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BIJECTIVE 1-COCYCLES, BRACES, AND NON-COMMUTATIVE PRIME FACTORIZATION

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Abstract. The structure group of an involutive set-theoretic solution to the Yang–Baxter equation is a generalized radical ring called a *brace*. The concept of brace is extended to that of a *quasiring* where the adjoint group is just a monoid. It is proved that a special class of lattice-ordered quasirings characterizes the divisor group A of a smooth non-commutative curve X. The multiplicative monoid A° of A is related to the additive group by a bijective 1-cocycle. Extending previous results on non-commutative arithmetic, the elements of A are represented as a class $\Phi(X)$ of self-maps of a universal cover of X. For affine subsets U of X, the regular functions on U form a hereditary order such that the monoid of fractional ideals embeds into A° as the class of monotone functions in $\Phi(U)$. The unit group of A is identified with the *annular symmetric group*, which occurred in connection with quasi-Garside groups of Euclidean type. The main part of the paper is self-contained and provides a quick approach to non-commutative prime factorization and its relationship to braces.

1. Introduction. The arithmetic of a classical smooth affine curve is given by the fractional ideals of a Dedekind domain. Since prime ideals correspond to points on the curve, the unique factorization of ideals into prime ideals allows one to relate ideals to *divisors*, that is, finite $\mathbb{Z}$-linear combinations of points. For non-commutative curves, Dedekind domains R with quotient field K have to be replaced by *hereditary orders*, finitely generated R-algebras Γ in a semisimple K-algebra. Their ideal theory has been developed some fifty years ago [8, 35, 36]. As in the commutative case, the ideal theory of Γ immediately reduces to the ideal theory of the localizations $\Gamma_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R \Gamma$, which are hereditary orders over a discrete valuation domain. For a local hereditary order $\Gamma_{\mathfrak{p}}$, the "points" (= maximal ideals) are no longer independent: they form a cyclic orbit under an automorphism of $\Gamma_{\mathfrak{p}}$. If there is just a single maximal ideal M, the ideals form a chain of powers of M, as in the commutative case.

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It remains true that ideals of a hereditary order Γ can be viewed as divisors, hence as elements of a free abelian group $\mathrm{Div}(\Gamma)$, but a divisor need not be an ideal. This was shown in [61], where divisors have been used in an essential way to give a generalized unique prime factorization of ideals. The non-commutative analogue of the fundamental theorem of arithmetic then consists in a normal form of the factorization into primes. Besides this, it was found that the fractional ideals can be associated in a unique fashion to monotone functions on the points, so that the multiplication of ideals corresponds to the composition of functions.

In this paper, we exhibit a close relationship between non-commutative arithmetic and *braces* [53], which have been introduced and studied [14, 56, 13, 40, 2, 1, 64, 3, 48, 54] in connection with set-theoretic solutions [24] to the Yang–Baxter equation [68, 31, 26, 44, 10, 30, 13, 57, 20, 27, 41]. Braces also arise in the theory of regular affine groups [43, 12, 11], groups of I-type [67, 31, 37, 55], finite solvable groups [14, 3, 54, 60], right-symmetric Lie algebras [56, 2], Hopf–Galois theory [28, 15, 32, 1], and Garside groups [16, 20, 57].

There are several equivalent definitions of braces; the simplest one perhaps is that of a bijective 1-cocycle $\gamma\colon G \to A$ from a group G onto an abelian group A with a right action of G. Identifying the two groups, we obtain a group A with a second group structure $G = (A; \circ)$, the *adjoint group* of A. For example, the Jacobson radical J of a ring is a brace with adjoint multiplication $a \circ b = ab + a + b$. Jacobson's equation still plays a role for braces, which thus also carry a ring-like structure.

Our aim is to completely describe the connection between multiplication of ideals and addition of divisors, which are not the same for a non-commutative curve. Indeed, this connection is given by a 1-cocycle, as in the case of a brace; this fact goes beyond former investigations [8, 9, 33, 34, 35, 36, 25] on hereditary arithmetics. If addition and multiplication coincide, we are in the classical setting of commutative curves. It remains to clarify the conditions which make a divisor correspond to an ideal.

Now let us give a more detailed account on the structures that arise with the adjoint multiplication of a divisor group. As there are non-invertible ideals, the adjoint group has to be replaced by a monoid. This leads to the concept of *quasiring*. Thus, a quasiring is equivalent to a bijective 1-cocycle $M \to A$ from a monoid M onto an M-module A. We show that a special class of lattice-ordered quasirings characterizes the divisor groups of non-commutative smooth algebraic curves. The adjoint monoid structure extends the multiplication of fractional ideals of a hereditary noetherian ring to the set of all divisors, as explained below.

Quasirings generalize *left* braces [13], where the underlying module over the adjoint group is a left module. Just as every ring is an algebra over its centre, a quasiring A is a one-sided algebra over an associative subring $R(A)$, the *quasicentre* of A, which can be identified with a ring of affine maps (Proposition 2).

The relationship between a hereditary R-order Γ and its base ring R corresponds to an affine piece of a non-commutative curve X' lying over a classical affine curve X. To locate the non-commutative monoid H of fractional ideals of X' within the divisor group $\mathrm{Div}(X')$, the lattice structure of $\mathrm{Div}(X')$ has to be taken into account. With respect to addition, $\mathrm{Div}(X')$ is an abelian lattice-ordered group, an abelian *ℓ-group* for short. Left multiplications are lattice endomorphisms (Proposition 3), while right multiplication only satisfies an inequality $(a \vee b) \circ c \leq a \circ c \vee b \circ c$. Because of these properties, we say that the divisors on X' form an *ℓ-quasiring*.

Thus, for an arbitrary ℓ-quasiring A, it is natural to associate an abstract curve X' consisting of the maximal $x < 0$ in A as its points. So the question arises which conditions on A characterize the divisor group of a concrete non-commutative curve X' in the above sense. An obvious necessary condition is that A has to be *noetherian*, which means that bounded increasing sequences are finite. We say that $c \in A$ is *normal* if the right multiplication $a \mapsto a \circ c$ is monotone. The normal elements form a submonoid and a sublattice $N(A)$ of A, and the invertible normal elements form a lattice-ordered subgroup $N^\times(A)$ (Corollaries of Proposition 3). By an old theorem of Birkhoff [5], the ℓ-group $N^\times(A)$ is a cardinal sum of infinite cyclic groups. If each element of A is bounded by elements of $N^\times(A)$, we say that A has *enough normal units*.

If A is noetherian with enough normal units, the maximal elements $\pi < 0$ in $N^\times(A)$ can be interpreted as points of a commutative curve X, and each $\pi \in X$ is majorized by finitely many $x \in X'$ which form a cycle with respect to conjugation with π (Corollary 1 of Proposition 6). If X consists of a single point π, we say that A is *local*.

In [61] the multiplicative structure of the monoid of fractional ideals was determined for concrete non-commutative curves. The resulting structure is a *hereditary arithmetic* (Definition 5). There is still a unique prime factorization in the sense that each fractional ideal has a unique normal form as a product of primes. As left or right modules, the invertible fractional ideals are *projective*, a module-theoretic property that can be expressed in terms of the multiplication and the partial order.

Extending abstract hereditary arithmetics to ℓ-quasirings of divisors, we call a noetherian ℓ-quasiring A with enough normal units *hereditary* if $X' \subset N(A)$ and each element of A is projective. For a hereditary ℓ-quasiring A, we show that $N(A)$ is a hereditary arithmetic (Corollary 2 of Proposition 6). The prime factorization in $N^\times(A)$ induces a decomposition $A = \boxplus_{\pi \in X} A_\pi$ into local (hereditary) ℓ-quasirings (Theorem 1).

Conversely, we show that any hereditary arithmetic H admits a natural embedding into a hereditary ℓ-quasiring A such that H coincides with the monoid $N(A)$ of normal elements in A (Theorem 2). We provide several characterizations of $N(A)$ and $N^{\times}(A)$ (Corollaries of Theorem 2 and Proposition 7).

As a final step, we give a very simple description for the multiplication of divisors, extending the functional representation of fractional ideals given in [61]. Let X be the point set of an abstract non-commutative curve, with a partition into finite cycles, given by a permutation τ of X. (So the finite orbits of τ correspond to the points of an underlying commutative curve.) Consider the *universal cover* $p\colon \widetilde{X} \twoheadrightarrow X$ which lifts each cycle of X to an infinite cyclic group. We give a new interpretation of divisors on X as functions $\widetilde{X} \to \widetilde{X}$ respecting the map p and the cycles of X, and show that these functions form a hereditary ℓ-quasiring $\Phi(X)$ (Theorem 3). So the normal elements in $\Phi(X)$ form a hereditary arithmetic $H = N(\Phi(X))$ associated with X. Coincidentally, the multiplication $\circ$ in $\Phi(X)$ (Jacobson's circle operation) is given by the composition of functions in $\Phi(X)$, usually denoted by $\circ$ as well. The hereditary arithmetic H consists of the monotone functions in $\Phi(X)$ (Theorem 3), and the quasicentre $R(\Phi(X))$ of $\Phi(X)$ is equal to the group $N^{\times}(\Phi(X))$ of normal units of $\Phi(X)$ (Corollary 2 of Theorem 2).

There is a certain analogy between ℓ-quasirings and *right ℓ-groups* [57], that is, groups with a right invariant lattice order. For example, the spherical Artin groups [7, 23], and more generally, all Garside groups [29, 22, 19, 21] are right ℓ-groups. Every right ℓ-group G has a subgroup $N(G)$ of *normal* elements. If $N(G)$ is noetherian, the above-mentioned theorem of Birkhoff [5] implies that $N(G)$ is a cardinal sum of cyclic groups, which yields a "decomposition" of G (a big crossed product) whenever G has *enough normal elements* [58, Theorem 2]. Extending a concept known for Artin groups [7], the ℓ-subgroup $N(G) = N(G)^{\times}$ is called the *quasicentre* of G, which explains our homonymous terminology for ℓ-quasirings.

As a main tool in their study of dual Artin monoids of Euclidean type [46, 47], McCammond and Sulway introduce a "middle group" $\mathrm{Mid}(B_n)$ between the Artin group of type $\mathbb{B}_n$ and its Coxeter group, also called the *annular symmetric group*. For a local hereditary ℓ-quasiring $\Phi(X)$ with $|X| = n$, the unit group $\Phi(X)^{\times}$ coincides with this group $\mathrm{Mid}(B_n)$ (Corollary of Theorem 3). Lusztig's representation [45] shows that $\mathrm{Mid}(B_n)$ contains the Coxeter group of Euclidean type $\widetilde{\mathbb{A}}_{n-1}$ as a subgroup, with factor group $\mathbb{Z}$.

2. Preliminaries. A group G with a lattice order $\leq$ is said to be a *right ℓ-group* [57] if

$$a \leq b \implies ac \leq bc \tag{1}$$

for $a, b, c \in G$. In particular, this implies that the equations $(a \wedge b)c = ac \wedge bc$ and $(a \vee b)c = ac \vee bc$ hold in G. Note that there is no essential difference between a right ℓ-group and a *left ℓ-group* where (1) is replaced by the implication

$$a \leq b \implies ca \leq cb. \tag{2}$$

Indeed, the bijection $a \mapsto a^{-1}$ carries each right lattice order into a left lattice order: $a \preccurlyeq b :\Leftrightarrow b^{-1} \leq a^{-1}$. If both partial orders coincide, the group is said to be *lattice-ordered*, or briefly an *ℓ-group* [4, 18]. Every *Garside group* [29, 22, 19, 21], hence in particular, every spherical Artin–Tits group [7, 23], is a right ℓ-group. Furthermore, the *right ordered groups* [17], which play an important role in low-dimensional topology [51, 63, 6], are right ℓ-groups.

A right ℓ-group is said to be *noetherian* [57] if each strictly ascending or strictly descending sequence is finite. Thus, for a noetherian right ℓ-group G, each element $a < 1$ is contained in a maximal element $x < 1$, an *atom* of G, and each $a \leq 1$ is a finite product of atoms. We call G *bounded atomic* [57] if the number of atoms x_i in such a product $a = x_1 \cdots x_n$ is bounded for each $a \leq 1$.

An element c of a right ℓ-group G is said to be *normal* if it satisfies (2) for all $a, b \in G$. The set $N(G)$ of normal elements is called the *quasicentre* [57] of G. By [58, Proposition 5], the quasicentre of G is an ℓ-group with respect to the induced partial order. For Garside groups, this concept was introduced by Brieskorn and Saito [7]. A normal element $u > 1$ such that each $a \in G$ is majorized by some power u^n with $n \in \mathbb{N}$ is said to be a *strong order unit* [57]. A bounded atomic right ℓ-group with a strong order unit is said to be a *quasi-Garside group* [21]. A quasi-Garside group with finitely many atoms is said to be *Garside*. The *Garside element* Δ of a Garside group [21] is a strong order unit. In the special case of ℓ-groups, the concept of strong order unit is well known [4, 18].

If the quasicentre $N(G)$ of a right ℓ-group G is noetherian, then G is said to be *quasinoetherian* [58]. The group G is said to have *enough normal elements* [58] if each $a \leq 1$ admits a normal element $n \leq a$. If $N(G)$ is cyclic, then G is called *quasicyclic* [58]. The maximal normal elements $p < 1$ of a quasinoetherian right ℓ-group G with enough normal elements are called *primes* [58], and the set of primes is denoted by $P(G)$. By [58, Theorem 2], the *negative cone* $G^- := \{a \in G \mid a \leq 1\}$ is then a *crossed product* $G^- = \boxtimes_{p \in P(G)} G_p^-$, where each G_p^- is the negative cone of a quasicyclic right ℓ-group G_p.

3. Non-commutative divisor groups. Let $K|\mathbb{C}$ be a field extension of transcendence degree 1, and let X be the corresponding smooth projective curve. Thus, any closed point $x \in X$ is associated with a discrete valuation ring $\mathscr{O}_x$, and any affine open set $U \subset X$ gives rise to a Dedekind ring $\mathscr{O}_U$, both with the same quotient field K. A non-commutative smooth curve over X can be conceived as a coherent sheaf $\mathscr{A}$ of hereditary $\mathscr{O}_X$-orders (see, e.g., [50, III.2.3]). The generic fibre then consists of a semisimple K-algebra $\mathscr{A}_0$, the $\mathscr{A}_x$ are hereditary $\mathscr{O}_x$-orders in $\mathscr{A}_0$, and almost all of the $\mathscr{A}_x$ are maximal orders.

The ideal theory of hereditary orders was investigated in the decade after 1963 by Brumer [8, 9], Harada [33, 34, 35], and Jacobinski [36]. In contrast to the commutative case, the existence of non-maximal local hereditary orders $\mathscr{A}_x$ leads to a linkage between prime ideals of $\mathscr{A}_x$, so that the union X' of the maximal spectra of the $\mathscr{A}_x$ splits into finite orbits under a permutation τ which reflects the Auslander–Reiten translate in the category $\mathrm{coh}(\mathscr{A})$ of coherent sheaves (see [39, 42] for a topical treatment). The prime ideals in a non-trivial clique are projective as modules but no longer invertible.

On the other hand, there is a free abelian group $\mathrm{Div}(X')$ of divisors on X' such that every fractional ideal of any hereditary order $\mathscr{A}(U)$ corresponds to a unique divisor. The lattice structure, as well as the multiplicative structure of $\mathscr{A}(U)$, can be extended from ideals to divisors [61]. By [61, Theorem 2], the additive abelian group structure of $\mathrm{Div}(X')$ is connected with the multiplicative monoid $\mathrm{Div}(X')$ by means of a bijective 1-cocycle. Taking this as a starting point, we will see below that the semigroup of fractional ideals can be retrieved from a ring-like structure of $\mathrm{Div}(X')$. Moreover, the unit group of $\mathrm{Div}(X')$ will turn out to be related to Euclidean quasi-Garside groups. As a first step, we give an algebraic description of non-commutative divisor groups.

Definition 1. We define a (*left*) *quasiring* to be an additive abelian group A with a multiplication $A \times A \to A$ (written as juxtaposition) satisfying $(\forall a, b, c \in A)$

$$0a = 0, \tag{QR1}$$

$$a(b+c) = ab + ac, \tag{QR2}$$

$$(ab + a + b)c = a(bc) + ac + bc. \tag{QR3}$$

For an associative (not necessarily unital) ring R, we define an *R-quasiring* to be a quasiring A with a right R-module structure satisfying $(ab)\gamma = a(b\gamma)$ for $a, b \in A$ and $\gamma \in R$. *Right quasirings* and *right R-quasirings* are defined dually.

Thus every quasiring A is a $\mathbb{Z}$-quasiring and vice versa. Moreover, it is readily checked that

$$R(A) := \{c \in A \mid \forall a, b \in A\colon (a+b)c = ac + bc\}$$

is an associative subring such that A is an $R(A)$-quasiring. We call $R(A)$ the *quasicentre* of A.

If the multiplication of a quasiring A is replaced by the operation

$$a \circ b := ab + a + b, \tag{3}$$

then (QR3) states that $(A;\circ)$ is associative, and the other axioms imply that $(A;\circ)$ is a monoid with neutral element 0. We call $A^\circ := (A;\circ)$ the *adjoint monoid* of the quasiring A. The group $A^\times$ of invertible elements of A° will be called the *adjoint group* of A. The inverse of an element $\alpha \in A^\times$ will be denoted by α'.

For any quasiring A, there is a left action $A^\circ \times A \to A$ of the adjoint monoid on the additive group $(A;+)$, given by

$$^a b := ab + b.$$

Then (QR3) becomes

$$^{a\circ b}c = {}^a({}^b c),$$

and (QR1) states that $^0 a = a$, while (QR2) says that A° acts by endomorphisms of $(A;+)$. In other words, (QR1)–(QR3) state that $(A;+)$ is a (*left*) *module* over A°. Equation (3) can be written as

$$a \circ b = a + {}^a b. \tag{4}$$

So the additive group is identified with the adjoint monoid by a 1-cocycle (4).

PROPOSITION 1. *Up to isomorphism, a quasiring A is equivalent to a module A over a monoid M together with a bijective 1-cocycle $\gamma\colon M \to A$.*

Proof. The bijection γ identifies M with A, which yields (4). Defining ab in M via (3), we deduce the correspondence by the preceding remarks. ∎

REMARKS. 1. Recall that a *skew-ring* [59] is a group $(A;+)$ with a binary operation $(a,b) \mapsto a^b$ satisfying $(a+b)^c = a^c + b^c$ and $a^{b\circ c} = (a^b)^c$ with $a^0 = a$, where $b \circ c := b^c + c$. Thus, a right quasiring is the same as a skew-ring with commutative additive group $(A;+)$. Note that every skew-brace [32] is a skew-ring, and by [59, Proposition 2], a unital near-ring is the same as a skew-ring A with an element 1 satisfying $1^a = a + 1$ for all $a \in A$.

2. The condition (QR1) in Definition 1 is not redundant. Indeed, any abelian group $A \neq 0$ can be endowed with a multiplication $ab := -b$ which satisfies (QR2) and (QR3), but not (QR1). If (QR1) is dropped from Definition 1, we may speak of a *generalized quasiring*. Then $b = 0$ in (QR3) implies that $a(0c) = -0c$ for all $a, c \in A$. Thus $0(c + 0c) = 0c + 0(0c) = 0$, and each $a \in A$ has a unique representation $a = (a + 0a) + 0(-a)$ where $a + 0a$ belongs to the quasiring $A_0 := \{b \in A \mid 0b = 0\}$. So there is a decomposition $A = A_0 \oplus 0A$ of abelian groups, with a quasiring A_0 and a subgroup $0A$ which satisfies (QR2) and (QR3).

3. Our terminology of Definition 1 is reminiscent of Jacobson's concept of *quasi*regularity, which means invertibility with respect to the *circle operation* (3). A right quasiring whose adjoint monoid is a group has been called a *brace* [53], because it unifies several structures including groups of I-type, regular group actions, and non-degenerate involutive set-theoretic solutions to the Yang–Baxter equation. In particular, *k-braces* over a field k (see [56, Definition 2], [12]) are just right k-quasirings A with $A^\circ = A^\times$. Recently, the term "brace" (over a field) has been used synonymously for generalized right quasirings over a field [11]. As this ring-like notion does not "embrace" any of the structures mentioned above, we reserve the now well-adopted concept of *brace* for quasirings A with $A^\circ = A^\times$.

Example 1. Every abelian group A gives rise to a quasiring A^A, consisting of all maps $f\colon A \to A$, with

$$(f+g)(a) := f(a) + g(a), \qquad (fg)(a) := g(a + f(a)) - g(a).$$

A straightforward calculation shows that the axioms (QR1)–(QR3) are satisfied. The derived operations in A^A are given by

$$(f \circ g)(a) = f(a) + g(a + f(a)), \qquad ({}^f g)(a) = g(a + f(a)).$$

The adjoint group $(A^A)^\times$ consists of the $f \in A^A$ for which $1_A + f$ is a permutation.

Let $\mathrm{Aff}(A)$ be the subgroup of *affine* maps $f \in A^A$, that is,

$$a - b = c - d \implies f(a) - f(b) = f(c) - f(d)$$

for all $a, b, c, d \in A$. Thus, $f \in \mathrm{Aff}(A)$ if and only if $a \mapsto f(a) - f(0)$ is linear. The next proposition shows that $\mathrm{Aff}(A)$ is a ring.

Proposition 2. *Let A be a quasiring. The map $a \mapsto e_a$ with $e_a(b) := {}^b a$ is an injective homomorphism $A \hookrightarrow A^A$ of quasirings with $R(A^A) = \mathrm{Aff}(A)$ and $R(A) = A \cap \mathrm{Aff}(A)$.*

Proof. For $a, b, c \in A$, we have $e_{a+b} = e_a + e_b$ by (QR2). Furthermore, (QR3) implies that $e_{bc}(a) = a(bc) + bc = (ab + a + b)c - ac = (a + e_b(a))c + c - ac - c = e_c(a + e_b(a)) - e_c(a) = (e_b e_c)(a)$. If $e_a = 0$, then $a = {}^0 a = e_a(0) = 0$. Thus $a \mapsto e_a$ is an injective homomorphism. Now $f \in R(A^A)$ if and only if $f(a + g(a) + h(a)) - f(a) = f(a + g(a)) - f(a) + f(a + h(a)) - f(a)$ for all $g, h \in A^A$ and $a \in A$. Thus $\mathrm{Aff}(A) \subset R(A^A)$. Conversely, the special case $a = 0$ with $c := g(0)$ and $d := h(0)$ shows that $f \in R(A^A)$ implies $f(c + d) - f(0) = f(c) - f(0) + f(d) - f(0)$ for all $c, d \in A$. Hence $R(A^A) = \mathrm{Aff}(A)$. The second equation is trivial. ■

Example 2. In particular, every abelian group A can be regarded as a ring with trivial multiplication. So there is an embedding $A = R(A) \hookrightarrow \mathrm{Aff}(A)$ which identifies the elements of A with the ideal of constant maps in A^A. For linear maps $f, g \in \mathrm{Aff}(A)$, we have $(fg)(a) = g(a+f(a))-g(a) = g(f(a))$. Thus $\mathrm{End}(A)^{\mathrm{op}}$ embeds into $\mathrm{Aff}(A)$. Since every affine map splits into a linear and a constant part, $\mathrm{Aff}(A) = A \rtimes \mathrm{End}(A)^{\mathrm{op}}$.

4. Lattice-ordered quasirings. Like classical divisor groups, the groups of divisors on a non-commutative curve X' over a smooth projective curve X are lattice-ordered. This leads to the following

DEFINITION 2. We say that a quasiring A is an *ℓ-quasiring* if its additive group is lattice-ordered with

$$a \circ (b \vee c) = a \circ b \vee a \circ c, \tag{5}$$

$$(a \vee b) \circ c \leq a \circ c \vee b \circ c \tag{6}$$

for all $a, b, c \in A$. If the additive ℓ-group is noetherian, we speak of a *noetherian ℓ-quasiring.* An element c of an ℓ-quasiring A will be called *normal* if it satisfies

$$(a \vee b) \circ c = a \circ c \vee b \circ c$$

for all $a, b \in A$. We write $N(A)$ for the monoid of normal elements, and $N^\times(A) := N(A) \cap A^\times$ for the group of normal units.

For $a, b \in A$ and $\gamma \in N^\times(A)$, we have $(a \circ \gamma' \vee b \circ \gamma') \circ \gamma = a \vee b$. Hence $(a \vee b) \circ \gamma' = a \circ \gamma' \vee b \circ \gamma'$, which shows that $N^\times(A)$ is a group. Note that (6) implies that an element $c \in A$ is normal if and only if it satisfies the implication

$$a \leq b \implies a \circ c \leq b \circ c.$$

Furthermore, Birkhoff's theorem [5, Theorem 37] shows that the additive group of a noetherian ℓ-quasiring is a *cardinal sum* [18] of infinite cyclic groups.

Recall that a set $(X; \cdot)$ with a binary operation $\cdot$ is said to be a *cycle set* [52] if the maps $y \mapsto x \cdot y$ are bijective and the equation

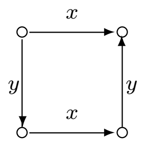
$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

holds in X. Finite cycle sets are *non-degenerate* [52] in the sense that the map $x \mapsto x \cdot x$ is bijective. Every (non-degenerate) cycle set gives rise to a (non-degenerate) set-theoretic solution to the Yang–Baxter equation [52, Propositions 1 and 2]. By [53, Proposition 5], braces are equivalent to cycle sets with an abelian group structure satisfying the equations

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c), \qquad (a + b) \cdot c = (a \cdot b) \cdot (a \cdot c).$$

Conversely, [52, Proposition 6] implies that every non-degenerate cycle set X extends to a brace with additive group $\mathbb{Z}^{(X)}$. The following example gives a brace on $\mathbb{Z}^{(X)}$ which does not come from a subcycle set of cardinality $|X|$.

Example 3. The fundamental group G of the Klein bottle is given by generators x, y with the relation $x = y \circ x \circ y$:



The group G acts on $\mathbb{Z}^2 = \mathbb{Z}x \oplus \mathbb{Z}y$ via

$$^{x}(mx + ny) := mx - ny$$

and $^{y}a = a$. So the group operation (4) is given by the Koszul 1-cocycle

$$(mx + ny) \circ (m'x + n'y) = (m + m')x + \big(n + (-1)^m n'\big)y,$$

which yields

$$mx + ny = y^{\circ n} \circ x^{\circ m}.$$

Thus G^{op} is the adjoint group of a brace A. In particular, A is a cycle set. However, $(A; +) = \mathbb{Z}x \oplus \mathbb{Z}y$ is not generated by a 2-element subcycle set Y of A. Indeed, there must be an element $mx + ny \in Y$ with m odd. Hence $^{mx+ny}(m'x + n'y) = m'x - n'y$ for all $m'x + n'y \in Y$. If $n' \neq 0$, then $Y = \{m'x + n'y, m'x - n'y\}$ does not generate $\mathbb{Z}x \oplus \mathbb{Z}y$. Thus, $n' = 0$, which is impossible.

Nevertheless, the group G is the adjoint group of a brace with additive group $\mathbb{Z}^{(Y)}$ generated by the two-element cycle set Y with $u \cdot v \neq v$ for $u, v \in Y$. (So the connection between G and $\mathbb{Z}^{(Y)}$ cannot be given by the Koszul 1-cocycle.) Indeed, the defining relation of G can be transformed into $x \circ x = z \circ z$ with $z := y \circ x$, so that $Y = \{x, z\}$ extends to a brace with adjoint group G (see [31, Example 1.7], [55, Section 3]).

On the other hand, with the lexicographic linear ordering

$$mx + ny \leq m'x + n'y \iff m < m' \text{ or } (m = m' \text{ and } n \leq n'),$$

G is a quasicyclic right ℓ-group with positive cone $G^+ = \mathbb{N}x \oplus \mathbb{N}y$ (cf. [21, I.3.2]) and strong order unit $2x$. In particular, G is quasinoetherian, but not noetherian, with quasicentre $N(G) = \mathbb{Z}2x \oplus \mathbb{Z}y$ and centre $\mathbb{Z}2x$.

Proposition 3. *Let A be an ℓ-quasiring. Then*

$$a \circ (b \wedge c) = a \circ b \wedge a \circ c \tag{7}$$

for $a, b, c \in A$. The adjoint action respects the lattice structure of A.

Proof. Since $(A; +)$ is an abelian ℓ-group, A satisfies the equation

$$(b \vee c) + (b \wedge c) = b + c. \tag{8}$$

Furthermore, (QR2) and (3) give $a \circ (b + c) = (a \circ b) + (a \circ c) - a$. Together with (5) and (8), this yields (7). By (4), this implies that $a + {}^{a}(b \wedge c) =$

$(a + {}^a b) \wedge (a + {}^a c) = a + ({}^a b \wedge {}^a c)$. Thus ${}^a(b \wedge c) = {}^a b \wedge {}^a c$. Similarly, ${}^a(b \vee c) = {}^a b \vee {}^a c$ follows by (5). ∎

COROLLARY 1. *The monoid $N(A)$ of normal elements of an ℓ-quasiring A is a sublattice.*

Proof. This follows immediately from (5) and (7). ∎

COROLLARY 2. *Let A be an ℓ-quasiring. The partial order of $(N(A); \circ)$ is isolated, that is, $a^{\circ n} \leq 0$ implies $a \leq 0$, and $a^{\circ n} \geq 0$ implies $a \geq 0$, for any integer $n > 0$.*

Proof. For $n = 1$, the assertion is trivial. In what follows, we write a^n instead of $a^{\circ n}$. Assume that $a^{n+1} \leq 0$. Then $a^n \leq (0 \vee a \vee \cdots \vee a^n) \circ a^i$ for $i \in \{0, \ldots, n\}$. Therefore, (7) gives $a^n \leq (0 \vee a \vee \cdots \vee a^n) \circ (0 \wedge a \wedge \cdots \wedge a^n)$. Since $a^i \circ (0 \wedge a \wedge \cdots \wedge a^n) \leq 0$ for $i \in \{0, \ldots, n\}$, Corollary 1 yields $a^n \leq 0$. Thus, by induction, the first implication follows. Similarly, $0 \leq (0 \vee a \vee \cdots \vee a^n) \circ (0 \wedge a \wedge \cdots \wedge a^n) \leq a^n$ if $0 \leq a^{n+1}$, which proves the second implication. ∎

COROLLARY 3. *Let A be an ℓ-quasiring. For any $\alpha \in N^{\times}(A)$,*

$$\alpha \wedge \alpha' \leq 0 \leq \alpha \vee \alpha'.$$

Proof. Eq. (7) gives $(\alpha \wedge \alpha') \circ (\alpha \wedge \alpha') = (\alpha \wedge \alpha') \circ \alpha \wedge (\alpha \wedge \alpha') \circ \alpha' \leq 0$. Hence $\alpha \wedge \alpha' \leq 0$ by Corollary 2. Similarly, $(\alpha \vee \alpha') \circ (\alpha \vee \alpha') = (\alpha \vee \alpha') \circ \alpha \vee (\alpha \vee \alpha') \circ \alpha' \geq 0$, which yields $0 \leq \alpha \vee \alpha'$. ∎

Note that by [18, Theorem 3.16], the lattice order of an ℓ-quasiring is distributive.

COROLLARY 4. *Let A be an ℓ-quasiring. Then $N^{\times}(A)$ is a sublattice, hence an ℓ-group.*

Proof. Let $\alpha, \beta \in N^{\times}(A)$. By Corollary 1, $\alpha \vee \beta, \alpha' \wedge \beta' \in N(A)$. Hence (7) and Corollary 3 give $(\alpha \vee \beta) \circ (\alpha' \wedge \beta') = (\alpha \vee \beta) \circ \alpha' \wedge (\alpha \vee \beta) \circ \beta' = (0 \vee \beta \circ \alpha') \wedge (\alpha \circ \beta' \vee 0) = 0 \vee (\beta \circ \alpha' \wedge \alpha \circ \beta') = 0$. On the other hand, the maps $c \mapsto c \circ \alpha$ and $c \mapsto c \circ \beta$ are order-isomorphisms, hence lattice isomorphisms. Thus $(\alpha' \wedge \beta') \circ (\alpha \vee \beta) = (\alpha' \wedge \beta') \circ \alpha \vee (\alpha' \wedge \beta') \circ \beta = (0 \wedge \beta' \circ \alpha) \vee (\alpha' \circ \beta \wedge 0) = 0 \wedge (\beta' \circ \alpha \vee \alpha' \circ \beta) = 0$. ∎

DEFINITION 3. We say that an ℓ-quasiring A has *enough normal units* if for every $a \in A$ there are normal units $\alpha, \beta \in N^{\times}(A)$ with $\alpha \leq a \leq \beta$.

If A is noetherian and has enough normal units, any pair $a, b \in A$ determines a pair of *residuals*

$$a \to b := \bigvee \{c \in A \mid c \circ a \leq b\},$$

$$a \rightsquigarrow b := \bigvee \{c \in N(A) \mid a \circ c \leq b\}.$$

In fact, let α be a normal unit with $\alpha \le a$. Then $c \circ a \le b$ implies that $c \circ \alpha \le c \circ a \le b$, which gives $c \le b \circ \alpha'$. Similarly, $a \circ c \le b$ with $c \in N(A)$ yields $c \le \alpha' \circ b$. So the existence of the residuals follows by the noetherian property. Since $N(A)$ is closed with respect to joins, $a \rightsquigarrow b$ is normal. In particular, $0 \rightsquigarrow a$ is the greatest normal element $\le a$. For brevity, we set

$$\overline{a} := a \to 0, \qquad \widetilde{a} := a \rightsquigarrow 0.$$

PROPOSITION 4. *Let A be a noetherian ℓ-quasiring with enough normal units. Then*

$$a \le b \to c \iff a \circ b \le c \iff b \le a \rightsquigarrow c$$

for $a, c \in A$ and $b \in N(A)$.

Proof. For $b \in N(A)$ and $c \in A$, the set of $a \in A$ with $a \circ b \le c$ is upward directed. Hence $(b \to c) \circ b \le c$, which yields the first equivalence. Similarly, the normal $b \in A$ with $a \circ b \le c$ form an upward directed set, which proves the second equivalence. ∎

COROLLARY. *Let A be a noetherian ℓ-quasiring with enough normal units. Then*

$$(a \rightsquigarrow a) \to a = a, \tag{9}$$

$$a \le (a \rightsquigarrow b) \to b, \tag{10}$$

$$a \rightsquigarrow b \le (c \circ a) \rightsquigarrow (c \circ b), \tag{11}$$

$$a \le b \implies b \rightsquigarrow c \le a \rightsquigarrow c, \tag{12}$$

$$b \le c \implies a \to b \le a \to c \tag{13}$$

for all $a, b, c \in A$. If $a \in N(A)$, then

$$(a \to a) \rightsquigarrow a = a, \tag{14}$$

$$a \le (a \to b) \rightsquigarrow b, \tag{15}$$

$$(b \circ a) \to c = b \to (a \to c). \tag{16}$$

Proof. Since $a \rightsquigarrow b$ is normal, we have $a \circ (a \rightsquigarrow b) \le b$, which gives (10) and the inequality "$\ge$" in (9). Furthermore, $c \circ a \circ (a \rightsquigarrow b) \le c \circ b$, which yields (11). Assume that $a \le b$. Then $a \circ (b \rightsquigarrow c) \le b \circ (b \rightsquigarrow c) \le c$, whence (12) follows. To prove (13), assume that $b \le c$. Then (6) gives $(a \to b) \circ a \le b \le c$. Hence $a \to b \le a \to c$.

Next, assume $a \in N(A)$. Then $(a \to a) \circ a \le a$, which gives $a \le (a \to a) \rightsquigarrow a$. The inequality (10) yields $0 \le a \to a \le ((a \to a) \rightsquigarrow a) \to a$. Hence $(a \to a) \rightsquigarrow a \le a$, which proves (14). Furthermore, $(a \to b) \circ a \le b$ yields (15). In particular, this implies $0 \le a \rightsquigarrow a \le ((a \rightsquigarrow a) \to a) \rightsquigarrow a$ for all $a \in A$. Hence $(a \rightsquigarrow a) \to a \le a$, which completes the proof of (9). Finally, Proposition 4 yields $(b \circ a) \to c = \bigvee\{d \in A \mid d \circ b \circ a \le c\} = \bigvee\{d \in A \mid d \circ b \le a \to c\} = b \to (a \to c)$, which proves (16). ∎

In what follows, let $X(A)$ denote the set of maximal elements $x < 0$ in A. The elements of $X(A)$ will be called *primes* of A.

Definition 4. Let A be a noetherian ℓ-quasiring with enough normal units. We say that $a \in A$ is *projective* if

$$a \to b = b \circ \overline{a}$$

for all $b \in A$. We call an ℓ-quasiring A *hereditary* if it is noetherian with enough normal units such that $X(A) \subset N(A)$ and every $a \in A$ is projective.

For example, every normal unit $\alpha \in N^{\times}(A)$ is projective. Indeed, the inverse α' coincides with $\overline{\alpha}$, and $\alpha \to b = b \circ \alpha'$ for all $b \in A$.

Proposition 5. *Let A be a noetherian ℓ-quasiring with enough normal units, and let $a \in A$ be projective. Then $b := \overline{a}$ is normal. If $a \in N(A)$, then $\widetilde{b} = a$.*

Proof. By assumption, we have $a \circ b = a \to a$. For $c \leq d$ in A, the implication (13) gives $c \circ \overline{a} = a \to c \leq a \to d = d \circ \overline{a}$. Thus $\overline{a}$ is normal. If a is normal, then (11) and (14) give $\widetilde{b} = b \rightsquigarrow 0 \leq (a \circ b) \rightsquigarrow (a \circ 0) = (a \to a) \rightsquigarrow a = a$. Hence $\widetilde{b} = a$ follows by the inequality (15). ∎

5. Hereditary arithmetics. In this section, we restrict our consideration to divisor groups of smooth projective (non-commutative) curves, which means that our ℓ-quasiring A is now assumed to be hereditary. By Proposition 5, this implies that $\tau(a) := \overline{\overline{a}}$ defines a map

$$\tau\colon N(A) \to N(A). \tag{17}$$

Proposition 6. *Let A be a hereditary ℓ-quasiring. Then $a \mapsto \overline{a}$ is an order-reversing bijection of $N(A)$. The map (17) is an automorphism of $N(A)$ as a lattice and as a monoid. Furthermore, $N^{\times}(A) = \{a \in N(A) \mid \tau(a) = a\}$.*

Proof. By Proposition 5, the map $a \mapsto \overline{a}$ is injective on $N(A)$. For $a \leq b$ in $N(A)$ we have $(b \to 0) \circ a \leq (b \to 0) \circ b \leq 0$. Hence

$$a \leq b \implies \overline{b} \leq \overline{a}. \tag{18}$$

Consider any $\gamma \in N^{\times}(A)$ with $\gamma < 0$. Then $0 < \overline{\gamma} = \gamma'$. So $a \mapsto \overline{a}$ is an order-reversing injection from $[\gamma, \gamma'] \cap N(A)$ to $[\gamma, \gamma'] \cap N(A)$. Since $[\gamma, \gamma']$ is distributive and of finite length, the interval $[\gamma, \gamma']$ is finite. Hence $a \mapsto \overline{a}$ is bijective on $N(A)$. In particular, τ is a lattice automorphism of $N(A)$. By (16), $(a \circ b) \to 0 = a \to (b \to 0) = a \to \overline{b}$. Hence $\overline{a \circ b} = \overline{b} \circ \overline{a}$, which implies that τ is a monoid automorphism.

Now assume that $\tau(a) = a$ for some $a \in N(A)$. Then $a \circ \overline{a} \leq 0 \leq a \to a = a \circ \overline{a}$, which gives $a \circ \overline{a} = 0$. Furthermore, $\overline{a} = \tau(\overline{a})$ implies that $\overline{a} \circ a = \overline{a} \circ \overline{\overline{a}} = 0$, whence $a \in N^{\times}(A)$. The converse is trivial. ∎

The second part of the following corollary resembles [58, Theorem 3].

Corollary 1. *Let A be a hereditary ℓ-quasiring. Then $N^\times(A)$ is a cardinal sum of infinite cyclic groups. The τ-orbits of $X(A)$ are finite, and $X \mapsto \bigwedge X$ gives a bijection between the τ-orbits of $X(A)$ and the maximal elements $\alpha < 0$ in $N^\times(A)$.*

Proof. By Corollary 4 of Proposition 3, $N^\times(A)$ is a noetherian ℓ-group. Hence Birkhoff's theorem [5] implies that $N^\times(A)$ is a cardinal sum of infinite cyclic groups. By Proposition 6, every τ-orbit X of $X(A)$ majorizes a normal unit. Since A is distributive, this implies that X is finite. Hence $\tau(\bigwedge X) = \bigwedge \tau(X) = \bigwedge X$. Thus $\bigwedge X \in N^\times(A)$ is maximal among the normal units $\alpha < 0$. Conversely, every maximal element $\alpha < 0$ in $N^\times(A)$ satisfies $\alpha < x$ for some $x \in X(A)$. Hence $\alpha = \bigwedge X$. ∎

Definition 5. We define a *hereditary arithmetic* [61] to be a lattice-ordered monoid $(H; \circ, 0, \leq, \overline{})$ with neutral element 0 and an anti-automorphism $a \mapsto \overline{a}$, that is, $\overline{a \circ b} = \overline{b} \circ \overline{a}$ and $a \leq b \Leftrightarrow \overline{b} \leq \overline{a}$, such that the following are satisfied:

(H1) $\overline{a} \circ a \leq 0 \leq a \circ \overline{a}$.
(H2) Every bounded ascending sequence becomes stationary.
(H3) For every $a \in H$, there are units $\alpha, \beta \in H^\times$ with $\alpha \leq a \leq \beta$.

Corollary 2. *Let A be a hereditary ℓ-quasiring. Then $N(A)$ is a hereditary arithmetic.*

Proof. That $a \mapsto \overline{a}$ is an anti-automorphism follows by Proposition 6 and (16). For $a \in N(A)$, the inequality $\overline{a} \circ a \leq 0$ follows by Proposition 4. Since a is projective, $0 \leq a \to a = a \circ \overline{a}$. The axioms (H2) and (H3) are trivial to check here. ∎

Remark. Differing from [61], our map (17) coincides with τ^{-1} in [61].

By (5), the anti-automorphism $a \mapsto \overline{a}$ yields a right-hand version of (7):

Corollary 3. *Let A be a hereditary ℓ-quasiring. For $a, b, c \in N(A)$,*

$$(a \wedge b) \circ c = a \circ c \wedge b \circ c.$$

Remark. Note that in a hereditary arithmetic A one can define residuals

$$a \to b := b \circ \overline{a}, \qquad a \rightsquigarrow b := \widetilde{a} \circ b,$$

where $a \mapsto \widetilde{a}$ is the inverse of $a \mapsto \overline{a}$. Then the equivalences

$$a \leq b \to c \iff a \circ b \leq c \iff b \leq a \rightsquigarrow c$$

follow by (H1). Furthermore, every $a \in A$ satisfies $a \circ \overline{a} = a \to a$ and the dual equation

$$\widetilde{a} \circ a = a \rightsquigarrow a.$$

So the concept of Definition 5 coincides with that in [61] (see [61, Propositions 2 and 10]).

EXAMPLE 4. Let R be a fully bounded hereditary noetherian semiprime ring [25]. By Goldie's theorem, R is an order in a semisimple ring Q (see [66]). For example, every classical hereditary order [49] is of that type. The *arithmetic* $A(R)$ consists of the finitely generated R-submodules I of Q with $IR = I$ and $QI = Q$. Thus $A(R)$ is a monoid with neutral element R. Any pair $I, J \in A(R)$ gives rise to left and right quotients

$$(I : J)_\ell := \{a \in Q \mid aJ \subset I\}, \quad (I : J)_r := \{a \in Q \mid Ja \subset I\},$$

which also belong to $A(R)$. In the notation of Proposition 4, we have $(I : J)_\ell = J \to I$ and $(I : J)_r = J \rightsquigarrow I$, and it is easily checked that $A(R)$ is a hereditary arithmetic. Conversely, every hereditary arithmetic is isomorphic to some $A(R)$, where R can be chosen to be a domain (see [65, Theorem 2.2 and Remark (ii)]).

DEFINITION 6. For a family A_i, $i \in I$, of ℓ-quasirings, we define the *cardinal sum* $\boxplus_{i\in I} A_i$ to be the cardinal sum of the additive ℓ-groups together with the componentwise multiplication.

Thus $\boxplus_{i\in I} A_i$ is again an ℓ-quasiring. If the A_i are noetherian, the same holds for $\boxplus_{i\in I} A_i$ since almost all components of an element $a \in \boxplus_{i\in I} A_i$ are zero. The circle multiplication $\circ$ and the residuals $\to$ and $\rightsquigarrow$ in $\boxplus_{i\in I} A_i$ are calculated componentwise. Thus, if the A_i are hereditary, the cardinal sum $\boxplus_{i\in I} A_i$ is hereditary, too.

Let A be a hereditary ℓ-quasiring. In what follows, we write $P(A)$ for the set of maximal elements $\pi < 0$ in $N^\times(A)$. The elements of $P(A)$ will be called *prime units*. For any $\pi \in P(A)$, define

$$A_\pi := \{a \in A \mid \exists n \in \mathbb{N}\colon n\pi \le a \le -n\pi\}.$$

By Corollary 1 of Proposition 6, the prime units correspond to the τ-orbits of $X(A)$. For distinct $\pi, \varrho \in P(A)$, Corollary 4 of Proposition 3 shows that $\pi \circ \varrho = \pi \wedge \varrho = \pi + \varrho$ since $\pi \vee \varrho = 0$. Thus (4) yields ${}^\pi\varrho = \varrho$ for prime units $\varrho \ne \pi$. By Proposition 3, the adjoint action respects the lattice structure of A. So the map $x \mapsto {}^\pi x$ permutes the primes $x \ge \varrho$ with $\varrho \ne \pi$. Consequently, it permutes the primes $x \ge \pi$. Hence ${}^\pi\pi = \pi$, and thus ${}^\pi\varrho = \varrho$ for all $\varrho \in P(A)$. By induction, this gives ${}^\alpha\varrho = \varrho$ for $\alpha \in N^\times(A)$ and $\varrho \in P(A)$, that is, $\alpha \circ \varrho = \alpha + {}^\alpha\varrho = \alpha + \varrho$. By (4), this yields ${}^\alpha(\beta \circ \varrho) = \beta \circ \varrho \Leftrightarrow {}^\alpha(\beta + \varrho) = \beta + \varrho \Leftrightarrow {}^\alpha\beta = \beta$ for all $\beta \in N^\times(A)$. Hence, by induction, ${}^\alpha\beta = \beta$, or equivalently,

$$\alpha \circ \beta = \alpha + \beta,$$

for all $\alpha, \beta \in N^\times(A)$.

Therefore, each A_π is an additive subgroup and a sublattice of A, and a submonoid of A°. Furthermore, A_π is closed with respect to the residuals $\to$ and $\rightsquigarrow$. Thus, with respect to the induced operations, A_π is a hereditary ℓ-quasiring.

THEOREM 1. *Let A be a hereditary ℓ-quasiring. Then $A = \boxplus_{\pi\in P(A)} A_\pi$.*

Proof. For $a \in A$ and $\pi \in P(A)$, we define

$$a_\pi := \bigwedge_{m\in\mathbb{Z}} \bigvee_{n\in\mathbb{Z}} ((a \wedge n\pi) \vee m\pi).$$

Since A is noetherian, the sequence of the $a\wedge n\pi$ becomes stationary for small $n \in \mathbb{Z}$. Hence a_π is well defined and belongs to A_π. So there is an integer $n \in \mathbb{N}$ with $n\pi \le a_\pi \le -n\pi$. By Corollary 1 of Proposition 6, π corresponds to a τ-orbit X of $X(A)$. Thus, if $a = \sum_{x\in X(A)} a(x)x$ with $a(x) \in \mathbb{Z}$, then $a_\pi = \sum_{x\in X} a(x)x$. So there is a normal unit $\beta \in N^\times(A)$ with $\beta \vee \pi = 0$, $\beta + n\pi \le a \le -\beta - n\pi$, and $\beta \le b \le -\beta$ for $b := a - a_\pi$. By induction, it is enough to show that $a = b \circ a_\pi$. By (4), the equation $b + a_\pi = b \circ a_\pi$ is equivalent to $a_\pi = {}^b(a_\pi)$. Since $c \mapsto {}^bc$ is additive, it is enough to verify ${}^bx = x$ for all primes $x \in X$.

Let us assume first that $b \le 0$. Since left or right multiplication by π is a lattice automorphism, the lengths of the intervals $[b, 0]$ and $[b \circ \pi, \pi]$ are equal. Thus $b\circ\pi \le b\wedge\pi$ implies that $b\circ\pi = b\wedge\pi$. By symmetry, this gives $b\circ\pi = b\wedge\pi = \pi\circ b$. Using (6), we obtain $b\wedge x = (\beta\vee\pi)\circ(b\wedge x) \le \beta\circ(b\wedge x)\vee \pi\circ(b\wedge x) \le \beta\circ x\vee\pi\circ b \le b\circ x\vee b\circ\pi = b\circ x \le b\wedge x$. Hence $b\circ x = b\wedge x = b+x$, and thus ${}^bx = x$. For the same reason, ${}^\beta x = x$, or equivalently, ${}^{\beta'}x = x$. Since $b \circ \beta \le 0$, it follows that ${}^bx = {}^{b\circ\beta\circ\beta'}x = {}^{b\circ\beta}x = x$. ∎

DEFINITION 7. We call a hereditary ℓ-quasiring A *local* if the group $N^\times(A)$ is cyclic.

Thus, Theorem 1 reduces the structure theory of hereditary ℓ-quasirings to the local case. The next result determines the circle operation for primes.

PROPOSITION 7. *Let A be a hereditary ℓ-quasiring and $x, y \in X(A)$ with $x \ne y$. Then $x \circ y = x \wedge y$ if and only if $\tau(x) \ne y$, and $x \circ x = x$ if and only if $x \notin P(A)$. Furthermore, $x \circ \tau(x) = x + (x \wedge \tau(x))$.*

Proof. Let X be the τ-orbit of x. By Corollary 1 of Proposition 6, $\pi := \bigwedge X \in P(A)$. If $y \notin X$, then $y \ne \tau(x)$, and Theorem 1 yields $x\circ y = x+y = x \wedge y$. So we can assume that $y \in X$. Assume first that $\tau(x) = y$. Then $n := |X| \ge 2$, and $y \le \overline{x}\to 0$, which gives $x\to y = y\circ\overline{x} \le 0$ by Proposition 4. If $x\circ y = x\wedge y$, then $y\circ x \le x\circ y$, which yields $y \le x\to(x\circ y) = x\circ y\circ\overline{x} \le x$, a contradiction. Hence $x \circ y < x \wedge y$.

Now n is equal to the length $\ell([\pi, 0])$ of the interval $[\pi, 0]$. If $n = 1$, then $x \in P(A)$ and $x \circ x = 2x$. Thus, assume that $n > 1$. The lattice homomorphism $a \mapsto x \circ a$ maps the Boolean lattice $[\pi, 0]$ into $[x \circ \pi, x]$. Hence $\sum_{z \in X} \ell([x \circ z, x]) = \ell([x \circ \pi, x]) = n$. Since $\ell([x \circ z, x]) \geq \ell([x \wedge z, x]) \geq 1$ for all $z \in X \smallsetminus \{x\}$ and $\ell([x \circ \tau(x), x]) \geq 2$, this is only possible if $x \circ x = x$ and $\ell([x \circ \tau(x), x]) = 2$, and $x \circ y = x \wedge y$ for all $y \in X \smallsetminus \{x, \tau(x)\}$. This proves the first assertion.

It remains to verify the equation $x \circ \tau(x) = x + (x \wedge \tau(x))$. For $\tau(x) = x$, this is obvious. Thus, assume that $\tau(x) \neq x$. Then

$$x + \pi = x \circ \pi = x \circ \bigwedge X = \bigwedge \{x \circ y \mid \tau(x) \neq y \in X\} \wedge (x \circ \tau(x)).$$

Hence $x \circ \tau(x) \not\geq \pi$, and thus $x \circ \tau(x) = x + (x \wedge \tau(x))$. ∎

In view of (3), Proposition 7 yields

Corollary 1. *Let A be a hereditary ℓ-quasiring. For $x \in X(A) \smallsetminus P(A)$ and $y \in X(A)$, we have*

$$xy = \begin{cases} 0 & \text{for } y \notin \{x, \tau(x)\}, \\ -x & \text{for } y = x, \\ x & \text{for } y = \tau(x). \end{cases}$$

In particular, we have the following characterization of normal units.

Corollary 2. *Let A be a hereditary ℓ-quasiring. For an element $\beta \in A$, the following are equivalent:*

(a) *β is a normal unit.*
(b) *${}^a\beta = \beta$ for all $a \in A$.*
(c) *$A\beta = 0$.*
(d) *$a \circ \beta = a + \beta$ for all $a \in A$.*
(e) *$\beta \in N(A)$, and $x \circ \beta = x + \beta$ for all $x \in X(A)$.*

Proof. By (3) and (4), it follows that (b), (c), and (d) are equivalent. The implication (d)⇒(e) is trivial.

(a)⇒(c). By (QR2), it is enough to verify $A\pi = 0$ for $\pi \in P(A)$. Suppose that $a\pi \neq 0$ for some $a \in A$. By Theorem 1, we can assume that $a \in A_\pi$ and $\pi \notin X(A)$. Since ${}^a\pi \neq \pi$ and $\ell([{}^a\pi, 0]) = \ell([a \circ \pi, a]) = |X(A_\pi)|$, there must be a prime $y \in X(A_\pi)$ with $y \not\geq {}^a\pi$ and $\tau(y) \geq {}^a\pi$. With ${}^a\pi = \sum_{x \in X(A)} n_x x$, Corollary 1 implies that $y({}^a\pi) = n_{\tau(y)} y < 0$. Hence ${}^{y \circ a}\pi = {}^y({}^a\pi) < {}^a\pi$, and thus $|X(A_\pi)| = \ell([{}^{y \circ a}\pi, 0]) > \ell([{}^a\pi, 0]) = |X(A_\pi)|$, a contradiction.

(e)⇒(a). Suppose that $0 < \beta \rightarrow \beta$. By Corollary 2 of Proposition 6, there is a prime $x \in X(A)$ with $(\beta \rightarrow \beta) \rightsquigarrow 0 \leq x$. Hence $\overline{x} \leq \beta \rightarrow \beta = \beta \circ \overline{\beta}$, which implies that $0 \leq x \circ \overline{x} \leq x \circ \beta \circ \overline{\beta} = (x + \beta) \circ \overline{\beta} = \beta \rightarrow (x + \beta)$. Thus $\beta \leq x + \beta$, a contradiction. So we have shown that $\beta \circ \overline{\beta} = \beta \rightarrow \beta = 0$. Hence $\beta \leq \overline{\beta} \rightarrow 0 = \tau(\beta)$. Since τ respects the length of intervals, we infer that $\tau(\beta) = \beta$. By Proposition 6, this yields $\beta \in N^\times(A)$. ∎

Corollary 3. *Let A be a hereditary ℓ-quasiring. Then $\beta \circ a \circ \beta' = {}^{\beta}a$ for $a \in A$ and $\beta \in N^{\times}(A)$. If $x \in X(A) \smallsetminus P(A)$ and $\pi \in P(A)$ with $\pi \le x$, then $\pi' \circ x \circ \pi = \tau(x)$.*

Proof. By Corollary 2, we have $a \circ \beta' = a + \beta'$. Since $\beta + {}^{\beta}\beta' = \beta \circ \beta' = 0$, this gives $\beta \circ a \circ \beta' = \beta + {}^{\beta}(a + \beta') = {}^{\beta}a$. Now assume that $x \in X(A) \smallsetminus P(A)$ and $x \ge \pi \in P(A)$. Then $\pi \circ \tau(x) \le x \circ \tau(x) \wedge \pi = (2x + \tau(x)) \wedge \pi = x + \pi$. Thus Corollary 2 yields $\pi \circ \tau(x) \le x \circ \pi$. Since $\ell([x \circ \pi, 0]) = \ell([\pi \circ \tau(x), 0])$, we obtain $\pi \circ \tau(x) = x \circ \pi$. ∎

Corollary 4. *Let A be a hereditary ℓ-quasiring. For $a \in N(A)$, the following are equivalent:*

(a) $a \notin N^{\times}(A)$.
(b) *There is a prime $x \in X(A)$ with $x \circ a = a$.*

Proof. (a)⇒(b). By Corollary 2, there is a prime x with $x \circ a \ne x + a$. Thus $x \circ a < x + a$. For $\pi \in P(A)$ with $\pi \le x$, Corollary 3 of Proposition 6 implies that $b \mapsto b \circ a$ is a lattice homomorphism from the Boolean lattice $[\pi, 0]$ to $[\pi \circ a, a]$. Hence $\pi \circ a = \bigwedge_{y \in X(A_\pi)} (y \circ a)$, and the pigeonhole principle implies that $x \circ a = a$ for some $x \in X(A)$. The converse (b)⇒(a) is trivial. ∎

6. The functional representation. In [61] we associate a *divisor group* $\mathrm{Div}(H)$ to any hereditary arithmetic H. As an additive group, $\mathrm{Div}(H)$ can be identified with the free abelian group generated by the set $\mathrm{Max}(H)$ of maximal elements $x < 0$ in H. (In [61], the unit element 0 is denoted by u.) By [61, Proposition 16], any interval $[b, a]$ of length 1 in H admits a unique $p \in \mathrm{Max}(H)$ with $p \circ a \le b$. Writing $p + a := b$ in this case, we can define a partial addition in H recursively, which leads to a unique embedding $H \hookrightarrow \mathrm{Div}(H)$. Interpreting $X := \mathrm{Max}(H)$ as a non-commutative curve, we observe that the divisors $a \in \mathrm{Div}(H)$ are finite $\mathbb{Z}$-linear combinations of points:

$$a = \sum_{x \in X} a(x)x. \tag{19}$$

By [61, Theorem 2], the multiplication $\circ$ in H can be extended to $\mathrm{Div}(H)$ in such a way that $(\mathrm{Div}(H); \circ)$ is a monoid with unit element 0, and there is a unique action $(a, b) \mapsto {}^{a}b$ of $(\mathrm{Div}(H); \circ)$ on $(\mathrm{Div}(H); +)$ such that the identity map $1 \colon (\mathrm{Div}(H); \circ) \to (\mathrm{Div}(H); +)$ is a bijective 1-cocycle:

$$a \circ b = a + {}^{a}b.$$

Taking into account that our τ is τ^{-1} in [61], we see that the action is given by

$${}^{a}b = \sum_{x \in X} b(\tau^{a(x)}(x))x. \tag{20}$$

THEOREM 2. *Let H be a hereditary arithmetic. With the action* (20)*, the divisor group $A := \mathrm{Div}(H)$ is a hereditary ℓ-quasiring such that $H = N(A)$.*

Proof. By Proposition 1 and [61, Theorem 2], A is a quasiring. Equation (20) shows that

$$^{a}(b \vee c) = {}^{a}b \vee {}^{a}c$$

in A. Since $(A; +)$ is a lattice-ordered group, this yields (5). By (4) and (20), we have

$$(a \circ b)(x) = a(x) + b(\tau^{a(x)}(x)) \tag{21}$$

for all $x \in X := \mathrm{Max}(H)$. Hence, if $a, b, c \in A$ and $a(x) \geq b(x)$, then $((a \vee b) \circ c)(x) = b(x) + c(\tau^{b(x)}(x)) \geq (a(x) + c(\tau^{a(x)}(x)) \wedge (b(x) + c(\tau^{b(x)}(x)) = ((a \circ c) \vee (b \circ c))(x)$, which gives the inequality (6). Thus A is a noetherian ℓ-quasiring. To show that A has enough normal units, we apply [61, Theorem 3], the analogue of Theorem 1, for hereditary arithmetics. So we can assume that H has a unique maximal invertible element $\pi < 0$. By [61, Proposition 13], the interval $[\pi, 0]$ is a Boolean lattice. Hence $\pi(x) = 1$ for all $x \in X$, and (21) yields $\pi^{\circ n}(x) = n$ for all $x \in X$. Thus, for each $a \in A$ there is an integer $n \in \mathbb{Z}$ with $\pi^n \leq a \leq \pi^{-n}$.

Next we show that each $y \in X$ is normal. Thus, let $a, b \in A$ with $a \leq b$ and $x \in X$ be given. Assume first that $y = \tau^{a(x)}(x)$. Then (21) gives $(a \circ y)(x) = a(x) + 1 \geq b(x) + 1 = (b \circ y)(x)$. Now assume that $y \neq \tau^{a(x)}(x)$. If $a(x) = b(x)$, then $(a \circ y)(x) = a(x) = b(x) = (b \circ y)(x)$. Otherwise, $a(x) > b(x)$ and $(a \circ y)(x) = a(x) \geq b(x) + 1 \geq (b \circ y)(x)$, whence $y \in N(A)$.

Now we show that each $a \in A$, represented by (19), is projective. For $a, b \in A$ and $x \in X$, we have $(a \to b)(x) = \bigwedge\{n \in \mathbb{Z} \mid n + a(\tau^n(x)) \geq b(x)\}$. Hence

$$\begin{aligned}(b \circ \overline{a})(x) &= b(x) + \bigwedge\{n \in \mathbb{Z} \mid n + a(\tau^{n+b(x)}(x)) \geq 0\} \\ &= \bigwedge\{b(x) + n \mid n + a(\tau^{n+b(x)}(x)) \geq 0\} \\ &= \bigwedge\{n \in \mathbb{Z} \mid n + a(\tau^n(x)) \geq b(x)\} = (a \to b)(x).\end{aligned}$$

This proves that A is a hereditary ℓ-quasiring.

It remains to verify that $H = N(A)$. By Theorem 1 and [61, Theorem 3], we can assume that A is local. So there is a unique prime unit $\pi \in N^{\times}(A)$. By [61, Proposition 21], an element $c \in A$ belongs to H if and only if it satisfies

$$c(x) \leq c(\tau(x)) + 1 \tag{22}$$

for all $x \in X$. On the other hand, c is normal in A if and only if $a \circ c \leq b \circ c$ for all $a, b \in A$ with $a \leq b$. By (21), this means that

$$a(x) + c(\tau^{a(x)}(x)) \geq b(x) + c(\tau^{b(x)}(x))$$

for all $x \in X$. With $a(x) = n$ and $b(x) = n-1$, this yields $n + c(\tau^n(x)) \geq n - 1 + c(\tau^{n-1}(x))$, which is equivalent to (22). Hence $N(A) = H$. ∎

By Theorem 2, any hereditary arithmetic comes from a hereditary ℓ-quasiring. On the other hand, Corollary 2 of Proposition 6 shows that each hereditary ℓ-quasiring A gives rise to a hereditary arithmetic $N(A)$. We do not know whether this correspondence is bijective, that is, whether there exists an "exotic" hereditary ℓ-quasiring A which is not isomorphic to $\mathrm{Div}(N(A))$.

As a first application, we get the following normality criterion.

COROLLARY 1. *Let H be a hereditary arithmetic. For a divisor $c \in \mathrm{Div}(H)$, the following are equivalent:*

(a) $c \in H$.
(b) $1 + c(\tau(x)) \geq c(x)$ *for all* $x \in \mathrm{Max}(H)$.
(c) $x \circ c \leq c$ *for all* $x \in \mathrm{Max}(H)$.

Proof. (a)⇔(b) follows by [61, Proposition 21].

(b)⇔(c). By (21), $(x \circ c)(y) = x(y) + c(\tau^{x(y)}(y))$ for $y \in \mathrm{Max}(H)$. So the inequality $(x \circ c)(y) \geq c(y)$ holds for $y \neq x$. For $y = x$, it reduces to (b). ∎

COROLLARY 2. *Let H be a hereditary arithmetic. The quasicentre of $A := \mathrm{Div}(H)$ is $R(A) = N^{\times}(A)$.*

Proof. Assume that $\beta \in R(A)$. Then $(2x)\beta - x\beta = x\beta$ for all $x \in X(A)$. Hence ${}^{2x}\beta - {}^{x}\beta = {}^{x}\beta - \beta$, and thus by (20),

$$\sum_{y \in X(A)} \big(\beta(\tau^{(2x)(y)}(y)) - \beta(\tau^{x(y)}(y))\big)y = \sum_{y \in X(A)} \big(\beta(\tau^{x(y)}(y)) - \beta(y)\big)y.$$

So we obtain $\beta(\tau^2(x)) - \beta(\tau(x)) = \beta(\tau(x)) - \beta(x)$. By induction, this shows that $\beta(x)$ is constant on each τ-orbit of $X(A)$. Thus $\beta \in N^{\times}(A)$. The converse follows by Corollary 2(c) of Proposition 7. ∎

COROLLARY 3. *Let A be a local hereditary ℓ-quasiring with $P(A) = \{\pi\}$. Then*

$$\tau(a) = \pi' \circ a \circ \pi$$

for all $a \in N(A)$.

Proof. By [61, Theorem 6], $a = \pi^m \circ x_1 \circ \cdots \circ x_n$ with $x_1, \ldots, x_n \in X(A)$ and some $n \in \mathbb{Z}$. So the corollary follows by Proposition 6 and Corollary 3 of Proposition 7. ∎

Theorem 2 enables us to extend the functional description of hereditary arithmetics [61, Section 8] to the divisor groups. To this end, we define a *τ-set* to be a set X with a bijection $\tau\colon X \to X$. A *morphism* between τ-sets is a map $f\colon X \to Y$ which satisfies $f(\tau x) = \tau f(x)$ for all $x \in X$. The disjoint union $\bigsqcup_{i \in I} X_i$ of τ-sets X_i is a coproduct in the category $\mathbf{Set}_\tau$ of τ-sets. Thus any τ-set X is a coproduct of indecomposable τ-sets, the orbits under τ. A morphism $f\colon X \to Y$ is epic if and only if it is surjective. Projective objects in $\mathbf{Set}_\tau$ are the τ-sets with infinite orbits. We endow them with a partial order:

$$x \le y :\Longleftrightarrow \exists n \in \mathbb{N}\colon y = \tau^n(x).$$

Up to isomorphism, any τ-set X admits a unique epimorphism $p\colon \widetilde{X} \twoheadrightarrow X$ with $\widetilde{X}$ projective. We call p the *universal cover* of X.

Now let H be a hereditary arithmetic. Then $X := \mathrm{Max}(H)$ is a τ-set with finite orbits. Consider the universal cover $p\colon \widetilde{X} \twoheadrightarrow X$. Regarding X as a non-commutative curve, we can view a divisor $a \in \mathrm{Div}(H)$ as a function $a\colon X \to \mathbb{Z}$ with finite support. Thus a defines a self-map $\widehat{a}\colon \widetilde{X} \to \widetilde{X}$ with $\widehat{a}(x) := \tau^{a(p(x))}(x)$ for $x \in \widetilde{X}$. Let $\tau^a\colon X \to X$ be the map given by $\tau^a(x) := \tau^{a(x)}(x)$. Then we have a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \widehat{a}\ } & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\ \tau^a\ } & X \end{array}$$

and (20) can be written as ${}^a b = b\tau^a$. In other words, the action $b \mapsto {}^a b$ coincides with the dual map

$$(\tau^a)^*\colon\ \mathrm{Div}(H) \to \mathrm{Div}(H).$$

Let $\varPhi(X)$ denote the set of all maps $f\colon \widetilde{X} \to \widetilde{X}$ which leave the τ-orbits invariant and factor through p. With respect to composition, $\varPhi(X)$ is a monoid $(\varPhi(X); \circ)$.

Extending [61, Theorem 5] to divisors, the next result gives a complete description of the monoid $\mathrm{Div}(H)$ in terms of composition of functions on the τ-set $\widetilde{X}$. Moreover, it gives a new interpretation of normal divisors.

Theorem 3. *Let H be a hereditary arithmetic with $X := \mathrm{Max}(H)$. The map $a \mapsto \widehat{a}$ is a bijection*

$$\mathrm{Div}(H) \xrightarrow{\sim} \varPhi(X)$$

which satisfies $\widehat{a \circ b} = \widehat{b} \circ \widehat{a}$ and $a \le b \Leftrightarrow \widehat{b} \le \widehat{a}$ for all $a, b \in \mathrm{Div}(H)$. Furthermore, $a \in H$ if and only if $\widehat{a}$ is monotone.

Proof. The bijectivity of the map $a \mapsto \widehat{a}$ follows by the representation (19). Indeed, every $f \in \varPhi(X)$ is of the form $f = gp$ with $g\colon X \to \widetilde{X}$. Thus, for each $x \in \widetilde{X}$, there is a unique $a(p(x)) \in \mathbb{Z}$ with $f(x) = \tau^{a(p(x))}(x)$, which means that $f = \widehat{a}$.

For $a, b \in \mathrm{Div}(H)$ and $x \in \widetilde{X}$, we have $\widehat{a}(x) = \tau^{a(p(x))}(x)$. Therefore, (21) gives

$$\begin{aligned}\widehat{b}(\widehat{a}(x)) &= \tau^{b(p(\widehat{a}(x)))}\tau^{a(p(x))}(x) = \tau^{b(\tau^{a}(p(x)))+a(p(x))}(x)\\ &= \tau^{a(p(x))+b(\tau^{a(p(x))}(p(x)))}(x) = \tau^{(a\circ b)(p(x))}(x) = \widehat{a\circ b}(x),\end{aligned}$$

which proves $\widehat{a\circ b} = \widehat{b}\circ\widehat{a}$. Furthermore, we have

$$\begin{aligned}a \le b &\iff \forall x\in X\colon a(x) \ge b(x)\\ &\iff \forall x\in \widetilde{X}\colon \tau^{a(p(x))}(x) \ge \tau^{b(p(x))}(x) \iff \widehat{a}\ge\widehat{b}.\end{aligned}$$

Finally, Corollary 1 of Theorem 2 yields

$$\begin{aligned}a\in H &\iff \forall x\in X\colon 1+a(\tau(x)) \ge a(x)\\ &\iff \forall x\in\widetilde{X}\colon \tau^{a(\tau(p(x)))+1}(x) \ge \tau^{a(p(x))}(x)\\ &\iff \forall x\in\widetilde{X}\colon \tau^{a(p(\tau(x)))}(\tau(x)) \ge \tau^{a(p(x))}(x)\\ &\iff \forall x\in\widetilde{X}\colon \widehat{a}(\tau(x)) \ge \widehat{a}(x),\end{aligned}$$

which proves that H consists of those divisors $a\in \mathrm{Div}(H)$ for which $\widehat{a}$ is monotone. ∎

In recent investigations on Euclidean Artin groups [46, 47], the *annular symmetric group* $\mathrm{Mid}(B_n)$, a factor group of the spherical Artin group $A(B_n)$ of type $\mathbb{B}_n$, has played an important part. Since $\mathrm{Mid}(B_n)$ has the Coxeter group $C(B_n)$ of type $\mathbb{B}_n$ as a factor group, the group $\mathrm{Mid}(B_n)$ has also been called the "middle group" [47]. The *raison d'être* for this group comes from the untypical embedding $A(\widetilde{\mathbb{A}}_{n-1}) \hookrightarrow A(B_n)$ of the Euclidean braid group into the spherical Artin group $A(B_n)$ which has no counterpart on the level of Coxeter groups. Instead of the Coxeter group of type $\mathbb{B}_n$, the middle group fits into a commutative diagram

$$\begin{array}{ccccc} A(\widetilde{\mathbb{A}}_{n-1}) & \hookrightarrow & A(B_n) & \twoheadrightarrow & \mathbb{Z}\\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \| \\ C(\widetilde{\mathbb{A}}_{n-1}) & \hookrightarrow & \mathrm{Mid}(B_n) & \twoheadrightarrow & \mathbb{Z}\end{array}$$

with short exact rows, which determines $\mathrm{Mid}(B_n)$ as a pushout of groups. In our context, the middle groups occurs as a unit group.

COROLLARY. *Let H be a hereditary arithmetic, and let the ℓ-quasiring $A := \mathrm{Div}(H)$ be local with $n := |X(A)|$. Then the unit group $A^{\times}$ is isomorphic to* $\mathrm{Mid}(B_n)$.

Proof. The annular braid group $A(B_n)$ is given by $n+1$ generators $\sigma_1,\dots,\sigma_n,\tau$ with relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i\ne j\pm 1$ and $i,j\in\mathbb{Z}/n\mathbb{Z}$, together with $\tau\sigma_i\tau^{-1} = \sigma_{i+1}$ for all i. So the σ_i generate the Euclidean braid group $A(\widetilde{\mathbb{A}}_{n-1})$, and $A(B_n)\cong A(\widetilde{\mathbb{A}}_{n-1})\rtimes\langle\tau\rangle$

(see [38]). The middle group $\mathrm{Mid}(B_n)$ is obtained by adding the relations $\sigma_i^2 = 1$ to those of the Artin group $A(B_n)$. By Lusztig's representation ([45, 3.6], [62, 4.1(ii)]), the Coxeter group $C(\widetilde{\mathbb{A}}_{n-1})$ can be identified with the group of n-periodic permutations w of $\mathbb{Z}$ with $\sum_{i=1}^{n}(w(i) - i) = 0$. If τ denotes the shift map $i \mapsto i + 1$, then every n-periodic permutation of $\mathbb{Z}$ is of the form $w\tau^k$ with $w \in C(\widetilde{\mathbb{A}}_{n-1})$ and a unique $k \in \mathbb{Z}$. Moreover, $\tau\sigma_i\tau^{-1} = \sigma_{i+1}$ for the transpositions $\sigma_i = (i\ i+1)$. So the middle group $\mathrm{Mid}(B_n)$ is isomorphic to the group of all n-periodic permutations of $\mathbb{Z}$.

On the other hand, Theorem 3 identifies $A^{\mathrm{op}} = \Phi(X)$ with the monoid of n-periodic maps $a\colon \mathbb{Z} \to \mathbb{Z}$, that is, $a(i + n) = a(i) + n$ for $i \in \mathbb{Z}$. Thus $A^\times \cong (A^{\mathrm{op}})^\times \cong \mathrm{Mid}(B_n)$. ∎

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