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ALMOST SPLIT TRIANGLES IN PRE-EXTRIANGULATED CATEGORIES

BY

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Abstract. We consider the notion of pre-extriangulated categories. We give some characterizations of almost split $\mathfrak{s}$-triangles in a pre-extriangulated category and provide necessary and sufficient conditions for the existence of such triangles. If $\mathcal{E}$ is an extension-closed subcategory of a pre-extriangulated category $\mathfrak{C}$ having almost split $\mathfrak{s}$-triangles, we obtain necessary and sufficient conditions for $\mathcal{E}$ to have almost split $\mathfrak{s}$-triangles using the approximation properties of a certain class of objects.

1. Introduction. Exact categories and triangulated categories are fundamental structures in different branches of mathematics. As expected, exact categories and triangulated categories are not independent of each other. For example, a triangulated category which is at the same time abelian must be semisimple [22]. Also, there are a series of ways to produce triangulated categories from abelian ones, such as taking the stable categories of Frobenius exact categories [8], or taking the homotopy categories or derived categories of complexes over abelian categories [22].

On the other hand, because of the recent development of the cluster theory, it becomes possible to produce abelian categories from triangulated ones: starting from a cluster category and taking a cluster tilting subcategory, one can get a suitable quotient category, which turns out to be abelian; see e.g. [15]. In [23], Nakaoka and Palu introduced the notion of extriangulated categories as a simultaneous generalization of exact categories and extension-closed subcategories of triangulated categories. The study of extriangulated categories has then become an active topic, and up to now, many results on exact categories and triangulated categories have been realized in the setting of extriangulated categories by J. Hu, Y. Liu, H. Nakaoka, Y. Palu, P. Zhou, B. Zhu and others; [5, 10–13, 18–21, 23–26].

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Auslander–Reiten theory, initiated in [2, 3], plays a crucial role in the representation theory of algebras and related topics, especially in understanding the structure of module categories of finite-dimensional algebras [1, 4] and the structure of exact and triangulated categories [14, 16, 17]. In particular, the existence of almost split sequences or triangles is closely related to this theory which has been investigated by many authors; for example, Lenzing and Zuazua [16] showed that the existence of Auslander–Reiten duality is equivalent to the existence of almost split sequences in Ext-finite Krull–Schmidt abelian categories, and Liu et al. [17] characterized almost split sequences in terms of linear forms in arbitrary exact categories. As a simultaneous generalization and enhancement of Auslander–Reiten theory in exact categories and triangulated categories, recently, Iyama, Nakaoka and Palu [13] investigated Auslander–Reiten theory in extriangulated categories. They gave two different sets of sufficient conditions in a k-linear Ext-finite Krull–Schmidt extriangulated category with k a field so that the existence of almost split extensions is equivalent to an Auslander–Reiten–Serre duality. In this paper, we will study the problem mentioned above in the most general setup, that is, we first introduce the notion of pre-extriangulated categories, and then investigate the existence of almost split $\mathfrak{s}$-triangles in arbitrary pre-extriangulated categories. This work generalizes some results of [13, 16, 17].

The paper is organized as follows. In Section 2, we first recall the notion of extriangulated category given by Nakaoka and Palu, and then define a notion of pre-extriangulated category by weakening the conditions. We also recall the notions of almost split extensions and $\mathfrak{s}$-triangles. In Section 3, we characterize almost split $\mathfrak{s}$-triangles in terms of linear forms on the stable endomorphism algebras of its ending terms in arbitrary pre-extriangulated categories. We obtain necessary and sufficient conditions for the existence of almost split $\mathfrak{s}$-triangles (Theorem 3.5). In Section 4, we investigate the relation between almost split $\mathfrak{s}$-triangles of a pre-extriangulated category and of its extension-closed full subcategories. We find that if the ambient category has almost split $\mathfrak{s}$-triangles, then the almost split $\mathfrak{s}$-triangles in an Ext-finite Krull–Schmidt extension-closed subcategory are precisely the minimal projectively or injectively stable approximations of almost split $\mathfrak{s}$-triangles in the ambient category (Theorem 4.7).

2. Preliminaries. Throughout, $\mathfrak{C}$ is an additive category and $\mathbb{E} : \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathfrak{Ab}$ is a biadditive functor, where $\mathfrak{Ab}$ is the category of abelian groups.

2.1. Extriangulated categories

Definition 2.1 ([23, Definitions 2.1 and 2.5]). For any $A, C \in \mathfrak{C}$, we have the abelian group $\mathbb{E}(C, A)$.

(1) An element $\delta \in \mathbb{E}(C,A)$ is called an $\mathbb{E}$-*extension*. More formally, an $\mathbb{E}$-extension is a triple (A,δ,C).
(2) The zero element 0 in $\mathbb{E}(C,A)$ is called the *split* $\mathbb{E}$-*extension*.

Let $a \in \mathfrak{C}(A,A')$ and $c \in \mathfrak{C}(C',C)$. Then we have the commutative diagram

$$\begin{array}{ccc} \mathbb{E}(C,A) & \xrightarrow{\mathbb{E}(C,a)} & \mathbb{E}(C,A') \\ {\scriptstyle \mathbb{E}(c,A)}\downarrow & \searrow^{\mathbb{E}(c,a)} & \downarrow{\scriptstyle \mathbb{E}(c,A')} \\ \mathbb{E}(C',A) & \xrightarrow{\mathbb{E}(C',a)} & \mathbb{E}(C',A') \end{array}$$

in $\mathfrak{Ab}$. For an $\mathbb{E}$-extension (A,δ,C), we briefly write $a_\star\delta := \mathbb{E}(C,a)(\delta)$ and $c^\star\delta := \mathbb{E}(c,A)(\delta)$. Then

$$\mathbb{E}(c,a)(\delta) = c^\star a_\star\delta = a_\star c^\star\delta.$$

Definition 2.2 ([23, Definition 2.3]). Given two $\mathbb{E}$-extensions (A,δ,C) and (A',δ',C'), a *morphism* from δ to δ' is a pair (a,c) of morphisms, where $a \in \mathfrak{C}(A,A')$ and $c \in \mathfrak{C}(C,C')$, such that $a_\star\delta = c^\star\delta'$. In this case, we write $(a,c) : \delta \to \delta'$.

Now let $A, C \in \mathfrak{C}$. Two sequences of morphisms

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A \xrightarrow{x'} B' \xrightarrow{y'} C$$

are said to be *equivalent* if there exists an isomorphism $b \in \mathfrak{C}(B,B')$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & \cong\downarrow b & & \| \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \end{array}$$

We denote by $[A \xrightarrow{x} B \xrightarrow{y} C]$ the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$. In particular, we write $0 := [A \xrightarrow{\binom{1}{0}} A \oplus C \xrightarrow{(0\ 1)} C]$.

Note that, for any pair $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$, since $\mathbb{E}$ is biadditive, there exists a natural isomorphism

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C,A) \oplus \mathbb{E}(C,A') \oplus \mathbb{E}(C',A) \oplus \mathbb{E}(C',A').$$

We define $\delta \oplus \delta'$ to be the element in $\mathbb{E}(C \oplus C', A \oplus A')$ corresponding to the element $(\delta, 0, 0, \delta')$ in $\mathbb{E}(C,A) \oplus \mathbb{E}(C,A') \oplus \mathbb{E}(C',A) \oplus \mathbb{E}(C',A')$ through the above isomorphism.

Definition 2.3 ([23, Definition 2.9]). Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to each $\mathbb{E}$-extension $\delta \in \mathbb{E}(C,A)$. The $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ provided that it satisfies the following condition:

(R) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C] \quad \text{and} \quad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

Then for any morphism $(a, c) : \delta \to \delta'$, there exists $b \in \mathfrak{C}(B, B')$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

Let $\mathfrak{s}$ be a realization of $\mathbb{E}$. If $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ for some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$, then we say that the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ *realizes* δ; and in condition (R), we say that the triple (a, b, c) *realizes* the morphism (a, c).

For any two equivalence classes $[A \xrightarrow{x} B \xrightarrow{y} C]$ and $[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$, we define

$$[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] := [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'].$$

Definition 2.4 ([23, Definition 2.10]). A realization $\mathfrak{s}$ of $\mathbb{E}$ is called *additive* if it satisfies the following conditions:

(1) For any $A, C \in \mathfrak{C}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C, A)$ satisfies $\mathfrak{s}(0) = 0$.
(2) For any pair of $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$, we have $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.

Definition 2.5 ([23, Definition 2.12]). Let $\mathfrak{C}$ be an additive category. The triple $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ is called an *externally triangulated* (or *extriangulated* for short) category if it satisfies the following conditions:

(ET1) $\mathbb{E} : \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathfrak{Ab}$ is a biadditive functor.
(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.
(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C] \quad \text{and} \quad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

For any commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

in $\mathfrak{C}$, there exists a morphism $(a, c) : \delta \to \delta'$ which is realized by the triple (a, b, c).
(ET3)$^{\mathrm{op}}$ Dual of (ET3).

(ET4) Let $\delta \in \mathbb{E}(C, A)$ and $\rho \in \mathbb{E}(F, B)$ be any pair of $\mathbb{E}$-extensions with

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C] \quad \text{and} \quad \mathfrak{s}(\rho) = [B \xrightarrow{u} D \xrightarrow{v} F].$$

Then there exist an object $E \in \mathfrak{C}$, an $\mathbb{E}$-extension ξ with $\mathfrak{s}(\xi) = [A \xrightarrow{z} D \xrightarrow{w} E]$, and a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle s} \\ A & \xrightarrow{z} & D & \xrightarrow{w} & E \\ & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle t} \\ & & F & = & F \end{array}$$

in $\mathfrak{C}$, which satisfy the following compatibilities:

(i) $\mathfrak{s}(y_\star \rho) = [C \xrightarrow{s} E \xrightarrow{t} F]$.
(ii) $s^\star \xi = \delta$.
(iii) $x_\star \xi = t^\star \rho$.

(ET4)$^{\mathrm{op}}$ Dual of (ET4).

For examples of extriangulated categories, see [23, Example 2.13], [10, Remark 3.3], and [24, Corollary 4.12 and Remark 4.13].

Definition 2.6. We call $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ (or $\mathfrak{C}$ when there is no confusion) a *pre-extriangulated* category if it satisfies (ET1), (ET2), (ET3) and (ET3)$^{\mathrm{op}}$.

Here is a natural way to produce pre-extriangulated categories. Let $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and let $\mathbb{F}$ be an additive subfunctor of $\mathbb{E}$. Then for any $a \in \mathrm{Hom}_{\mathfrak{C}}(A, A')$, $c \in \mathrm{Hom}_{\mathfrak{C}}(C', C)$, and $\delta \in \mathbb{F}(C, A)$, we have $a_\star \delta \in \mathbb{F}(C, A')$ and $c^\star \delta \in \mathbb{F}(C', A)$. Define $\mathfrak{s}_{\mathbb{F}}$ to be the restriction of $\mathfrak{s}$ to $\mathbb{F}$, that is, $\mathfrak{s}_{\mathbb{F}}(\delta) = \mathfrak{s}(\delta)$ for any $\mathbb{F}$-extension δ. By [9, Claim 3.8], the triplet $(\mathfrak{C}, \mathbb{F}, \mathfrak{s}_{\mathbb{F}})$ satisfies (ET1), (ET2), (ET3) and (ET3)$^{\mathrm{op}}$, that is, $(\mathfrak{C}, \mathbb{F}, \mathfrak{s}_{\mathbb{F}})$ is a pre-extriangulated category.

In fact, the following example shows that the category of finitely generated modules over an Artin algebra always admits a proper pre-extriangulated structure.

Example 2.7. Let Λ be an Artin algebra, and $\mathfrak{C} = \mathrm{mod}\,\Lambda$ the category of finitely generated Λ-modules. Note that $\mathbb{E}(C, A) = \mathrm{Ext}^1_\Lambda(C, A)$ for any $A, C \in \mathrm{mod}\,\Lambda$. Let $\mathbb{F}$ be the additive subfunctor generated by all almost split sequences in $\mathrm{mod}\,\Lambda$. Then $(\mathfrak{C}, \mathbb{F}, \mathfrak{s}_{\mathbb{F}})$ is a pre-extriangulated category but not an extriangulated category; see [6] for more details.

Definition 2.8 ([13, Definition 1.16]). Let $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1) and (ET2).

(1) If a sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes an $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$, then the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ is called an $\mathfrak{s}$-*triangle*, and we write

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}.$$

In this case, x is called an $\mathfrak{s}$-*inflation*, and y is called an $\mathfrak{s}$-*deflation*.

(2) Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any $\mathfrak{s}$-triangles. If a triple (a, b, c) realizes $(a, c) : \delta \to \delta'$ as in condition (R), then we write

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} & \end{array} \tag{2.1}$$

and call the triple (a, b, c) a *morphism of* $\mathfrak{s}$-*triangles*.

REMARK 2.9. Let $(\mathfrak{C}, \mathbb{E}, \mathfrak{s})$ be a triple satisfying (ET1) and (ET2), and let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be an $\mathfrak{s}$-triangle.

(1) For any $a \in \mathfrak{C}(A, A')$, there exists a morphism of $\mathfrak{s}$-triangles

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\ \downarrow{\scriptstyle a} & & \downarrow & & \| & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C & \overset{a_\star\delta}{\dashrightarrow} & \end{array}$$

(2) For any $c \in \mathfrak{C}(C', C)$, there exists a morphism of $\mathfrak{s}$-triangles

$$\begin{array}{ccccccc} A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{c^\star\delta}{\dashrightarrow} & \\ \| & & \downarrow & & \downarrow{\scriptstyle c} & & \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \end{array}$$

The following lemma is frequently used in this paper.

LEMMA 2.10 ([23, Corollary 3.5]). *Assume that $\mathfrak{C}$ is a pre-extriangulated category and consider the morphism* (2.1) *of $\mathfrak{s}$-triangles. Then the following statements are equivalent:*

(1) *a factors through x.*
(2) $a_\star\delta = c^\star\delta' = 0$.
(3) *c factors through y'.*

In particular, in the case $\delta = \delta'$ and $(a, b, c) = (\mathrm{Id}, \mathrm{Id}, \mathrm{Id})$, we have

$$x \textit{ is a section} \iff \delta \textit{ is split} \iff y \textit{ is a retraction.}$$

LEMMA 2.11 ([23, Corollary 3.5]). *Assume that $\mathfrak{C}$ is a pre-extriangulated category and consider the morphism* (2.1) *of $\mathfrak{s}$-triangles. If any two of a, b, c are isomorphisms, then so is the third.*

DEFINITION 2.12. Let $\mathfrak{C}$ be an additive category and $\mathbb{E} : \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathfrak{Ab}$ a biadditive functor.

(1) Let $f \in \mathfrak{C}(C', C)$ be a morphism. We call f an $\mathbb{E}$-*projective morphism* if $\mathbb{E}(f, A) = 0$ for any $A \in \mathfrak{C}$, and an $\mathbb{E}$-*injective morphism* if $\mathbb{E}(A, f) = 0$ for any $A \in \mathfrak{C}$.
(2) Let $C \in \mathfrak{C}$. We call C an $\mathbb{E}$-*projective object* if the identity morphism Id_C is $\mathbb{E}$-projective, and an $\mathbb{E}$-*injective object* if Id_C is $\mathbb{E}$-injective.

EXAMPLE 2.13. Consider Example 2.7. Then

$$\text{the class of } \mathbb{E}\text{-projective morphisms} = \langle \Lambda\text{-proj} \rangle,$$
$$\text{the class of } \mathbb{E}\text{-injective morphisms} = \langle \Lambda\text{-inj} \rangle.$$

Let $\mathbb{F}$ be an additive functor defined in Example 2.7. By [7, Theorem 46],

$$\text{the class of } \mathbb{F}\text{-projective morphisms} = \operatorname{rad} \Lambda + \langle \Lambda\text{-proj} \rangle,$$
$$\text{the class of } \mathbb{F}\text{-injective morphisms} = \operatorname{rad} \Lambda + \langle \Lambda\text{-inj} \rangle.$$

Here $\operatorname{rad} \Lambda$ is the Jacobson radical of Λ, and $\langle \Lambda\text{-proj} \rangle$ (respectively, $\langle \Lambda\text{-inj} \rangle$) denotes the ideal of morphisms factoring through a projective (respectively, injective) object in $\operatorname{mod} \Lambda$.

We call an $\mathfrak{s}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ *split* if δ is a split $\mathbb{E}$-extension.

LEMMA 2.14. *Let $\mathfrak{C}$ be a pre-extriangulated category, and let $f \in \mathfrak{C}(C', C)$ be a morphism. Then the following statements are equivalent.*

(1) *f is $\mathbb{E}$-projective.*
(2) *f factors through any $\mathfrak{s}$-deflation $y : B \to C$.*
(3) *For any $\mathfrak{s}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, if there exists a morphism of $\mathfrak{s}$-triangles*

$$\begin{array}{ccccccc} A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{f^{\star}\delta}{\dashrightarrow} & \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \end{array} \tag{2.2}$$

then the top $\mathfrak{s}$-triangle is split.

We denote by $\mathcal{P}$ (respectively, $\mathcal{I}$) the ideal of $\mathfrak{C}$ consisting of all $\mathbb{E}$-projective (respectively, $\mathbb{E}$-injective) morphisms. The *stable category* (respectively, *costable category*) of $\mathfrak{C}$ is defined as the ideal quotient

$$\underline{\mathfrak{C}} := \mathfrak{C}/\mathcal{P} \ (\text{respectively, } \overline{\mathfrak{C}} := \mathfrak{C}/\mathcal{I}).$$

2.2. Almost split extensions or $\mathfrak{s}$-triangles. In [13], Iyama, Nakaoka and Palu introduced the notion of almost split $\mathbb{E}$-extensions.

Definition 2.15 ([13, Definition 2.1]). Let $\mathfrak{C}$ be an additive category and $\mathbb{E} : \mathfrak{C}^{\mathrm{op}} \times \mathfrak{C} \to \mathfrak{A}b$ a biadditive functor. A non-split (i.e. non-zero) $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$ is said to be *almost split* if it satisfies the following conditions:

(AS1) $a_\star\delta = 0$ for any non-section $a \in \mathfrak{C}(A, A')$.
(AS2) $c^\star\delta = 0$ for any non-retraction $c \in \mathfrak{C}(C', C)$.

A non-zero object $A \in \mathfrak{C}$ is called *endo-local* if $\mathrm{End}_{\mathfrak{C}}(A)$ is local [13, Definition 2.4].

Definition 2.16 ([13, Definition 2.7]). Let $\mathfrak{C}$ satisfy (ET1) and (ET2), and $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be an $\mathfrak{s}$-triangle in $\mathfrak{C}$. It is called *almost split* if δ is an almost split $\mathbb{E}$-extension.

The following class of morphisms is basic to understand almost split $\mathfrak{s}$-triangles.

Definition 2.17 ([13, Definition 2.8]). Let $\mathfrak{C}$ be an additive category and A an object in $\mathfrak{C}$. A morphism $a : A \to B$ which is not a section is called *left almost split* if

- any morphism $A \to B'$ which is not a section factors through a.

Dually, a morphism $a : B \to A$ which is not a retraction is called *right almost split* if

- any morphism $B' \to A$ which is not a retraction factors through a.

Let $\mathfrak{C}$ be a pre-extriangulated category. It was shown in [13] that an $\mathfrak{s}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ is almost split if and only if x is left almost split and y is right almost split.

Recall that a morphism $f : X \to Y$ is called *right minimal* if each $g \in \mathrm{End}_{\mathfrak{C}}(X)$ with $f \circ g = f$ is an automorphism. Dually, a morphism $f : X \to Y$ is called *left minimal* if each $g \in \mathrm{End}_{\mathfrak{C}}(Y)$ with $g \circ f = f$ is an automorphism.

Let $\mathfrak{C}$ be a pre-extriangulated category. By [13, Propositions 2.5 and 2.10], for an $\mathfrak{s}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, we have

- if x is left almost split, then A is endo-local and y is right minimal;
- if y is right almost split, then C is endo-local and x is left minimal.

Lemma 2.18 ([13, Theorem 2.15]). *Let $\mathfrak{C}$ be a pre-extriangulated category. Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be a non-split $\mathfrak{s}$-triangle with A, C endo-local. Then the following conditions are equivalent:*

(1) *δ is almost split.*
(2) (AS1) *holds.*
(3) (AS2) *holds.*

In the following, let R be a commutative ring and $\mathfrak{C}$ be an additive R-category, that is, an additive category in which every morphism set is an R-module and the composition operation of morphisms is R-bilinear. A non-zero object $X \in \mathfrak{C}$ is called *Krull–Schmidt* if it is a finite direct sum of endo-local objects, and the category $\mathfrak{C}$ is called *Krull–Schmidt* if every non-zero object is Krull–Schmidt.

LEMMA 2.19 ([17, Proposition 2.1]). *Let $X \in \mathfrak{C}$. Then X is Krull–Schmidt which decomposes into $X = X_1 \oplus \cdots \oplus X_n$ with each X_i endo-local if and only if* $\mathrm{End}_{\mathfrak{C}}(X)$ *is semiperfect and all its idempotents split. In this case, each direct summand of X admits a decomposition as a direct sum of objects of a subfamily of $\{X_1, \ldots, X_n\}$, which is its unique* (*up to isomorphism and permutation*) *decomposition into a direct sum of indecomposable objects.*

LEMMA 2.20 ([17, Proposition 2.2]). *Let $\mathcal{J}$ be an ideal of $\mathfrak{C}$ and let $X \in \mathfrak{C}$.*

(1) *If X is endo-local, then either $\mathcal{J}(X,X) = \mathrm{End}_{\mathfrak{C}}(X)$ or $\mathcal{J}(X,X) \subseteq \mathrm{rad}(\mathrm{End}_{\mathfrak{C}}(X))$.*
(2) *If $\mathfrak{C}$ is Krull–Schmidt, then so is $\mathfrak{C}/\mathcal{J}$.*

3. The existence of almost split $\mathfrak{s}$-triangles. Throughout this section, unless otherwise stated, $\mathfrak{C}$ is a pre-extriangulated R-category with R a commutative ring.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z \overset{\delta}{\dashrightarrow}$ be an almost split $\mathfrak{s}$-triangle. Then both the $\mathrm{End}_{\mathfrak{C}}(X)$-socle and the $\mathrm{End}_{\mathfrak{C}}(Z)$-socle of $\mathbb{E}(Z,X)$ are simple generated by δ. Moreover, assume that $X \xrightarrow{f'} Y' \xrightarrow{g'} Z \overset{\delta'}{\dashrightarrow}$ is an almost split $\mathfrak{s}$-triangle. By definition, there are the following morphisms of $\mathfrak{s}$-triangles:

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \overset{\delta}{\dashrightarrow} & \\
\Vert & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle h} & & \\
X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z & \overset{\delta'}{\dashrightarrow} & \\
\Vert & & \downarrow{\scriptstyle v} & & \downarrow & & \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \overset{\delta}{\dashrightarrow} &
\end{array}$$

Since f and f' are left minimal, vu and uv are isomorphisms, and hence u is an isomorphism. By Lemma 2.11, h is an isomorphism. This means that the almost split $\mathfrak{s}$-triangle which is starting at X and ending at Z is unique up to isomorphism. Thus we may write $X = \tau Z$ and $Z = \tau^{-} X$.

Based on [13, Definition 2.6], we say that $\mathfrak{C}$ has *right almost split $\mathfrak{s}$-triangles* if each indecomposable object is either $\mathbb{E}$-projective or the ending term of an almost split $\mathfrak{s}$-triangle, while $\mathfrak{C}$ has *left almost split $\mathfrak{s}$-triangles* if each indecomposable object is either $\mathbb{E}$-injective or the starting term of an almost split $\mathfrak{s}$-triangle. We say that $\mathfrak{C}$ has *almost split $\mathfrak{s}$-triangles* if it has both right almost split $\mathfrak{s}$-triangles and left almost split $\mathfrak{s}$-triangles.

Now let I be an injective cogenerator for the category $\operatorname{Mod} R$ of all R-modules. Then there is an exact functor $D := \operatorname{Hom}_R(-, I) : \operatorname{Mod} R \to \operatorname{Mod} R$.

For any $U, V \in \operatorname{Mod} R$, an R-bilinear form

$$\langle -, - \rangle : U \times V \to I$$

is called *non-degenerate* if it satisfies the following conditions:

- For any $0 \neq u \in U$, there exists some $v \in V$ such that $\langle u, v \rangle \neq 0$.
- For any $0 \neq v \in V$, there exists some $u \in U$ such that $\langle u, v \rangle \neq 0$.

We have two well-known facts:

FACT 3.1. *Every R-linear form $\varphi : \mathbb{E}(Z, X) \to I$ determines, for each $L \in \mathfrak{C}$, two R-bilinear forms:*

$$_{\varphi}\langle -, - \rangle : \mathbb{E}(L, X) \times \underline{\operatorname{Hom}}_{\mathfrak{C}}(Z, L) \to I, \qquad (\sigma, \underline{g}) \mapsto \varphi(g^{\star}\sigma),$$

and

$$\langle -, - \rangle_{\varphi} : \overline{\operatorname{Hom}}_{\mathfrak{C}}(L, X) \times \mathbb{E}(Z, L) \to I, \qquad (\overline{f}, \delta) \mapsto \varphi(f_{\star}\delta).$$

FACT 3.2. *If $\delta \in \mathbb{E}(Z, X)$ is a non-zero $\mathbb{E}$-extension, then there exists an R-linear form $\varphi : \mathbb{E}(Z, X) \to I$ such that $\varphi(\delta) \neq 0$.*

PROPOSITION 3.3. *Let $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ be an almost split $\mathfrak{s}$-triangle and let $\varphi : \mathbb{E}(Z, X) \to I$ be an R-linear form. If $\varphi(\delta) \neq 0$, then the R-bilinear forms*

$$_{\varphi}\langle -, - \rangle : \mathbb{E}(L, X) \times \underline{\operatorname{Hom}}_{\mathfrak{C}}(Z, L) \to I$$

and

$$\langle -, - \rangle_{\varphi} : \overline{\operatorname{Hom}}_{\mathfrak{C}}(L, X) \times \mathbb{E}(Z, L) \to I$$

defined as above are both non-degenerate for each $L \in \mathfrak{C}$.

Proof. Let $L \in \mathfrak{C}$ and $0 \neq \underline{g} \in \underline{\operatorname{Hom}}_{\mathfrak{C}}(Z, L)$, that is, g is not $\mathbb{E}$-projective. Then there exists some $\zeta \in \mathbb{E}(L, M)$ such that $\eta := g^{\star}\zeta \neq 0$. Let $M \to N \overset{p}{\to} Z \overset{\eta}{\dashrightarrow}$ be an $\mathfrak{s}$-triangle. Then p is not a retraction, and hence we have a morphism of $\mathfrak{s}$-triangles

$$\begin{array}{ccccccc} M & \longrightarrow & N & \overset{p}{\longrightarrow} & Z & \overset{\eta}{\dashrightarrow} & \\ \downarrow{\scriptstyle h} & & \downarrow & & \| & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \overset{\delta}{\dashrightarrow} & \end{array}$$

which means that $\delta = h_{\star}\eta = h_{\star}g^{\star}\zeta$. Thus

$$_{\varphi}\langle h_{\star}\zeta, \underline{g} \rangle = \varphi(g^{\star}(h_{\star}\zeta)) = \varphi(h_{\star}g^{\star}\zeta) = \varphi(\delta) \neq 0.$$

On the other hand, let $0 \neq \xi \in \mathbb{E}(L,X)$. Assume that $X \xrightarrow{i} E \to L \overset{\xi}{\dashrightarrow}$ is an $\mathfrak{s}$-triangle. In particular, i is not a section. Then we have a morphism of $\mathfrak{s}$-triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \overset{\delta}{\dashrightarrow} & \\ \| & & \downarrow & & \downarrow{\scriptstyle g} & & \\ X & \xrightarrow{i} & E & \longrightarrow & L & \overset{\xi}{\dashrightarrow} & \end{array}$$

which shows that $\delta = g^\star \xi$. Then $\underline{g} \in \underline{\mathrm{Hom}}_{\mathfrak{C}}(Z,L)$ and

$${}_{\varphi}\langle \xi, \underline{g} \rangle = \varphi(g^\star \xi) = \varphi(\delta) \neq 0.$$

Thus ${}_{\varphi}\langle -,- \rangle$ is non-degenerate.

Similarly, we can prove that $\langle -,- \rangle_{\varphi}$ is non-degenerate. ∎

Let $F : \mathfrak{C} \to \operatorname{Mod} R$ and $G : \mathfrak{C} \to \operatorname{Mod} R$ be two R-linear functors. Recall from [17] that a natural transformation $\phi : F \to G$ is a *functorial monomorphism* if $\phi_X : F(X) \to G(X)$ is injective for each $X \in \mathfrak{C}$. Let Λ be an R-algebra. A non-zero R-linear form $\psi : \Lambda \to I$ is called *almost vanishing* if $\psi(\operatorname{rad} \Lambda) = 0$, where $\operatorname{rad} \Lambda$ is the Jacobson radical of Λ.

Now we give a characterization of almost split $\mathfrak{s}$-triangles.

THEOREM 3.4. *Let $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ be an $\mathfrak{s}$-triangle with X, Z being endo-local. The following statements are equivalent:*

(1) *The $\mathfrak{s}$-triangle is almost split.*
(2) *There exists a functorial monomorphism $\phi : \mathbb{E}(Z,-) \to D\overline{\mathrm{Hom}}_{\mathfrak{C}}(-,X)$ such that $\phi_X(\delta)$ is almost vanishing on $\overline{\mathrm{End}}_{\mathfrak{C}}(X)$.*
(3) *There exists a functorial monomorphism $\psi : \mathbb{E}(-,X) \to D\underline{\mathrm{Hom}}_{\mathfrak{C}}(Z,-)$ such that $\psi_Z(\delta)$ is almost vanishing on $\underline{\mathrm{End}}_{\mathfrak{C}}(Z)$.*

Proof. Notice that in each of the statements, $\delta \neq 0$, thus we may assume that X is not $\mathbb{E}$-injective and Z is not $\mathbb{E}$-projective. By Lemma 2.20, $\operatorname{rad}(\overline{\mathrm{End}}_{\mathfrak{C}}(X)) = \operatorname{rad}(\mathrm{End}_{\mathfrak{C}}(X))/\mathcal{I}(X,X)$.

(1)⇒(2) Assume that $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ is an almost split $\mathfrak{s}$-triangle. By Fact 3.2, there exists an R-linear form $\varphi : \mathbb{E}(Z,X) \to I$ such that $\varphi(\delta) \neq 0$. For each $L \in \mathfrak{C}$, by Proposition 3.3, there is a non-degenerate bilinear form

$$\langle -,- \rangle_{\varphi} : \overline{\mathrm{Hom}}_{\mathfrak{C}}(L,X) \times \mathbb{E}(Z,L) \to I, \qquad (\overline{f}, \sigma) \mapsto \varphi(f_\star \sigma).$$

This induces an R-linear monomorphism

$$\phi_L : \mathbb{E}(Z,L) \to D\overline{\mathrm{Hom}}_{\mathfrak{C}}(L,X), \qquad \eta \mapsto \langle -, \eta \rangle_{\varphi},$$

which is natural in L. In particular, since ϕ_X is injective, $\phi_X(\delta) \neq 0$. Let $\overline{f} \in \operatorname{rad}(\overline{\mathrm{End}}_{\mathfrak{C}}(X))$. Then $f \in \operatorname{rad}(\mathrm{End}_{\mathfrak{C}}(X))$ and consider the following commutative diagram:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \overset{\delta}{\dashrightarrow} & \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow & & \Big\| & & \\ X & \longrightarrow & Y' & \longrightarrow & Z & \overset{f_\star\delta}{\dashrightarrow} & \end{array}$$

Since f is not a section, by Definition 2.15 we have $f_\star\delta = 0$. Thus

$$\phi_X(\delta)(\overline{f}) = \langle \overline{f}, \delta \rangle_\varphi = \varphi(f_\star\delta) = 0$$

and hence $\phi_X(\delta)$ is almost vanishing on $\overline{\mathrm{End}}_{\mathfrak{C}}(X)$.

(2)⇒(1) Let $\phi : \mathbb{E}(Z,-) \to D\overline{\mathrm{Hom}}_{\mathfrak{C}}(-,X)$ be a functorial monomorphism such that $\phi_X(\delta)$ is almost vanishing on $\overline{\mathrm{End}}_{\mathfrak{C}}(X)$. In particular, $\phi_X(\delta) \neq 0$, and since ϕ_X is injective, it follows that $\delta \neq 0$. For any non-section $u \in \mathrm{Hom}_{\mathfrak{C}}(X,L)$, let $v \in \mathrm{Hom}_{\mathfrak{C}}(L,X)$. Then $vu \in \mathrm{rad}(\mathrm{End}_{\mathfrak{C}}(X))$, and hence $\overline{vu} \in \mathrm{rad}(\overline{\mathrm{End}}_{\mathfrak{C}}(X))$. This means that $\phi_X(\delta)(\overline{vu}) = 0$, that is, $((D\overline{\mathrm{Hom}}_{\mathfrak{C}}(u,X))\phi_X)(\delta) = 0$. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{E}(Z,X) & \xrightarrow{\quad\mathbb{E}(Z,u)\quad} & \mathbb{E}(Z,L) \\ {\scriptstyle \phi_X}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi_L} \\ D\overline{\mathrm{Hom}}_{\mathfrak{C}}(X,X) & \xrightarrow{D\overline{\mathrm{Hom}}_{\mathfrak{C}}(u,X)} & D\overline{\mathrm{Hom}}_{\mathfrak{C}}(L,X) \end{array}$$

Then $\phi_L(u_\star\delta) = \phi_L(\mathbb{E}(Z,u)(\delta)) = 0$. Since ϕ_L is injective, we have $u_\star\delta = 0$. By Lemma 2.18, $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ is an almost split $\mathfrak{s}$-triangle.

Similarly, we can prove (1)⇔(3). ∎

Let $X, Y \in \mathfrak{C}$. Then $D\overline{\mathrm{Hom}}_{\mathfrak{C}}(X,Y)$ is an $\mathrm{End}_{\mathfrak{C}}(X)$-$\mathrm{End}_{\mathfrak{C}}(Y)$-bimodule with multiplications defined by

$$f\theta g : \overline{\mathrm{Hom}}_{\mathfrak{C}}(X,Y) \to I, \qquad \overline{h} \mapsto \theta(\overline{ghf}),$$

for any $f \in \mathrm{End}_{\mathfrak{C}}(X)$, $\theta \in D\overline{\mathrm{Hom}}_{\mathfrak{C}}(X,Y)$ and $g \in \mathrm{End}_{\mathfrak{C}}(Y)$. Similarly, $D\underline{\mathrm{Hom}}_{\mathfrak{C}}(X,Y)$ is an $\mathrm{End}_{\mathfrak{C}}(X)$-$\mathrm{End}_{\mathfrak{C}}(Y)$-bimodule.

THEOREM 3.5. *Let X, Z be endo-local objects in $\mathfrak{C}$. The following statements are equivalent:*

(1) *There is an almost split $\mathfrak{s}$-triangle $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ in $\mathfrak{C}$.*
(2) *The $\mathrm{End}_{\mathfrak{C}}(X)$-socle of $\mathbb{E}(Z,X)$ is non-zero and there is a functorial monomorphism $\phi : \mathbb{E}(Z,-) \to D\overline{\mathrm{Hom}}_{\mathfrak{C}}(-,X)$.*
(3) *The $\mathrm{End}_{\mathfrak{C}}(Z)$-socle of $\mathbb{E}(Z,X)$ is non-zero and there is a functorial monomorphism $\psi : \mathbb{E}(-,X) \to D\underline{\mathrm{Hom}}_{\mathfrak{C}}(Z,-)$.*

Proof. (1)⇒(2) Assume that there is an almost split $\mathfrak{s}$-triangle $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ in $\mathfrak{C}$. Then δ is a non-zero element in the $\mathrm{End}_{\mathfrak{C}}(X)$-socle of

$\mathbb{E}(Z,X)$. Moreover, by Theorem 3.4, there is a functorial monomorphism $\phi : \mathbb{E}(Z,-) \to D\overline{\mathrm{Hom}}_{\mathfrak{C}}(-,X)$.

(2)⇒(1) Assume that there is a non-zero $\mathbb{E}$-extension δ lying in the $\mathrm{End}_{\mathfrak{C}}(X)$-socle of $\mathbb{E}(Z,X)$. Let $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ be an $\mathfrak{s}$-triangle. Then X is not $\mathbb{E}$-injective and Z is not $\mathbb{E}$-projective. Let $\phi : \mathbb{E}(Z,-) \to D\overline{\mathrm{Hom}}_{\mathfrak{C}}(-,X)$ be a functorial monomorphism. Then $\theta = \phi_X(\delta) \in D\overline{\mathrm{End}}_{\mathfrak{C}}(X)$ is a non-zero element which satisfies $\mathrm{rad}(\mathrm{End}_{\mathfrak{C}}(X))\theta = 0$. If $\overline{f} \in \mathrm{rad}(\overline{\mathrm{End}}_{\mathfrak{C}}(X))$, then $f \in \mathrm{rad}(\mathrm{End}_{\mathfrak{C}}(X))$ by Lemma 2.20(1), and thus $\theta(\overline{f}) = (f\theta)(\overline{\mathrm{Id}_X}) = 0$, which shows that θ is almost vanishing. By Theorem 3.4, $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ is an almost split $\mathfrak{s}$-triangle.

Similarly, we can prove (1)⇔(3). ∎

In the rest of this section, we assume that $\mathfrak{C}$ is a pre-extriangulated R-category with R a commutative Artin ring. We fix I to be the minimal injective cogenerator for $\mathrm{Mod}\, R$. In this case, there is a duality

$$D = \mathrm{Hom}_R(-, I) : \mathrm{mod}\, R \to \mathrm{mod}\, R,$$

where $\mathrm{mod}\, R$ denotes the category of modules of finite R-length.

Lemma 3.6. *Let $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ be an almost split $\mathfrak{s}$-triangle, and $L \in \mathfrak{C}$.*

(1) *$\mathbb{E}(Z,L) \in \mathrm{mod}\, R$ if and only if $\overline{\mathrm{Hom}}_{\mathfrak{C}}(L,X) \in \mathrm{mod}\, R$. In this case, $\mathbb{E}(Z,L) \cong D\overline{\mathrm{Hom}}_{\mathfrak{C}}(L,X)$.*
(2) *$\mathbb{E}(L,X) \in \mathrm{mod}\, R$ if and only if $\underline{\mathrm{Hom}}_{\mathfrak{C}}(Z,L) \in \mathrm{mod}\, R$. In this case, $\mathbb{E}(L,X) \cong D\underline{\mathrm{Hom}}_{\mathfrak{C}}(Z,L)$.*

Proof. (1) By Proposition 3.3, there is a non-degenerate R-bilinear form

$$\langle -,- \rangle_{\varphi} : \ \overline{\mathrm{Hom}}_{\mathfrak{C}}(L,X) \times \mathbb{E}(Z,L) \to I.$$

If $\mathbb{E}(Z,L) \in \mathrm{mod}\, R$ or $\overline{\mathrm{Hom}}_{\mathfrak{C}}(L,X) \in \mathrm{mod}\, R$, then we have an R-linear isomorphism

$$\mathbb{E}(Z,L) \to D\overline{\mathrm{Hom}}_{\mathfrak{C}}(L,X) : \xi \mapsto \langle -, \xi \rangle_{\varphi}.$$

(2) is proved similarly. ∎

The following is a local version of [13, Theorem 3.4] in a pre-extriangulated category.

Theorem 3.7. *Let X be an endo-local non-$\mathbb{E}$-injective object and Z an endo-local non-$\mathbb{E}$-projective object.*

(1) *Assume $\overline{\mathrm{Hom}}_{\mathfrak{C}}(L,X) \in \mathrm{mod}\, R$ for any $L \in \mathfrak{C}$. Then there is an almost split $\mathfrak{s}$-triangle $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ if and only if $\mathbb{E}(Z,-) \cong D\overline{\mathrm{Hom}}_{\mathfrak{C}}(-,X)$.*
(2) *Assume $\underline{\mathrm{Hom}}_{\mathfrak{C}}(Z,L) \in \mathrm{mod}\, R$ for any $L \in \mathfrak{C}$. Then there is an almost split $\mathfrak{s}$-triangle $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ if and only if $\mathbb{E}(-,X) \cong D\underline{\mathrm{Hom}}_{\mathfrak{C}}(Z,-)$.*

Proof. (1) *The "if" part.* Since $\overline{\mathrm{Hom}}_{\mathfrak{C}}(L,X) \in \operatorname{mod} R$ for any $L \in \mathfrak{C}$, then there is a functorial isomorphism $\phi : \mathbb{E}(Z,-) \to D\overline{\mathrm{Hom}}_{\mathfrak{C}}(-,X)$. Since X is not $\mathbb{E}$-injective, $\overline{\mathrm{End}}_{\mathfrak{C}}(X) \neq 0$. Since ϕ_X is bijective, $\mathbb{E}(Z,X) \neq 0$. In particular, the $\mathrm{End}_{\mathfrak{C}}(X)$-socle of $\mathbb{E}(Z,X)$ is non-zero. By Theorem 3.5, there is an almost split $\mathfrak{s}$-triangle $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ in $\mathfrak{C}$.

(2) is proved similarly. ∎

Recall from [13] that the extriangulated category $\mathfrak{C}$ is *Ext-finite* (respectively, *Hom-finite*) if $\mathbb{E}(X,Y)$ (respectively, $\mathrm{Hom}_{\mathfrak{C}}(X,Y)$) is a module of finite R-length for all $X,Y \in \mathfrak{C}$.

COROLLARY 3.8. *Let $\mathfrak{C}$ be a Krull–Schmidt pre-extriangulated R-category. If $\mathfrak{C}$ has almost split $\mathfrak{s}$-triangles, then $\mathfrak{C}$ is Ext-finite if and only if $\overline{\mathfrak{C}}$ is Hom-finite if and only if $\underline{\mathfrak{C}}$ is Hom-finite.*

Proof. Assume that $\overline{\mathfrak{C}}$ is Hom-finite. Let $X,Y \in \mathfrak{C}$ be any indecomposable objects. If X is $\mathbb{E}$-projective, then $\mathbb{E}(X,-) = 0$. Otherwise, since $\overline{\mathrm{Hom}}_{\mathfrak{C}}(Y,\tau X) \in \operatorname{mod} R$, by Lemma 3.6, $\mathbb{E}(X,Y) \cong D\overline{\mathrm{Hom}}_{\mathfrak{C}}(Y,\tau X)$, and hence $\mathbb{E}(X,Y) \in \operatorname{mod} R$. Thus $\mathfrak{C}$ is Ext-finite.

Conversely, assume $\mathfrak{C}$ is Ext-finite. If Y is $\mathbb{E}$-injective, then $\overline{\mathrm{Hom}}_{\mathfrak{C}}(X,Y) = 0$. Otherwise, since $\mathbb{E}(\tau^{-}Y,X) \in \operatorname{mod} R$, by Lemma 3.6, $D\overline{\mathrm{Hom}}_{\mathfrak{C}}(X,Y) \cong \mathbb{E}(\tau^{-}Y,X)$, and hence $\overline{\mathrm{Hom}}_{\mathfrak{C}}(X,Y) \in \operatorname{mod} R$. Thus $\overline{\mathfrak{C}}$ is Hom-finite.

Similarly, $\mathfrak{C}$ is Ext-finite if and only if $\underline{\mathfrak{C}}$ is Hom-finite. ∎

4. Almost split $\mathfrak{s}$-triangles in extension-closed subcategories. Let $\mathfrak{C}$ be an additive category and $\mathcal{E}$ be a full subcategory of $\mathfrak{C}$. Let $C \in \mathfrak{C}$. Recall that a morphism $f : E \to C$ with $E \in \mathcal{E}$ is called a *right $\mathcal{E}$-approximation* of C if the induced map $\mathrm{Hom}_{\mathfrak{C}}(E',E) \to \mathrm{Hom}_{\mathfrak{C}}(E',C)$ is surjective for each $E' \in \mathcal{E}$. Dually, one has the notion of *left $\mathcal{E}$-approximation.*

Throughout this section, unless otherwise stated, $\mathfrak{C}$ is a pre-extriangulated R-category with R a commutative ring and $\mathcal{E}$ an extension-closed full subcategory of $\mathfrak{C}$ (i.e. $\mathcal{E}$ is a full subcategory of $\mathfrak{C}$ such that for any given $\mathfrak{s}$-triangle $X \to Y \to Z \overset{\delta}{\dashrightarrow}$, the condition $X,Z \in \mathcal{E}$ implies $Y \in \mathcal{E}$). In this case, we define $\mathbb{E}_{\mathcal{E}}$ to be the restriction of $\mathbb{E}$ to $\mathcal{E}^{\mathrm{op}} \times \mathcal{E}$, and define $\mathfrak{s}_{\mathcal{E}}$ by restricting $\mathfrak{s}$. Then $(\mathcal{E},\mathbb{E}_{\mathcal{E}},\mathfrak{s}_{\mathcal{E}})$ is also a pre-extriangulated category. In the following, we mainly study when an almost split $\mathfrak{s}$-triangle in $\mathfrak{C}$ induces an almost split $\mathfrak{s}_{\mathcal{E}}$-triangle in $\mathcal{E}$.

We denote by $\widetilde{\mathcal{E}}$ the full subcategory of $\overline{\mathfrak{C}}$ generated by the objects in $\mathcal{E}$. Then $\overline{\mathcal{E}}$ is a quotient category of $\widetilde{\mathcal{E}}$.

DEFINITION 4.1. Let $X \in \mathfrak{C}$. A morphism $f : E \to X$ in $\mathfrak{C}$ with $E \in \mathcal{E}$ is called a *right injectively stable $\mathcal{E}$-approximation* of X if the induced morphism $\overline{f}$ is a right $\widetilde{\mathcal{E}}$-approximation of X in $\overline{\mathfrak{C}}$, and a *minimal right* *injectively stable $\mathcal{E}$-approximation* if, in addition, $\overline{f}$ is right minimal in $\overline{\mathfrak{C}}$ and E has no non-zero summands which are $\mathbb{E}$-injective in $\mathfrak{C}$.

The notions of *left projectively stable $\mathcal{E}$-approximations* and *minimal left projectively stable $\mathcal{E}$-approximations* are defined dually.

LEMMA 4.2 ([17, Lemma 4.2]). *Let $X \in \mathfrak{C}$ with a right injectively stable $\mathcal{E}$-approximation $f : E \to X$. If E is a Krull–Schmidt object, then f decomposes as $f = (g, h) : E_1 \oplus E_2 \to X$, where g is a minimal right injectively stable $\mathcal{E}$-approximation of X.*

LEMMA 4.3. *Let $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ be an almost split $\mathfrak{s}$-triangle in $\mathfrak{C}$.*

(1) *If $Z \in \mathcal{E}$, then Z is $\mathbb{E}_{\mathcal{E}}$-projective if and only if $0 \to X$ is a right injectively stable $\mathcal{E}$-approximation of X.*
(2) *If $X \in \mathcal{E}$, then X is $\mathbb{E}_{\mathcal{E}}$-injective if and only if $Z \to 0$ is a left projectively stable $\mathcal{E}$-approximation of Z.*

Proof. (1) By assumption, $\delta \neq 0$. For any $L \in \mathcal{E}$, by Facts 3.1, 3.2 and Proposition 3.3, there exists a non-degenerate R-bilinear form

$$\langle -, - \rangle : \overline{\mathrm{Hom}}_{\mathfrak{C}}(L, X) \times \mathbb{E}(Z, L) \to I.$$

If $Z \in \mathcal{E}$, then $\mathbb{E}(Z, L) = \mathbb{E}_{\mathcal{E}}(Z, L)$. Thus, Z is $\mathbb{E}_{\mathcal{E}}$-projective if and only if $\mathbb{E}_{\mathcal{E}}(Z, L) = 0$ for each $L \in \mathcal{E}$ if and only if $\overline{\mathrm{Hom}}_{\mathfrak{C}}(L, X) = 0$ for each $L \in \mathcal{E}$, which is equivalent to $0 \to X$ being a right injectively stable $\mathcal{E}$-approximation of X.

(2) is proved similarly. ∎

LEMMA 4.4. *Let $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ be an almost split $\mathfrak{s}$-triangle in $\mathfrak{C}$ and $Z \in \mathcal{E}$. Assume that there is a minimal right injectively stable $\mathcal{E}$-approximation $f : E \to X$.*

(1) *The map $\mathbb{E}(L, f) : \mathbb{E}(L, E) \to \mathbb{E}(L, X)$ is injective for each $L \in \mathcal{E}$.*
(2) *Assume that $u : L \to E$ lies in $\mathcal{E}$. Then u is $\mathbb{E}_{\mathcal{E}}$-injective if and only if fu is $\mathbb{E}$-injective.*
(3) *If Z is not $\mathbb{E}_{\mathcal{E}}$-projective, then E is indecomposable.*

Proof. (1) Given any $\mathfrak{s}$-triangle $E \xrightarrow{r} G \to L \overset{\xi}{\dashrightarrow}$, consider the following commutative diagram:

$$\begin{array}{ccccccc} E & \xrightarrow{r} & G & \longrightarrow & L & \overset{\xi}{\dashrightarrow} & \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| & & \\ X & \xrightarrow{s} & F & \longrightarrow & L & \overset{f_\star \xi}{\dashrightarrow} & \end{array}$$

Assume $\mathbb{E}(L, f)(\xi) = f_\star \xi = 0$. Then f factors through r, that is, there exists $e : G \to X$ such that $f = er$. Moreover, since $E, L \in \mathcal{E}$, we have $G \in \mathcal{E}$, and hence there exists $h : G \to E$ such that $\overline{e} = \overline{f}\ \overline{h}$. Thus $\overline{f} = \overline{er} = \overline{f}\ \overline{hr}$. By assumption, $\overline{f}$ is right minimal in $\overline{\mathfrak{C}}$, hence $\overline{hr}$ is an automorphism. Without loss of generality, we may assume $\overline{hr} = \overline{\mathrm{Id}_E}$, that is, $\mathrm{Id}_E - hr$ is $\mathbb{E}$-injective. Consider the following commutative diagram:

$$\begin{array}{ccccccc} E & \xrightarrow{r} & G & \longrightarrow & L & \overset{\xi}{\dashrightarrow} & \\ {\scriptstyle \mathrm{Id}_E - hr}\downarrow & & \downarrow{\scriptstyle g} & & \| & & \\ E & \longrightarrow & E' & \longrightarrow & L & \overset{\rho}{\dashrightarrow} & \end{array}$$

Since $\mathrm{Id}_E - hr$ is $\mathbb{E}$-injective, the bottom $\mathfrak{s}$-triangle is split. Then there exists $p : G \to E$ such that $\mathrm{Id}_E - hr = pr$. Thus $\mathrm{Id}_E = (h + p)r$, which shows that r is a section. It follows that $\xi = 0$.

(2) *The "only if" part.* Assume that fu is not $\mathbb{E}$-injective. Let $\varphi : \mathbb{E}(Z, X) \to I$ be an R-linear form such that $\varphi(\delta) \neq 0$. By Proposition 3.3, there exists $\xi \in \mathbb{E}(Z, L) = \mathbb{E}_{\mathcal{E}}(Z, L)$ such that $\varphi(f_\star u_\star \xi) = \varphi((fu)_\star \xi) \neq 0$. In particular, $u_\star \xi \neq 0$, which shows that u is not $\mathbb{E}_{\mathcal{E}}$-injective.

The "if" part. Assume that fu is $\mathbb{E}$-injective. For any $\eta \in \mathbb{E}_{\mathcal{E}}(N, L)$, let $L \xrightarrow{r} G \to N \overset{\eta}{\dashrightarrow}$, then $f_\star(u_\star \eta) = (fu)_\star \eta = 0$. By (1), $u_\star \eta = 0$, which shows that u is $\mathbb{E}_{\mathcal{E}}$-injective.

(3) Since the right $\mathrm{End}_{\mathfrak{C}}(Z)$-module $\mathbb{E}(Z, X)$ has a simple socle, every non-zero $\mathrm{End}_{\mathfrak{C}}(Z)$-submodule of $\mathbb{E}(Z, X)$ is indecomposable. Suppose that Z is not $\mathbb{E}_{\mathcal{E}}$-projective. In particular, Z is not $\mathbb{E}$-projective. By Lemma 4.3, f is not $\mathbb{E}$-injective. By Proposition 3.3, $\mathbb{E}(Z, E) \neq 0$. By (1), $\mathbb{E}(Z, f) : \mathbb{E}(Z, E) \to \mathbb{E}(Z, X)$ is injective. Thus $\mathbb{E}(Z, E)$ is isomorphic to a non-zero $\mathrm{End}_{\mathfrak{C}}(Z)$-submodule of $\mathbb{E}(Z, X)$, and hence it is an indecomposable right $\mathrm{End}_{\mathfrak{C}}(Z)$-module. Assume that $E = E_1 \oplus E_2$ with non-zero injections $e_i : E_i \to E$, $i = 1, 2$. By assumption, E_1 and E_2 are non-zero in $\overline{\mathfrak{C}}$. Since $\overline{f}$ is right minimal in $\overline{\mathfrak{C}}$, we have $\overline{fe_i} \neq 0$. By Proposition 3.3, $\mathbb{E}(Z, E_i) \neq 0$ for $i = 1, 2$. Moreover,

$$\mathbb{E}(Z, E) \cong \mathbb{E}(Z, E_1) \oplus \mathbb{E}(Z, E_2),$$

which is a contradiction. Therefore, E is indecomposable. ∎

Proposition 4.5. *Let $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ be an almost split $\mathfrak{s}$-triangle in $\mathfrak{C}$ and $Z \in \mathcal{E}$. Assume that there is a minimal right injectively stable $\mathcal{E}$-approximation $f : E \to X$. If E is Krull–Schmidt in $\mathcal{E}$, then there is a commutative diagram*

$$\begin{array}{ccccccc} E & \longrightarrow & N & \longrightarrow & Z & \overset{\xi}{\dashrightarrow} & \\ \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \overset{\delta}{\dashrightarrow} & \end{array}$$

such that the top row is an almost split $\mathfrak{s}_{\mathcal{E}}$-triangle in $\mathcal{E}$.

Proof. Assume that E is Krull–Schmidt in $\mathcal{E}$. By Definition 4.1, E is not $\mathbb{E}$-injective, that is, $E \neq 0$ in $\overline{\mathfrak{C}}$, and $\overline{f} \neq 0$ in $\overline{\mathfrak{C}}$. By Lemma 4.3, Z is not $\mathbb{E}_{\mathcal{E}}$-projective. By Lemma 4.4(3), E is indecomposable. Now for each $L \in \mathcal{E}$, by Lemma 4.4(2), there is an exact sequence

$$0 \to \mathcal{I}_{\mathcal{E}}(L, E) \to \operatorname{Hom}_{\mathcal{E}}(L, E) \xrightarrow{f \circ -} \overline{\operatorname{Hom}}_{\mathfrak{C}}(L, X) \to 0,$$

where $\mathcal{I}_{\mathcal{E}}(L, E)$ denotes the class of $\mathbb{E}_{\mathcal{E}}$-injective morphisms from L to E. This shows that there is an R-linear isomorphism

$$\overline{\operatorname{Hom}}_{\mathcal{E}}(L, E) \to \overline{\operatorname{Hom}}_{\mathfrak{C}}(L, X), \qquad \overline{u} \mapsto \overline{fu},$$

which is natural in L. Then we obtain a functorial isomorphism

$$D\overline{\operatorname{Hom}}_{\mathfrak{C}}(-, X)|_{\mathcal{E}} \to D\overline{\operatorname{Hom}}_{\mathcal{E}}(-, E).$$

On the other hand, there is a functorial monomorphism

$$\mathbb{E}(Z, -) \to D\overline{\operatorname{Hom}}_{\mathfrak{C}}(-, X)$$

by Theorem 3.4. Thus there is a functorial monomorphism

$$\mathbb{E}(Z, -)|_{\mathcal{E}} \to D\overline{\operatorname{Hom}}_{\mathcal{E}}(-, E).$$

Moreover, since $\mathbb{E}_{\mathcal{E}}(Z, -) = \mathbb{E}(Z, -)|_{\mathcal{E}}$, we obtain a functorial monomorphism

$$\phi : \mathbb{E}_{\mathcal{E}}(Z, -) \to D\overline{\operatorname{Hom}}_{\mathcal{E}}(-, E).$$

Since f is not $\mathbb{E}$-injective, we have a non-split $\mathfrak{s}$-triangle $E \to G_1 \to Z \overset{\zeta}{\dashrightarrow}$ such that the $\mathfrak{s}$-triangle $X \xrightarrow{r} G_2 \to Z \overset{f_\star\zeta}{\dashrightarrow}$ is not split. In particular, r is not a section, and hence we have a commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \overset{\delta}{\dashrightarrow} & \\
\| & & \downarrow & & \downarrow{\scriptstyle g} & & \\
X & \xrightarrow{r} & G_2 & \longrightarrow & Z & \overset{f_\star\zeta}{\dashrightarrow} &
\end{array}$$

This means that $\delta = g^\star f_\star \zeta = f_\star g^\star \zeta$. Let $\xi := g^\star \zeta$ and $E \to N \to Z \overset{\xi}{\dashrightarrow}$ be an $\mathfrak{s}$-triangle. We claim that it is almost split in $\mathcal{E}$. Indeed, suppose that $u_\star \xi \neq 0$ for some $u \in \operatorname{rad}(\operatorname{End}_{\mathcal{E}}(E))$. Let $E \to N' \to Z \overset{u_\star\xi}{\dashrightarrow}$ be an $\mathfrak{s}$-triangle. Then we have a commutative diagram

$$\begin{array}{ccccccc}
E & \longrightarrow & N' & \longrightarrow & Z & \overset{u_\star\xi}{\dashrightarrow} & \\
\downarrow{\scriptstyle v} & & \downarrow & & \| & & \\
X & \longrightarrow & Y & \longrightarrow & Z & \overset{\delta}{\dashrightarrow} &
\end{array}$$

that is, $\delta = v_\star u_\star \xi$. Since $\overline{f}$ is a right $\widetilde{\mathcal{E}}$-approximation of X, there exists $w : E \to E$ such that $\overline{v} = \overline{fw}$, and hence $f_\star \xi = \delta = f_\star((wu)_\star \xi)$. By

Lemma 4.4(1), $\xi = (wu)_\star \xi$. Since $wu \in \mathrm{rad}(\mathrm{End}_{\mathcal{E}}(E))$, we have $\xi = 0$, which is a contradiction. Thus ξ lies in the $\mathrm{End}_{\mathcal{E}}(E)$-socle of $\mathbb{E}(Z, E)$. Since ϕ_E is an $\mathrm{End}_{\mathcal{E}}(E)$-linear monomorphism, $\phi_E(\xi)$ is almost vanishing on $\overline{\mathrm{End}}_{\mathcal{E}}(E)$. By Theorem 3.4, $E \to N \to Z \overset{\xi}{\dashrightarrow}$ is an almost split $\mathfrak{s}_{\mathcal{E}}$-triangle in $\mathcal{E}$. ∎

In the rest of this section, we assume that $\mathfrak{C}$ is a pre-extriangulated R-category with R a commutative Artin ring.

Proposition 4.6. *Let $X \to Y \to Z \overset{\delta}{\dashrightarrow}$ be an almost split $\mathfrak{s}$-triangle in $\mathfrak{C}$ and $E \to N \to Z \overset{\eta}{\dashrightarrow}$ be an almost split $\mathfrak{s}_{\mathcal{E}}$-triangle in $\mathcal{E}$. Consider the following commutative diagram:*

$$\begin{array}{ccccccc}
E & \xrightarrow{r} & N & \xrightarrow{s} & Z & \overset{\eta}{\dashrightarrow} & \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| & & \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \overset{\delta}{\dashrightarrow} &
\end{array}$$

If $\overline{\mathrm{Hom}}_{\mathcal{E}}(L, E) \in \mathrm{mod}\, R$ for any $L \in \mathcal{E}$, then f is a minimal right injectively stable $\mathcal{E}$-approximation of X, and g is a right injectively stable $\mathcal{E}$-approximation of Y.

Proof. Fix an R-linear form $\varphi : \mathbb{E}(Z, X) \to I$ such that $\varphi(\delta) \neq 0$. This implies an R-linear form

$$\psi : \mathbb{E}(Z, E) \to I, \quad \zeta \mapsto \varphi(f_\star \zeta),$$

such that $\psi(\eta) = \varphi(f_\star \eta) = \varphi(\delta) \neq 0$. Let $L \in \mathcal{E}$. By Proposition 3.3, we have non-degenerate R-bilinear forms

$$\langle -, - \rangle_\varphi : \overline{\mathrm{Hom}}_{\mathfrak{C}}(L, X) \times \mathbb{E}(Z, L) \to I, \quad (\overline{q}, \zeta) \mapsto \varphi(q_\star \zeta),$$

and

$$\langle -, - \rangle_\psi : \overline{\mathrm{Hom}}_{\mathcal{E}}(L, E) \times \mathbb{E}_{\mathcal{E}}(Z, L) \to I, \quad (\widetilde{p}, \zeta) \mapsto \psi(p_\star \zeta).$$

Let $q : L \to X$ be a morphism in $\mathfrak{C}$. Since $\mathbb{E}(Z, L) = \mathbb{E}_{\mathcal{E}}(Z, L)$, we have an R-linear form

$$\langle \overline{q}, - \rangle_\varphi : \mathbb{E}(Z, L) \to I, \quad \zeta \mapsto \langle \overline{q}, \zeta \rangle_\varphi.$$

By assumption, $\overline{\mathrm{Hom}}_{\mathcal{E}}(L, E) \in \mathrm{mod}\, R$ for any $L \in \mathcal{E}$, thus there exists $p : L \to E$ in $\mathcal{E}$ such that $\langle \overline{q}, - \rangle_\varphi = \langle \widetilde{p}, - \rangle_\psi$. Hence

$$\langle \overline{q}, \zeta \rangle_\varphi = \langle \widetilde{p}, \zeta \rangle_\psi = \psi(p_\star \zeta) = \varphi(f_\star(p_\star \zeta)) = \varphi((fp)_\star \zeta) = \langle \overline{fp}, \zeta \rangle_\varphi.$$

Since $\langle -, - \rangle_\varphi$ is non-degenerate, $\overline{q} = \overline{fp}$. This implies that f is a right injectively stable $\mathcal{E}$-approximation of X, which is minimal since E is endo-local.

On the other hand, consider a morphism $h : L \to Y$ with $L \in \mathcal{E}$. Since v is not a retraction, vh is not a retraction. Thus there exists $w : L \to N$ such that $vh = sw = vgw$, that is, $v(h - gw) = 0$. Hence there exists $t : L \to X$ such that $h - gw = ut$. Since f is a right injectively stable $\mathcal{E}$-approximation of X, there exists $j : L \to E$ such that $\overline{t} = \overline{fj}$. Thus $\overline{h - gw} = \overline{ufj} = \overline{grj}$. This shows that $\overline{h} = \overline{g}(\overline{w} + \overline{rj})$. By definition, g is a right injectively stable $\mathcal{E}$-approximation of Y. ∎

Now we give the main result of this section.

THEOREM 4.7. *Let $\mathcal{E}$ be an extension-closed subcategory of $\mathfrak{C}$ which is Ext-finite and Krull–Schmidt.*

(1) *If $\mathfrak{C}$ has right almost split $\mathfrak{s}$-triangles, then $\mathcal{E}$ has right almost split $\mathfrak{s}_{\mathcal{E}}$-triangles if and only if τZ has a right injectively stable $\mathcal{E}$-approximation for any indecomposable non-$\mathbb{E}_{\mathcal{E}}$-projective object Z in $\mathcal{E}$.*
(2) *If $\mathfrak{C}$ has left almost split $\mathfrak{s}$-triangles, then $\mathcal{E}$ has left almost split $\mathfrak{s}_{\mathcal{E}}$-triangles if and only if $\tau^{-}X$ has a left projectively stable $\mathcal{E}$-approximation for any indecomposable non-$\mathbb{E}_{\mathcal{E}}$-injective object X in $\mathcal{E}$.*

Proof. (1) Assume that $\mathfrak{C}$ has right almost split $\mathfrak{s}$-triangles.

The "if" part. Let Z be an indecomposable non-$\mathbb{E}_{\mathcal{E}}$-projective object. In particular, Z is an indecomposable non-$\mathbb{E}$-projective object. Thus there is an almost split $\mathfrak{s}$-triangle $\tau Z \to Y \to Z \overset{\delta}{\dashrightarrow}$ in $\mathfrak{C}$. By assumption and Lemma 4.2, τZ has a minimal right injectively stable $\mathcal{E}$-approximation, and hence by Proposition 4.5, there is an almost split $\mathfrak{s}_{\mathcal{E}}$-triangle in $\mathcal{E}$ ending at Z.

The "only if" part. Assume that $\mathcal{E}$ has right almost split $\mathfrak{s}_{\mathcal{E}}$-triangles. Let Z be an indecomposable non-$\mathbb{E}_{\mathcal{E}}$-projective object. Then there is an almost split $\mathfrak{s}_{\mathcal{E}}$-triangle $E \to Y \to Z \overset{\delta}{\dashrightarrow}$ in $\mathcal{E}$. For any $L \in \mathcal{E}$, by assumption $\mathbb{E}_{\mathcal{E}}(Z, L) \in \operatorname{mod} R$, and hence $\overline{\operatorname{Hom}}_{\mathcal{E}}(L, E) \in \operatorname{mod} R$ by Lemma 3.6. Thus, τZ has a right injectively stable $\mathcal{E}$-approximation by Proposition 4.6.

(2) is proved similarly. ∎

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