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Helson operators and coinvariant subspaces

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Abstract. We give an explicit expression of the functions in the finite-dimensional coinvariant subspace of the Hardy and weighted Bergman spaces over the infinite polydisc. As an application, an alternative form of the Perfekt–Pushnitski theorem characterizing Helson operators with finite rank is obtained. Interestingly, it turns out that a Helson operator is automatically bounded if it is of finite rank.

1. Introduction. Let $\mathbb{D}$ denote the open unit disc with boundary $\mathbb{T}$. Let dA be the normalized area measure on $\mathbb{D}$. For $\alpha > 1$, the weighted area measure dA_α on $\mathbb{D}$ is defined by

$$dA_\alpha(z) = (\alpha - 1)(1 - |z|^2)^{\alpha-2}\, dA(z),$$

and

$$dV_\alpha = dA_\alpha \times dA_\alpha \times \cdots$$

is the associated infinite product measure on the (countably) infinite polydisc $\mathbb{D}^\infty$. Let $d\sigma$ be the Haar measure of the (countably) infinite torus $\mathbb{T}^\infty$. Let $\mathscr{P}_\infty$ be the polynomial ring of infinite many variables. For $\alpha > 1$ and $p > 0$, the weighted Bergman space $A^p_\alpha(\mathbb{D}^\infty)$ of the infinite polydisc is defined to be the closure in $L^p(\mathbb{D}^\infty, dV_\alpha)$ of $\mathscr{P}_\infty$. The Hardy space $H^p(\mathbb{T}^\infty)$ of the infinite torus is defined as the closure in $L^p(\mathbb{T}^\infty, d\sigma)$ of $\mathscr{P}_\infty$. Similarly to the case of finite polydisc, the Hardy space $H^p(\mathbb{T}^\infty)$ is the limit space of $A^p_\alpha(\mathbb{D}^\infty)$ as $\alpha \to 1^+$, in the sense that

$$\lim_{\alpha\to 1^+} \|f\|_{A^p_\alpha(\mathbb{D}^\infty)} = \|f\|_{H^p(\mathbb{T}^\infty)}, \quad f \in \mathscr{P}_\infty.$$

For convenience, we use $A^p_1(\mathbb{D}^\infty)$ to stand for $H^p(\mathbb{T}^\infty)$ and use L^p_α to stand for $L^p(\mathbb{T}^\infty, d\sigma)$ if $\alpha = 1$ and $L^p(\mathbb{D}^\infty, dV_\alpha)$ if $\alpha > 1$. In the following contexts, we always assume that the parameter α associated with $A^p_\alpha(\mathbb{D}^\infty)$ is greater

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than or equal to 1. It is shown in [11, Theorem 8.1] that $A^2_\alpha(\mathbb{D}^\infty)$ can be regarded as reproducing kernel Hilbert spaces on the Hilbert multidisc $\mathbb{D}^\infty_2$ defined by

$$\mathbb{D}^\infty_2 := \ell^2 \cap \mathbb{D}^\infty = \{\zeta = (\zeta_1, \zeta_2, \dots) \in \ell^2 : |\zeta_j| < 1,\ j = 1, 2, \dots\}.$$

The reproducing kernel of $A^2_\alpha(\mathbb{D}^\infty)$ at $\lambda \in \mathbb{D}^\infty_2$ is given by

$$K^\alpha_\lambda(\zeta) = \prod_{j=1}^{\infty} \frac{1}{(1 - \bar{\lambda}_j \zeta_j)^\alpha},$$

where the infinite product converges almost everywhere with respect to the measure dV_α ($d\sigma$ if $\alpha = 1$) and in the norm of $A^2_\alpha(\mathbb{D}^\infty)$ by the martingale convergence theorem [11, Lemma 3.2]. Every function f in $A^2_\alpha(\mathbb{D}^\infty)$ is an analytic function on $\mathbb{D}^\infty_2$ admitting a power series expansion

$$f = \sum_{\kappa \in \mathbb{N}^\infty_0} c_\kappa z^\kappa,$$

where $\mathbb{N}^\infty_0$ is the additive semigroup of nonnegative integer-valued sequences with finite support and $z^\kappa = \prod_{j=1}^\infty z_j^{\kappa_j}$. For more information about these function spaces, we refer to [2, 4, 11, 19, 24].

Let $p = (p_1, p_2, \dots)$ be the ordered prime numbers. According to the fundamental theorem of arithmetic, each integer $n \in \mathbb{N}$ can be uniquely written as a product of prime numbers

$$n = p^{\kappa(n)} := \prod_{j=1}^{\infty} p_j^{\kappa_j}, \quad \kappa(n) \in \mathbb{N}^\infty_0.$$

Clearly, the map κ is a semigroup isomorphism from $(\mathbb{N}, \times)$ onto $(\mathbb{N}^\infty_0, +)$. The Bohr lift [5] associates a formal Dirichlet series with a formal power series of infinitely many variables:

$$\sum_{n=1}^{\infty} a_n n^{-s} \mapsto \sum_{n=1}^{\infty} a_n z^{\kappa(n)}.$$

By means of the Bohr lift, the Hardy and Bergman spaces over the infinite polydisc are isometrically isomorphic to the corresponding spaces of Dirichlet series [2, 3, 4, 19]. Due to this connection, the spaces $A^p_\alpha(\mathbb{D}^\infty)$ received considerable attention in recent years. In particular, these function spaces and related operator theory have been widely studied; see [2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 19, 21, 22, 24, 27, 31] and the references therein.

An interesting aspect of the Hardy space over the infinite torus is the study of Helson forms (multiplicative Hankel forms) initiated by Helson [21]. A Helson form is induced by a Helson matrix (multiplicative Hankel matrix) $\{\varrho_{nm}\}_{n,m=1}^\infty$ determined by a sequence $(\varrho_j)_{j=1}^\infty$. We refer to [27] for an explicit comparison of Helson matrices, Hankel matrices, and Hankel matrices on finitely many variables. In [4], the weighted Helson form related to the Bergman space of the infinite polydisc was studied. As observed by Helson [21], (weighted) Helson forms are naturally realized as small Hankel operators on the Hardy or Bergman spaces of the infinite polydisc (see (2.1)). Let P_α be the orthogonal projection from L^2_α onto $A^2_\alpha(\mathbb{D}^\infty)$. For $\varphi \in A^2_\alpha(\mathbb{D}^\infty)$, the *Helson operator* (small Hankel operator) H_φ in $A^2_\alpha(\mathbb{D}^\infty)$ is densely defined by

$$H_\varphi q = P_\alpha(\varphi V q), \quad q \in \mathscr{P}_\infty,$$

where V is given by $(Vf)(z) = f(\bar{z})$. Naturally, there are many interesting problems concerning Helson operators, such as boundedness, compactness, membership of Schatten classes, positivity etc. Much valuable work related to these problems is presented in [4, 6, 8, 9, 10, 21, 22, 25, 27, 28], and a detailed overview can be found in [27]. In particular, Perfekt and Pushnitski [27, Theorem 6.6] obtained a complete description of Helson matrices of finite rank and a complete characterization of bounded Helson forms of finite rank in terms of certain factorizable differential operators. Definitely, Perfekt and Pushnitski's approach can be applied to the weighted Bergman spaces $A^2_\alpha(\mathbb{D}^\infty)$ of the infinite polydisc. In this paper, we will provide an alternative characterization for finite rank Helson operators in $A^2_\alpha(\mathbb{D}^\infty)$ by studying finite-dimensional coinvariant subspaces of $A^2_\alpha(\mathbb{D}^\infty)$.

A closed subspace M of $A^2_\alpha(\mathbb{D}^\infty)$ is said to be an *invariant subspace* if it is invariant under multiplication by all coordinate functions. Similarly, a closed subspace N of $A^2_\alpha(\mathbb{D}^\infty)$ is said to be a *coinvariant subspace* if N is invariant under the action of all operators $T_{\bar{z}_j}$ with $j \in \mathbb{N}$, where $T_{\bar{z}_j} f = P_\alpha(\bar{z}_j f)$. Clearly, N is coinvariant if and only if $N^\perp$ is invariant. The theory of invariant and coinvariant subspaces is one of the most important parts of operator theory [15, 20]. On the Hardy and Bergman spaces of finite polydiscs, functions in finite-dimensional coinvariant subspaces are characterized by certain polynomial spaces [17, Theorem 3.3]. However, in the setting of the infinite polydisc, the situation turns out to be more complicated. On the one hand, the polynomial space is not enough to characterize finite-dimensional coinvariant subspaces. On the other hand, a description of finite rank Helson operators with homogeneous symbols is needed, and this is the content of Section 2. In Section 3, we obtain an explicit expression of functions in finite-dimensional coinvariant subspaces of $A^2_\alpha(\mathbb{D}^\infty)$ via functions admitting a homogeneous expansion of finite degree.

Applying the results in Section 3, we provide an alternative characterization of finite rank Helson operators on the Hardy and Bergman spaces of the infinite polydisc. Based on this characterization, it is shown that finite rank Helson operators are automatically bounded. In the case of finite polydiscs, the boundedness of finite rank Hankel operators is a direct corollary of the description of finite rank Hankel operators [17]. However, in the case of infinitely many variables, some new difficulties show up. Indeed, we need to show that the products $H_\varphi C_{\Phi_\lambda}$ of the Helson operators H_φ with homogeneous symbols and the composition operators C_{Φ_λ} are bounded (see Lemma 4.3), while C_{Φ_λ} may be unbounded. This difficulty is overcome by exploring the domain of the adjoint of the unbounded operator C_{Φ_λ}.

NOTATION. For a function f, we write $\widetilde{f}(z) = \overline{f(\bar{z})}$. For a subset M of $A^2_\alpha(\mathbb{D}^\infty)$, we write $\widetilde{M} = \{\widetilde{f} : f \in M\}$, and $[M]$ denotes the closure of M.

2. Helson operators with homogeneous symbols and multilinear forms. For a nonnegative integer m, let $\mathscr{P}_m$ denote the linear space of analytic m-homogeneous polynomials. Every element in $[\mathscr{P}_m]$ admits an m-homogeneous expansion $\sum_{|\kappa|=m} c_\kappa z^\kappa$. We begin with an interesting example which has been used in [8, 25] to show the existence of Schatten class Helson operators on $H^2(\mathbb{T}^\infty)$ lacking bounded symbols.

EXAMPLE 2.1. If $\varphi \in [\mathscr{P}_1]$, then H_φ has rank 2.

For $\varphi \in [\mathscr{P}_m]$ with $m \geq 2$, the Helson operator H_φ may not be of finite rank, for example, $\varphi = \sum_{j=1}^\infty c_j z_j^2 \in [\mathscr{P}_2]$ with $c_j \neq 0$ for infinitely many j. This can be derived from the next proposition which plays an important role in this paper.

PROPOSITION 2.2. *For $\varphi \in A^2_\alpha(\mathbb{D}^\infty)$, the following are equivalent:*

(a) *The Helson operator H_φ is of finite rank.*
(b) *φ lies in $M^\perp$ for some invariant subspace M with finite codimension.*
(c) *The coinvariant subspace generated by φ is finite-dimensional.*

Proof. (a)⇔(c): According to

$$H_\varphi \widetilde{p} = P_\alpha(\varphi V \widetilde{p}) = P_\alpha(\varphi \bar{p}) = T_{\bar{p}}\varphi, \quad p \in \mathscr{P}_\infty, \tag{2.1}$$

the Helson operator H_φ maps $\mathscr{P}_\infty$ onto the coinvariant subspace generated by φ.

(b)⇔(c): It is clear. ∎

According to (2.1), if H_φ is bounded, then

$$\ker H_\varphi = (\operatorname{ran} H_\varphi^*)^\perp = (\operatorname{ran} H_{\widetilde{\varphi}})^\perp = \mathfrak{N}_{\widetilde{\varphi}}^\perp, \tag{2.2}$$

where $\mathfrak{N}_{\widetilde{\varphi}}$ is the coinvariant subspace generated by $\widetilde{\varphi}$. In particular, $\ker H_\varphi$ is an invariant subspace. Moreover, if both H_{φ_1} and H_{φ_2} are bounded, then $\ker H_{\varphi_1} = \ker H_{\varphi_2}$ if and only if $\mathfrak{N}_{\varphi_1} = \mathfrak{N}_{\varphi_2}$.

In this section, it will be convenient to use the notation of symmetric bilinear form. Given $\varphi \in A^2_\alpha(\mathbb{D}^\infty)$, the corresponding weighted Helson form $[\cdot,\cdot]_\varphi$ is defined by

$$[f, g]_\varphi = \langle fg, \widetilde{\varphi}\rangle, \quad f, g \in \mathscr{P}_\infty.$$

There exists a natural connection between the weighted Helson form $[\cdot,\cdot]_\varphi$ and the Helson operator H_φ:

$$[f,g]_\varphi = \langle H_\varphi g,\, \widetilde{f}\rangle = \langle g,\, H_{\widetilde{\varphi}}\widetilde{f}\rangle, \quad f,g\in\mathscr{P}_\infty. \tag{2.3}$$

The weighted Helson form is said to be *of finite rank* if its kernel

$$\ker[\cdot,\cdot]_\varphi = \{f\in\mathscr{P}_\infty : [f,g]_\varphi = 0,\ \forall g\in\mathscr{P}_\infty\}$$

has finite codimension. By (2.3), the weighted Helson form $[\cdot,\cdot]_\varphi$ is bounded if and only if the Helson operator H_φ is bounded, and $[\cdot,\cdot]_\varphi$ is of finite rank if and only if H_φ is of finite rank.

For $\alpha\ge 1$, let $d_\alpha(n)$ denote the nth coefficient of the Dirichlet series determined by ζ^α, where ζ is the Riemann zeta function. Writing $n=\prod_{j=1}^\infty p_j^{\kappa_j}$, we have

$$d_\alpha(n) = \prod_{j=1}^\infty \binom{\kappa_j+\alpha-1}{\kappa_j} \quad\text{and}\quad \|z^{\kappa(n)}\|^2_{A^2_\alpha(\mathbb{D}^\infty)} = \frac{1}{d_\alpha(n)}. \tag{2.4}$$

See [4] for details.

Lemma 2.3. *If $\varphi\in[\mathscr{P}_m]$, then H_φ is a Hilbert–Schmidt operator. In particular, the weighted Helson form $[\cdot,\cdot]_\varphi$ is bounded.*

Proof. Let $e_j(z)=\sqrt{d_\alpha(j)}\,z^{\kappa(j)}$. Then $\{e_j\}_{j=1}^\infty$ forms an orthonormal basis of $A^2_\alpha(\mathbb{D}^\infty)$. Given $f=\sum_{j\ge1}\widehat{f}(j)z^{\kappa(j)}$ in $A^2_\alpha(\mathbb{D}^\infty)$, we have

$$\begin{aligned}\|H_f\|^2_{S_2} &= \sum_{i\ge1}\sum_{j\ge1}|\langle H_f e_i,\, e_j\rangle|^2 = \sum_{i\ge1}\sum_{j\ge1}\frac{|\widehat{f}(ij)|^2 d_\alpha(i)d_\alpha(j)}{d_\alpha(ij)^2}\\ &= \sum_{n\ge1}\frac{|\widehat{f}(n)|^2}{d_\alpha(n)^2}\sum_{ij=n}d_\alpha(i)d_\alpha(j) = \sum_{n\ge1}|\widehat{f}(n)|^2\,\frac{d_{2\alpha}(n)}{d_\alpha(n)^2},\end{aligned} \tag{2.5}$$

where the last equation is obtained by comparing the coefficients of $\zeta^\alpha\zeta^\alpha$ and $\zeta^{2\alpha}$.

Now assume $\varphi=\sum_{|\kappa|=m}\widehat{\varphi}(\kappa)z^\kappa\in[\mathscr{P}_m]$. Equation (2.5) gives

$$\|H_\varphi\|^2_{S_2} = \sum_{|\kappa|=m}|\widehat{\varphi}(\kappa)|^2\frac{d_{2\alpha}(p^\kappa)}{d_\alpha(p^\kappa)^2} \le C_{m,\alpha}\sum_{|\kappa|=m}\frac{|\widehat{\varphi}(\kappa)|^2}{d_\alpha(p^\kappa)} = C_{m,\alpha}\|\varphi\|^2_{A^2_\alpha(\mathbb{D}^\infty)},$$

where $C_{m,\alpha}=\sup_{|\kappa|=m}d_{2\alpha}(p^\kappa)<\infty$. ∎

For $\varphi\in[\mathscr{P}_m]$, define a symmetric m-linear form Λ_φ on $[\mathscr{P}_1]\times\cdots\times[\mathscr{P}_1]$ by

$$\Lambda_\varphi(f_1,\ldots,f_m) = \Big\langle \prod_{j=1}^m f_j,\, \widetilde{\varphi}\Big\rangle.$$

It turns out that Λ_φ is bounded, that is, there is a constant $C > 0$ such that

$$|\Lambda_\varphi(f_1, \dots, f_m)| \leq C\|f_1\| \cdots \|f_m\|.$$

Indeed, for $f \in [\mathscr{P}_1]$, we have $\|f\|^2_{A^2_\alpha(\mathbb{D}^\infty)} = \alpha^{-1}\|f\|^2_{H^2(\mathbb{T}^\infty)}$. Writing f_j in the form of power series, an easy calculation shows

$$\Big\|\prod_{j=1}^m f_j\Big\|^2_{A^2_\alpha} \leq \Big\|\prod_{j=1}^m f_j\Big\|^2_{H^2} \leq m! \prod_{j=1}^m \|f_j\|^2_{H^2} = \alpha^m m! \prod_{j=1}^m \|f_j\|^2_{A^2_\alpha}.$$

Then the boundedness of Λ_φ follows from Lemma 2.3 and the Cauchy–Schwarz inequality.

For functions g_j $(1 \leq j \leq m)$ in $[\mathscr{P}_1]$, the tensor product $g_1 \otimes \cdots \otimes g_m$ is the m-linear form on $[\mathscr{P}_1] \times \cdots \times [\mathscr{P}_1]$ defined by

$$(g_1 \otimes \cdots \otimes g_m)(f_1, \dots, f_m) = \prod_{j=1}^m \langle f_j, g_j\rangle.$$

According to the semigroup isomorphism κ, the power series expansion of a function φ in $A^2_\alpha(\mathbb{D}^\infty)$ can be written as

$$\varphi = \sum_{n=1}^\infty \widehat{\varphi}(n) z^{\kappa(n)}.$$

In particular, if $\varphi \in [\mathscr{P}_m]$, then φ can be uniquely represented as

$$\varphi = \sum_{i_1 \leq \cdots \leq i_m} \widehat{\varphi}(p_{i_1} \cdots p_{i_m}) z_{i_1} \cdots z_{i_m}.$$

The main result of this section is the following theorem which characterizes finite rank Helson operators with m-homogeneous symbols.

Theorem 2.4. *Suppose $m \geq 1$ and*

$$\varphi = \sum_{i_1 \leq \cdots \leq i_m} \widehat{\varphi}(p_{i_1} \cdots p_{i_m}) z_{i_1} \cdots z_{i_m} \in [\mathscr{P}_m].$$

The following are equivalent:

(a) *H_φ is a finite rank Helson operator on $A^2_\alpha(\mathbb{D}^\infty)$.*

(b) *There is a positive integer N and elements g_{lj} in $[\mathscr{P}_1]$ with $1 \leq l \leq m$ and $1 \leq j \leq N$ such that*

$$\Lambda_\varphi = \sum_{j=1}^N g_{1j} \otimes g_{2j} \otimes \cdots \otimes g_{mj}. \tag{2.6}$$

(c) *There is a positive integer N and $x_{lj} = (x_{lj}(1), x_{lj}(2), \dots)$ in ℓ^2 with $1 \leq l \leq m$ and $1 \leq j \leq N$ such that*

$$\widehat{\varphi}(p_{i_1} \cdots p_{i_m}) = \sum_{j=1}^N \frac{d_\alpha(\prod_{k=1}^m p_{i_k})}{m!\alpha^m} \sum_{\sigma \in S_m} x_{\sigma(1)j}(i_1) \cdots x_{\sigma(m)j}(i_m),$$

where S_m is the symmetric group of the set $\{1, \dots, m\}$.

Proof. (a)⇒(b): We will prove (b) by induction. The case of $m = 1$ is trivially reduced to the definition and Example 2.1. We assume that $m \geq 2$ and the implication (a)⇒(b) holds for $m-1$. For each $f_1 \in [\mathscr{P}_1]$, $\varLambda_\varphi(f_1, \cdot, \ldots, \cdot)$ defines a bounded $(m-1)$-linear form on $[\mathscr{P}_1] \times \cdots \times [\mathscr{P}_1]$. This induces a linear map $\varPhi_m$ from $[\mathscr{P}_1]$ to the Banach space of all bounded $(m-1)$-linear forms on $[\mathscr{P}_1] \times \cdots \times [\mathscr{P}_1]$ by

$$f_1 \mapsto \varLambda_\varphi(f_1, \cdot, \ldots, \cdot).$$

The kernel of $\varPhi_m$ is

$$\ker \varPhi_m = \{f_1 \in [\mathscr{P}_1] : \varLambda_\varphi(f_1, f_2, \ldots, f_m) = 0,\ \forall f_j \in [\mathscr{P}_1],\ 2 \leq j \leq m\}.$$

By Lemma 2.3, the weighted Helson form $[\cdot, \cdot]_\varphi$ can be continuously extended over $A^2_\alpha(\mathbb{D}^\infty) \times A^2_\alpha(\mathbb{D}^\infty)$ with

$$\ker\, [\cdot, \cdot]_\varphi = \{f \in A^2_\alpha(\mathbb{D}^\infty) : [f, g]_\varphi = 0,\ \forall g \in A^2_\alpha(\mathbb{D}^\infty)\}.$$

Clearly,

$$\ker\, [\cdot, \cdot]_\varphi \cap [\mathscr{P}_1] \subseteq \ker \varPhi_m.$$

Hence $\ker \varPhi_m$ has finite codimension in $[\mathscr{P}_1]$, and the image of $\varPhi_m$ is finite-dimensional. As a result, a basis of the image of $\varPhi_m$, denoted by $\{\varLambda_j\}_{j=1}^n$, can be chosen as a finite subset of $\{\varPhi_m(z_k)\}_{k=1}^\infty$.

We claim that each $\varLambda_j$ admits the form (2.6) (in the $(m-1)$-version). Actually, the claim holds for all $\varPhi_m(z_k)$. For any $f_j \in [\mathscr{P}_1]$ with $2 \leq j \leq m$, we have

$$\begin{aligned} \varLambda_\varphi(z_k, f_2, \ldots, f_m) &= \langle z_k f_2 \cdots f_m,\ \widetilde{\varphi}\rangle = \langle f_2 \cdots f_m,\ T_{\bar{z}_k}\widetilde{\varphi}\rangle \\ &= \varLambda_{T_{\bar{z}_k}\varphi}(f_2, \ldots, f_m). \end{aligned} \tag{2.7}$$

This implies $\varPhi_m(z_k) = \varLambda_{T_{\bar{z}_k}\varphi}$. By (a) and Proposition 2.2, $H_{T_{\bar{z}_k}\varphi}$ is of finite rank. Note that $T_{\bar{z}_k}\varphi$ belongs to $[\mathscr{P}_{m-1}]$. The claim follows from the induction hypothesis. Moreover, there exist linear functionals α_j on $[\mathscr{P}_1]$ such that

$$\varPhi_m(\cdot) = \sum_{j=1}^n \alpha_j(\cdot)\varLambda_j.$$

Since $\varLambda_\varphi$ is bounded, so is $\varPhi_m$. It follows that each linear functional α_j is bounded. Then the conclusion follows from the Riesz representation theorem and the previous claim.

(b)⇒(a): We use induction again. For $m = 1$, (a) holds by Example 2.1. Assume $m \geq 2$ and the implication (b)⇒(a) holds for $m-1$. By Proposition 2.2, we only need to show that the coinvariant subspace generated by φ is finite-dimensional. According to (2.7) and (b), for each $k \in \mathbb{N}$ we have

$$\varLambda_{T_{\bar{z}_k}\varphi}(\cdot, \ldots, \cdot) = \varLambda_\varphi(z_k, \cdot, \ldots, \cdot) = \sum_{j=1}^N \langle z_k,\ g_{1j}\rangle g_{2j} \otimes \cdots \otimes g_{mj}. \tag{2.8}$$

In particular, each $\Lambda_{T_{\bar{z}_k}\varphi}$ lies in $\operatorname{span}\{g_{2j} \otimes \cdots \otimes g_{mj}\}_{j=1}^N$. Note that the linear map $f \mapsto \Lambda_f$ is injective from $[\mathscr{P}_{m-1}]$ to the Banach space of all $(m-1)$-linear forms on $[\mathscr{P}_1] \times \cdots \times [\mathscr{P}_1]$. By considering the inverse, (2.8) gives

$$\dim \operatorname{span}\{T_{\bar{z}_k}\varphi : k \in \mathbb{N}\} \leq N. \tag{2.9}$$

Furthermore, combining the formula (2.8) and the induction hypothesis, we conclude that $H_{T_{\bar{z}_k}\varphi}$ is of finite rank. Then it follows from Proposition 2.2 that, for each fixed k,

$$\dim \operatorname{span}\{T_{\bar{z}}^{\kappa} T_{\bar{z}_k}\varphi : \kappa \in \mathbb{N}_0^{\infty}\} < \infty, \tag{2.10}$$

where $T_{\bar{z}}^{\kappa} = \prod_{j=1}^{\infty} T_{\bar{z}_j}^{\kappa_j}$. We deduce from (2.9) and (2.10) that the dimension of the coinvariant subspace generated by φ is finite.

(b)$\Rightarrow$(c): Assume the homogeneous expansion of g_{lj} is

$$g_{lj}(z) = \sum_{k=1}^{\infty} x_{lj}(k) z_k. \tag{2.11}$$

Since Λ is symmetric, one has

$$\Lambda_{\varphi} = \sum_{j=1}^{N} \frac{1}{m!} \sum_{\sigma \in S_m} g_{\sigma(1)j} \otimes \cdots \otimes g_{\sigma(m)j}.$$

Then (c) can be deduced from the following equalities:

$$\begin{aligned} \widehat{\varphi}(p_{i_1} \cdots p_{i_m}) &= d_{\alpha}(p_{i_1} \cdots p_{i_m}) \langle z_{i_1} \cdots z_{i_m}, \widetilde{\varphi} \rangle \\ &= d_{\alpha}(p_{i_1} \cdots p_{i_m}) \Lambda_{\varphi}(z_{i_1}, \cdots, z_{i_m}). \end{aligned}$$

(c)$\Rightarrow$(b): Let g_{lj} be defined by (2.11). It is easy to see that Λ_{φ} admits the form (2.6). ■

Recall that each function f in $A_{\alpha}^2(\mathbb{D}^{\infty})$ admits a unique homogeneous expansion [11, Section 9]:

$$f = \sum_{m=0}^{\infty} F_m, \quad F_m \in [\mathscr{P}_m]. \tag{2.12}$$

A function f in $A_{\alpha}^2(\mathbb{D}^{\infty})$ is said to be *of finite degree* if all but finitely many m homogeneous terms F_m in (2.12) vanish. In this case, the largest m satisfying $F_m \neq 0$ is called the *degree* of f. For the sake of simplicity, we introduce the following notation.

Definition 2.5. For $\alpha \geq 1$ and a positive integer d, we denote by $\mathscr{X}_d^{\alpha}$ the set of all functions f of the form

$$f = \sum_{m=0}^{d} F_m, \quad F_m \in [\mathscr{P}_m],$$

where all homogeneous terms F_m satisfy the conditions in Theorem 2.4.

The next corollary will be used in Section 3.

COROLLARY 2.6. *If $\varphi = \sum_{m=0}^{d} F_m$ with $F_m \in [\mathscr{P}_m]$, then H_φ is of finite rank on $A^2_\alpha(\mathbb{D}^\infty)$ if and only if each H_{F_m} is of finite rank on $A^2_\alpha(\mathbb{D}^\infty)$, or equivalently, φ lies in $\mathscr{X}^\alpha_d$.*

Proof. By Lemma 2.3, H_φ is bounded. For every nonnegative integer m, let P_m denote the orthogonal projection from $A^2_\alpha(\mathbb{D}^\infty)$ onto $[\mathscr{P}_m]$. The conclusion follows immediately once we verify that

$$H_{F_m} = \sum_{j=0}^{m} P_j H_\varphi P_{m-j}. \tag{2.13}$$

Let $f = \sum_j f_j$ and $g = \sum_j g_j$ be the homogeneous expansions of two polynomials f and g. Easy computation gives

$$\begin{aligned}\Big\langle \sum_{j=0}^{m} P_j H_\varphi P_{m-j} f,\, g\Big\rangle &= \sum_{j=0}^{m} \langle H_\varphi f_{m-j}, g_j\rangle = \Big\langle \varphi,\, \sum_{j=0}^{m} \widetilde{f}_{m-j} g_j\Big\rangle \\ &= \Big\langle F_m,\, \sum_{j=0}^{m} \widetilde{f}_{m-j} g_j\Big\rangle = \langle F_m, \widetilde{f} g\rangle = \langle F_m V f,\, g\rangle = \langle H_{F_m} f,\, g\rangle.\end{aligned}$$

Since both f and g are arbitrary, we obtain (2.13). ∎

3. Finite-dimensional coinvariant subspaces. In this section, we characterize finite-dimensional coinvariant subspaces of $A^2_\alpha(\mathbb{D}^\infty)$. On the Hardy and Bergman spaces of finite polydiscs, functions in a finite-dimensional coinvariant subspace can be determined by some kind of polynomials and unitary operators involving Möbius transformations [17], and in particular they are rational functions with poles outside $\overline{\mathbb{D}}^n$. As we shall see, in the case of the infinite polydisc, the role of the polynomials will be played by the functions in $\mathscr{X}^\alpha_d$ discussed in Section 2.

For an ideal $\mathcal{I}$ of $\mathscr{P}_\infty$, let $\mathcal{Z}(\mathcal{I})$ denote the zero variety of $\mathcal{I}$,

$$\mathcal{Z}(\mathcal{I}) = \{\lambda \in \mathbb{C}^\infty : p(\lambda) = 0,\ \forall p \in \mathcal{I}\}.$$

Similarly, for a subspace M of $A^2_\alpha(\mathbb{D}^\infty)$ we define the zero variety of M by

$$\mathcal{Z}(M) = \{\lambda \in \mathbb{D}^\infty_2 : f(\lambda) = 0,\ \forall f \in M\}.$$

Note that the reproducing kernel K^α_λ lies in $M^\perp$ if and only if $\lambda \in \mathcal{Z}(M)$.

We begin with an Ahern–Clark type theorem associated to the spaces $A^2_\alpha(\mathbb{D}^\infty)$; see [1, 14] for the case of finitely many variables.

LEMMA 3.1. *Let M be an invariant subspace of $A^2_\alpha(\mathbb{D}^\infty)$ with finite codimension k. Then $M \cap \mathscr{P}_\infty$ is an ideal of $\mathscr{P}_\infty$ satisfying*

(a) *$M \cap \mathscr{P}_\infty$ is dense in M;*
(b) $\dim \mathscr{P}_\infty/(M \cap \mathscr{P}_\infty) = k$;
(c) *$\mathcal{Z}(M \cap \mathscr{P}_\infty)$ is a finite subset of $\mathbb{D}^\infty_2$.*

Proof. The proof is similar to the case of the Hardy space over the infinite torus [13, Theorem 3.2]. ∎

For $\lambda \in \mathbb{C}^\infty$, let $\mathcal{I}_\lambda$ denote the maximal ideal of $\mathscr{P}_\infty$ consisting of all polynomials vanishing at λ. As shown in [13, Proposition 2.1], each maximal ideal of $\mathscr{P}_\infty$ is of this form. Combining this result with Lemma 3.1, we find that $\mathcal{Z}(M)$ is nonempty if M is a nontrivial finite-codimensional invariant subspace. For an ideal $\mathcal{I}$ of $\mathscr{P}_\infty$, its *radical* $r(\mathcal{I})$ is defined as the ideal of all polynomials p such that p^k belongs to $\mathcal{I}$ for some positive integer k.

The well-known Lasker–Noether theorem states that every ideal of $\mathbb{C}[z_1, \ldots, z_d]$ admits a primary decomposition. Although this statement is not true for a general ideal of $\mathscr{P}_\infty$ [13, 16], it is shown in [13, Proposition 2.4] that every ideal of $\mathscr{P}_\infty$ with finite zero variety admits a unique primary decomposition. This result enables us to obtain the following lemma.

Lemma 3.2. *If M is a finite-codimensional invariant subspace of $A^2_\alpha(\mathbb{D}^\infty)$ with zero variety $\mathcal{Z}(M) = \{\lambda^{(1)}, \ldots, \lambda^{(n)}\}$, then M can be uniquely represented as*

$$M = \bigcap_{j=1}^{n} M_j,$$

where each M_j is an invariant subspace and $\mathcal{Z}(M_j) = \{\lambda^{(j)}\}$.

Proof. Set $\mathcal{I} = M \cap \mathscr{P}_\infty$. Then $\mathcal{I}$ is an ideal of $\mathscr{P}_\infty$ satisfying conditions (a)–(c) in Lemma 3.1. By [13, Proposition 2.4], $\mathcal{I}$ admits a unique primary decomposition

$$\mathcal{I} = \bigcap_{j=1}^{n} \mathfrak{p}_j, \tag{3.1}$$

where each $\mathfrak{p}_j$ is a primary ideal satisfying $r(\mathfrak{p}_j) = \mathcal{I}_{\lambda^{(j)}}$.

We claim that $[\mathfrak{p}_j] \cap \mathscr{P}_\infty = \mathfrak{p}_j$ for each j. To prove this, we use the idea from [14, Lemma 2.3(b)]. Since $\dim \mathscr{P}_\infty/\mathcal{I} < \infty$, we have $\dim \mathfrak{p}_j/\mathcal{I} < \infty$. Then there exists a finite-dimensional subspace F of $\mathscr{P}_\infty$ such that $\mathfrak{p}_j = \mathcal{I} + F$. This implies that

$$\mathfrak{p}_j \subset [\mathcal{I}] + F \subset [\mathfrak{p}_j].$$

The fact that $[\mathcal{I}] + F$ is closed gives $[\mathcal{I}] + F = [\mathfrak{p}_j]$. Using this equality and Lemma 3.1(a), we obtain

$$[\mathfrak{p}_j] \cap \mathscr{P}_\infty = ([\mathcal{I}] + F) \cap \mathscr{P}_\infty = ([\mathcal{I}] \cap \mathscr{P}_\infty) + F = \mathcal{I} + F = \mathfrak{p}_j,$$

as claimed.

The intersection $N := \bigcap_{j=1}^{n} [\mathfrak{p}_j]$ is an invariant subspace and $M \subseteq N$. Let $\mathcal{J} = N \cap \mathscr{P}_\infty$. Since N is of finite codimension, Lemma 3.1 shows that

$\mathcal{J}$ is an ideal of $\mathscr{P}_\infty$ with $[\mathcal{J}] = N$. By the claim and (3.1), we have

$$\mathcal{I} = \bigcap_{j=1}^{n} \mathfrak{p}_j = \bigcap_{j=1}^{n} ([\mathfrak{p}_j] \cap \mathscr{P}_\infty) = \Big(\bigcap_{j=1}^{n} [\mathfrak{p}_j]\Big) \cap \mathscr{P}_\infty = \mathcal{J}. \tag{3.2}$$

Taking closure on both sides of (3.2) gives $M = N = \bigcap_{j=1}^{n} [\mathfrak{p}_j]$. Existence is proved by letting $M_j = [\mathfrak{p}_j]$.

For uniqueness, assume that M admits another representation $M = \bigcap_{i=1}^{m} L_i$ with $\mathcal{Z}(L_i) = \{\mu^{(i)}\}$. By (a) and (c) in Lemma 3.1, we find that $\mathcal{Z}(L_i \cap \mathscr{P}_\infty) = \{\mu^{(i)}\}$. Applying a theorem due to S. Lang [23], we deduce that $r(L_i \cap \mathscr{P}_\infty) = \mathcal{I}_{\mu^{(i)}}$. In particular, $L_i \cap \mathscr{P}_\infty$ is primary and hence

$$\bigcap_{I=1}^{m} (L_i \cap \mathscr{P}_\infty) = \mathcal{I} = \bigcap_{j=1}^{n} \mathfrak{p}_j.$$

The proof is finished by the uniqueness of the primary decomposition of $\mathcal{I}$. ∎

Recall the notation $\mathscr{X}_d^\alpha$ is given by Definition 2.5, and the functions in $\mathscr{X}_d^\alpha$ are completely characterized in Theorem 2.4.

Lemma 3.3. *Let M be an invariant subspace of $A_\alpha^2(\mathbb{D}^\infty)$ with finite codimension d. If $\mathcal{Z}(M) = \{\mathbf{0}\}$, then $M^\perp \subseteq \mathscr{X}_d^\alpha$.*

Proof. Let A_j be the restriction of $T_{\bar{z}_j}$ on $M^\perp$. Then $\{A_j\}_{j=1}^\infty$ can be considered as a family of commutative d-by-d matrices. We will show that each A_j is nilpotent. For any given eigenvalue λ_1 of A_1, we tend to prove $\lambda_1 = 0$. Let E_{λ_1} be the eigenspace of A_1 associated with λ_1. Since each A_j commutes with A_1, E_{λ_1} is a common invariant subspace of $\{A_j\}_{j=2}^\infty$. As a result, there is a nonzero f in E_{λ_1} such that f is a common eigenvector of $\{A_j|_{E_{\lambda_1}}\}_{j=1}^\infty$. For $j \geq 2$, assume λ_j is the eigenvalue of A_j such that

$$A_j f = T_{\bar{z}_j} f = \lambda_j f. \tag{3.3}$$

Writing $f = \sum_{\kappa \in \mathbb{N}_0^\infty} \widehat{f}(\kappa) z^\kappa$, equation (3.3) gives

$$\begin{aligned} \widehat{f}(\kappa + e_j) \|z^{\kappa + e_j}\|^2_{A_\alpha(\mathbb{D}^\infty)} &= \langle f, z^{\kappa+e_j} \rangle = \langle T_{\bar{z}_j} f, z^\kappa \rangle \\ &= \langle \lambda_j f, z^\kappa \rangle = \lambda_j \widehat{f}(\kappa) \|z^\kappa\|^2_{A_\alpha(\mathbb{D}^\infty)}, \end{aligned} \tag{3.4}$$

where $e_j \in \mathbb{N}_0^\infty$ is defined by $e_j(k) = \delta_{jk}$. Because of $\|T_{\bar{z}_j}\| \leq 1$, $\lambda = (\lambda_1, \lambda_2, \dots)$ lies in $\overline{\mathbb{D}}^\infty$. By iteration in (3.4), one obtains

$$\widehat{f}(\kappa) \|z^\kappa\|^2_{A_\alpha(\mathbb{D}^\infty)} = \lambda^\kappa \widehat{f}(\mathbf{0}).$$

Since f is nonzero, the above equality implies $\widehat{f}(\mathbf{0}) \neq 0$. Then Parseval's identity, equality (2.4), and Euler's product formula yield

$$\begin{aligned}\|f\|^2_{A^2_\alpha(\mathbb{D}^\infty)} &= \sum_{\kappa\in\mathbb{N}_0^\infty} |\widehat{f}(\kappa)|^2 \|z^\kappa\|^2_{A^2_\alpha(\mathbb{D}^\infty)} = |\widehat{f}(\mathbf{0})|^2 \sum_{\kappa\in\mathbb{N}_0^\infty} \frac{|\lambda^\kappa|^2}{\|z^\kappa\|^2_{A^2_\alpha(\mathbb{D}^\infty)}} \\ &= |\widehat{f}(\mathbf{0})|^2 \sum_{n\geq 1} d_\alpha(n)|\lambda^{\kappa(n)}|^2 = |\widehat{f}(\mathbf{0})|^2 \Big(\prod_{j\in\mathbb{N}} \frac{1}{1-|\lambda_j|^2}\Big)^\alpha .\end{aligned}$$

The fact that $f \in A^2_\alpha(\mathbb{D}^\infty)$ forces $\lambda \in \mathbb{D}_2^\infty$ and

$$f = \widehat{f}(\mathbf{0}) \sum_{\kappa\in\mathbb{N}_0^\infty} \frac{\lambda^\kappa z^\kappa}{\|z^\kappa\|^2_{A_\alpha(\mathbb{D}^\infty)}} = \widehat{f}(\mathbf{0}) K^\alpha_{\overline{\lambda}}(z).$$

Since $f \in M^\perp$ and $\mathcal{Z}(M) = \{\mathbf{0}\}$, λ has to be $\mathbf{0}$. In particular, $\lambda_1 = 0$. We get $\sigma(A_1) = \{0\}$ because λ_1 is arbitrary. Similar arguments show that $\sigma(A_j) = \{0\}$ for all $j \geq 2$. Therefore, each A_j is nilpotent.

As a commuting family of d-by-d nilpotent matrices, $\{A_j\}_{j=1}^\infty$ can be simultaneously triangularized as strictly upper-triangular matrices. Then

$$\prod_{j=1}^\infty A_j^{\kappa_j} = 0 \quad \text{whenever} \quad |\kappa| = \sum_{j=1}^\infty \kappa_j > d.$$

This shows that each function φ in $M^\perp$ has degree at most d. Now the conclusion follows from Proposition 2.2 and Corollary 2.6. ∎

Given a point $\lambda \in \mathbb{D}^\infty$, let Φ_λ denote the self-homeomorphism of $\overline{\mathbb{D}}^\infty$ defined by

$$\Phi_\lambda(z) = (\varphi_{\lambda_1}(z_1), \varphi_{\lambda_2}(z_2), \dots), \quad z \in \overline{\mathbb{D}}^\infty,$$

where $\varphi_{\lambda_i}(z_i) = \frac{\lambda_i - z_i}{1-\overline{\lambda}_i z_i}$. Clearly, Φ_λ maps $\mathbb{T}^\infty$ onto $\mathbb{T}^\infty$. As mentioned in the introduction, Cole and Gamelin [11, Sections 2 and 3] proved that for $\alpha \geq 1$ and $\lambda \in \mathbb{D}_2^\infty$ the normalized reproducing kernel $k^\alpha_\lambda = K^\alpha_\lambda/\|K^\alpha_\lambda\|$ is the $A^2_\alpha(\mathbb{D}^\infty)$-limit and the almost everywhere limit of the martingale $\{G_N\}$, where

$$G_N = \prod_{j=1}^N \frac{(1-|\lambda_j|^2)^{\alpha/2}}{(1-\overline{\lambda}_j z_j)^\alpha}.$$

They also proved that for $\lambda \in \mathbb{D}_2^\infty$ the pull back of the Haar measure $d\sigma$ of $\mathbb{T}^\infty$ under the transformation Φ_λ, denoted by $d\Phi^*_\lambda\sigma$, is mutually absolutely continuous with respect to $d\sigma$, and the corresponding Radon–Nikodym derivative is given by

$$\frac{d\Phi^*_\lambda\sigma}{d\sigma} = |k^1_\lambda|^2 = \prod_{j=1}^\infty \frac{1-|\lambda_j|^2}{|1-\overline{\lambda}_j z_j|^2} \quad \text{a.e. } d\sigma.$$

We need similar results concerning the weighted Bergman spaces $A^2_\alpha(\mathbb{D}^\infty)$ $(\alpha > 1)$.

PROPOSITION 3.4. *Let $d\Phi^*_\lambda V_\alpha$ be the pull back of dV_α associated to Φ_λ. If $\lambda \in \mathbb{D}^\infty_2$, then $d\Phi^*_\lambda V_\alpha$ is mutually absolutely continuous with respect to dV_α and*

$$\frac{d\Phi^*_\lambda V_\alpha}{dV_\alpha} = |k^\alpha_\lambda|^2 \quad a.e.\ dV_\alpha.$$

Furthermore,

$$k^\alpha_\lambda \circ \Phi_\lambda\, k^\alpha_\lambda = 1 \quad a.e.\ dV_\alpha. \tag{3.5}$$

Proof. The first statement can be proved by similar arguments to those in [11, Sections 2 and 3]. We only prove (3.5). Recall that G_N converges to k^α_λ almost everywhere with respect to dV_α. By the first statement of this proposition, we conclude that the preimage of a null set under Φ_λ is also null. Consequently, $G_N \circ \Phi_\lambda$ converges to $k^\alpha_\lambda \circ \Phi_\lambda$ almost everywhere, and hence $G_N \circ \Phi_\lambda\, G_N$ converges to $k^\alpha_\lambda \circ \Phi_\lambda\, k^\alpha_\lambda$ almost everywhere. Note that $G_N \circ \Phi_\lambda\, G_N = 1$. The conclusion follows. ∎

By Proposition 3.4, for $f \in L^2_\alpha$, making a change of variables gives

$$\int_{\mathbb{D}^\infty} |f|^2\, dV_\alpha = \int_{\mathbb{D}^\infty} |f \circ \Phi_\lambda|^2 |k^\alpha_\lambda|^2\, dV_\alpha. \tag{3.6}$$

The versions of (3.5) and (3.6) corresponding to the case $\alpha = 1$ can be derived similarly. For $\alpha \geq 1$ and $\lambda \in \mathbb{D}^\infty_2$, let U_λ be the operator on L^2_α defined by

$$U_\lambda f = f \circ \Phi_\lambda\, k^\alpha_\lambda.$$

It follows from the above discussion that $U^2_\lambda = I$ and U_λ is unitary. Clearly, $A^2_\alpha(\mathbb{D}^\infty)$ is a reducing subspace of U_λ.

THEOREM 3.5. *If M is an invariant subspace of $A^2_\alpha(\mathbb{D}^\infty)$ with finite codimension d and $\mathcal{Z}(M) = \{\lambda^{(1)}, \ldots, \lambda^{(n)}\}$, then each function f in $M^\perp$ can be represented as*

$$f = \sum_{j=1}^n U_{\lambda^{(j)}} f_j, \quad f_j \in \mathscr{X}^\alpha_d. \tag{3.7}$$

Conversely, if f is of the form (3.7) for some distinct points $\lambda^{(1)}, \ldots, \lambda^{(n)}$ in $\mathbb{D}^\infty_2$ and $d \in \mathbb{N}$, then the coinvariant subspace generated by f is of finite dimension.

Proof. By Lemma 3.2, M can be written as

$$M = \bigcap_{j=1}^n M_j,$$

where each M_j is an invariant subspace with $\mathcal{Z}(M_j) = \lambda^{(j)}$. Then

$$M^\perp = \sum_{j=1}^{n} M_j^\perp. \tag{3.8}$$

Set $N_j = U_{\lambda^{(j)}} M_j$. Then N_j is also an invariant subspace. Actually, for any $g \in M_j$ and $p \in \mathscr{P}_\infty$, we have $p \circ \Phi_{\lambda^{(j)}} g \in M_j$ and

$$pU_{\lambda^{(j)}} g = p \circ \Phi_{\lambda^{(j)}} \circ \Phi_{\lambda^{(j)}} g \circ \Phi_{\lambda^{(j)}} k^\alpha_{\lambda^{(j)}} = U_{\lambda^{(j)}}(p \circ \Phi_{\lambda^{(j)}} g).$$

Clearly, each N_j is finite-codimensional with $\mathcal{Z}(N_j) = \{\mathbf{0}\}$. We deduce from Lemma 3.3 that $N_j^\perp \subseteq \mathscr{X}^\alpha_d$. Because $U_{\lambda^{(j)}}$ is unitary and $U^2_{\lambda^{(j)}} = I$, one has $M_j^\perp = U_{\lambda^{(j)}} N_j^\perp$. Combining this and (3.8) gives the first part of the theorem.

For the converse, since the image of a coinvariant subspace under U_λ is also coinvariant, the conclusion follows from Proposition 2.2 and Theorem 2.4. ∎

The argument in this section can also be applied to the case of finitely many variables. In that case, the problem considered in Section 2 and the analogous version of Proposition 3.4 are trivial, and the space $\mathscr{X}^\alpha_d$ is reduced to the space of polynomials of degree at most d.

4. Finite rank Helson operators. By means of the results in Section 3, we characterize finite rank Helson operators on $A^2_\alpha(\mathbb{D}^\infty)$. For the case $\alpha = 1$, it is an alternative form of [27, Theorem 6.6(ii)].

THEOREM 4.1. *For $\varphi \in A^2_\alpha(\mathbb{D}^\infty)$, the Helson operator H_φ is of finite rank on $A^2_\alpha(\mathbb{D}^\infty)$ if and only if φ is of the following form:*

$$\varphi = \sum_{j=1}^{n} f_j \circ \Phi_{\lambda^{(j)}} k^\alpha_{\lambda^{(j)}}, \tag{4.1}$$

where $\lambda^{(j)} \in \mathbb{D}^\infty_2$ and $f_j \in \mathscr{X}^\alpha_d$ for some $d \in \mathbb{N}$.

Proof. This follows from Proposition 2.2 and Theorem 3.5. ∎

In view of (4.1), the symbols of finite rank Helson operators on $A^2_\alpha(\mathbb{D}^\infty)$ are represented by functions in $\mathscr{X}^\alpha_d$ and normalized reproducing kernels. Clearly, both the space $\mathscr{X}^\alpha_d$ and the normalized reproducing kernels depend on α. Nevertheless, in the case of finitely many variables, $\mathscr{X}^\alpha_d$ should be replaced by the space of polynomials of degree at most d, which does not depend on α.

Another main result of this section is Theorem 4.5 which states that a Helson operator of finite rank is automatically bounded. The proof is based on Theorem 4.1 and the properties of a class of composition operators.

For $\lambda \in \mathbb{D}_2^\infty$, let C_{Φ_λ} denote the densely defined composition operator in $A^2_\alpha(\mathbb{D}^\infty)$ defined by

$$C_{\Phi_\lambda} q = q \circ \Phi_\lambda, \quad q \in \mathscr{P}_\infty.$$

It follows from Proposition 3.4 that C_{Φ_λ} extends to a bounded linear operator from $A^2_\alpha(\mathbb{D}^\infty)$ into $A^2_\alpha(\mathbb{D}^\infty)$ if and only if $\lambda \in \mathbb{D}_2^\infty \cap \ell^1$. Thus C_{Φ_λ} may be unbounded. In the next proposition, we are concerned with the domain of $C^*_{\Phi_\lambda}$.

PROPOSITION 4.2. *If $p > 2$, then for any $\lambda \in \mathbb{D}_2^\infty$ the space $A^p_\alpha(\mathbb{D}^\infty)$ is contained in the domain of the operator $C^*_{\Phi_\lambda}$ in $A^2_\alpha(\mathbb{D}^\infty)$.*

Proof. We only provide the proof for the case of the Bergman spaces ($\alpha > 1$); the proof for the Hardy space is similar. It is sufficient to show that for any $g \in A^p_\alpha(\mathbb{D}^\infty)$ with $p > 2$ the linear functional

$$q \mapsto \langle q \circ \Phi_\lambda, g\rangle, \quad q \in \mathscr{P}_\infty,$$

extends to a bounded linear functional on $A^2_\alpha(\mathbb{D}^\infty)$. By Proposition 3.4, making a change of variables gives

$$\begin{aligned}
|\langle q \circ \Phi_\lambda, g\rangle| &= \Big| \int_{\mathbb{D}^\infty} q \circ \Phi_\lambda(\zeta) \overline{g(\zeta)}\, dV_\alpha(\zeta) \Big| \\
&= \Big| \int_{\mathbb{D}^\infty} q(\zeta) \overline{g \circ \Phi_\lambda(\zeta)} |k^\alpha_\lambda(\zeta)|^2 \, dV_\alpha(\zeta) \Big| \\
&\le \Big(\int_{\mathbb{D}^\infty} |g \circ \Phi_\lambda(\zeta)|^2 |k^\alpha_\lambda(\zeta)|^4 \, dV_\alpha(\zeta) \Big)^{1/2} \|q\|_{A^2_\alpha(\mathbb{D}^\infty)}.
\end{aligned}$$

Making a change of variables again and using (3.5), we obtain

$$\begin{aligned}
\int_{\mathbb{D}^\infty} |g \circ \Phi_\lambda(\zeta)|^2 |k^\alpha_\lambda(\zeta)|^4 \, dV_\alpha(\zeta) &= \int_{\mathbb{D}^\infty} |g(\zeta)|^2 |k^\alpha_\lambda \circ \Phi_\lambda(\zeta)|^4 |k^\alpha_\lambda(\zeta)|^2 \, dV_\alpha(\zeta) \\
&= \int_{\mathbb{D}^\infty} |g(\zeta)|^2 |k^\alpha_\lambda(\zeta)|^{-2} \, dV_\alpha(\zeta) \\
&= \|K^\alpha_\lambda\|^2 \int_{\mathbb{D}^\infty} |g(\zeta)|^2 \Big| \prod_{j=1}^\infty (1 - \bar{\lambda}_j \zeta_j) \Big|^{2\alpha} \, dV_\alpha(\zeta) \\
&\le C_{\lambda,\alpha} \|K^\alpha_\lambda\|^2 \|g\|_p,
\end{aligned}$$

where the inequality follows from the Hölder inequality, and

$$C_{\lambda,\alpha} = \Big(\int_{\mathbb{D}^\infty} \Big| \prod_{j=1}^\infty (1 - \bar{\lambda}_j \zeta_j) \Big|^{2\alpha p/(p-2)} \, dV_\alpha(\zeta) \Big)^{(p-2)/p}$$

is finite [11, Lemma 3.2]. ∎

Recall that Helson operators with homogeneous symbols are Hilbert–Schmidt operators (Lemma 2.3), and the notation $\mathscr{X}_d^\alpha$ is introduced in Definition 2.5.

Lemma 4.3. *If $\varphi \in \mathscr{X}_d^\alpha$ with $d \in \mathbb{N}$, then for all $\lambda \in \mathbb{D}_2^\infty$ the product $H_\varphi C_{\Phi_\lambda}$ extends to a bounded linear operator on $A_\alpha^2(\mathbb{D}^\infty)$.*

Proof. Since H_φ is a bounded operator with finite rank, it admits a canonical form

$$H_\varphi = \sum_{j=1}^n v_j \otimes u_j$$

for some orthogonal sets $\{v_j\}_{j=1}^n$ and $\{u_j\}_{j=1}^n$. Let f be a nonzero function in $A_\alpha^2(\mathbb{D}^\infty)$. For each j, one has

$$\|v_j\|^2 f \otimes u_j = (f \otimes v_j) H_\varphi,$$

and hence

$$\|v_j\|^2 \langle g, u_j\rangle f = (f \otimes v_j) H_\varphi g = 0, \quad g \in \ker H_\varphi.$$

Since f is nonzero, u_j belongs to $(\ker H_\varphi)^\perp$. By (2.2), the degree of u_j is at most d. Writing u_j in the form of a homogeneous expansion, we deduce that u_j belongs to $A_\alpha^p(\mathbb{D}^\infty)$ for all $p > 0$ from Khinchin's theorem for higher degree [11, Theorem 9.1]. Then the conclusion follows immediately from Proposition 4.2. ∎

The proof of the next lemma is similar to the case of finitely many variables [17]. We include the proof for completeness.

Lemma 4.4. *For $\varphi \in A_\alpha^2(\mathbb{D}^\infty)$ and $\lambda \in \mathbb{D}_2^\infty$, we have*

$$U_\lambda H_{U_\lambda \varphi} q = H_\varphi C_{\Phi_{\bar\lambda}} q, \quad q \in \mathscr{P}_\infty.$$

Proof. First of all, we show that for any $f \in A_\alpha^2(\mathbb{D}^\infty)$,

$$U_\lambda H_f = H_{(k_\lambda^\alpha/\overline{k_\lambda^\alpha}) f\circ\Phi_\lambda} U_{\bar\lambda}. \tag{4.2}$$

Indeed, for any $q \in \mathscr{P}_\infty$ one has

$$\begin{aligned} U_\lambda H_f q &= U_\lambda P_\alpha(fVq) = P_\alpha U_\lambda(fVq) = P_\alpha[f\circ\Phi_\lambda (Vq)\circ\Phi_\lambda k_\lambda^\alpha] \\ &= P_\alpha[f\circ\Phi_\lambda\, V(q\circ\Phi_{\bar\lambda})k_\lambda^\alpha] = P_\alpha[f\circ\Phi_\lambda\, V(q\circ\Phi_{\bar\lambda}k_{\bar\lambda}^\alpha)\, k_\lambda^\alpha/\overline{k_\lambda^\alpha}] \\ &= H_{(k_\lambda^\alpha/\overline{k_\lambda^\alpha}) f\circ\Phi_\lambda} U_{\bar\lambda} q. \end{aligned}$$

We deduce from (4.2) that

$$\begin{aligned} U_\lambda H_{U_\lambda\varphi} q = U_\lambda H_{\varphi\circ\Phi_\lambda k_\lambda^\alpha} q &= H_{(k_\lambda^\alpha/\overline{k_\lambda^\alpha})\varphi k_\lambda^\alpha\circ\Phi_\lambda} U_{\bar\lambda} q \\ &= P_\alpha[(k_\lambda^\alpha/\overline{k_\lambda^\alpha})\varphi k_\lambda^\alpha\circ\Phi_\lambda V(q\circ\Phi_{\bar\lambda}k_{\bar\lambda}^\alpha)] \\ &= P_\alpha[\varphi V(q\circ\Phi_{\bar\lambda})k_\lambda^\alpha\circ\Phi_\lambda k_\lambda^\alpha] = P_\alpha[\varphi V(q\circ\Phi_{\bar\lambda})] = H_\varphi C_{\Phi_{\bar\lambda}} q \end{aligned}$$

for all q in $\mathscr{P}_\infty$. ∎

THEOREM 4.5. *For $\varphi \in A^2_\alpha(\mathbb{D}^\infty)$, the Helson operator H_φ is bounded if it is of finite rank.*

Proof. If H_φ is of finite rank, we deduce from Theorem 4.1 that φ is of the form (4.1), namely,

$$\varphi = \sum_{j=1}^{n} U_{\lambda^{(j)}} f_j, \quad f_j \in \mathscr{X}^\alpha_d.$$

By Lemma 4.3, for each j the product $H_{f_j} C_{\Phi_{\lambda^{(j)}}}$ is bounded. It follows from Lemma 4.4 that $H_{U_{\lambda^{(j)}} f_j}$ is bounded. Hence H_φ is bounded. ∎

Based on some results of this paper and a theorem on the moment problem, a positive Helson operator of finite rank can be determined in terms of its symbol; see [27, Corollary 6.8] for a description of positive finite rank Helson forms and [29] for the case of finitely many variables.

THEOREM 4.6. *For $\varphi \in A^2_\alpha(\mathbb{D}^\infty)$, the Helson operator H_φ is of finite rank and positive if and only if φ is of the form*

$$\varphi = \sum_{j=1}^{n} c_j K^\alpha_{\lambda^{(j)}}, \tag{4.3}$$

where $c_j > 0$ and $\lambda^{(j)} \in \mathbb{D}^\infty_2 \cap (-1,1)^\infty$.

Proof. If φ is of the form (4.3), applying Theorem 4.1 shows that H_φ is of finite rank, and the positivity of H_φ follows from the inequality

$$\langle H_\varphi f, f\rangle = \langle \varphi, \widetilde{f} f\rangle = \sum_{j=1}^{n} c_j |f(\lambda^{(j)})|^2 \geq 0, \quad f \in \mathscr{P}_\infty.$$

For the converse, we begin with the case $\alpha = 1$. Assume that H_φ is of finite rank and positive. By Theorem 4.5, H_φ is bounded. Note that φ admits a power series

$$\varphi = \sum_{n\in\mathbb{N}} \widehat{\varphi}(n) z^{\kappa(n)}. \tag{4.4}$$

Then the Helson matrix $\{\widehat{\varphi}(nm)\}_{n,m=1}^\infty$ is positive. It follows from a theorem of [27, Theorem 3.3] on the moment problem that there is a positive measure μ on $(-1,1)^\infty$ such that

$$\widehat{\varphi}(n) = \int_{(-1,1)^\infty} t^{\kappa(n)}\, d\mu(t), \quad n \in \mathbb{N}. \tag{4.5}$$

Then for $f = \sum_{n\in\mathbb{N}} a_n z^{\kappa(n)} \in \mathscr{P}_\infty$ we have

$$\langle H_\varphi f, f\rangle = \sum_{n,m\in\mathbb{N}} \widehat{\varphi}(nm) a_n \bar{a}_m = \int_{(-1,1)^\infty} |f(t)|^2\, d\mu(t).$$

As a result,

$$\mathscr{P}_\infty \cap \ker H_\varphi = \Big\{ f \in \mathscr{P}_\infty : \int_{(-1,1)^\infty} |f(t)|^2\, d\mu(t) = 0 \Big\}. \tag{4.6}$$

Since $\ker H_\varphi$ is an invariant subspace of finite codimension, an application of Lemma 3.1 shows that the zero variety of $\mathscr{P}_\infty \cap \ker H_\varphi$ is a finite subset of $\mathbb{D}_2^\infty$. By (4.6), each function in $\mathscr{P}_\infty \cap \ker H_\varphi$ vanishes μ-almost everywhere on $(-1,1)^\infty$. This implies that the support of μ is a subset of $\mathcal{Z}(\mathscr{P}_\infty \cap \ker H_\varphi)$. Consequently, μ is supported on a finite number of points $\{\lambda^{(j)}\}_{j=1}^n$ in $\mathbb{D}_2^\infty \cap (-1,1)^\infty$. From (4.5), we deduce that φ is of the form (4.3).

For $\alpha > 1$, assume that $\varphi \in A^2_\alpha(\mathbb{D}^\infty)$ has a power series (4.4) and that H_φ is of finite rank and positive. Let $\psi(z) = \sum_{n=1}^\infty \frac{\widehat{\varphi}(n)}{d_\alpha(n)} z^{\kappa(n)}$. Then it is easy to see that ψ lies in $H^2(\mathbb{T}^\infty)$. For any f and g in $\mathscr{P}_\infty$ given by

$$f(z) = \sum_{n\in\mathbb{N}} a_n z^{\kappa(n)}, \quad g(z) = \sum_{n\in\mathbb{N}} b_n z^{\kappa(n)},$$

we have

$$\begin{aligned}\langle H_\varphi f, g\rangle_{A^2_\alpha(\mathbb{D}^\infty)} &= \langle \varphi, \widetilde{f g}\rangle_{A^2_\alpha(\mathbb{D}^\infty)} = \sum_{n,m\in\mathbb{N}} a_n \bar{b}_m \frac{\widehat{\varphi}(nm)}{d_\alpha(nm)} \\ &= \langle \psi, \widetilde{f g}\rangle_{H^2(\mathbb{T}^\infty)} = \langle \mathbb{H}_\psi f, g\rangle_{H^2(\mathbb{T}^\infty)},\end{aligned}$$

where $\mathbb{H}_\psi$ is the Helson operator on $H^2(\mathbb{T}^\infty)$ with symbol ψ. Then the conclusion for $\alpha > 1$ can be obtained by applying the result for $\alpha = 1$ to $\mathbb{H}_\psi$. The proof is finished. ∎

We remark that characterizing positive Hankel operators in terms of their symbols is a long-standing open problem even in the case of the Hardy space over the unit circle [18, Problem 8.14]. Some useful comments on this problem can be found in [30].

To end this section, we make a comparison between the results of Perfekt and Pushnitski [27, Theorem 6.6] and that of this paper. Actually, Perfekt and Pushnitski give a complete description of Helson matrices of finite rank [27, Theorem 6.6(i)]), and they obtain a criterion for a finite rank Helson matrix to be bounded [27, Theorem 6.6(ii)]. In this paper, Theorem 4.1 characterizes finite rank Helson operators on $A^2_\alpha(\mathbb{D}^\infty)$, and it can be viewed as an alternative form of [27, Theorem 6.6(ii)]. Furthermore, it is worth mentioning that the "automatic boundedness" phenomenon (Theorem 4.5) for finite rank Helson operators on $A^2_\alpha(\mathbb{D}^\infty)$ does not hold for the general Helson matrices of finite rank considered in [27, Theorem 6.6(i)].

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References

[1] P. Ahern and D. Clark, *Invariant subspaces and analytic continuation in several variables*, J. Math. Mech. 19 (1970), 963–969.

[2] M. Bailleul and P. Lefèvre, *Some Banach spaces of Dirichlet series*, Studia Math. 226 (2015), 17–55.

[3] F. Bayart, *Hardy spaces of Dirichlet series and their composition operators*, Monatsh. Math. 136 (2002), 203–236.

[4] F. Bayart, O. F. Brevig, A. Haimi, J. Ortega-Cerdà and K.-M. Perfekt, *Contractive inequalities for Bergman spaces and multiplicative Hankel forms*, Trans. Amer. Math. Soc. 371 (2019), 681–707.

[5] H. Bohr, *Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen* $\sum a_n/n^s$, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. 1913, 441–488.

[6] H. Bommier-Hato, *Volterra operators and Hankel forms on Bergman spaces of Dirichlet series*, Monatsh. Math. 193 (2020), 247–289.

[7] A. Bondarenko, O. F. Brevig, E. Saksman and K. Seip, *Linear space properties of* H^p *spaces of Dirichlet series*, Trans. Amer. Math. Soc. 372 (2019), 6677–6702.

[8] O. F. Brevig and K.-M. Perfekt, *Failure of Nehari's theorem for multiplicative Hankel forms in Schatten classes*, Studia Math. 228 (2015), 101–108.

[9] O. F. Brevig, K.-M. Perfekt and K. Seip, *Volterra operators on Hardy spaces of Dirichlet series*, J. Reine Angew. Math. 754 (2019), 179–223.

[10] O. F. Brevig, K.-M. Perfekt, K. Seip, A. G. Siskakis and D. Vukotić, *The multiplicative Hilbert matrix*, Adv. Math. 302 (2016), 410–432.

[11] B. J. Cole and T. W. Gamelin, *Representing measures and Hardy spaces for the infinite polydisk algebra*, Proc. London Math. Soc. (3) 53 (1986), 112–142.

[12] H. Dan and K. Guo, *The periodic dilation completeness problem: cyclic vectors in the Hardy space over the infinite-dimensional polydisk*, J. London Math. Soc. (2) 103 (2021), 1–34.

[13] H. Dan, K. Guo and H. Huang, *Submodules of the Hardy module in infinitely many variables*, J. Operator Theory 80 (2018), 375–397.

[14] R. G. Douglas, V. I. Paulsen, C. H. Sah and K. Yan, *Algebraic reduction and rigidity for Hilbert modules*, Amer. J. Math. 117 (1995), 75–92.

[15] S. R. Garcia, J. Mashreghi and W. T. Ross, *Introduction to Model Spaces and Their Operators*, Cambridge Stud. Adv. Math. 148, Cambridge Univ. Press, Cambridge, 2016.

[16] R. Gilmer and H. Heinzer, *The Laskerian property, power series rings and Noetherian spectra*, Proc. Amer. Math. Soc. 79 (1980), 13–16.

[17] K. Guo and D. Zheng, *Invariant subspaces, quasi-invariant subspaces, and Hankel operators*, J. Funct. Anal. 187 (2001), 308–342.

[18] W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory*, 50th anniversary edition, Problem Books in Mathematics, Springer, Cham, 2019.

[19] H. Hedenmalm, P. Lindqvist and K. Seip, *A Hilbert space of Dirichlet series and systems of dilated functions in* $L^2(0,1)$, Duke Math. J. 86 (1997), 1–37.

[20] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[21] H. Helson, *Hankel forms and sums of random variables*, Studia Math. 176 (2006), 85–92.

[22] H. Helson, *Hankel forms*, Studia Math. 198 (2010), 79–84.

[23] S. Lang, *Hilbert's Nullstellensatz in infinite-dimensional space*, Proc. Amer. Math. Soc. 3 (1952), 407–410.

[24] N. Nikolski, *In a shadow of the RH: cyclic vectors of Hardy spaces on the Hilbert multidisc*, Ann. Inst. Fourier (Grenoble) 62 (2012), 1601–1626.

[25] J. Ortega-Cerdà and K. Seip, *A lower bound in Nehari's theorem on the polydisc*, J. Anal. Math. 118 (2012), 339–342.

[26] V. V. Peller, *Hankel Operators and Their Applications*, Springer Monogr. Math., Springer, New York, 2003.

[27] K.-M. Perfekt and A. Pushnitski, *On Helson matrices: moment problems, non-negativity, boundedness, and finite rank*, Proc. London Math. Soc. (3) 116 (2018), 101–134.

[28] K.-M. Perfekt and A. Pushnitski, *On the spectrum of the multiplicative Hilbert matrix*, Ark. Mat. 56 (2018), 163–183.

[29] S. C. Power, *Finite rank multivariable Hankel forms*, Linear Algebra Appl. 48 (1982), 237–244.

[30] S. C. Power, *Hankel operators on Hilbert space*, Bull. London Math. Soc. 12 (1980), 422–442.

[31] H. Queffélec and M. Queffélec, *Diophantine Approximation and Dirichlet Series*, 2nd ed., Texts and Readings in Math. 80, Hindustan Book Agency, New Delhi, and Springer, Singapore, 2020.

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