{d-\alpha_F},$$

we obtain

$$\begin{aligned}
\int_{Y_M} |f_\lambda(x)|^2 \, d\mu_F(x) &\lesssim \Big(\int_{Y_M} |f_\lambda|^p\Big)^{\frac{2}{p}} \Big(\int_{Y_M} |\mu_F * \eta_{1/r}|^{\frac{p}{p-2}}\Big)^{1-\frac{2}{p}} \\
&\lessapprox \Big(\frac{M}{|\mathbb{W}_\lambda|}\Big)^{1-\frac{2}{p}} \Big(\sum_{T\in\mathbb{W}_\lambda} \|f_T\|_{L^p}^2\Big) r^{2(d-\alpha_F)/p} \mu_F\big(N_{r^{-1/2}}(Y_M)\big)^{1-\frac{2}{p}} \\
&\lesssim \Big(\frac{r^{-\frac{d}{4}+c(\alpha_E,\alpha_F)\delta}}{\mu_F(N_{r^{-1/2}}(Y_M))}\Big)^{1-\frac{2}{p}} \\
&\quad\times \Big(\sum_{T\in\mathbb{W}_\lambda} \|f_T\|_{L^p}^2\Big) r^{2(d-\alpha_F)/p} \mu_F(N_{r^{-1/2}}(Y_M))^{1-\frac{2}{p}} \\
&\lesssim r^{(-\frac{d}{4}+c(\alpha_E,\alpha_F)\delta)\frac{p-2}{p}+\frac{2(d-\alpha_F)}{p}} \sum_{T\in\mathbb{W}_\lambda} \|f_T\|_{L^p}^2 \\
&\lesssim r^{O(\delta)+(\frac{5}{2p}-\frac{1}{4})d-\frac{2\alpha_F}{p}} \sum_{T\in\mathbb{W}_\lambda} \|f_T\|_{L^p}^2.
\end{aligned} \tag{4.15}$$

On the other hand, by the definition of f_T in (4.9), Hausdorff–Young's inequality and Hölder's inequality, we have

$$\begin{aligned}
\|f_T\|_{L^p} &\lesssim \|f_T\|_{L^\infty}|T|^{1/p} \lesssim \|\eta_1(M_T\mu_E^{2^{-k}} * \widehat{\sigma_r})\|_{L^\infty}|T|^{1/p} \\
&\lesssim \|\widehat{M_T\mu_E^{2^{-k}}}\|_{L^1(d\sigma_r)}|T|^{1/p} \lesssim \sigma_r(\operatorname{supp}\widehat{M_T\mu_E^{2^{-k}}})^{1/2}\|\widehat{M_T\mu_E^{2^{-k}}}\|_{L^2(d\sigma_r)}|T|^{1/p} \\
&\lesssim r^{-(\frac{1}{2p}+\frac{1}{4})(d-1)+O(\delta)}\|\widehat{M_T\mu_E^{2^{-k}}}\|_{L^2(d\sigma_r)}.
\end{aligned}$$

Plugging this into (4.15) yields

$$\begin{aligned}
\int_{Y_M} |f_\lambda(x)|^2 \, d\mu_F(x) &\lesssim r^{O(\delta)+(\frac{3}{2p}-\frac{3}{4})d-\frac{2\alpha_F}{p}+\frac{1}{p}+\frac{1}{2}} \sum_{T\in\mathbb{W}_\lambda} \|\widehat{M_T\mu_E^{2^{-k}}}\|_{L^2(d\sigma_r)}^2 \\
&\lesssim r^{-\frac{d}{2(d+1)}-\frac{(d-1)\alpha_F}{d+1}+\epsilon} r^{-(d-1)} \int |\widehat{\mu_E^{2^{-k}}}(\xi)|^2 \psi_r(\xi)\, d\xi.
\end{aligned} \tag{4.16}$$

This together with (4.13) and (4.10) implies (4.8) in the case $r > 10R_0(k)$.

CASE 2: $r \le 10R_0(k)$. By Hölder's inequality and Hausdorff–Young's inequality, we get

$$\int_F |\mu_{E,\,\mathrm{good}}^{2^{-k}} * \widehat{\sigma_r}(x)|^2\, d\mu_F(x) \le \|\mu_{E,\,\mathrm{good}}^{2^{-k}} * \widehat{\sigma_r}\|_{L^\infty}^2 \le \|\widehat{\mu_{E,\,\mathrm{good}}^{2^{-k}}}\|_{L^1(d\sigma_r)}^2$$
$$\le \|\widehat{\mu_{E,\,\mathrm{good}}^{2^{-k}}}\|_{L^2(d\sigma_r)}^2.$$

By a similar argument to [9, p. 812], we have

$$\|\widehat{\mu_{E,\,\mathrm{good}}^{2^{-k}}}\|_{L^2(d\sigma_r)}^2 \lesssim r^{-(d-1)} \int |\widehat{\mu_{E,\,\mathrm{good}}^{2^{-k}}}(\xi)|^2 \psi_r(\xi)\, d\xi.$$

Hence

$$\int_F |\mu_{E,\,\mathrm{good}}^{2^{-k}} * \widehat{\sigma_r}(x)|^2\, d\mu_F(x) \lesssim r^{-(d-1)} \int |\widehat{\mu_{E,\,\mathrm{good}}^{2^{-k}}}(\xi)|^2 \psi_r(\xi)\, d\xi$$
$$\lesssim C(R_0(k)) r^{-\frac{d}{2(d+1)} - \frac{(d-1)\alpha_F}{d+1} + \epsilon} r^{-(d-1)} \int |\widehat{\mu_{E,\,\mathrm{good}}^{2^{-k}}}(\xi)|^2 \psi_r(\xi)\, d\xi,$$

which yields (4.8) in the case $r \le 10R_0(k)$.

Thus, we conclude the proof of Lemma 4.5. ∎

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