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COMPLETE SPACELIKE HYPERSURFACES IN THE ANTI-DE SITTER SPACE: RIGIDITY, NONEXISTENCE AND CURVATURE ESTIMATES

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Abstract. Our purpose is to investigate the geometry of complete spacelike hypersurfaces immersed in the anti-de Sitter space $\mathbb{H}_1^{n+1}$. We start by proving rigidity results for such hypersurfaces under suitable constraints on their higher order mean curvatures. We also obtain a lower estimate for the index of minimum relative nullity for r-maximal spacelike hypersurfaces and a nonexistence result for 1-maximal spacelike hypersurfaces of $\mathbb{H}_1^{n+1}$. Finally, we employ a technique due to Aledo and Alías (2000) to prove some curvature estimates for complete spacelike hypersurface of $\mathbb{H}_1^{n+1}$; as a consequence, we get further nonexistence results. In particular, we show the nonexistence of complete maximal spacelike hypersurfaces in certain open regions of $\mathbb{H}_1^{n+1}$. Our approach is mainly based on a suitable extension of the generalized maximum principle of Omori and Yau due to Alías, Impera and Rigoli (2012).

1. Introduction. The geometric behavior of spacelike hypersurfaces immersed in a Lorentzian space is of importance from both the physical and mathematical points of view. For example, Marsden and Tipler [25] and Stumbles [31] pointed out that spacelike hypersurfaces immersed with constant mean curvature in a Lorentz manifold play an important role in general relativity, serving as convenient initial data for the Cauchy problem for Einstein's equations. From the mathematical viewpoint, a basic related question is the existence and uniqueness of spacelike hypersurfaces in Lorentz manifolds under the assumption of some reasonable geometric properties, like the constancy of the mean or scalar curvature. A first relevant result in this direction was the proof of the famous conjecture due to Calabi [11] for maximal hypersurfaces (that is, hypersurfaces with vanishing mean curvature) in

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the Lorentz–Minkowski space, given by Cheng and Yau [17]. In the de Sitter space, Goddard [22] conjectured that every complete spacelike hypersurface with constant mean curvature is totally umbilical. Although the conjecture turned out to be false, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under additional hypotheses (see, for example, [2, 26]).

Here, we deal with complete hypersurfaces immersed in a special Lorentz space form with negative constant sectional curvature, known as the *anti-de Sitter space*. It is obtained in the following way: Let $\mathbb{R}_2^{n+2}$ be the $(n+2)$-dimensional real vector space $\mathbb{R}^{n+2}$ endowed with the metric of index 2,

$$ds^2 = -dx_1^2 - dx_2^2 + \sum_{i=3}^{n+2} dx_i^2.$$

The (unit) *anti-de Sitter space* $\mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$ is the hyperquadric

$$\mathbb{H}_1^{n+1} = \{x \in \mathbb{R}_2^{n+2} : ds^2(x,x) = -1\}.$$

We observe that an interesting feature of the four-dimensional anti-de Sitter space $\mathbb{H}_1^4$ is that, as a cosmological model, this spacetime is a *maximally symmetric universe* with constant negative curvature, which is conformally related to the half of the Einstein static universe. Consequently, $\mathbb{H}_1^4$ represents (locally) a unique solution to Einstein's equation in the absence of any ordinary matter or gravitational radiation. So, this spacetime may be thought of as a ground state of general relativity (see, for instance, [32, Chapter 6] and [33, Chapter 14]).

Concerning complete spacelike hypersurfaces immersed in the anti-de Sitter space, Choi, Ki and Kim [18] used the generalized maximum principle of Omori [28] and Yau [34] to show that if the height function with respect to a timelike vector of a complete maximal spacelike hypersurface Σ^n of $\mathbb{H}_1^{n+1}$ has a certain boundedness property, then Σ^n must be totally geodesic. Later, by extending a technique due to Yau [35], the second author jointly with Camargo [13] obtained other rigidity results for complete maximal spacelike hypersurfaces of $\mathbb{H}_1^{n+1}$, imposing suitable conditions on both the norm of the second fundamental form and a certain height function naturally attached to the hypersurface. They also characterized complete maximal spacelike graphs satisfying a certain assumption on the gradient of the function which determines the graph. Working with a suitable warped product model of $\mathbb{H}_1^{n+1}$, the same authors jointly with Caminha and Parente [12] extended the main result of [13] by showing that if Σ^n is a complete spacelike hypersurface with constant mean curvature and bounded scalar curvature in $\mathbb{H}_1^{n+1}$, such that the gradient of its height function with respect to a timelike vector has integrable norm, then Σ^n is totally umbilical. Next, the second author jointly with Aquino [8] obtained another characterization theorem concerning complete constant mean curvature spacelike hypersurfaces of $\mathbb{H}_1^{n+1}$, under suitable constraints on the behavior of the Gauss mapping. Furthermore, the same authors jointly with the fourth author [9] obtained similar results for complete spacelike hypersurfaces with constant scalar curvature in $\mathbb{H}_1^{n+1}$.

In higher codimensions, Ishihara [23] proved that an n-dimensional complete maximal spacelike submanifold immersed in the anti-de Sitter space $\mathbb{H}_p^{n+p}$ of index p must have the squared norm of the second fundamental form bounded from above by np. Moreover, the only ones that attain this estimate are the maximal hyperbolic cylinders $\mathbb{H}^{k_1}(-n/k_1) \times \cdots \times \mathbb{H}^{k_{p+1}}(-n/k_{p+1})$, where $k_1 + \cdots + k_{p+1} = n$. Cao and Wei [15] showed that if $n \geq 3$ then every n-dimensional complete maximal spacelike hypersurface in $\mathbb{H}_1^{n+1}$ with exactly two principal curvatures everywhere is isometric to some hyperbolic cylinder under an additional condition on these curvatures. Perdomo [30] studied the 2-dimensional case and constructed new examples of complete maximal surfaces in $\mathbb{H}_1^3$. More recently, Chaves, Sousa and Valério [16] studied complete maximal spacelike hypersurfaces in $\mathbb{H}_1^{n+1}$ with either constant scalar curvature or constant non-zero Gauss–Kronecker curvature. They characterized the hyperbolic cylinders as the only such hypersurfaces with $n-1$ principal curvatures having the same sign everywhere.

In this paper we study the geometry of complete spacelike hypersurfaces immersed in the anti-de Sitter space $\mathbb{H}_1^{n+1}$ in terms of the behavior of higher order mean curvatures. We prove rigidity and nonexistence results and we also obtain suitable estimates for the index of minimum relative nullity and for higher order mean curvatures, the scalar curvature and the Ricci curvature of these hypersurfaces. In particular, we show the nonexistence of complete maximal spacelike hypersurfaces in certain open regions of $\mathbb{H}_1^{n+1}$. Our approach is based on a suitable extension of the generalized maximum principle of Omori and Yau due to Alías, Impera and Rigoli [6] (see Lemma 2.4).

This article is organized in the following way: In Section 2 we recall some basic facts related to spacelike hypersurfaces immersed in the anti-de Sitter space, focusing on their higher order mean curvature, new transformations and their corresponding linearized operators. In particular, we quote some auxiliary lemmas to be used in our proofs. Section 3 is devoted to stating and proving our rigidity results concerning complete spacelike hypersurfaces of $\mathbb{H}_1^{n+1}$ under suitable constraints on higher order mean curvatures. In Section 4 we obtain a lower estimate for the index of minimum relative nullity for r-maximal spacelike hypersurfaces and a nonexistence result for 1-maximal spacelike hypersurfaces of $\mathbb{H}_1^{n+1}$. Finally, in Section 5 we extend a technique due to Aledo and Alías [3] to prove estimates for higher order mean curvatures, which yield further nonexistence results.

2. Preliminaries. As introduced before, the *anti-de Sitter spacetime* $\mathbb{H}_1^{n+1}$ is the hyperquadric

$$\mathbb{H}_1^{n+1} = \{p \in \mathbb{R}_2^{n+2} : \langle p, p\rangle = -1\},$$

in the indefinite index 2 flat space $\mathbb{R}_2^{n+2}$. Topologically, $\mathbb{H}_1^{n+1}$ is $\mathbb{S}^1 \times \mathbb{R}^n$ and the semi-Euclidean metric on $\mathbb{R}_2^{n+2}$ induces a Lorentzian metric of constant sectional curvature -1 on $\mathbb{H}_1^{n+1}$. Moreover, the universal covering manifold $\widetilde{\mathbb{H}}_1^{n+1}$ of $\mathbb{H}_1^{n+1}$ is topologically $\mathbb{R}^{n+1}$ and is thus a Lorentzian analogue of the usual Riemannian hyperbolic space of negative curvature -1, which is called the *universal anti-de Sitter spacetime* (see, for instance, [10, Section 5.3] or [29, Section 8.6]).

Let $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$ be a connected *spacelike hypersurface* immersed into $\mathbb{H}_1^{n+1}$, which means that the metric induced via ψ is a Riemannian metric on Σ^n. We denote by $C^\infty(\Sigma^n)$ the ring of real C^∞ functions defined on Σ^n and by $\mathfrak{X}(\Sigma^n)$ the $C^\infty(\Sigma^n)$-module of C^∞ vector fields on Σ^n. We will denote by ∇^0, $\overline{\nabla}$ and ∇ the Levi-Civita connections of $\mathbb{R}_2^{n+2}$, $\mathbb{H}_1^{n+1}$ and Σ^n, respectively. The *Gauss* and *Weingarten formulas* corresponding to Σ^n are, respectively,

$$\nabla^0_X Y = \nabla_X Y - \langle AX, Y\rangle N + \langle X, Y\rangle \psi \tag{2.1}$$

and

$$A(X) = -\overline{\nabla}_X N = -\nabla^0_X N, \tag{2.2}$$

for all $X, Y \in \mathfrak{X}(\Sigma^n)$, where $A : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ stands for the *Weingarten operator* of $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ with respect to a choice of time orientation N for Σ^n.

As in [29], the curvature tensor $\mathrm{R} : \mathfrak{X}(\Sigma^n) \times \mathfrak{X}(\Sigma^n) \times \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ of $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is given by

$$\mathrm{R}(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[\,,]$ denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma^n)$. So, the Gauss equation reads

$$\mathrm{R}(X, Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X - \langle AX, Z\rangle AY + \langle AY, Z\rangle AX \tag{2.3}$$

for any tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma^n)$.

On the other hand, for each $p \in \Sigma^n$, the Weingarten operator A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$. For $0 \le r \le n$, let $S_r(p)$ denote the rth *elementary symmetric function* of the eigenvalues of A_p. Thus, one gets n smooth functions $S_r : \Sigma^n \to \mathbb{R}$ such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by convention. If $p \in \Sigma^n$ and $\{e_i\}_{i=1}^n$ is a basis of $T_p\Sigma$

formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_i\}_{i=1}^n$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the rth elementary symmetric polynomial in the indeterminates $X_1, \dots, X_n$. This allows us to define the *rth mean curvature* H_r of Σ^n, $0 \le r \le n$, by

$$\binom{n}{r} H_r = (-1)^r S_r. \tag{2.4}$$

We observe that $H_0 = 1$, while $H_1 = -(1/n)S_1$ is the usual *mean curvature* H of Σ^n. It also follows from the Gauss equation that H_2 is, up to a constant, the *normalized scalar curvature* R of Σ^n. Indeed, from (2.3) we find that the *Ricci curvature* Ric of Σ^n is given by

$$\mathrm{Ric}(X,Y) = -(n-1)\langle X,Y\rangle - \mathrm{tr}(A)\langle A(X),Y\rangle + \langle A(X),A(Y)\rangle \tag{2.5}$$

for all $X, Y \in \mathfrak{X}(\Sigma^n)$. Hence, we obtain the relation

$$|A|^2 = n^2H^2 - n(n-1)H_2 = n^2H^2 + n(n-1)(R+1). \tag{2.6}$$

For $0 \le r \le n$, we define the *rth Newton transformation* $P_r : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ on Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$, via the recurrence relation

$$P_r = \binom{n}{r} H_r I + AP_{r-1}. \tag{2.7}$$

From (2.7) we can verify that

$$P_r = \sum_{k=0}^{r} (-1)^k \binom{n}{k} H_k A^{r-k}. \tag{2.8}$$

Consequently, the Cayley–Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Hence, all bases of $T_p\Sigma$ diagonalizing A at $p \in \Sigma^n$ also diagonalize all the P_r at p. So, let $\{e_i\}_{i=1}^n$ be an orthonormal frame on $T_p\Sigma$ which diagonalizes A_p, $A_p(e_i) = \lambda_i(p)e_i$. Then from (2.8) it is not difficult to verify that (see, for instance, [4])

$$(P_r)_p(e_i) = (-1)^r \sum_{i_1<\dots<i_r,\, i_j \ne i} \lambda_{i_1}(p) \dots \lambda_{i_r}(p) e_i. \tag{2.9}$$

Hence, from (2.4) and (2.9) we get the trace formulas

$$\mathrm{tr}(P_r) = c_r H_r, \qquad \mathrm{tr}(AP_r) = -c_r H_{r+1}, \tag{2.10}$$

where $c_r = (n-r)\binom{n}{r}$.

Given $f \in C^\infty(\Sigma^n)$, for each $0 \le r \le n$, a second order differential operator $L_r : C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$ is defined by

$$L_r(f) = \operatorname{tr}(P_r \nabla^2 f). \tag{2.11}$$

Here, $\nabla^2 f : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f, that is,

$$\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X \nabla f, Y\rangle = \operatorname{Hess} f(X,Y)$$

for all $X, Y \in \mathfrak{X}(\Sigma^n)$. For a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ and $f \in C^\infty(\Sigma^n)$, it follows from the properties of Hessian that

$$L_r(\varphi \circ f) = \varphi'(f) L_r(f) + \varphi''(f)\langle P_r \nabla f, \nabla f\rangle. \tag{2.12}$$

The next lemma provides a geometric condition which guarantees the ellipticity of L_1 (cf. [5, Lemma 3.2]).

Lemma 2.1. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}_1$ be a spacelike hypersurface. If $H_2 > 0$ on Σ^n, then L_1 is elliptic, or equivalently P_1 is positive definite* (*for an appropriate choice of the orientation N*).

When $r \ge 2$, the following lemma establishes sufficient conditions for the ellipticity of L_r (see [5, Lemma 3.3]).

Lemma 2.2. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}_1$ be a spacelike hypersurface. If there exists an elliptic point of Σ^n, with respect to an appropriate choice of orientation N, and $H_{r+1} > 0$ on Σ^n for $2 \le r \le n-1$, then for all $k \in \{1, \dots, r\}$ the operator L_k is elliptic, or equivalently P_k is positive definite* (*for an appropriate choice of N if k is odd*).

Here, by an *elliptic point* we mean a point $p_0 \in \Sigma^n$ where all principal curvatures $\lambda(p_0)$ are negative. The next lemma is a consequence of one due to Alías, Brasil Jr. and Colares [4] who studied the existence of an elliptic point in a spacelike hypersurface of a conformally stationary spacetime (see [4, Lemma 5.4]).

Lemma 2.3. *Let $X \in \mathfrak{X}(\mathbb{H}^{n+1}_1)$ be a complete closed conformal vector field and let $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}_1$ be a complete spacelike hypersurface. Suppose that the divergence of X on $\mathbb{H}^{n+1}_1$, $\operatorname{Div} X$, does not vanish at a point of Σ^n where the restriction $|X|_{\Sigma^n} = \sqrt{-\langle X, X\rangle}|_{\Sigma^n}$ of $|X|$ to Σ^n attains a local minimum. Then there exists an elliptic point in Σ^n.*

We conclude this section by quoting the maximum principle which will be used to prove our rigidity results for complete spacelike hypersurfaces of $\mathbb{H}^{n+1}_1$. Let Σ^n be a complete Riemannian manifold and let $\mathcal{P} : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ denote a self-adjoint operator. Extending the idea of the linearized operator L_r defined in (2.11), we consider a second order linear differential operator $\mathcal{L} : C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$ naturally associated to $\mathcal{P}$, given by

$$\mathcal{L}(f) = \operatorname{tr}(\mathcal{P}\nabla^2 f). \tag{2.13}$$

From [6, Lemma 4.2], we have the following generalized maximum principle, which is an extension of the generalized maximum principle of Omori [28] and Yau [34] (see also [7] for a modern and accessible reference to the generalized maximum principle of Omori–Yau).

LEMMA 2.4. *Let Σ^n be a complete Riemannian manifold with sectional curvature bounded from below, and $f \in C^\infty(\Sigma^n)$ be a function which is bounded from above on Σ^n. If $\mathcal{P}$ is positive semidefinite and* $\operatorname{tr}(\mathcal{P})$ *is bounded from above on Σ^n, then there exists a sequence $\{p_k\}_{k\geq 1}$ of points of Σ^n such that*

$$\lim_k f(p_k) = \sup_{\Sigma^n} f, \qquad \lim_k |\nabla f(p_k)| = 0, \qquad \limsup_k \mathcal{L}(f)(p_k) \leq 0,$$

where the differential operator $\mathcal{L}$ is given by (2.13).

3. Rigidity of complete spacelike hypersurfaces. We start this section by recalling the description of the totally umbilical hypersurfaces of $\mathbb{H}_1^{n+1}$ (see, for instance, [1, Section 4] or [24, Example 2]). Fix a nonzero vector $a \in \mathbb{R}_2^{n+2}$ with $\langle a, a\rangle \in \{-1, 0, 1\}$ and consider the smooth function $h_a : \mathbb{H}_1^{n+1} \to \mathbb{R}$ defined by $h_a(p) = \langle p, a\rangle$. A straightforward computation shows that for every real number τ with $\langle a, a\rangle + \tau^2 \neq 0$, the level set

$$M_\tau = h_a^{-1}(\tau) = \{p \in \mathbb{H}_1^{n+1} : \langle p, a\rangle = \tau\}$$

is a totally umbilical hypersurface in $\mathbb{H}_1^{n+1}$ with the Gauss mapping

$$N_\tau(p) = \frac{1}{\sqrt{|\langle a, a\rangle + \tau^2|}}(a + \tau p). \tag{3.1}$$

Hence, the shape operator A_τ of M_τ is given by

$$A_\tau(X) = -\frac{\tau}{\sqrt{|\langle a, a\rangle + \tau^2|}} X \tag{3.2}$$

for all smooth vector fields X tangent to M_τ. Consequently, we have the following possibilities:

(1) if a is a unit spacelike vector, then M_τ is isometric to the anti-de Sitter space $\mathbb{H}_1^n(-\sqrt{1+\tau^2})$ of constant sectional curvature $-\frac{1}{1+\tau^2}$;
(2) if a is a nonzero null vector, then $\tau \neq 0$ and M_τ is isometric to the Lorentz–Minkowski space $\mathbb{R}_1^n$;
(3) if a is a unit timelike vector, then either $|\tau| > 1$ and M_τ is isometric to a de Sitter space $\mathbb{S}_1^n(\sqrt{\tau^2 - 1})$ of constant sectional curvature $\frac{1}{\tau^2-1}$, or $|\tau| < 1$ and M_τ is isometric to a hyperbolic space $\mathbb{H}^n(-\sqrt{1-\tau^2})$ of constant sectional curvature $-\frac{1}{1-\tau^2}$.

For a fixed vector $a \in \mathbb{R}^{n+2}_2$, define the *height* and *angle* functions attached to a spacelike hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}_1$ by $l_a = \langle \psi, a\rangle$ and $f_a = \langle N, a\rangle$. A direct computation shows that

$$\nabla l_a = a^\top, \tag{3.3}$$

$$\nabla f_a = -A(a^\top), \tag{3.4}$$

where A is the Weingarten operator of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}_1$ and $a^\top$ is the orthogonal projection of a onto the tangent bundle $T\Sigma^n$, that is,

$$a^\top = a + f_a N + l_a \psi. \tag{3.5}$$

Using the Gauss and Weingarten formulas (2.1) and (2.2), from (3.3) it is not difficult to verify that

$$\nabla_X \nabla l_a = -f_a A(X) + l_a X \tag{3.6}$$

for all $X \in \mathfrak{X}(\Sigma^n)$. Thus, it follows from (2.10), (2.11) and (3.6) that

$$L_r(l_a) = c_r(H_{r+1} f_a + H_r l_a). \tag{3.7}$$

When $a \in \mathbb{R}^{n+2}_2$ is a fixed unit timelike vector, we obtain the following useful formula.

PROPOSITION 3.1. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}_1$ be a spacelike hypersurface such that H_r is positive on Σ^n, and let $a \in \mathbb{R}^{n+2}_2$ be a fixed unit timelike vector. Then*

$$\begin{aligned} L_r(l_a^2) = {} & c_r\left(\sqrt{H_r}\, f_a + \frac{H_{r+1}}{\sqrt{H_r}} l_a\right)^2 + c_r\left(H_r - \frac{H_{r+1}^2}{H_r}\right) l_a^2 \\ & + c_r H_r |\nabla l_a|^2 + c_r H_r (1 - 2f_a^2) + 2\langle P_r(\nabla l_a), \nabla l_a\rangle, \end{aligned} \tag{3.8}$$

where $c_r = (n-r)\binom{n}{r}$.

Proof. From (2.12) and (3.7), we have

$$\begin{aligned} L_r(l_a^2) &= 2l_a L_r(l_a) + 2\langle P_r(\nabla l_a), \nabla l_a\rangle \\ &= 2l_a\{c_r H_{r+1} f_a + c_r H_r l_a\} + 2\langle P_r(\nabla l_a), \nabla l_a\rangle \\ &= 2c_r H_{r+1} l_a f_a + 2c_r H_r l_a^2 + 2\langle P_r(\nabla l_a), \nabla l_a\rangle. \end{aligned} \tag{3.9}$$

By adding and subtracting the terms $c_r \frac{H_{r+1}^2}{H_r} l_a^2$ and $c_r H_r f_a^2$, we obtain

$$\begin{aligned} L_r(l_a^2) = {} & c_r\left(\sqrt{H_r} f_a + \frac{H_{r+1}}{\sqrt{H_r}} l_a\right)^2 + c_r\left(H_r - \frac{H_{r+1}^2}{H_r}\right) l_a^2 \\ & + c_r H_r (l_a^2 - f_a^2) + 2\langle P_r(\nabla l_a), \nabla l_a\rangle. \end{aligned} \tag{3.10}$$

On the other hand, taking into account that $a \in \mathbb{R}^{n+2}_2$ is a unit timelike vector, from (3.5) we get

$$l_a^2 = |\nabla l_a|^2 + 1 - f_a^2. \tag{3.11}$$

Inserting (3.11) in (3.10), we conclude the proof of (3.8). ∎

Motivated by the description of the totally umbilical hypersurfaces of $\mathbb{H}_1^{n+1}$, in our next results we infer the rigidity of spacelike hypersurfaces $\psi : \Sigma^n \hookrightarrow \mathbb{H}_1^{n+1}$. For this, we will assume that the orientation N of such a spacelike hypersurface is in the time-orientation of a certain fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$, which means that $f_a < 0$.

THEOREM 3.2. *Let $\psi : \Sigma^n \hookrightarrow \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface such that $f_a^2 \leq 1/2$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose that the mean curvature H is positive, bounded and that the second mean curvature H_2 satisfies*

$$0 \leq H_2 \leq 1. \tag{3.12}$$

If

$$|a^\top| \leq C \inf_{\Sigma^n}(H - H_2) \tag{3.13}$$

for some positive constant C, then $\psi : \Sigma^n \hookrightarrow \mathbb{H}_1^{n+1}$ is a totally umbilical spacelike hypersurface M_τ with $\tau^2 = 1/2$.

Proof. Straightforward computations show that the totally umbilical spacelike hypersurfaces $M_{-\sqrt{2}/2}$ and $M_{\sqrt{2}/2}$ satisfy all the hypotheses of the theorem. Now, we shall see that these are the only such hypersurfaces. To do so, let $\psi : \Sigma^n \hookrightarrow \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface satisfying the hypotheses of the theorem. Taking into account the algebraic relation

$$\sum_{i=1}^{n} \lambda_i^2 = |A|^2 = n^2H^2 - n(n-1)H_2, \tag{3.14}$$

since H is bounded and (3.12) guarantees that H_2 is also bounded, we see that all the principal curvatures λ_i of $\psi : \Sigma^n \hookrightarrow \mathbb{H}_1^{n+1}$ are also bounded. Consequently, from the Gauss equation

$$K_\Sigma(e_i, e_j) = -1 - \lambda_i\lambda_j, \tag{3.15}$$

the sectional curvature K_Σ of ψ is bounded from below.

On the other hand, using again hypothesis (3.12), [20, Lemma 3.10] implies that the self-adjoint operator $\mathcal{P}_1 := HP_1$ is positive semidefinite, with $\operatorname{tr}(\mathcal{P}_1) = n(n-1)H^2$ being bounded. So, we consider the differential operator $\mathcal{L}_1 : C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$ given by

$$\mathcal{L}_1(f) = \operatorname{tr}(\mathcal{P}_1\nabla^2 f). \tag{3.16}$$

From (3.12), we also have

$$H_2^2 \leq H_2 \leq H^2.$$

Since we are assuming that $f_a^2 \leq 1/2$, from Proposition 3.1 we get

$$L_r(l_a) \geq n(n-1)\left(H - \frac{H_2^2}{H}\right)l_a^2. \tag{3.17}$$

Hence, from (3.16) and (3.17) we obtain

$$\mathcal{L}_1(l_a^2) \geq n(n-1)(H^2 - H_2^2)l_a^2 \geq 0. \tag{3.18}$$

In view of (3.11), our hypothesis (3.13) implies that the function l_a is bounded. Thus, Lemma 2.4 yields a sequence $\{p_k\}_{k\geq 1}$ of points in Σ^n such that

$$\lim_k l_a^2(p_k) = \sup_{\Sigma^n} l_a^2, \quad \lim_k |\nabla l_a^2(p_k)| = 0, \quad \limsup_k \mathcal{L}_1(\, l_a^2\,)(p_k) \leq 0. \tag{3.19}$$

Consequently, from (3.18) and (3.19) we have

$$\begin{aligned} 0 &\geq \limsup_k \mathcal{L}_1(l_a^2)(p_k) \\ &\geq n(n-1)\Big(\sup_{\Sigma^n} l_a^2\Big) \limsup_k\, (H^2 - H_2^2)(p_k) \geq 0. \end{aligned} \tag{3.20}$$

Hence, since (3.11) and our constraint on f_a imply in particular that $\sup_{\Sigma^n} l_a^2 > 0$, from (3.20) we get

$$\limsup_k\, (H^2 - H_2^2)(p_k) = 0,$$

and consequently

$$\inf_{\Sigma^n}\, (H - H_2) = 0. \tag{3.21}$$

Therefore, by (3.13) and (3.21), $a^\top$ vanishes identically on Σ^n, which means that l_a is constant on Σ^n, and hence $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is a totally umbilical spacelike hypersurface M_τ of $\mathbb{H}_1^{n+1}$. Since we must have $H = H_2$, we also get

$$\frac{\tau}{\sqrt{1-\tau^2}} = \left(\frac{\tau}{\sqrt{1-\tau^2}}\right)^2,$$

which yields $\tau^2 = \frac{1}{2}$. ■

Under suitable control of the sectional curvature of the spacelike hypersurface, we obtain an extension of Theorem 3.2 to the case of higher order mean curvatures.

Theorem 3.3. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature K_Σ bounded from below and satisfying $K_\Sigma \leq -1$, and such that $f_a^2 \leq 1/2$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose that, for some $r \in \{1, \ldots, n-1\}$, H_r is bounded and*

$$0 \leq H_{r+1} \leq H_r. \tag{3.22}$$

If

$$|a^\top| \leq C \inf_{\Sigma^n}\, (H_r - H_{r+1}) \tag{3.23}$$

for some positive constant C, then $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is a totally umbilical spacelike hypersurface M_τ with $\tau^2 = 1/2$.

Proof. We define a self-adjoint operator $\mathcal{P}_r : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ by

$$\mathcal{P}_r = H_r P_r. \tag{3.24}$$

For each $p \in \Sigma^n$, we take a local orthonormal frame $\{e_1, \ldots, e_n\}$ such that $A(e_i) = \lambda_i e_i$. So, from (2.9) we have

$$P_r(e_i) = (-1)^r \sum_{i_1 < \cdots < i_r,\, i_j \neq i} \lambda_{i_1} \ldots \lambda_{i_r} e_i.$$

Thus, for any $i \in \{1, \ldots, n\}$,

$$\langle \mathcal{P}_r(e_i), e_i \rangle = \binom{n}{r}^{-1} \sum_{\substack{i_1 < \cdots < i_r,\, i_j \neq i \\ j_1 < \cdots < j_r}} (\lambda_{i_1}\lambda_{j_1}) \ldots (\lambda_{i_r}\lambda_{j_r}). \tag{3.25}$$

Moreover, from the Gauss equation (3.15) and taking into account our constraint on the sectional curvature K_Σ of Σ^n, we have

$$\lambda_i \lambda_j = -1 - K_\Sigma(e_i, e_j) \geq 0$$

for all $i, j \in \{1, \ldots, n\}$, with $i \neq j$. Hence, from (3.25) we get

$$\langle \mathcal{P}_r(e_i), e_i \rangle \geq 0$$

for any $i \in \{1, \ldots, n\}$, which implies that $\mathcal{P}_r$ is positive semidefinite. In addition, since we are assuming that H_r is bounded on Σ^n, from (2.10) and (3.24) we see that the same is true for $\operatorname{tr}(\mathcal{P}_r) = c_r H_r^2$.

Now, extending the idea of the proof of Theorem 3.2, we consider the differential operator $\mathcal{L}_r : C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$ given by

$$\mathcal{L}_r(f) = \operatorname{tr}(\mathcal{P}_r \nabla^2 f). \tag{3.26}$$

Since $\mathcal{P}_r$ is positive semidefinite and $f_a^2 \leq 1/2$, from Proposition 3.1, (3.22) and (3.26) we get

$$\mathcal{L}_r(l_a^2) \geq c_r (H_r^2 - H_{r+1}^2) l_a^2 \geq 0. \tag{3.27}$$

Furthermore, taking into account (3.11) once more, hypothesis (3.23) implies that l_a is bounded. Thus, Lemma 2.4 yields a sequence $\{p_k\}_{k\geq 1} \subset \Sigma^n$ such that

$$\lim_k l_a^2(p_k) = \sup_{\Sigma^n} l_a^2, \quad \lim_k |\nabla l_a^2(p_k)| = 0, \quad \limsup_k \mathcal{L}_r(l_a^2)(p_k) \leq 0. \tag{3.28}$$

Consequently, from (3.27) and (3.28) we have

$$\begin{aligned} 0 &\geq \limsup_k \mathcal{L}_r(l_a^2)(p_k) \\ &\geq c_r \Big(\sup_{\Sigma^n} l_a^2\Big) \limsup_k (H_r^2 - H_{r+1}^2)(p_k) \geq 0. \end{aligned} \tag{3.29}$$

Since $\sup_{\Sigma^n} l_a^2 > 0$, from (3.29) we get

$$\limsup_k (H_r^2 - H_{r+1}^2)(p_k) = 0,$$

and in particular

$$\inf_{\Sigma^n}(H_r - H_{r+1}) = 0. \tag{3.30}$$

From (3.23) and (3.30) we find that $a^\top$ vanishes identically on Σ^n, which means that l_a is constant on Σ^n, and hence $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is a totally umbilical spacelike hypersurface M_τ of $\mathbb{H}_1^{n+1}$. Furthermore, since in this case $H_r = H_{r+1}$, we must have

$$\left(\frac{\tau}{\sqrt{1-\tau^2}}\right)^r = \left(\frac{\tau}{\sqrt{1-\tau^2}}\right)^{r+1},$$

which implies $\tau^2 = 1/2$. ∎

Remark 3.4. We point out that the assumption $K_\Sigma \leq -1$ in Theorem 3.3 is compatible with our previous conclusion. Indeed, from item (3) of the description of the totally umbilical hypersurfaces of $\mathbb{H}_1^{n+1}$ quoted at the beginning of this section, for $\tau^2 = 1/2$ we have

$$K_\Sigma = K_{M_\tau} = -\frac{1}{1-\tau^2} = -2.$$

We say that a spacelike hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is *locally tangent from below* to a totally umbilical hypersurface M_τ of $\mathbb{H}_1^{n+1}$ when there exists a point $p \in \Sigma^n$ and a neighborhood $\mathcal{U} \subset \Sigma^n$ of p such that $l_a(p) = \tau$ and $l_a(q) \leq \tau$ for all $q \in \mathcal{U}$ (see Figure 1).

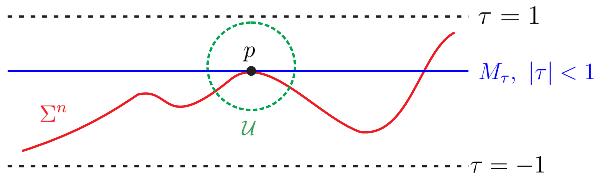

Fig. 1. $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is locally tangent from below to a hypersurface M_τ.

On the other hand, we say that $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is *locally tangent from above* to M_τ when there exists a point $p \in \Sigma^n$ and a neighborhood $\mathcal{U} \subset \Sigma^n$ of p such that $l_a(p) = \tau$ and $l_a(q) \geq \tau$ for all $q \in \mathcal{U}$ (see Figure 2).

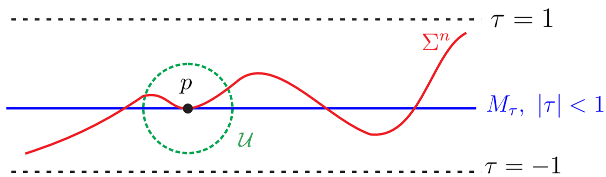

Fig. 2. $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is locally tangent from above to a hypersurface M_τ.

In our next result, for a constant $0 < \rho \leq 1$, we will also consider the open regions

$$\Omega^+(a,\rho) = \{p \in \mathbb{H}_1^{n+1} : 0 < \langle p, a\rangle < \rho\}$$

of the chronological past, and

$$\Omega^-(a,\rho) = \{p \in \mathbb{H}_1^{n+1} : -\rho < \langle p, a\rangle < 0\}$$

of the chronological future of $\mathbb{H}_1^{n+1}$, with respect to a fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$ (see Figure 3).

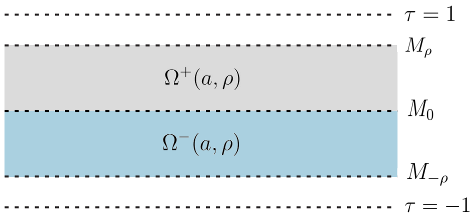

Fig. 3. The open regions $\Omega^+(a,\rho)$ and $\Omega^-(a,\rho)$ of $\mathbb{H}_1^{n+1}$ determined by a fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$.

Theorem 3.5. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface such that $f_a^2 \leq 1/2$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose that H is bounded, and for some $r \in \{1, \dots, n-1\}$, H_{r+1} is positive with*

$$H_{r+1} \leq H_r. \tag{3.31}$$

Assume in addition that

$$|a^\top| \leq C \inf_{\Sigma^n} (H_r - H_{r+1}), \tag{3.32}$$

for some positive constant C.

(i) *If $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is contained in $\Omega^+(a,\rho)$ for some $1/\sqrt{2} < \rho \leq 1$, and it is locally tangent from below to a totally umbilical spacelike hypersurface M_τ with $0 < \tau < \rho$, then $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is isometric to M_τ and $\tau = 1/\sqrt{2}$.*

(ii) *If $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is contained in $\Omega^-(a,\rho)$ for some $-1 \leq \rho < -1/\sqrt{2}$, and it is locally tangent from above to a totally umbilical spacelike hypersurface M_τ with $\rho < \tau < 0$, then $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is isometric to M_τ and $\tau = -1/\sqrt{2}$.*

Proof. Let us consider the vector field $X \in \mathfrak{X}(\mathbb{H}_1^{n+1})$ defined by

$$X(p) = \langle p, a\rangle p + a.$$

From [27, Example 3] we know that X is a complete closed conformal vector field with

$$\operatorname{Div} X(p) = (n+1)\langle p, a\rangle, \tag{3.33}$$

$$|X|_{\Sigma^n} = \sqrt{-\langle X, X\rangle}|_{\Sigma^n} = \sqrt{1 - l_a^2}. \tag{3.34}$$

Thus, supposing for instance that $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is contained in $\Omega^+(a,\rho)$ and that it is locally tangent from below to M_τ with $0 < \tau < \rho$, from (3.34) we see that $|X|_{\Sigma^n}$ attains a local minimum on Σ^n. Consequently, by Lemma 2.3, there exists an elliptic point on Σ^n. Since we are also supposing that $H_{r+1} > 0$, it follows from Lemma 2.2 that the jth Newton transformation $P_j : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ is positive definite, and consequently H_j is positive for all $j \in \{1,\ldots,r\}$.

Moreover, from (3.14) and $H_2 > 0$, we get

$$\sum_i \lambda_i^2 \leq n^2 H^2.$$

Consequently, the boundedness of H implies the boundedness of all principal curvatures of $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$. So, from the Gauss equation (3.15) we conclude that K_Σ is bounded from below. Moreover, H_r is bounded, and hence $\mathrm{tr}(P_r) = c_r H_r$ is bounded.

From Proposition 3.1 we get

$$L_r(l_a^2) \geq c_r \left(\frac{H_r^2 - H_{r+1}^2}{H_r} \right) l_a^2 \geq 0. \tag{3.35}$$

Thus, Lemma 2.4 yields a sequence $\{p_k\}_{k\geq 1} \subset \Sigma^n$ such that

$$\lim_k l_a^2(p_k) = \sup_{\Sigma^n} l_a^2, \quad \lim_k |\nabla l_a^2(p_k)| = 0, \quad \limsup_k L_r(\, l_a^2 \,)(p_k) \leq 0. \tag{3.36}$$

Consequently, from (3.35) and (3.36) we have

$$0 \geq \limsup_k L_r(l_a^2)(p_k) \geq c_r \left(\sup_{\Sigma^n} l_a^2 \right) \limsup_k \frac{H_r^2 - H_{r+1}^2}{H_r} \geq 0. \tag{3.37}$$

Since $\sup_{\Sigma^n} l_a^2 > 0$, from (3.37) we obtain

$$\limsup_k \frac{H_r^2 - H_{r+1}^2}{H_r} = 0. \tag{3.38}$$

In particular,

$$\inf_{\Sigma^n} \frac{H_r - H_{r+1}}{H_r} = 0, \tag{3.39}$$

and since $H_r > 0$, we conclude that

$$\inf_{\Sigma^n} (H_r - H_{r+1}) = 0. \tag{3.40}$$

We finish the proof of item (i) reasoning as in the last part of the proof of Theorem 3.3. The proof of (ii) is similar. ■

4. Nullity and nonexistence of r-maximal spacelike hypersurfaces. Let $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ be a spacelike hypersurface with second fundamental form $A : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$. Following [19], for $p \in \Sigma^n$, we define the *space of relative nullity* of Σ^n at p by

$$\mathcal{N}(p) = \{v \in T_p\Sigma : v \in \ker(A_p)\},$$

where $\ker(A_p)$ denotes the kernel of $A_p : T_p\Sigma \to T_p\Sigma$. The *index of relative nullity* of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ at p is

$$\nu(p) = \dim \mathcal{N}(p),$$

and the *index of minimum relative nullity* of $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}$ is

$$\nu_0 = \min_{p \in \Sigma^n} \nu(p).$$

We also recall that a spacelike hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is said to be r-*maximal* if H_{r+1} vanishes identically on Σ^n.

THEOREM 4.1. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ be a complete r-maximal* ($2 \le r \le n-1$) *spacelike hypersurface with sectional curvature bounded from below and satisfying $K_\Sigma \le -1$, and such that $f_a^2 \le \frac{1}{2}$ and $|a^\top|$ is bounded for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. If H_r is a nonnegative constant, then the index of minimum relative nullity of ψ is at least $n-r+1$. Moreover, if H_{r-1} does not vanish on Σ^n, then through every point of Σ^n there passes an $(n-r+1)$-dimensional hyperbolic space $\mathbb{H}^{n-r+1} \hookrightarrow \mathbb{H}_1^{n+1}$ totally contained in Σ^n.*

Proof. Suppose that $H_r > 0$. Reasoning as in the proof of Theorem 3.3, we conclude that $\inf_{\Sigma^n}(H_r - H_{r+1}) = 0$. Thus, since Σ^n is supposed to be r-maximal, we get $H_r = 0$. Hence, from [14, Proposition 2.3(c)], we see that $H_j = 0$ for all $j \ge r$, and hence $\nu_0 \ge n - r + 1$.

Now, assume that H_{r-1} does not vanish on Σ^n. From [19, Theorem 5.3] (see also [21]), the distribution $\Sigma^n \ni p \mapsto \mathcal{N}(p)$ of minimal relative nullity of $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is smooth and integrable with complete leaves, totally geodesic in Σ^n and in $\mathbb{H}_1^{n+1}$. Therefore, the result follows from the characterization of complete totally geodesic submanifolds of $\mathbb{H}_1^{n+1}$ as hyperbolic spaces of suitable dimension. ∎

We close this section with the following nonexistence result.

THEOREM 4.2. *There does not exist a complete* 1*-maximal spacelike hypersurface $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ with nonnegative constant mean curvature and such that $f_a^2 \le \frac{1}{2}$ and $|a^\top|$ is bounded for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$.*

Proof. Assume there exists such a hypersurface. We have two cases to consider:

If $H = 0$, from (2.6) we get $|A| = 0$, and consequently $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ is a totally geodesic spacelike hypersurface M_0. But, from (3.2) and (3.11) we get $f_a^2 = 1$, contrary to assumption.

If $H > 0$, we can reason as in the proof of Theorem 3.2 to conclude that $H = 0$, a contradiction. ∎

5. Curvature estimates and further nonexistence results. In order to prove our further results, we need Omori's classical generalized maximum principle [28].

LEMMA 5.1. *Let Σ^n be a complete Riemannian manifold with sectional curvature bounded from below and let $u : \Sigma^n \to \mathbb{R}$ be a smooth function bounded from above. Then for each $\epsilon > 0$ there exists a point $p_\epsilon \in \Sigma^n$ such that*

(i) $|\nabla u(p_\epsilon)| < \epsilon$;
(ii) $\operatorname{Hess} u(v, v) < \epsilon$ *for all unit tangent vectors* $v \in T_p\Sigma$;
(iii) $\sup_{\Sigma^n} u - \epsilon < u(p_\epsilon) \le \sup_{\Sigma^n} u$.

Now we are in a position to present the following curvature estimates for complete spacelike hypersurfaces immersed in the anti-de Sitter space.

THEOREM 5.2. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below. If $\psi(\Sigma^n)$ is contained either in $\Omega^-(a, \rho)$ or in $\Omega^+(a, \rho)$, for some unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and some $0 < \rho < 1$, then*

$$\sup_{\Sigma^n} H_r \ge \left(\frac{\sup_{\Sigma^n} u^\pm}{\sqrt{1 - (\sup_{\Sigma^n} u^\pm)^2}} \right)^r \quad \text{for all } r \in \{1, \dots, n\},$$

where $u^\pm \in C^\infty(\Sigma^n)$ is defined by $u^\pm = \pm l_a$ with $\psi(\Sigma^n) \subset \Omega^\mp(a, p)$.

Proof. Assume that $\psi(\Sigma^n) \subset \Omega^+(a, \rho)$ for some unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and some $0 < \rho < 1$. In that case, we choose the orientation N of ψ to be the same as the time-orientation of a, so that $f_a < 0$. On Σ^n, we define $u = l_a$ (for simplicity, we write u instead of u^+). From (3.6) we obtain

$$\operatorname{Hess} u(X, X) = -f_a \langle A(X), X \rangle + u \langle X, X \rangle. \tag{5.1}$$

Since u is a smooth function on Σ^n bounded from above, we know from Lemma 5.1 that for each $j \in \mathbb{N}$ there exists a point $p_j \in \Sigma^n$ such that

$$|\nabla u(p_j)| < 1/j, \tag{5.2}$$

$$\operatorname{Hess} u(p_j)(v, v) < 1/j, \tag{5.3}$$

for each tangent vector $v \in T_{p_j}\Sigma$ with $|v| = 1$, and

$$\sup_{\Sigma^n} u - 1/j < u(p_j) \le \sup_{\Sigma^n} u. \tag{5.4}$$

Let $\{e_i^j\}_{i=1}^n$ be an orthonormal basis of principal directions at p_j satisfying $A_{p_j}(e_i^j) = \lambda_i(p_j) e_i^j$. From (5.1) and (5.3), we arrive at

$$\operatorname{Hess} u(p_j)(e_i^j, e_i^j) = -f_a(p_j)\lambda_i(p_j) + u(p_j) < 1/j.$$

Since N is in the same time-orientation as a, it follows that

$$\lambda_i(p_j) < -\frac{1/j - u(p_j)}{f_a(p_j)}. \tag{5.5}$$

On the other hand, it follows from (5.4) that $1/j - u(p_j) \to -\sup_{\Sigma^n} u < 0$ as $j \to \infty$, so that

$$\lambda_i(p_j) < 0 \tag{5.6}$$

for all j large enough. Thus, for large j,

$$\binom{n}{r} H_r(p_j) > \binom{n}{r} \left(\frac{u(p_j) - 1/j}{-f_a(p_j)} \right)^r. \tag{5.7}$$

From (3.11) we deduce that $|\nabla u|^2 = -1 + f_a^2 + u^2$ and, using (5.2) and (5.4), we get

$$\lim_{j\to\infty} -f_a(p_j) = \sqrt{1 - (\sup_{\Sigma^n} u)^2}. \tag{5.8}$$

Letting $j \to \infty$ and using (5.4) and (5.8), we deduce from (5.7) that

$$\sup_{\Sigma^n} H_r \geq \left(\frac{\sup_{\Sigma^n} u}{\sqrt{1 - (\sup_{\Sigma^n} u)^2}} \right)^r.$$

If $\psi(\Sigma^n) \subset \Omega^-(a, \rho)$ for some unit timelike vector $a \in \mathbb{R}^{n+2}_2$ and some $0 < \rho < 1$, we choose N representing the opposite of the time-orientation of a. Then, arguing similarly, we arrive at

$$\sup_{\Sigma^n} H_r \geq \left(\frac{\sup_{\Sigma^n} u^-}{\sqrt{1 - (\sup_{\Sigma^n} u^-)^2}} \right)^r$$

for all $r \in \{1, \ldots, n\}$. The proof is now complete. ∎

From Theorem 5.2 we obtain the following sufficient conditions for the nonexistence of complete spacelike hypersurfaces in $\Omega^+(a, \rho)$ (or $\Omega^-(a, \rho)$).

Corollary 5.3. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}^{n+1}_1$ be a complete spacelike hypersurface with sectional curvature bounded from below.*

(i) *If $H_r \leq 0$ for some even r, then $\psi(\Sigma^n)$ cannot be contained in $\Omega^-(a, \rho)$ or $\Omega^+(a, \rho)$ for any unit timelike vector $a \in \mathbb{R}^{n+2}_2$ and any $0 < \rho < 1$.*

(ii) *If $H_r \leq 0$ for some odd r (for an appropriate choice of orientation N), then $\psi(\Sigma^n)$ cannot be contained in $\Omega^+(a, \rho)$ for any unit timelike vector $a \in \mathbb{R}^{n+2}_2$ and any $0 < \rho < 1$.*

As mentioned in the introduction, Ishihara [23] proved that a complete maximal spacelike hypersurface immersed in $\mathbb{H}^{n+1}_1$ must have the squared norm of the second fundamental form bounded from above by n, and this bound is reached only by the maximal hyperbolic cylinders $\mathbb{H}^m(-\frac{n}{m}) \times \mathbb{H}^{n-m}(-\frac{n}{n-m})$ with $1 \leq m \leq n-1$ (see [23, Theorems 1.2 and 1.3]).

This result jointly with the Gauss equation guarantees that a complete maximal spacelike hypersurface of $\mathbb{H}_1^{n+1}$ must have sectional curvature bounded from below. Thus, taking into account (2.6), it is not difficult to verify that Corollary 5.3(i) yields the following nonexistence result.

COROLLARY 5.4. *There do not exist complete maximal spacelike hypersurfaces $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ contained in $\Omega^-(a,\rho)$ or $\Omega^+(a,\rho)$ for any unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and any $0 < \rho < 1$.*

As a consequence of (2.6), the normalized scalar curvature R of $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ satisfies $R = -1 - H_2$. Hence, under the assumptions of Theorem 5.2,

$$\inf_{\Sigma^n} R \le \frac{1}{(\sup_{\Sigma^n} u^{\pm})^2 - 1}. \tag{5.9}$$

In view of this estimate, Corollary 5.3(i) also yields the following result.

COROLLARY 5.5. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below. If the normalized scalar curvature R of $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ satisfies $R > -1$, then $\psi(\Sigma^n)$ cannot be contained in $\Omega^-(a,\rho)$ or $\Omega^+(a,\rho)$ for any unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and any $0 < \rho < 1$.*

We now obtain an estimate for the Ricci curvature of a complete spacelike hypersurface in $\mathbb{H}_1^{n+1}$.

THEOREM 5.6. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below. If $\psi(\Sigma^n)$ is contained either in $\Omega^-(a,\rho)$ or in $\Omega^+(a,\rho)$, for some unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and some $0 < \rho < 1$, then its Ricci curvature* Ric *satisfies*

$$\inf_{\Sigma^n} \mathrm{Ric} = \inf_{\substack{p\in\Sigma^n \\ v\in T_p\Sigma \\ |v|=1}} \mathrm{Ric}_p(v,v) \le \frac{n-1}{(\sup_{\Sigma^n} u^{\pm})^2 - 1},$$

where $u^{\pm} \in C^{\infty}(\Sigma^n)$ is defined by $u^{\pm} = \pm l_a$ with $\psi(\Sigma^n) \subset \Omega^{\mp}(a,p)$.

Proof. As in the proof of Theorem 5.2, we first assume that $\psi(\Sigma^n) \subset \Omega^+(a,\rho)$ for some unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and some $0 < \rho < 1$. In that case, we choose the orientation N of ψ in the same time-orientation as a. It follows from (2.5) and (5.6) that

$$\begin{aligned} \mathrm{Ric}(e_j^k, e_j^k) &= -(n-1) - \sum_{i=1}^{n} \lambda_i(p_k)\lambda_j(p_k) + \lambda_j^2(p_k) \\ &= -(n-1) - \sum_{i\neq j} \lambda_i(p_k)\lambda_j(p_k) \\ &\le -(n-1) - (n-1)\left(\frac{1/k - u(p_k)}{-f_a(p_k)}\right)^2, \end{aligned} \tag{5.10}$$

where $u = l_a$ (for simplicity, we write u instead of u^+). Letting $k \to \infty$ and using (5.4) and (5.8), from (5.10) we get

$$\inf_{\substack{p\in\Sigma^n\\ v\in T_p\Sigma\\ |v|=1}} \mathrm{Ric}_p(v,v) \le -(n-1)\left[1+\left(\frac{-\sup_{\Sigma^n} u}{\sqrt{1-(\sup_{\Sigma^n} u)^2}}\right)^2\right] = \frac{n-1}{(\sup_{\Sigma^n} u)^2-1}.$$

When $\psi(\Sigma^n) \subset \Omega^-(a,\rho)$, we consider the function $u^- \in C^\infty(\Sigma^n)$ defined by $u^- = -l_a$, which is smooth and bounded from above. A straightforward computation shows that

$$\mathrm{Hess}\, u^-(X,X) = f_a\langle A(X),X\rangle + u^-\langle X,X\rangle$$

for all $X \in \mathfrak{X}(\Sigma^n)$. Let $\{q_k\} \subset \Sigma^n$ be a maximizing sequence for u^- in the sense of Lemma 5.1. For each $k \in \mathbb{N}$, let $\{e_j^k\}_{j=1}^n$ be an orthonormal basis of principal directions at p_k satisfying $A_{p_k}(e_j^k) = \lambda_j(p_k)e_j^k$. Then

$$\lambda_j(p_k) < \frac{1/k - u^-(p_k)}{f_a(p_k)} < 0 \tag{5.11}$$

for all k large enough, since $1/k - u^-(p_k) \to -\sup_{\Sigma^n} u^- < 0$ as $k \to \infty$, and $f_a > 0$ on Σ^n. Moreover, it can be easily seen that

$$\lim_{k\to\infty} f_a(p_k) = 1 - \left(\sup_{\Sigma^n} u^-\right)^2. \tag{5.12}$$

On the other hand, from (2.5) and (5.11) we get

$$\mathrm{Ric}(e_j^k, e_j^k) \le -(n-1) - (n-1)\left(\frac{1/k - u^-(p_k)}{f_a(p_k)}\right)^2. \tag{5.13}$$

Therefore, letting $k \to \infty$ and using (5.12), from (5.13) we obtain

$$\inf_{\substack{p\in\Sigma^n\\ v\in T_p\Sigma\\ |v|=1}} \mathrm{Ric}_p(v,v) \le -(n-1)\left[1+\left(\frac{-\sup_{\Sigma^n} u^-}{\sqrt{1-(\sup_{\Sigma^n} u^-)^2}}\right)^2\right] = \frac{n-1}{(\sup_{\Sigma^n} u^-)^2-1}.$$

This finishes the proof of Theorem 5.6. ∎

We conclude the paper with the following consequence of Theorem 5.6.

COROLLARY 5.7. *Let $\psi : \Sigma^n \looparrowright \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below. If its Ricci curvature* Ric *satisfies* Ric $> -(n-1)$, *then $\psi(\Sigma^n)$ cannot be contained in $\Omega^-(a,\rho)$ or $\Omega^+(a,\rho)$ for any unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and any $0 < \rho < 1$.*

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