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Upper bounds for fractional joint moments of the Riemann zeta function

by

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1. Introduction. In the past two decades, conjectural connections between the zeros of the Riemann zeta function $\zeta(s)$ and eigenvalues of random unitary matrices have led to many interesting developments in understanding the moments of the zeta function. In the recent random matrix theory literature, there has been a fair bit of interest in understanding the joint moments of the characteristic polynomial of a random unitary matrix with its derivative. In this paper, the primary objects are the joint moments of $\zeta(s)$, given by

$$\mathcal{I}_T(k,h) = \int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k-2h} \left|\zeta'\left(\tfrac{1}{2}+it\right)\right|^{2h} dt,$$

as well as the joint moments of the Hardy Z function

$$\mathcal{J}_T(k,h) = \int_T^{2T} |Z(t)|^{2k-2h} |Z'(t)|^{2h}\, dt,$$

where

$$Z(t) = e^{i\theta(t)}\zeta\left(\tfrac{1}{2}+it\right), \qquad \theta(t) = \arg\left(\Gamma\left(\tfrac{1}{4}+\tfrac{it}{2}\right)\right) - \frac{\log \pi}{2}t.$$

Note in particular that $|Z(t)| = |\zeta(\frac{1}{2}+it)|$, and that $Z(t)$ is real-valued for $t \in \mathbb{R}$. The work of Keating and Snaith [23, 24], Hughes [20], and Hall [17] has led to the conjecture that whenever $k > -\frac{1}{2}$ and $-\frac{1}{2} < h \le k + \frac{1}{2}$,

$$\begin{aligned} \mathcal{I}_T(k,h) &\sim \mathfrak{C}_\zeta(k,h) T (\log T)^{k^2+2h}, \\ \mathcal{J}_T(k,h) &\sim \mathfrak{C}_Z(k,h) T (\log T)^{k^2+2h} \end{aligned} \tag{1.1}$$

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for certain constants $\mathfrak{C}_\zeta(k,h), \mathfrak{C}_Z(k,h)$ as $T \to \infty$. There are conjectured values for the constants $\mathfrak{C}_Z(k,h)$ for general real h,k, but values for $\mathfrak{C}_\zeta(k,h)$ are only conjectured for integral h,k. In both cases, the constants split as a product of an arithmetic factor and a random matrix factor. The arithmetic factor is a well understood product over primes. The random matrix factor has many different expressions including combinatorial sums [13, 14, 20], a multiple contour integral in the case $h = k$ [12], and a determinant of Bessel functions [2, 12]. For h,k not necessarily equal, the random matrix factor can be solved for finite N and is related to the solution of a Painlevé V type differential equation [4]. Furthermore, the limit as $N \to \infty$ is related to the solution of a certain Painlevé III equation [1, 2, 4, 15].

Previously, the asymptotics (1.1) for $\mathcal{I}_T(h,k)$ and $\mathcal{J}_T(h,k)$ were known for $h,k \in \{0,1,2\}$ with $h \le k$ due to Ingham [22] and Conrey [8]. Assuming the Riemann hypothesis, Conrey and Ghosh [11] established the conjectured asymptotic (1.1) for $\mathcal{J}_T\big(1,\frac{1}{2}\big)$ and came close to establishing the corresponding asymptotic for $\mathcal{I}_T\big(1,\frac{1}{2}\big)$. Upper bounds of the conjectured order of magnitude were only known for very few values of h and k. Upper bounds for half-integer-valued $h,k \le 2$ are available due to work [8, 11]. In the case $h = 0$, quite a bit more is known. Unconditionally, upper bounds of the correct order of magnitude are known for all real $0 \le k \le 2$ due to work of Heap, Radziwiłł, and Soundararajan [19]. Upper bounds of the correct order for all real $k \ge 0$ are also known conditionally on the Riemann hypothesis due to work of Harper [18], which builds on the work of Soundararajan [28]. The aim of this paper is to establish upper bounds for $\mathcal{I}_T(k,h)$ and $\mathcal{J}_T(k,h)$ of the expected order in a larger range of h and k.

THEOREM 1.1. *Let $1 \le k \le 2$ and $0 \le h \le 1$. Then for large T,*

$$\mathcal{I}_T(k,h) \ll T(\log T)^{k^2+2h},$$

and the same bound holds for $\mathcal{J}_T(k,h)$.

The proof we give is based on [19], which in turn is based on a method introduced by Radziwiłł and Soundararajan [27]. The general principle in these works is that if one can compute the $2k$th moment of a given L-function twisted by an arbitrary Dirichlet polynomial, then one can find upper bounds of the right order for all of its lower order moments. In particular, Heap, Radziwiłł, and Soundararajan [19] used this approach to prove Theorem 1.1 in the case $h = 0$. We combine the ideas of [19] with twisted joint moment calculations to deduce Theorem 1.1 in the case of $h = 1$ and then deduce the result from Hölder's inequality—the bounds we obtain are of the conjectured order of magnitude since the exponent of $\log T$ in (1.1) is linear in h. We are forced to take $k \in [1,2]$ because $2k - 2h$ is only nonnegative when $k \ge 1$ at the boundary case $h = 1$. It is likely that one could establish sharp bounds on $\mathcal{I}_T(k,h)$ and $\mathcal{J}_T(k,h)$ in the full range $k > -\frac{1}{2}$ and $-\frac{1}{2} < h \le k + \frac{1}{2}$ assuming the Riemann hypothesis.

2. Outline of the proof. We will deduce Theorem 1.1 from the following.

PROPOSITION 2.1. *Let T be large and $1 \le k \le 2$. Then*

$$\int_T^{2T} \big|\zeta\big(\tfrac{1}{2}+it\big)\big|^{2k-2}\big|\zeta'\big(\tfrac{1}{2}+it\big)\big|^2\,dt \ll T(\log T)^{k^2+2},$$

and the same bound holds when $\zeta\big(\frac{1}{2}+it\big)$ is replaced by $Z(t)$.

Proof of Theorem 1.1. Recall that [19, Theorem 1.1] gives, for $0 \le k \le 2$,

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \ll T(\log T)^{k^2}.$$

Therefore by Hölder's inequality with $p = \frac{1}{h}$ and $q = \frac{1}{1-h}$, this estimate and Theorem 1.1 give

$$\begin{aligned}\mathcal{I}_T(k,h) &\le \Big(\int_T^{2T} \big|\zeta\big(\tfrac{1}{2}+it\big)\big|^{2k-2}|\zeta'(\tfrac{1}{2}+it)|^2\,dt\Big)^h \Big(\int_T^{2T} \big|\zeta\big(\tfrac{1}{2}+it\big)\big|^{2k}\,dt\Big)^{1-h}\\ &\ll T(\log T)^{k^2+2h}.\end{aligned}$$

The case of the joint moments of $Z(t)$ is similar since $|Z(t)| = \big|\zeta\big(\frac{1}{2}+it\big)\big|$. ∎

REMARK 2.2. The bound for $J_T(k,1)$ in Proposition 2.1 can be deduced from the corresponding bound for $\mathcal{I}_T(k,1)$. The following argument is due to an anonymous referee: The product rule implies

$$Z'(t) = i\theta'(t)e^{i\theta(t)}\zeta\big(\tfrac{1}{2}+it\big) + ie^{i\theta(t)}\zeta'\big(\tfrac{1}{2}+it\big).$$

Combining this with the inequality $|a+b|^2 \le 2|a|^2 + 2|b|^2$ implies

$$|Z'(t)|^2 \ll |\theta'(t)|^2\big|\zeta\big(\tfrac{1}{2}+it\big)\big|^2 + \big|\zeta'\big(\tfrac{1}{2}+it\big)\big|^2.$$

An application of Stirling's formula gives $\theta'(t) \sim \frac{1}{2}\log t$ and thus

$$|Z'(t)|^2 \ll (\log T)^2\big|\zeta\big(\tfrac{1}{2}+it\big)\big|^2 + \big|\zeta'\big(\tfrac{1}{2}+it\big)\big|^2$$

for $t \in [T, 2T]$. Then Proposition 2.1 for $Z(t)$ readily follows from [19, Theorem 1.1] and the conclusion of Proposition 2.1 for $\zeta\big(\frac{1}{2}+it\big)$.

To prove Proposition 2.1, we will approximate the logarithm of $\zeta(s)$ by a truncated sum over primes $\sum_{p \le X} p^{-s}$. Following the works [18, 26, 28], we will break up this sum into increments that have progressively smaller variance. This in turn allows us to work with a Dirichlet polynomial of length T^θ for some small but fixed $\theta > 0$, which is long enough to give a good enough approximation of $\zeta(s)$.

We follow the notation introduced in [19]. Denote by $\log_j$ the j-fold iterated logarithm, and take ℓ to be the largest integer with $\log_\ell T \geq 10^4$. Now define a sequence T_j for $1 \leq j \leq \ell$ by $T_1 = e^2$ and

$$T_j = \exp\left(\frac{\log T}{(\log_j T)^2}\right)$$

for $2 \leq j \leq \ell$, and for $2 \leq j \leq \ell$ and $s \in \mathbb{C}$ set

$$\mathcal{P}_j(s) = \sum_{T_{j-1} \leq p < T_j} \frac{1}{p^s} \quad \text{and} \quad P_j = \sum_{T_{j-1} \leq p < T_j} \frac{1}{p}.$$

The hope is then that on average $\log \zeta(s)$ will be controlled by the sum of the increments $\mathcal{P}_j(s)$, where P_j is the variance of the jth increment on the half-line. In essence, we are approximating $\zeta(s)$ with a truncated Euler product (though we are omitting the terms p^{-ks} with $k \geq 2$ when truncating the Dirichlet series for $\log \zeta(s)$). A modern theme in analytic number theory is that for $\sigma \geq 1/2$ the zeta function $\zeta(\sigma + it)$ ought to behave like a truncated Euler product—for example, see the work of Gonek [16] assuming the Riemann hypothesis. Thus we expect that $\exp(\alpha \mathcal{P}(s))$ should be a good approximation to $\zeta(s)^\alpha$ so long as s is not near a zero of ζ. Because the zeros of zeta do not contribute to the moments of zeta, one might expect that this approximation will be sufficient for estimating moments of zeta.

To facilitate computations, it will be convenient to replace $\exp(\alpha \mathcal{P}(s))$ with a Dirichlet polynomial. We will accomplish this by expanding the exponential as a Taylor series. By Mertens' second estimate

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + O(1),$$

it follows that

$$P_j \sim 2 \log_j T - 2 \log_{j+1} T.$$

We then define for $2 \leq j \leq \ell$ the truncated Taylor expansion

$$\mathcal{N}_j(s;\alpha) = \sum_{\substack{p \mid n \Rightarrow T_{j-1} \leq p < T_j \\ \Omega(n) \leq 500 P_j}} \frac{\alpha^{\Omega(n)} g(n)}{n^s}$$

where $\Omega(n)$ is the number of prime divisors of n counting multiplicity, and g is the multiplicative function given by $g(p^m) = 1/m!$ on prime powers. For most $t \in [T, 2T]$ the size of $\mathcal{P}_j(s)$ will be smaller than $50P_j$, say, so $\mathcal{N}_j(s;\alpha)$ will be a good proxy for $\exp(\alpha \mathcal{P}_j(s))$. Therefore we expect $\prod_{2 \leq j \leq \ell} \mathcal{N}_j(\frac{1}{2} + it; \alpha)$ to behave similarly to $e^{\alpha \mathcal{P}(s)}$ which, as mentioned earlier, should be a good approximation for $\zeta(s)^\alpha$. Now each $\mathcal{N}_j$ is a Dirichlet polynomial of length at most $T_j^{500 P_j}$ so $\prod_{2 \leq j \leq \ell} \mathcal{N}_j(\frac{1}{2} + it; \alpha)$

is a Dirichlet polynomial of length at most $T^{1/10}$, which is amenable to analysis.

We will deduce Proposition 2.1 in two steps. First we bound the integrand by a product of integral powers of ζ and ζ' with short Dirichlet polynomials.

PROPOSITION 2.3. *For* $1 \le k \le 2$ *and* $s = \frac{1}{2} + it$ *with* $t \in \mathbb{R}$,

$$\begin{aligned}|\zeta(s)|^{2k-2}|\zeta'(s)|^2 \le 2k|\zeta(s)|^2|\zeta'(s)|^2 \prod_{2\le j\le \ell} |\mathcal{N}_j(s;k-2)|^2 \\ + (4-2k)|\zeta'(s)|^2 \prod_{2\le j\le \ell} |\mathcal{N}_j(s;k-1)|^2 \\ + \sum_{2\le v\le \ell} \Big(2k|\zeta(s)|^2|\zeta'(s)|^2 \prod_{2\le j< v} |\mathcal{N}_j(s;k-2)|^2 \\ + (4-2k)|\zeta'(s)|^2 \prod_{2\le j<v} |\mathcal{N}_j(s;k-1)|^2\Big) \left|\frac{\mathcal{P}_v(s)}{50P_v}\right|^{2\lceil 50P_v\rceil}.\end{aligned}$$

The same bound holds when $\zeta(s)$ *is replaced by* $Z(t)$.

The proof of Proposition 2.3 is almost identical to that of [19, Proposition 2.1], so it is omitted. The only difference is that one uses the conjugate exponents $p = \frac{1}{k-1}$ and $q = \frac{1}{2-k}$, and then one multiplies the resulting inequality by $|\zeta'(s)|^2$ or $|Z'(t)|^2$. This reduces the proof of Proposition 2.1 to the calculation of two types of twisted moments.

PROPOSITION 2.4. *For* $1 \le k \le 2$,

$$\int_T^{2T} \big|\zeta'\big(\tfrac{1}{2}+it\big)\big|^2 \prod_{2\le j\le \ell} \big|\mathcal{N}_j\big(\tfrac{1}{2}+it;k-1\big)\big|^2\,dt \ll T(\log T)^{k^2+2}, \tag{2.1}$$

and for $2 \le v \le \ell$ *and* $0 \le r \le \lceil 50P_v\rceil$,

$$\begin{aligned}\int_T^{2T} \big|\zeta'\big(\tfrac{1}{2}+it\big)\big|^2 \prod_{2\le j<v} \big|\mathcal{N}_j\big(\tfrac{1}{2}+it;k-1\big)\big|^2 \big|\mathcal{P}_v\big(\tfrac{1}{2}+it\big)\big|^{2r}\,dt \\ \ll T(\log T)^3(\log T_{v-1})^{k^2-1}(2^r r! P_v^r \exp(P_v)),\end{aligned} \tag{2.2}$$

and the same bounds hold when $\zeta\big(\frac{1}{2}+it\big)$ *is replaced by* $Z(t)$.

PROPOSITION 2.5. *For* $1 \le k \le 2$,

$$\int_T^{2T} \big|\zeta\big(\tfrac{1}{2}+it\big)\big|^2\big|\zeta'\big(\tfrac{1}{2}+it\big)\big|^2 \prod_{2\le j\le \ell} \big|\mathcal{N}_j\big(\tfrac{1}{2}+it;k-2\big)\big|^2\,dt \ll T(\log T)^{k^2+2}, \tag{2.3}$$

and for $2 \le v \le \ell$ *and* $0 \le r \le \lceil 50P_v \rceil$,

$$\int_T^{2T} \big|\zeta\big(\tfrac{1}{2}+it\big)\big|^2 \big|\zeta'\big(\tfrac{1}{2}+it\big)\big|^2 \prod_{2\le j<v} \big|\mathcal{N}_j\big(\tfrac{1}{2}+it;k-2\big)\big|^2 \big|\mathcal{P}_v\big(\tfrac{1}{2}+it\big)\big|^{2r}\,dt \ll T(\log T)^6(\log T_{v-1})^{k^2-4}(18^r r! P_v^r \exp(P_v)), \tag{2.4}$$

and the same bounds hold when $\zeta\big(\frac{1}{2}+it\big)$ *is replaced by* $Z(t)$.

We will derive estimates for general twisted joint moments of ζ in the following section, and then use them to prove Propositions 2.4 and 2.5 in the final section. Before we undertake this, let us see how these estimates imply Proposition 2.1.

Proof of Proposition 2.1. Our estimates give

$$\int_T^{2T} \big|\zeta\big(\tfrac{1}{2}+it\big)\big|^{2k-2} \big|\zeta'\big(\tfrac{1}{2}+it\big)\big|^2\,dt \ll T(\log T)^{k^2+2}$$
$$+ \sum_{2\le v\le \ell} T(\log T_{v-1})^{k^2+2}\bigg(\bigg(\frac{\log T}{\log T_{v-1}}\bigg)^3 \frac{2^{\lceil 50P_v\rceil}\lceil 50P_v\rceil! P_v^{\lceil 50P_v\rceil}\exp(P_v)}{(50P_v)^{2\lceil 50P_v\rceil}}$$
$$+ \bigg(\frac{\log T}{\log T_{v-1}}\bigg)^6 \frac{18^{\lceil 50P_v\rceil}\lceil 50P_v\rceil! P_v^{\lceil 50P_v\rceil}\exp(P_v)}{(50P_v)^{2\lceil 50P_v\rceil}}\bigg) \ll T(\log T)^{k^2+2},$$

where the final bound follows by the same reasoning used in [19]. The reasoning for the Z function is the same. ∎

3. Twisted moment formulae. We will derive the necessary twisted joint moment formulae from formulae for twisted moments of $\zeta(s)$ with small shifts off of the critical line. Fortunately there are many known formulae for computing twisted moments of ζ due to connections with the proportion of zeros of ζ lying on the critical line [7, 25]. Then following work of Young [29], we can differentiate these formulae with respect to the shifts to obtain the desired twisted joint moments. The formula in [29] is valid for Dirichlet polynomials of length $T^{1/2-\varepsilon}$, and we note that work of Bettin, Chandee, and Radziwiłł [6] provides asymptotics for the twisted second moment without shifts for any Dirichlet polynomial of length at most $T^{17/33-\varepsilon}$. The twisted fourth moment formula we use was first proven by Hughes and Young [21] for Dirichlet polynomials of length at most $T^{1/11-\varepsilon}$, which was later increased to $T^{1/4-\varepsilon}$ by Bettin, Bui, Li, and Radziwiłł [5].

Following these works, we will bound the desired twisted moments by introducing a smooth cutoff. Going forward, we fix a smooth nonnegative $\phi:\mathbb{R}\to\mathbb{R}$ such that $\operatorname{supp}\phi \subset [3/4, 9/4]$ and $\phi(t)=1$ for all $t\in[1,2]$.

LEMMA 3.1. *Given a Dirichlet polynomial* $A(s) = \sum_{h\le T^\theta} a_h/h^s$ *with* $\theta < 1/2$, *if*

$$F(z_1, z_2) = \sum_{h,k\le T^\theta} \frac{a_h\overline{a_k}}{[h,k]} \frac{(h,k)^{z_1+z_2}}{h^{z_2}k^{z_1}},$$

then

$$\widetilde{I}^{(1)}(T) := \int_{\mathbb{R}} \left|\zeta'\left(\tfrac{1}{2}+it\right)\right|^2 \left|A\left(\tfrac{1}{2}+it\right)\right|^2 \phi(t/T)\,dt \ll T(\log T)^3 \max_{|z_j|=3^j/\log T} |F(z_1,z_2)|,$$

and the same bound holds when $\zeta\left(\frac{1}{2}+it\right)$ *is replaced by* $Z(t)$.

Proof. Let $\alpha,\beta\in\mathbb{C}$ have modulus less than $1/\log T$. Then by using [29, Lemma 5] and reasoning in a similar way to [29, proof of Lemma 6] we may write

$$\begin{aligned} I_T(\alpha,\beta) &:= \int_{\mathbb{R}} \zeta\left(\tfrac{1}{2}+\alpha+it\right)\zeta\left(\tfrac{1}{2}+\beta-it\right)\left|A\left(\tfrac{1}{2}+it\right)\right|^2\phi(t/T)\,dt \\ &= \sum_{h,k\le T^\theta} \frac{a_h\overline{a_k}}{[h,k]} \int_{\mathbb{R}} \bigg(\frac{(h,k)^{\alpha+\beta}}{h^\beta k^\alpha}\zeta(1+\alpha+\beta) \\ &\quad + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{(h,k)^{-\alpha-\beta}}{h^{-\alpha}k^{-\beta}}\zeta(1-\alpha-\beta)\bigg)\phi(t/T)\,dt + O(T^{1-\delta}) \end{aligned}$$

for some $\delta > 0$. The main term is holomorphic in α,β sufficiently small since the principal parts of $\zeta(1+\alpha+\beta)$ and $\zeta(1-\alpha-\beta)$ cancel. We may express the main term as a multiple contour integral around α and $-\beta$: by [10, Lemma 2.5.1]

$$\begin{aligned} I_T(\alpha,\beta) = -\frac{1}{(2\pi i)^2} \oint_{|z_2|=9/\log T} \oint_{|z_1|=3/\log T} & F(z_1,-z_2) \\ & \times \frac{\zeta(1+z_1-z_2)(z_1-z_2)^2}{(z_1-\alpha)(z_1+\beta)(z_2-\alpha)(z_2+\beta)} \\ \times \bigg(\int_{\mathbb{R}} \left(\frac{t}{2\pi}\right)^{z_1-z_2-\beta-\alpha/2} & \phi(t/T)\,dt \bigg)\,dz_1\,dz_2 + O(T^{1-\delta}). \end{aligned}$$

The shifts α and $-\beta$ are enclosed in these contours because we have assumed $|\alpha|,|\beta| < 1/\log T$. Now since $I_T(\alpha,\beta)$ is holomorphic with respect to small α and β, as in [29] the derivatives of $I_T(\alpha,\beta)$ with respect to α and β can be obtained via Cauchy's theorem as contour integrals along circles of radii $\asymp 1/\log T$. Since the error term holds uniformly on these contours, we conclude

$$\begin{aligned}\widetilde{I}_T(\alpha,\beta) &:= \int_{\mathbb{R}} \zeta'\big(\tfrac12+\alpha+it\big)\zeta'\big(\tfrac12+\beta+it\big)\big|A\big(\tfrac12+it\big)\big|^2\phi(t/T)\,dt \\ &= \frac{d}{d\alpha}\,\frac{d}{d\beta}\bigg[-\frac{1}{(2\pi i)^2}\oint_{|z_2|=9/\log T}\oint_{|z_1|=3/\log T} F(z_1,-z_2) \\ &\qquad\times\frac{\zeta(1+z_1-z_2)(z_1-z_2)^2}{(z_1-\alpha)(z_1+\beta)(z_2-\alpha)(z_2+\beta)} \\ &\qquad\times\bigg(\int_{\mathbb{R}}\Big(\frac{t}{2\pi}\Big)^{(z_1-z_2-\beta-\alpha)/2}\phi(t/T)\,dt\bigg)\,dz_1\,dz_2\bigg] + O(T^{1-\delta}).\end{aligned}$$

To compute $\widetilde{I}^{(1)}(T)$, we evaluate these derivatives and then set $\alpha=\beta=0$, obtaining

$$\begin{aligned}\widetilde{I}^{(1)}(T) &= \frac{1}{(2\pi i)^2}\oint_{|z_2|=9/\log T}\oint_{|z_1|=3/\log T} F(z_1,-z_2)\zeta(1+z_1-z_2)(z_1-z_2)^2 \\ &\times\int_{\mathbb{R}}\bigg[\Big(z_1+z_2+\frac{z_1z_2}{2}\log\frac{t}{2\pi}\Big) \\ &\times\Big(z_1+z_2-\frac{z_1z_2}{2}\log\frac{t}{2\pi}\Big)\Big(\frac{t}{2\pi}\Big)^{(z_1-z_2)/2}\phi(t/T)\,dt\bigg]\frac{dz_1}{z_1^4}\,\frac{dz_2}{z_2^4} + O(T^{1-\delta}).\end{aligned}$$

Finally, since $|z_j| = 3^j/\log T$ and $\operatorname{supp}\phi\subset[3/4,9/4]$, notice that

$$\zeta(1+z_1-z_2)\ll\log T,\qquad (z_1-z_2)^2\ll(\log T)^{-2},$$

and

$$\int_{\mathbb{R}}\Big(z_1+z_2+\frac{z_1z_2}{2}\log\frac{t}{2\pi}\Big)\Big(z_1+z_2-\frac{z_1z_2}{2}\log\frac{t}{2\pi}\Big)\Big(\frac{t}{2\pi}\Big)^{(z_1-z_2)/2}\phi(t/T)\,dt \ll T(\log T)^{-2},$$

so the claim now follows.

The case for twisted moments of Z is similar. This time following the argument of [29] gives the more symmetric formula (up to terms of order $O(T^{1-\delta})$)

$$\begin{aligned}&\int_{\mathbb{R}} Z(\alpha+t)Z(\beta-t)\big|A\big(\tfrac12+it\big)\big|^2\phi(t/T)\,dt \\ &\quad= \sum_{h,k\le T^\theta}\frac{a_h\overline{a_k}}{[h,k]}\int_{\mathbb{R}}\bigg(\Big(\frac{t}{2\pi}\Big)^{(\alpha+\beta)/2}\frac{(h,k)^{\alpha+\beta}}{h^\beta k^\alpha}\zeta(1+\alpha+\beta) \\ &\qquad+\Big(\frac{t}{2\pi}\Big)^{(-\alpha-\beta)/2}\frac{(h,k)^{-\alpha-\beta}}{h^{-\alpha}k^{-\beta}}\zeta(1-\alpha-\beta)\bigg)\phi(t/T)\,dt.\end{aligned}$$

Now applying [10, Lemma 2.5.1] gives, up to a power savings, the simpler formula

$$-\frac{1}{(2\pi i)^2}\oint_{|z_2|=9/\log T}\oint_{|z_1|=3/\log T} F(z_1,-z_2)\frac{\zeta(1+z_1-z_2)(z_1-z_2)^2}{(z_1-\alpha)(z_1+\beta)(z_2-\alpha)(z_2+\beta)} \times\bigg(\int_{\mathbb{R}}\Big(\frac{t}{2\pi}\Big)^{(z_1-z_2)/2}\phi(t/T)\,dt\bigg)\,dz_1\,dz_2.$$

Then differentiating with respect to α and β and setting the shifts to zero we obtain

$$\frac{1}{(2\pi i)^2}\oint_{|z_2|=9/\log T}\oint_{|z_1|=3/\log T} F(z_1,-z_2)\zeta(1+z_1-z_2)(z_1^2-z_2^2)^2 \times\bigg(\int_{\mathbb{R}}\Big(\frac{t}{2\pi}\Big)^{(z_1-z_2)/2}\phi(t/T)\,dt\bigg)\frac{dz_1}{z_1^4}\frac{dz_2}{z_2^4},$$

which satisfies the same bound. ∎

Lemma 3.2. *Given a Dirichlet polynomial* $A(s)=\sum_{h\le T^\theta} a_h/h^s$ *with* $\theta<1/4$, *if*

$$G(z_1,z_2,z_3,z_4)=\sum_{h,k\le T^\theta}\frac{a_h\overline{a_k}}{[h,k]}B_{z_1,z_2,z_3,z_4}\Big(\frac{h}{(h,k)}\Big)B_{z_3,z_4,z_1,z_2}\Big(\frac{k}{(h,k)}\Big),$$

where

$$B_{z_1,z_2,z_3,z_4}(n) =\prod_{p^m\|n}\bigg(\sum_{j\ge0}\frac{\sigma_{z_1,z_2}(p^{j+m})\sigma_{z_3,z_4}(p^j)}{p^j}\bigg)\bigg(\sum_{j\ge0}\frac{\sigma_{z_1,z_2}(p^j)\sigma_{z_3,z_4}(p^j)}{p^j}\bigg)^{-1}$$

and $\sigma_{z_1,z_2}(n)=\sum_{ab=n}a^{-z_1}b^{-z_2}$, *then*

$$\widetilde{I}^{(2)}(T):=\int_{\mathbb{R}}\big|\zeta\big(\tfrac12+it\big)\big|^2\big|\zeta'\big(\tfrac12+it\big)\big|^2\big|A\big(\tfrac12+it\big)\big|^2\phi(t/T)\,dt \ll T(\log T)^6\max_{|z_j|=3^j/\log T}|G(z_1,z_2,z_3,z_4)|.$$

The same bound holds when $\zeta\big(\frac12+it\big)$ *is replaced by* $Z(t)$.

Proof. This is similar to the proof of Lemma 3.1. Using the twisted 4th moment formula with shifts in [5] and [10, Lemma 2.5.1], we can write, up to a power savings in T,

$$\int_{\mathbb{R}} \zeta\big(\tfrac{1}{2}+\alpha+it\big)\zeta\big(\tfrac{1}{2}+it\big)\zeta\big(\tfrac{1}{2}+\beta-it\big)\zeta\big(\tfrac{1}{2}-it\big)\big|A\big(\tfrac{1}{2}+it\big)\big|^2\phi(t/T)\,dt$$

$$= \frac{1}{4(2\pi i)^4} \oint_{|z_j|=3^j/\log T} A(z_1,z_2,-z_3,-z_4)G(z_1,z_2,-z_3,-z_4)\Delta(z_1,z_2,z_3,z_4)^2$$

$$\times \bigg(\int_{\mathbb{R}} \bigg(\frac{t}{2\pi}\bigg)^{\frac{z_1+z_2-z_3-z_4-\alpha-\beta}{2}} \phi(t/T)\,dt\bigg) \prod_{m=1}^{4} \frac{dz_m}{z_m^2(z_m-\alpha)(z_m+\beta)},$$

where $\Delta(z_1,z_2,z_3,z_4) = \prod_{1\le j<k\le 4}(z_k-z_j)$ is the Vandermonde determinant and

$$A(z_1,z_2,z_3,z_4) = \frac{\zeta(1+z_1+z_3)\zeta(1+z_1+z_4)\zeta(1+z_2+z_3)\zeta(1+z_2+z_4)}{\zeta(2+z_1+z_2+z_3+z_4)}.$$

Now differentiating with respect to α and β and then setting $\alpha=\beta=0$ gives, up to a power savings in T,

$$\widetilde{I}^{(2)}(T)$$

$$= \frac{1}{4(2\pi i)^4} \oint_{|z_j|=3^j/\log T} A(z_1,z_2,-z_3,-z_4)G(z_1,z_2,-z_3,-z_4)\Delta(z_1,z_2,z_3,z_4)^2$$

$$\times \bigg[\int_{\mathbb{R}} \bigg(z_1^2z_2^2z_3^2z_4^2\bigg(\log\frac{t}{2\pi}\bigg)^2 - (z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4)^2\bigg)$$

$$\times \bigg(\frac{t}{2\pi}\bigg)^{(z_1+z_2-z_3-z_4)/2} \phi(t/T)\,dt\bigg] \prod_{m=1}^{4}\frac{dz_m}{z_m^6}.$$

To deduce the claim, notice that

$$A(z_1,z_2,-z_3,-z_4) \ll (\log T)^4, \qquad \Delta(z_1,z_2,z_3,z_4)^2 \ll (\log T)^{-12},$$

and

$$\int_{\mathbb{R}} \bigg(z_1^2z_2^2z_3^2z_4^2\bigg(\log\frac{t}{2\pi}\bigg)^2 - (z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4)^2\bigg)$$

$$\times \bigg(\frac{t}{2\pi}\bigg)^{(z_1+z_2-z_3-z_4)/2} \phi(t/T)\,dt \ll T(\log T)^{-6}$$

for $|z_j| = 3^j/\log T$ and $t\in[3T/4, 9T/4]$. As in the previous proof, the analysis for the Z function is simpler, and the same bound holds. ∎

4. Proof of Propositions 2.4 and 2.5. The proofs of Propositions 2.4 and 2.5 are straightforward modifications of [19, proof of Proposition 3 (↔ 2.3)]. In fact we will see that Proposition 2.5 is an immediate consequence of Lemma 3.2 and a bound for $G(z_1, z_2, z_3, z_4)$ proven in [19]. This will then conclude the proof of Theorem 1.1.

Proof of Proposition 2.4. We will apply Lemma 3.1 to the Dirichlet polynomials

$$\prod_{2\le j\le \ell} \mathcal{N}_j(s; k-1) \quad \text{and} \quad \Big(\prod_{2\le j<v} \mathcal{N}_j\big(\tfrac12+it; k-1\big)\Big)\mathcal{P}_v\big(\tfrac12+it\big)^r.$$

By multiplicativity, it suffices to bound the sums

$$\sum_{\substack{p\mid mn\Rightarrow T_{j-1}\le p<T_j\\ \Omega(m),\Omega(n)\le 500P_j}} \frac{(k-1)^{\Omega(m)+\Omega(n)} g(n)g(m)}{[n,m]}\cdot\frac{(m,n)^{z_1+z_2}}{m^{z_2}n^{z_1}} \tag{4.1}$$

arising from the $\mathcal{N}_j(s; k-1)$ terms and

$$\sum_{\substack{p\mid mn\Rightarrow T_{v-1}\le p<T_v\\ \Omega(m)=\Omega(n)=r}} \frac{r!^2 g(n)g(m)}{[n,m]}\cdot\frac{(m,n)^{z_1+z_2}}{m^{z_2}n^{z_1}} \tag{4.2}$$

coming from the $\mathcal{P}_v(\frac12+it)^r$ term. In both cases, we will use the estimate

$$\frac{(m,n)^{z_1+z_2}}{m^{z_2}n^{z_1}} \ll 1,$$

which holds under the assumptions $|z_j| \le 9/\log T$ and $m,n\le T^{1/10}$.

First we handle (4.1). We drop the condition $\Omega(m),\Omega(n)\le 500P_j$ using Rankin's trick: Since $|k-1|\le 1$ and $\exp(\Omega(m)+\Omega(n)-500P_j)\ge 1$ when either $\Omega(m)$ or $\Omega(n)$ is larger than $500P_j$, the terms with either $\Omega(m)$ or $\Omega(n)$ exceeding $500P_j$ contribute an error of size at most

$$\begin{aligned} e^{-500P_j} &\sum_{p\mid mn\Rightarrow T_{j-1}\le p<T_j} \frac{((k-1)e)^{\Omega(n)+\Omega(m)}}{[n,m]} \\ &\ll e^{-500P_j} \prod_{T_{j-1}\le p<T_j}\bigg(1+\frac{e+e+e^2}{p}+O\Big(\frac{1}{p^2}\Big)\bigg) \ll e^{-100P_j} \end{aligned}$$

Now write

$$\begin{aligned} &\sum_{p\mid mn\Rightarrow T_{j-1}\le p<T_j} \frac{(k-1)^{\Omega(n)+\Omega(m)} g(n)g(m)}{[n,m]} \\ &\qquad = \prod_{T_{j-1}\le p<T_j}\bigg(1+\frac{2(k-1)+(k-1)^2}{p}+O\Big(\frac{1}{p^2}\Big)\bigg) \\ &\qquad = \prod_{T_{j-1}\le p<T_j}\bigg(1+\frac{k^2-1}{p}+O\Big(\frac{1}{p^2}\Big)\bigg). \end{aligned}$$

Therefore by Lemma 3.1 we conclude that the integral in (2.1) is

$$\ll T(\log T)^3 \prod_{2\le j\le \ell}\bigg(\prod_{T_{j-1}\le p<T_j}\bigg(1+\frac{k^2-1}{p}+O\Big(\frac{1}{p^2}\Big)\bigg)+O(e^{-100P_j})\bigg)$$
$$\ll T(\log T)^{k^2+2}.$$

Now we handle the sums (4.2). Write

$$\sum_{\substack{p|mn\Rightarrow T_{v-1}\le p<T_v\\ \Omega(m)=\Omega(n)=r}}\frac{r!^2 g(n)g(m)}{[n,m]}\le r!^2\sum_{j=0}^{r}\sum_{\substack{p|d\Rightarrow T_{v-1}\le p<T_v\\ \Omega(d)=j}}\frac{1}{d}\bigg(\sum_{\substack{p|n\Rightarrow T_{v-1}\le p<T_v\\ \Omega(n)=r-j}}\frac{g(nd)}{n}\bigg)^2.$$

By the inequalities $\binom{r}{j}\le 2^r$ and $g(nd)\le g(n)g(d)$, we may further bound this by

$$r!^2\sum_{j=0}^{r}\Big(\frac{1}{j!}P_v^j\Big)\Big(\frac{1}{(r-j)!}P_v^{r-j}\Big)^2 = r!P_v^r\sum_{j=0}^{r}\binom{r}{j}\frac{P_v^{r-j}}{(r-j)!}\le 2^r r!P_v^r\exp(P_v).$$

The claim now readily follows by Lemma 3.1. ∎

Proof of Proposition 2.5. This is a direct consequence of Lemma 3.2 and proof of [19, Proposition 3 (↔ 2.3)], where it is shown in the first case that

$$\max_{|z_j|=3^j/\log T}|G(z_1,z_2,z_3,z_4)|\ll T(\log T)^{k^2-4},$$

and in the second case that

$$\max_{|z_j|=3^j/\log T}|G(z_1,z_2,z_3,z_4)|\ll (\log T_{v-1})^{k^2-4}(18^r r!P_v^r\exp(P_v)).\ \blacksquare$$

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