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# SOME NEW CHARACTERIZATIONS OF THE WULFF SHAPE

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**Abstract.** We consider characterizations of a closed orientable connected hypersurface $X : M \to \mathbb{R}^{n+1}$ in terms of anisotropic higher order mean curvature, which is an extension of the usual mean curvature. On the one hand, we improve the result of Leyla Onat (2010) and remove the convexity condition there. On the other hand, we generalize the results on rigidity of standard spheres to the anisotropic setting and show that if $M$ satisfies some assumptions involving constant linear combinations of anisotropic mean curvatures or other relations, then it must be the Wulff shape, up to translations and homotheties.

**1. Introduction.** A fundamental question in differential geometry is whether the geodesic sphere is the only compact orientable hypersurface in Euclidean space satysfying certain conditions on higher order mean curvatures $H_r$, $r = 1, \dots, n$, as in the famous Alexandrov theorem. Recall that $H_1$, $H_2$ and $H_n$ are the mean curvature, the scalar curvature and the Gauss–Kronecker curvature respectively. This type of rigidity theorem has aroused much interest and there are numerous relevant studies.

First of all, let us recall some classical results on rigidity of the sphere. In 1956, Alexandrov [A56] proved that the only compact embedded hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature is the sphere. Then in [R88] and [R87], Ros proved successively that the only compact embedded hypersurface with constant scalar curvature or any $r$th mean curvature $H_r$ is also the sphere. (See also Montiel and Ros [MR91] for another proof.) In the immersion case, though there exist compact non-spherical examples with constant mean curvature (see examples constructed by Hsiang–Teng–Yu [HTY83] and Wente [W86] in higher dimensions and in $\mathbb{R}^3$ respectively), one can guarantee a closed immersed hypersurface with constant (higher order) mean curvature to be a sphere by adding some other conditions. For example, in 1984, Barbosa and do Carmo [BD84] proved that the only closed

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stable hypersurface with $H_1 =$ constant is the sphere, and in 1993, Alencar, do Carmo and Rosenberg [ADR93] proved the same for general $H_r$. Additionally, Koh [K00] proved that a closed oriented hypersurface with constant ratio of two mean curvature fuctions is also a sphere. In [H89], Hopf showed that if the dimension of $M$ is 2 and the genus of $M$ is zero, then $M$ must be homeomorphic to $\mathbb{S}^2$, which partially answered the well-known Hopf conjecture. As for the constant scalar curvature, a closed immersed hypersurface can also be a sphere under additional assumptions on curvatures or norm of the total umbilicity tensor of $M$, and so on; see [CY77, L96, AMP18] and the references therein. In 1989, Ecker and Huisken [EH89] proved that a closed, connected hypersurface in a space form whose sectional curvature is nonnegative is an umbilical sphere provided a certain positive and symmetric function of the principal curvature is constant; the function involves $H_r^{1/r}$.

Apart from assuming that the $r$th mean curvature $H_r$ is constant for some $r$, there are various other theorems on rigidity of the sphere in $\mathbb{R}^{n+1}$ under other conditions, e.g. involving distinct higher order mean curvatures. For instance, based on the Minkowski integral formula, Feeman and Hsiung [FH59] proved that if there exists an integer $1 \leq s \leq n-1$ such that $H_s$ is positive and the support function $p$ satisfies either $p \leq -H_{s-1}/H_s$ or $p \geq -H_{s-1}/H_s$, then $M$ is an $n$-sphere. In [S60], Stong proved other characterizations of the sphere and three of them are the following:

THEOREM 1.1. *Let $M^n$ denote a closed orientable hypersurface immersed in $\mathbb{R}^{n+1}$. If there are integers $i$ and $s$, $1 \leq i < s \leq n$, with $H_i, \dots, H_s > 0$ and constants $C_j \geq 0$ for $i \leq j \leq s-1$ such that at all points of $M$ one has $H_s = \sum_{j=i}^{s-1} C_j H_j$, then $M$ is a sphere.*

THEOREM 1.2. *Let $M^n$ denote a closed orientable hypersurface immersed in $\mathbb{R}^{n+1}$. Suppose there is an integer $s$, $1 < s \leq n$, with $H_i > 0$ for $i = 1, \dots, s$ and a constant $c$ with*

$$H_{s-1}^{\frac{1}{s-1}} \geq c \geq H_s^{1/s}$$

*at all points of $M$. Assume additionally that the support function is of fixed sign throughout $M$. Then $M$ is a sphere.*

THEOREM 1.3. *Let $M^n$ denote a closed orientable hypersurface immersed in $\mathbb{R}^{n+1}$. Suppose there is an integer $s$, $1 < s \leq n$, with $H_{s-1}, H_s > 0$ and a constant $c$ with*

$$\frac{H_{s-1}}{H_s} \geq c \geq \frac{H_{s-2}}{H_{s-1}}$$

*at all points of $M$. Assume additionally that the support function is of fixed sign throughout $M$. Then $M$ is a sphere.*

In 2014, Wu and Xia [WX14] obtained the following theorem in space forms which involves linear combinations of different higher order mean curvatures similarly to Theorem 1.1:

THEOREM 1.4. *Let $0 \le k \le n$ be an integer and $M^n$ a closed, $k$-convex embedded hypersurface in $\mathbb{R}^{n+1}$. Suppose one of the following cases holds:*

(i) *There are nonnegative constants $\{a_j\}_{j=l}^{k}$ and $\{b_i\}_{i=1}^{l-1}$, $2 \le l < k \le n$, not all vanishing, such that*

$$\sum_{j=l}^{k} a_j H_j = \sum_{i=1}^{l-1} b_i H_i.$$

(ii) *There are nonnegative constants $a_0$ and $\{b_i\}_{i=1}^{k}$, not all vanishing, such that*

$$a_0 = \sum_{i=1}^{k} b_i H_i.$$

*Then $M$ is a geodesic hypersphere.*

Motivated by the previous work in the usual isotropic case, our aim is to investigate rigidity of closed orientable hypersurfaces in Euclidean space via the so-called anisotropic higher order mean curvatures.

Here is a brief introduction to the anisotropic setting.

Let $F : \mathbb{S}^n \to \mathbb{R}^+$ be a smooth, positive function which satisfies the convexity condition

$$(D^2F + FI)_x > 0, \quad \forall x \in \mathbb{S}^n, \tag{1.1}$$

where $D^2F$ is the intrinsic Hessian of $F$ on $\mathbb{S}^n$ and $I$ is the identity on $T_x\mathbb{S}^n$; "$> 0$" means the matrix is positive definite. We consider the map

$$\Phi : \mathbb{S}^n \to \mathbb{R}^{n+1}, \quad x \mapsto F(x)x + DF(x). \tag{1.2}$$

Its image $\mathcal{W}_F = \Phi(\mathbb{S}^n)$ is a smooth strictly convex hypersurface in $\mathbb{R}^{n+1}$ which is called the *Wulff shape* of $F$ [C04, HL08a, HL08b, HL08c, HL$^+$09, KP05, P98, W06]. For $F \equiv 1$, we observe that the Wulff shape of $F$ is exactly the standard sphere $\mathbb{S}^n$.

Using the concept of the Wulff shape, we can define the anisotropic Gauss map, anisotropic shape operator, and anisotropic principal curvatures of $M$ similarly to the isotropic case (see Section 2) and then make some studies in the anisotropic setting.

Actually, from the definition above, we can view the Wulff shape in the anisotropic setting as a natural generalization of the sphere. So one can expect that the Wulff shape has rigidity properties similar to the sphere. In fact, numerous classical characterizations of the sphere have been generalized to the anisotropic case.

As early as 1998, Palmer [P98] proved that the only closed stable hypersurface with constant anisotropic mean curvature is the Wulff shape. (See also Winklmann [W06] for a different proof.) This result is a typical anisotropic version of the isotropic result of [BD84]. Moreover, in [HL08b] and [HL08c], He and Li showed respectively that if $M$ is a closed orientable hypersurface with constant anisotropic higher order mean curvature or constant quotient of two consecutive anisotropic higher order mean curvatures, then $M$ is stable if and only if $M$ is the Wulff shape, which are counterparts of [BC97] and [K00].

Further, He and Li [HL08a] obtained a series of characterizations of the Wulff shape by proving the so-called anisotropic Minkowski formula. One of them is

THEOREM 1.5. *Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional compact convex hypersurface, and $F : \mathbb{S}^n \to \mathbb{R}^+$ a smooth function which satisfies* (1.1). *If $H_r^F/H_k^F$ is constant for some $k$ and $r$ with $0 \le k < r \le n$, where $H_r^F$ denotes the $r$th anisotropic mean curvature, then up to translations and homotheties, $X(M)$ is the Wulff shape.*

Also in [HL$^+$09], He et al. gave an anisotropic version of the Alexandrov theorem [A56]. Namely, they proved that the only closed orientable embedded hypersurface of dimension $\ge 2$ in Euclidean space is the Wulff shape, up to translations and homotheties. (In 2013, Ma and Xiong [MX13] gave another proof by applying the evolution method.) Moreover, Koiso and Palmer [KP10] derived an anisotropic Hopf theorem that the only closed genus zero hypersurfaces with constant anisotropic mean curvature are rescalings of the Wulff shape.

In 2010, Onat [O10] proved the following generalization of Theorem 1.5.

THEOREM 1.6. *Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional compact convex hypersurface, and $F : \mathbb{S}^n \to \mathbb{R}^+$ a smooth function which satisfies* (1.1). *If there are nonnegative constants $C_1, \dots, C_{r-1}$, $1 \le r \le n$, such that*

$$H_r^F = \sum_{i=1}^{r-1} C_i H_i^F,$$

*then $X(M)$ is the Wulff shape, up to translations and homotheties.*

In the proof of this theorem, the condition of convexity only served in interpreting the $r$-convexity of $M$. But in fact, we do not need such a strong assumption and are able to derive the conclusion for $2 \le r \le n$ by making use of the compactness of $M$ and the condition on the linear combination of anisotropic higher order mean curvatures—see Theorem 1.7(1).

Moreover, in [CD13], Colares and da Silva proved that a closed oriented stable hypersurface with $\frac{(r+1)\binom{n}{r+1}H^F_{r+1}}{a(k+1)\binom{n}{k+1}H^F_{k+1}-b}$ = constant for some $0 \leq k < r \leq n-1$ and some restricted $a, b$ must be the Wulff shape. In 2018, da Silva et al. [DDV18] showed that if there exist nonnegative real numbers $a_k$, $k \in \{r, \ldots, s\}$, $0 \leq r \leq s \leq n-1$, not all zero, such that $H^F_{s+1}$ is positive and $a_r b_r H^F_{r+1} + \cdots + a_s b_s H^F_{s+1}$ = constant with $b_k = (k+1)\binom{n}{k+1}$, then the closed hypersurface is stable if and only if it is the Wulff shape. Later, Roth and Upadhyay [RU19] also showed that the only closed, connected hypersurface with non-vanishing higher order anisotropic mean curvature $H^F_r$, $r \in \{2, \ldots, n\}$, and the relation $H^F_r = aH^F + b$ for some real constants $a \geq 0, b > 0$ embedded in Euclidean space must be the Wulff shape, up to translations and homotheties. These results are anisotropic analogues of the results of [DDV16] and [D18] respectively, and both were obtained using a similar method to the isotropic case.

However, due to the asymmetry of the anisotropic Weingarten operator (see Section 2) and some other difficulties, some generalizations of standard results to the anisotropic case are not trivial and are still to be investigated.

In this paper, we will prove several characterizations of the Wulff shape in terms of anisotropic higher order mean curvatures. We remove the convexity condition in Theorem 1.6 and generalize Theorem 1.4 to the anisotropic setting. We also weaken the assumption in Theorem 1.1. Here we mention that our ideas are greatly inspired by [KY04] and [AC05], where the authors proved similar characterizations of ellipsoids and umbilical hypersurfaces in de Sitter space respectively.

More precisely, our results are the following:

Theorem 1.7. (1) *Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional compact hypersurface, and $F : \mathbb{S}^n \to \mathbb{R}^+$ a smooth function which satisfies* (1.1). *If there are nonnegative constants $C_1, \ldots, C_{r-1}$, $2 \leq r \leq n$, such that*

$$H^F_r = \sum_{i=1}^{r-1} C_i H^F_i \tag{1.3}$$

*then $X(M)$ is the Wulff shape, up to translations and homotheties.*

(2) *Let $X : M \to \mathbb{R}^{n+1}$ be a closed $r$-convex hypersurface in $\mathbb{R}^{n+1}$ ($1 \leq r \leq n$), and $F : \mathbb{S}^n \to \mathbb{R}^+$ a smooth function which satisfies* (1.1). *Suppose one of the following cases holds:*

(i) *There are nonnegative constants $\{a_j\}_{j=l}^r$ and $\{b_i\}_{i=1}^{l-1}$, $2 \leq l < r \leq n$, not all vanishing, such that*

$$\sum_{j=l}^{r} a_j H^F_j = \sum_{i=1}^{l-1} b_i H^F_i. \tag{1.4}$$

(ii) *$M$ is embedded and there are nonnegative constants $a_0$ and $\{b_i\}_{i=1}^r$, not all vanishing, such that*

$$a_0 = \sum_{i=1}^{r} b_i H_i^F. \tag{1.5}$$

*Then $M$ is the Wulff shape, up to translations and homotheties.*

Theorem 1.7(1) has a natural corollary which also extends Theorem 1.5.

COROLLARY 1.8. *Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional compact hypersurface, and $F : \mathbb{S}^n \to \mathbb{R}^+$ a smooth function which satisfies* (1.1). *If $H_r^F/H_k^F$ is constant for some $k$ and $r$ with $1 \le k < r \le n$, where $H_r^F$ denotes the $r$th anisotropic mean curvature, then up to translations and homotheties, $X(M)$ is the Wulff shape.*

We also give an anisotropic analogue of Theorems 1.2 and 1.3.

THEOREM 1.9. *Let $X : M \to \mathbb{R}^{n+1}$ be an $n$-dimensional compact hypersurface, and $F : \mathbb{S}^n \to \mathbb{R}^+$ a smooth function which satisfies* (1.1).

(1) *Suppose there exists an integer $s$, $1 < s \le n$, and a constant $c$ such that $H_s^F > 0$ and*

$$(H_{s-1}^F)^{\frac{1}{s-1}} \ge c \ge (H_s^F)^{1/s} \tag{1.6}$$

*at all points of $M^n$. Assume additionally that $\langle X, \nu \rangle$ has fixed sign, where $\nu$ denotes the unit inner normal of $M$. Then up to translations and homotheties, $X(M)$ is the Wulff shape.*

(2) *Suppose there exists an integer $s$, $1 < s \le n$, and a constant $c$ such that $H_{s-1}^F, H_s^F > 0$ and*

$$\frac{H_{s-1}^F}{H_s^F} \ge c \ge \frac{H_{s-2}^F}{H_{s-1}^F} \tag{1.7}$$

*at all points of $M$. Assume additionally that $\langle X, \nu \rangle$ has fixed sign. Then up to translations and homotheties, $X(M)$ is the Wulff shape.*

The rest of the paper is organized as follows.

In Section 2, we introduce some basic concepts in anisotropic geometry and some key lemmas such as the anisotropic Newton–Maclaurin formula and anisotropic Heintze–Karcher inequality which will be essential in proving our results. In Section 3, we give a proof for Theorem 1.7 and its Corollary 1.8. In Section 4, we prove Theorem 1.9.

**2. Preliminaries.** In this section, we first recall some basic geometric quantities in anisotropic geometry.

Let $X : M^n \to \mathbb{R}^{n+1}$ $(n \ge 2)$ be a smooth isometric immersion of an $n$-dimensional closed, connected and orientable hypersurface. Let $\nu : M \to \mathbb{S}^n$

denote its Gauss map, that is, $\nu$ is the unit inner normal vector field globally defined on $M$. Then we can define the *anisotropic Gauss map* of $M$ as

$$\nu_F : M \to \mathcal{W}, \quad \nu_F = -\Phi \circ \nu.$$

Let $A_F := D^2F + FI$, $S_F := -d(\Phi \circ \nu) = -A_F \circ d\nu$. The operator $S_F$ is usually called the *anisotropic Weingarten operator* (or $F$-Weingarten operator) and we call the eigenvalues of $S_F$ *anisotropic principal curvatures*, denoted by $\lambda_1^F, \dots, \lambda_n^F$. From the positive definiteness of $A_F$, there exists a nonsingular matrix $C$ such that $A_F = C^T C$, so $S_F = -A_F \circ d\nu$ has the same eigenvalues as the real symmetric matrix $-C \circ d\nu \circ C^T$. Thus, $\lambda_i^F$ $(i = 1, \dots, n)$ are all real. Moreover, if the principal curvatures of $X$ are positive, so are the anisotropic principal curvatures. This fact will be used frequently in the later proof. But note that though $A_F$ and $d\nu$ are symmetric, $S_F$ is symmetric if and only if $A_F$ and $d\nu$ commute, which does not occur in general. If $\lambda_1^F = \cdots = \lambda_n^F$ holds everywhere on $M$, then $M$ is called *anisotropic umbilical* for $F$.

Denote by $\sigma_r$ the elementary symmetric functions of the anisotropic principal curvatures $\lambda_1^F, \dots, \lambda_n^F$, i.e.,

$$\sigma_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1}^F \cdots \lambda_{i_r}^F, \quad 1 \le r \le n. \tag{2.1}$$

For convenience, we set $\sigma_0 = 1$. The normalized *anisotropic mean curvature* $H_r^F$ is defined by $H_r^F = \sigma_r / \binom{n}{r}$. We note that when $F = 1$, $H_r^F$ is the standard $r$th normalized mean curvature $H_r$.

Now, we give some auxiliary lemmas which are not only necessary in our proof but also useful in other anisotropic problems.

First, we recall inequalities concerning the $(r+1)$th anisotropic mean curvatures which are derived from the classical Newton–Maclaurin formula and Gårding's result [G59, HL08b, DDV18, CD13].

LEMMA 2.1. *Let $X : M^n \to \mathbb{R}^{n+1}$ be a closed orientable hypersurface, and $F : \mathbb{S}^n \to \mathbb{R}^+$ a smooth function which satisfies* (1.1). *Assume $H_{r+1}^F > 0$ holds at every point of $M$.*

(i) *If $1 \le k \le r$, then $H_k^F > 0$ at every point of $M$.*
(ii) $H_{r+1}^F / H_r^F \le \cdots \le H_{j+1}^F / H_j^F \le \cdots \le H_2^F / H_1^F \le H_1^F$, *and hence*

$$H_{j+1}^F H_r^F - H_j^F H_{r+1}^F \ge 0, \quad \forall 0 \le j \le r.$$

*Equality holds (for all $j$) if and only if $X(M)$ is the Wulff shape, up to translations and homotheties.*
(iii) $(H_{r+1}^F)^{1/(r+1)} \le (H_r^F)^{1/r} \le \cdots \le (H_2^F)^{1/2} \le H_1^F$. *Equality holds throughout if and only if $X(M)$ is the Wulff shape, up to translations and homotheties.*

The following famous integral formula, proved by He and Li [HL08a], generalizes the classical Hsiung–Minkowski formula. This formula plays a crucial role in the rigidity results involving anisotropic $r$th mean curvatures.

LEMMA 2.2. *Let $X : M \to \mathbb{R}^{n+1}$ be a compact hypersurface, and $F : \mathbb{S}^n \to \mathbb{R}^+$ a smooth function which satisfies* (1.1). *Then*

$$\int_M \big(F(\nu)H_r^F + H_{r+1}^F\langle X,\nu\rangle\big)\, dA_X = 0, \qquad r = 0,1,\ldots,n-1, \tag{2.2}$$

*where $dA_X$ is the induced area element of $M$.*

Similar to the classical hypersurface theory, we have the following characterization for anisotropic umbilical hypersurfaces in $\mathbb{R}^{n+1}$ [HL$^+$09].

LEMMA 2.3. *Let $X : M \to \mathbb{R}^{n+1}$ be a compact hypersurface, and $F : \mathbb{S}^n \to \mathbb{R}^+$ a smooth function which satisfies* (1.1). *If $n \geq 2$ and $\lambda_1^F = \cdots = \lambda_n^F \neq 0$, then $\lambda_1^F = \cdots = \lambda_n^F =$ constant, so up to translations and homotheties, $X(M)$ is the Wulff shape.*

Finally, we will also need the following anisotropic Heintze–Karcher type inequality from [HL$^+$09, MX13].

LEMMA 2.4. *Let $X : M \to \mathbb{R}^{n+1}$ be a closed embedded hypersurface, and $F : \mathbb{S}^n \to \mathbb{R}^+$ a smooth function which satisfies* (1.1). *If the anisotropic mean curvature $H_1^F$ with respect to the inner normal $\nu$ is everywhere positive on $M$, then*

$$\int_M \frac{F(\nu)}{H_1^F}\, dA_X \geq (n+1)V, \tag{2.3}$$

*where $V$ is the algebraic $(n+1)$-volume of the compact domain determined by $M$ and is given by*

$$V = -\frac{1}{n+1}\int_M \langle X,\nu\rangle\, dA_X. \tag{2.4}$$

*Moreover, equality holds in* (2.3) *if and only if $M$ is anisotropic umbilical.*

**3. Proof of Theorem 1.7.** In this section, we give a proof of Theorem 1.7 and Corollary 1.8.

**3.1. Proof of Theorem 1.7(1).** First of all, we present two key lemmas to be used in the proof of Theorem 1.7.

LEMMA 3.1. *Under the assumptions of Theorem* 1.7(1), *$M$ is $r$-convex, i.e. $H_r^F(p) > 0$ for all $p \in M$.*

*Proof.* Since $M$ is a compact, orientable hypersurface, there exists at least one point $p_0 \in M$ such that all the principal curvatures with respect to $\nu$ are positive. By the definition of the anisotropic Weingarten operator $S_F$

and the positivity of $A_F$, all the anisotropic principal curvatures are positive at $p_0$. So, $H_r^F(p_0) > 0$.

Let $U$ be the connected component of the set $\{p \in M \mid H_r^F(p) > 0\}$ containing $p_0$. Since $p_0 \in U$ and $H_r^F$ is a continuous function on $M$, $U$ is a nonempty open subset of $M$. Note that

$$H_r^F(p_0) = \sum_{i=1}^{r-1} C_i H_i^F(p_0) > 0, \tag{3.1}$$

so at least one $C_i$ is positive, say $C_k$. Then following the ideas of Montiel and Ros [MR91, Lemma 1] or Barbosa and Colares [BC97, Lemma 3.3], for any point $p \in U$ we have

$$(H_i^F)^{r/i}(p) \geq H_r^F(p) > 0, \quad \forall i = 1, \ldots, r-1.$$

This implies

$$H_i^F(p) > 0, \quad i = 1, \ldots, r-1, \tag{3.2}$$

and then

$$(H_k^F)^{r/k}(p) \geq H_r^F(p) = \sum_{i=1}^{r-1} C_i H_i^F(p) \geq C_k H_k^F(p) > 0, \quad \forall p \in U. \tag{3.3}$$

$$H_k^F(p) \geq C_k^{\frac{k}{r-k}}. \tag{3.4}$$

As a result,

$$H_r^F(p) \geq C_k H_k^F(p) \geq C_k C_k^{\frac{k}{r-k}} = C_k^{\frac{r}{r-k}} > 0. \tag{3.5}$$

Denote

$$C_0 = \min\{C_k^{\frac{r}{r-k}} \mid 1 \leq k < r \leq n,\ k, r \in \mathbb{Z}^+, C_k > 0\};$$

then

$$H_r^F(p) \geq C_0 > 0. \tag{3.6}$$

This means that for any point $p \in U$,

$$p \in \{q \in M \mid H_r^F(q) \geq C_0 > 0\},$$

which implies that $U$ is also closed. Therefore $M = U$ and Lemma 3.1 is proved. ∎

Lemma 3.2. *Under the conditions of Theorem* 1.7(1),

$$H_{r-1}^F \geq \sum_{i=1}^{r-1} C_i H_{i-1}^F. \tag{3.7}$$

*Furthermore, if equality holds, then all the anisotropic principal curvatures are the same.*

*Proof.* From Lemmas 3.1 and 2.1(i), we have

$$H_i^F > 0, \quad i = 1, \ldots, r. \tag{3.8}$$

Since by (1.3),

$$H_r^F = \sum_{i=1}^{r-1} C_i H_i^F,$$

we derive

$$1 = \sum_{i=1}^{r-1} C_i \frac{H_i^F}{H_r^F} \geq \sum_{i=1}^{r-1} C_i \frac{H_{i-1}^F}{H_{r-1}^F}$$

by use of Lemma 2.1(ii). In turn,

$$H_{r-1}^F - \sum_{i=1}^{r-1} C_i H_{i-1}^F \geq 0. \tag{3.9}$$

If equality holds, we have

$$\frac{H_i^F}{H_r^F} = \frac{H_{i-1}^F}{H_{r-1}^F},$$

which implies that all the anisotropic principal curvatures are the same as a result of Lemma 2.1(ii). ∎

*Proof of Theorem 1.7(1).* Since $F$ is positive, from (3.9) we have

$$F H_{r-1}^F \geq \sum_{i=1}^{r-1} C_i F H_{i-1}^F. \tag{3.10}$$

Then by (2.2),

$$\begin{aligned} 0 &\leq \int_M \Big( F H_{r-1}^F - \sum_{i=1}^{r-1} C_i F H_{i-1}^F \Big)\, dA_X \\ &= -\int_M H_r^F \langle X, \nu \rangle \, dA_X + \sum_{i=1}^{r-1} C_i \int_M H_i^F \langle X, \nu \rangle \, dA_X \\ &= -\int_M \Big( H_r^F - \sum_{i=1}^{r-1} C_i H_i^F \Big) \langle X, \nu \rangle \, dA_X = 0. \end{aligned} \tag{3.11}$$

So

$$\int_M F \Big( H_{r-1}^F - \sum_{i=1}^{r-1} C_i H_{i-1}^F \Big)\, dA_X = 0,$$

which implies that

$$H_{r-1}^F = \sum_{i=1}^{r-1} C_i H_{i-1}^F.$$

Hence, by Lemmas 3.2 and 2.3, all the anisotropic principal curvatures are equal and so $X(M)$ is the Wulff shape, up to translations and homotheties. ■

### 3.2. Proof of Theorem 1.7(2)(i)

LEMMA 3.3. *Under the assumption of Theorem* 1.7(2)(i), *we have*

$$\sum_{j=l}^{r} a_j H_{j-1}^F \geq \sum_{i=1}^{l-1} b_i H_{i-1}^F, \tag{3.12}$$

*and equality holds if and only if all the anisotropic principal curvatures are equal.*

*Proof.* Since $M$ is $r$-convex and at least one of $\{b_i\}_{i=1}^{l-1}$ is not zero, we can easily see that

$$\sum_{i=1}^{l-1} b_i H_i^F > 0. \tag{3.13}$$

Meanwhile, we have

$$H_i^F H_{j-1}^F \geq H_{i-1}^F H_j^F, \quad 1 \leq i < j \leq r, \tag{3.14}$$

by Lemma 2.1(ii). Multiplying (3.14) by $a_j$ and $b_i$ and summing over $j$ and $i$, we get

$$\sum_{j=l}^{r} a_j H_{j-1}^F \sum_{i=1}^{l-1} b_i H_i^F \geq \sum_{j=l}^{r} a_j H_j^F \sum_{i=1}^{l-1} b_i H_{i-1}^F. \tag{3.15}$$

Then we find from (1.4), (3.13) and (3.15) that (3.12) holds. Equality holds if and only if

$$H_i^F H_{j-1}^F = H_{i-1}^F H_j^F, \quad 1 \leq i < j \leq r.$$

Therefore, from Lemma 2.1(ii) we can see that $X(M)$ must be anisotropic umbilical. ■

*Proof of Theorem 1.7(2)(i).* First, using (1.4) and (2.2), we get

$$\begin{aligned} 0 &= \int_M \Big( \sum_{j=l}^{r} a_j H_j^F - \sum_{i=1}^{l-1} b_i H_i^F \Big) \langle X, \nu \rangle \, dA_X \\ &= \int_M \Big( -\sum_{j=l}^{r} a_j H_{j-1}^F F + \sum_{i=1}^{l-1} b_i H_{i-1}^F F \Big) \, dA_X \\ &= -\int_M F \Big( \sum_{j=l}^{r} a_j H_{j-1}^F - \sum_{i=1}^{l-1} b_i H_{i-1}^F \Big) \, dA_X. \end{aligned} \tag{3.16}$$

Since $F$ is positive, from (3.12) and (3.16) we see that (3.12) must hold. Therefore, from Lemmas 3.3 and 2.3, all the anisotropic curvatures of $M$ are the same, which means that $X(M)$ is the Wulff shape, up to translations and homotheties. ■

**3.3. Proof of Theorem 1.7(2)(ii).** Using (1.5), (2.2), (2.3), (2.4) and Lemma 2.2(ii), we have

$$\begin{aligned}
-a_0(n+1)V(M) &= \int_M a_0\langle X,\nu\rangle\, dA_X = \int_M \Big(\sum_{i=1}^r b_i H_i^F\Big)\langle X,\nu\rangle\, dA_X \\
&= -\int_M F\Big(\sum_{i=1}^r b_i H_{i-1}^F\Big)\, dA_X = -\int_M F\sum_{i=1}^r b_i H_{i-1}^F \frac{H_1^F}{H_1^F}\, dA_X \\
&\le -\int_M F\sum_{i=1}^r b_i \frac{H_i^F}{H_1^F}\, dA_X = -\int_M F\frac{a_0}{H_1^F}\, dA_X \\
&\le -a_0(n+1)V(M).
\end{aligned}$$

So

$$-\int_M \frac{F}{H_1^F} a_0\, dA_X = -a_0(n+1)V(M),$$

i.e.,

$$\int_M \frac{F}{H_1^F}\, dA_X = (n+1)V(M).$$

From Lemmas 2.4 and 2.3 we get the assertion.

**3.4. Proof of Corollary 1.8.** As mentioned in the proof of Lemma 3.1, there exists a point $p_0$ such that the anisotropic principal curvatures at $p_0$ are positive and therefore $H_r^F(p_0)$ and $H_k^F(p_0)$ are positive.

Denote

$$H_r^F/H_k^F \equiv C.$$

Since $\frac{H_r^F}{H_k^F}(p_0) > 0$, we have $C > 0$ and $H_r^F = CH_k^F$. So from Theorem 1.7(1), $X(M)$ is the Wulff shape, up to translations and homotheties.

## 4. Proof of Theorem 1.9

**4.1. Proof of Theorem 1.9(1).** Set $r = 0$ in (2.2) we get

$$\int_M \big(F(\nu) + H_1^F\langle X,\nu\rangle\big)\, dA_X = 0. \tag{4.1}$$

Since $H_s^F > 0$ for some $1 < s \le n$, $H_1^F$ must be positive due to Lemma 2.1(i) and therefore

$$H_1^F \ge (H_{s-1}^F)^{\frac{1}{s-1}} \ge c. \tag{4.2}$$

Since

$$c \geq (H_s^F)^{1/s} \geq 0$$

and there exists at least one point $p_0$ such that $H_s^F(p_0) > 0$, $c$ must be strictly positive. Then by the positivity of $F$ and the condition that $\langle X, \nu \rangle$ has fixed sign, we can deduce from (4.1) that

$$\langle X, \nu \rangle < 0. \tag{4.3}$$

Then on the one hand, using (4.1) and (4.2) we have

$$\int_M F H_{s-1}^F \, dA_X \geq \int_M F c^{s-1} dA_X = -\int_M c^{s-1} H_1^F \langle X, \nu \rangle \, dA_X. \tag{4.4}$$

On the other hand, from (1.6), (2.2), (4.2) and (4.3),

$$\begin{aligned} \int_M F H_{s-1}^F dA_X &= -\int_M H_s^F \langle X, \nu \rangle \, dA_X \\ &\leq -c^s \int_M \langle X, \nu \rangle \, dA_X = -c^{s-1} c \int_M \langle X, \nu \rangle \, dA_X \\ &\leq -\int_M c^{s-1} H_1^F \langle X, \nu \rangle \, dA_X. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) we get

$$\int_M F(H_{s-1}^F - c^{s-1}) \, dA_X = 0, \tag{4.6}$$

$$\int_M c^{s-1}(H_1^F - c)\langle X, \nu \rangle \, dA_X = 0. \tag{4.7}$$

Thus again use the positivity of $F$ and (4.3), (4.6), (4.7), we get

$$H_{s-1}^F = c^{s-1}, \qquad H_1^F = c.$$

This implies that

$$H_1^F = c = (H_{s-1}^F)^{\frac{1}{s-1}}.$$

By Lemmas 2.1(iii) and 2.3, $X(M)$ is the Wulff shape, up to translations and homotheties.

**4.2. Proof of Theorem 1.9(2).** Setting $r = s-1$ and $r = s-2$ in (2.2) we have

$$\int_M \big(F(\nu) H_{s-1}^F + H_s^F \langle X, \nu \rangle\big) \, dA_X = 0, \tag{4.8}$$

$$\int_M \big(F(\nu) H_{s-2}^F + H_{s-1}^F \langle X, \nu \rangle\big) \, dA_X = 0. \tag{4.9}$$

Since $H_{s-1}^F, H_s^F, F > 0$ and $\langle X, \nu \rangle$ has fixed sign, from (4.8) we obtain

$$\langle X, \nu \rangle < 0. \tag{4.10}$$

From (1.7), (4.9) and (4.10), we get

$$\int_M F(\nu)H^F_{s-2}dA_X = -\int_M H^F_{s-1}\langle X,\nu\rangle\, dA_X \tag{4.11}$$
$$\geq -\int_M cH^F_s\langle X,\nu\rangle\, dA_X = \int_M cH^F_{s-1}F(\nu)\, dA_X$$
$$\geq \int_M H^F_{s-2}F(\nu)\, dA_X = -\int_M H^F_{s-1}\langle X,\nu\rangle\, dA_X.$$

This implies that

$$\int_M (H^F_{s-1} - cH^F_s)\langle X,\nu\rangle\, dA_X = 0.$$

So from (1.7) and (4.10) we obtain

$$H^F_{s-1} = cH^F_s,$$

and from Corollary 1.8, the assertion follows.

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