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Kato's inequality for the strong $p(\cdot)$-Laplacian

by

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Abstract. Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be open. Consider the strong $p(\cdot)$-Laplacian

$$\widetilde{\Delta}_{p(\cdot)} u := |\nabla u|^{p(\cdot)-4}[(p(\cdot)-2)\Delta_\infty u + |\nabla u|^2 \Delta u],$$

where

$$\Delta_\infty u := \sum_{i,j=1}^{d} (\partial_i u)(\partial_j u)\partial_{ij}^2 u.$$

We show that

$$\widetilde{\Delta}_{p(\cdot)} |u| \geq (\operatorname{sgn} u)\widetilde{\Delta}_{p(\cdot)} u$$

in the sense of distributions for a certain exponent $p \in C^1(\Omega)$ with $1 < p^- < p^+ < \infty$ and for functions u belonging to an admissible class. This extends the well-known Kato's inequality for strongly elliptic second-order differential operators to the strong $p(\cdot)$-Laplacian.

1. Introduction. The strong $p(\cdot)$-Laplacian

$$\widetilde{\Delta}_{p(\cdot)} u := |\nabla u|^{p(\cdot)-4}[(p(\cdot)-2)\Delta_\infty u + |\nabla u|^2 \Delta u], \tag{1.1}$$

where

$$\Delta_\infty u := \sum_{i,j=1}^{d} (\partial_i u)(\partial_j u)\partial_{ij}^2 u,$$

was first introduced by Adamowicz and Hästö [AH10]. When $u \in C^2(\Omega)$ one can recast (1.1) in the form

$$\widetilde{\Delta}_{p(\cdot)} u = \operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u) - |\nabla u|^{p(\cdot)-2}\log(|\nabla u|)\nabla u \cdot \nabla p \tag{1.2}$$

which is suitable for a weak formulation. The appearance of the log-term in (1.2) poses a technical challenge as it is of supercritical growth. That is, $|\nabla u|^{p(\cdot)-2}\log(|\nabla u|)\nabla u$ grows with order $|\nabla u|^{p(\cdot)-1}|\log(|\nabla u|)|$ whereas the

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main term $|\nabla u|^{p(\cdot)-2}\nabla u$ has order $|\nabla u|^{p(\cdot)-1}$. This leads one to think that the strong $p(\cdot)$-Laplacian has a disadvantage over the standard $p(\cdot)$-Laplacian

$$\Delta_{p(\cdot)}u = \operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u).$$

On the contrary, Adamowicz and Hästö [AH10, AH11] showed that the weak solutions of

$$\widetilde{\Delta}_{p(\cdot)}u = 0$$

enjoy certain nice properties such as scalability, geometric regularity and a scale-invariant Harnack inequality which are not shared by the weak solutions to $\Delta_{p(\cdot)}u = 0$. Gradient estimates for weak solutions of equations involving the strong $p(\cdot)$-Laplacian were then studied in [ZZ12, ZZZ17]. Besides these, the theory of the strong $p(\cdot)$-Laplacian remains incomplete and its further properties are open to explore.

In this paper we wish to add one interesting remark to the aforementioned list of to-be-explored properties of the strong $p(\cdot)$-Laplacian. To be specific, we show that the well-known Kato inequality holds for $\widetilde{\Delta}_{p(\cdot)}$. The background on the Kato inequality is as follows. Kato [Kat72] introduced the inequality

$$H|u| \geq \operatorname{Re}(\overline{\operatorname{sgn} u}\, Hu), \tag{1.3}$$

which holds in the distributional sense for all $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $Hu \in L^1_{\mathrm{loc}}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is open and

$$Hu = \sum_{j,k=1}^{d} \partial_j(a_{jk}\partial_k u).$$

Here $a_{jk} \in C^1(\Omega, \mathbb{R})$ for all $j, k \in \{1, \ldots, d\}$ and the matrix $(a_{jk})_{1\leq j,k\leq d}$ is symmetric and strongly elliptic. This inequality has many far-reaching consequences. Kato first used the inequality in proving the self-adjointness of the Schrödinger operator $-\Delta + V$ in the same paper [Kat72], and then provided an L_p-analysis of $-\Delta + V$ in a subsequent paper [Kat86]. The latter contains the most surprising result when $p = 1$. Since then Kato's inequality has been generalized to various settings. In the realm of linear operators, [Sim77] and [Are84] used Kato's inequality to characterize generators of positive C_0-semigroups, and in [Ouh05, Chapter 3] the sub-Markovian property of a C_0-semigroup is described via Kato's inequality. In another direction, Brezis and Ponce [BP04] extended (1.3) with $H = \Delta$ in order to allow functions $u \in L^1_{\mathrm{loc}}(\Omega)$ such that Δu is a Radon measure. This enabled the study in [BMP07] of the existence of solutions of the nonlinear equation $-\Delta u + g(u) = \mu$, where μ is a measure and $g : \mathbb{R} \to \mathbb{R}$ is a non-decreasing continuous function. Regarding non-linear extensions, [LH19] and [Hor01] developed the p-Laplacian version (with p being a constant) and more generally the quasi-linear version of Kato's inequality, generalizing [Kat72] and [BP04].

These two papers [LH19, Hor01] are of special interest for us. Another nonlinear extension of Kato's inequality includes a Kato's inequality up to the boundary in [BP08].

Having the strong $p(\cdot)$-Laplacian and Kato's inequality in mind, our main result in this paper can be thought of as a further nonlinear development of Kato's inequality. In fact, it provides a blended taste of [Kat72, Hor01, AH11] together.

Now we would like to formulate our problem precisely. Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ be open. We emphasize that Ω need not be bounded. Let $p \in \mathcal{P}(\Omega)$, where $\mathcal{P}(\Omega)$ is the set of measurable functions $\Omega \to [1, \infty]$. Set

$$p^- := \operatorname*{ess\,inf}_{x \in \Omega} p(x) \quad \text{and} \quad p^+ := \operatorname*{ess\,sup}_{x \in \Omega} p(x).$$

Equation (1.2) states that $\widetilde{\Delta}_{p(\cdot)}$ operates well on C^2-functions. A broader class on which the strong $p(\cdot)$-Laplacian acts is given next.

Definition 1.1. The class $\mathcal{A}_{p(\cdot)}(\Omega)$ of admissible functions is defined to consist of all $u \in W^{2,p(\cdot)}_{\mathrm{loc}}(\Omega)$ such that $|\nabla u|^{p(\cdot)-2}|\nabla^2 u| \in L^1_{\mathrm{loc}}(\Omega)$.

Hereafter, we denote

$$a \vee b := \max\{a, b\} \quad \text{and} \quad a \wedge b := \min\{a, b\},$$

where $a, b \in \mathbb{R}$. Two remarks on the class $\mathcal{A}_{p(\cdot)}(\Omega)$ are in order.

Remark 1.2. Suppose $p^+ < \infty$ and $u \in \mathcal{A}_{p(\cdot)}(\Omega)$. Then it follows from (1.1) that

$$|\widetilde{\Delta}_{p(\cdot)} u| \le [(p^+ - 1) \vee 2] |\nabla u|^{p(\cdot)-2} |\nabla^2 u|,$$

from which we may conclude that $\widetilde{\Delta}_{p(\cdot)} u \in L^1_{\mathrm{loc}}(\Omega)$. Here and in what follows,

$$|\nabla^2 u| := \sum_{i,j=1}^{d} |\partial^2_{ij} u|.$$

Remark 1.3. We require a higher integrability for functions in $\mathcal{A}_{p(\cdot)}(\Omega)$ compared to those in the class K_p of [Hor01]. This is necessary for the second term on the right hand side of (1.2) (involving ∇p) to be integrable. Note that [Hor01] deals with the classical p-Laplacian with p being a constant, whence ∇p is trivial.

It is convenient to use the following notation. Let $\mathcal{D}'(\Omega)$ denote the dual space of $C^\infty_c(\Omega)$ and

$$(\operatorname{sgn} u)(x) := \begin{cases} \dfrac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\[2ex] 0 & \text{otherwise} \end{cases}$$

for all $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R})$ and $x \in \Omega$. Given a measurable function f on Ω, we write

$$[f \neq 0] := \{x \in \Omega : f(x) \neq 0\}.$$

A similar convention applies to $[f = 0]$, $[f > 0]$ and $[f < 0]$. Also define

$$\mathcal{B} := \overline{[p(\cdot) < 2]} \cap [p(\cdot) \geq 2], \tag{1.4}$$

where the bar denotes the closure in Ω. The appearance of the boundary set $\mathcal{B}$ is due to a technical reason in our subsequent analysis.

Kato's inequality for the strong $p(\cdot)$-Laplacian is as follows.

THEOREM 1.4. *Let $d \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^d$ be open. Let $p \in C^1(\Omega)$ be such that $1 < p^- < p^+ < \infty$ and $\nabla p = 0$ on $\mathcal{B}$. Let $u \in \mathcal{A}_{p(\cdot)}(\Omega)$. Then*

$$\widetilde{\Delta}_{p(\cdot)}|u| \geq (\operatorname{sgn} u)\widetilde{\Delta}_{p(\cdot)}u \quad \text{in } \mathcal{D}'(\Omega).$$

That is,

$$\begin{aligned} -\int_{[\nabla u \neq 0]} \big|\nabla|u|\big|^{p(\cdot)-2}\nabla|u| \cdot \nabla\varphi \\ -\int_{[\nabla u \neq 0]} \big|\nabla|u|\big|^{p(\cdot)-2} \log\big(\big|\nabla|u|\big|\big)\nabla|u| \cdot \nabla p\,\varphi \\ \geq \int_{[\nabla u \neq 0]} (\operatorname{sgn} u)(\widetilde{\Delta}_{p(\cdot)}u)\varphi \end{aligned} \tag{1.5}$$

for all $0 \leq \varphi \in C_c^\infty(\Omega)$.

We have three remarks concerning the main result.

REMARK 1.5. In Theorem 1.4, the function u is in $\mathcal{A}_{p(\cdot)}(\Omega)$, which implies $\widetilde{\Delta}_{p(\cdot)}u \in L^1_{\mathrm{loc}}(\Omega)$ by Remark 1.2. However, $|u|$ need not be in $\mathcal{A}_{p(\cdot)}(\Omega)$. This explains the weak formulation on the left hand side of (1.5).

REMARK 1.6. Technically, the integrals in (1.5) should be taken on the set $[\nabla|u| \neq 0]$. Nevertheless, $[\nabla|u| \neq 0] = [\nabla u \neq 0]$ in view of [GT83, Lemma 7.6].

REMARK 1.7. The integrals in (1.5) are well-defined. The technical details are presented in Proposition 3.1 when a sufficient background has been developed.

The paper is outlined as follows. In Section 2 we provide appropriate settings of function spaces for our problem. Theorem 1.4 is proved in Section 3.

2. Function spaces. This section provides a brief summary of definitions and properties for Lebesgue and Sobolev spaces with variable exponents to be used subsequently. A thorough account can be found in [DH$^+$11].

Let $\Omega \subset \mathbb{R}^d$ be open and $p \in \mathcal{P}(\Omega)$ with $p^+ < \infty$.

DEFINITION 2.1. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ is defined to consist of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\varrho_{L^{p(\cdot)}(\Omega)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx < \infty.$$

We endow $L^{p(\cdot)}(\Omega)$ with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\{\lambda > 0 : \varrho_{L^{p(\cdot)}(\Omega)}(u/\lambda) \le 1\}.$$

It is well-known that $L^{p(\cdot)}(\Omega)$ is a Banach space. We will implicitly make use of the following convenient facts.

(i) For all $u \in L^{p(\cdot)}(\Omega)$,

$$\begin{aligned}\varrho_{L^{p(\cdot)}(\Omega)}(u)^{1/p^-} \wedge \varrho_{L^{p(\cdot)}(\Omega)}(u)^{1/p^+} &\le \|u\|_{L^{p(\cdot)}(\Omega)}\\ &\le \varrho_{L^{p(\cdot)}(\Omega)}(u)^{1/p^-} \vee \varrho_{L^{p(\cdot)}(\Omega)}(u)^{1/p^+}.\end{aligned}$$

(ii) For all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$ the following version of the Hölder inequality holds:

$$\|uv\|_{L^{w(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{q(\cdot)}(\Omega)},$$

where $p, q, w \in \mathcal{P}(\Omega)$ are such that

$$\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{w(x)} \quad \text{for a.e. } x \in \Omega.$$

(iii) Let $p, q \in \mathcal{P}(\Omega)$ be such that $p \ge q$. Define $r \in \mathcal{P}(\Omega)$ by

$$\frac{1}{r(\cdot)} = \frac{1}{q(\cdot)} - \frac{1}{p(\cdot)}.$$

Then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ provided that $1 \in L^{r(\cdot)}(\Omega)$. The condition $1 \in L^{r(\cdot)}(\Omega)$ is automatic when $|\Omega| < \infty$ due to [DH$^+$11, Lemma 3.2.12].

(iv) For all $u \in L^{p(\cdot)}(\Omega)$ and all $s \in (0, p^-]$,

$$\big\||u|^s\big\|_{L^{p(\cdot)/s}(\Omega)} = \|u\|^s_{L^{p(\cdot)}(\Omega)}$$

(cf. [DH$^+$11, Lemma 3.2.6]).

A generalization of property (iv) above is given next.

LEMMA 2.2. *Let $p \in \mathcal{P}(\Omega)$ be such that $p^+ < \infty$. Let q be measurable on Ω and $0 < q(x) \le p(x)$ for all $x \in \Omega$. Let $f \in L^{p(\cdot)}(\Omega)$. Then $|f|^q \in L^{p(\cdot)/q(\cdot)}(\Omega)$ and*

$$\big\||f|^q\big\|_{L^{p(\cdot)/q(\cdot)}(\Omega)} \le \max_{x\in\Omega} \|f\|^{q(x)}_{L^{p(\cdot)}(\Omega)} \le \|f\|^{q^-}_{L^{p(\cdot)}(\Omega)} \vee \|f\|^{q^+}_{L^{p(\cdot)}(\Omega)}.$$

Proof. The second inequality is clear, so we focus on the first, from which it also follows that $|f|^q \in L^{p(\cdot)/q(\cdot)}(\Omega)$.

Obviously the first inequality holds when $f = 0$. Suppose $f \neq 0$. We consider two cases.

CASE 1: $\|f\|_{L^{p(\cdot)}(\Omega)} = 1$. We have

$$\big\||f|^q\big\|_{L^{p(\cdot)/q(\cdot)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big(\frac{|f(x)|^{q(x)}}{\lambda}\Big)^{p(x)/q(x)} dx \le 1\Big\}$$
$$= \inf\Big\{\lambda > 0 : \int_\Omega \frac{|f(x)|^{p(x)}}{\lambda^{p(x)/q(x)}}\, dx \le 1\Big\}.$$

For $\lambda = 1$ we obtain

$$\int_\Omega \frac{|f(x)|^{p(x)}}{\lambda^{p(x)/q(x)}}\, dx = \int_\Omega |f(x)|^{p(x)}\, dx = 1,$$

where we use [DH$^+$11, Lemma 3.2.4] in the last step. Consequently, we have $\big\||f|^q\big\|_{L^{p(\cdot)/q(\cdot)}(\Omega)} \le 1$.

CASE 2: $\|f\|_{L^{p(\cdot)}(\Omega)} \neq 1$. We set $g = f/\|f\|_{L^{p(\cdot)}(\Omega)}$. Then $\|g\|_{L^{p(\cdot)}(\Omega)} = 1$. It follows from Case 1 that

$$1 \ge \big\||g|^q\big\|_{L^{p(\cdot)/q(\cdot)}(\Omega)} = \bigg\|\frac{|f|^q}{\|f\|^q_{L^{p(\cdot)}(\Omega)}}\bigg\|_{L^{p(\cdot)/q(\cdot)}(\Omega)}$$
$$\ge \bigg\|\frac{|f|^q}{\max_{x\in\Omega}\|f\|^{q(x)}_{L^{p(\cdot)}(\Omega)}}\bigg\|_{L^{p(\cdot)/q(\cdot)}(\Omega)} = \frac{1}{\max_{x\in\Omega}\|f\|^{q(x)}_{L^{p(\cdot)}(\Omega)}}\big\||f|^q\big\|_{L^{p(\cdot)/q(\cdot)}(\Omega)}.$$

Hence the claim follows. ■

LEMMA 2.3. *Let q be measurable on Ω with values in $(0,\infty)$. Let $\alpha \in (0, q^-)$ and $u \in L^{q(\cdot)}_{\mathrm{loc}}(\Omega)$. Then*

$$|u|^{q(\cdot)-\alpha} \le |u|^{q(\cdot)} + 1$$

on Ω. In particular, $|u|^{q(\cdot)-\alpha} \in L^1_{\mathrm{loc}}(\Omega)$.

Proof. This follows directly from Young's inequality with exponents $\big(\frac{q}{q-\alpha}, \frac{q}{\alpha}\big)$. ■

In the proof of Theorem 1.4, we will work with a mollification. To state its properties, we require the following smoothness condition on the exponent $p(\cdot)$ which is taken from [DH$^+$11, Definition 4.1.1].

DEFINITION 2.4. We say that $\alpha : \Omega \to \mathbb{R}$ is *locally log-Hölder continuous* if there exists a $c_1 > 0$ such that

$$|\alpha(x) - \alpha(y)| \le \frac{c_1}{\log(e + 1/|x-y|)} \quad \text{for all } x, y \in \Omega.$$

We say that $\alpha : \Omega \to \mathbb{R}$ satisfies the *log-Hölder decay condition* if there exist constants $\alpha_\infty \in \mathbb{R}$ and $c_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| \le \frac{c_2}{\log(e + |x|)} \quad \text{for all } x \in \Omega.$$

We say that $\alpha : \Omega \to \mathbb{R}$ is *globally log-Hölder continuous* if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.

The class $\mathcal{P}^{\log}(\Omega)$ is defined to consist of all $p \in \mathcal{P}(\Omega)$ such that $1/p$ is globally log-Hölder continuous.

Let $\{J_n\}_{n\in\mathbb{N}}$ be a sequence of standard Friedrichs mollifiers. That is, let $J \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ be defined by

$$J(x) = \begin{cases} A\exp\left(-\dfrac{1}{1-|x|^2}\right) & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where A is the normalizing constant such that $\int_{\mathbb{R}^d} J = 1$. For each $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ define

$$J_n(x) = n^d J(nx). \tag{2.1}$$

Observe that J_n has a bell-shaped curve for all $n \in \mathbb{N}$. With this in mind, an application of [DH$^+$11, Lemma 4.6.3 and Theorem 4.6.4] yields the following result.

Lemma 2.5. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$ and $u \in L^{p(\cdot)}(\mathbb{R}^d)$. For each $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ define*

$$u_n(x) = u * J_n(x) = \int_{\mathbb{R}^d} u(x-y)J_n(y)\,dy.$$

Then the following properties hold:

(i) *There exists a $C > 0$ such that*

$$\|u_n\|_{L^{p(\cdot)}(\mathbb{R}^d)} \le C\|u\|_{L^{p(\cdot)}(\mathbb{R}^d)} \quad \textit{for all } n \in \mathbb{N}.$$

(ii) *$u_n \to u$ pointwise in $\mathbb{R}^d$ as $n \to \infty$.*

(iii) *$u_n \to u$ in $L^{p(\cdot)}(\mathbb{R}^d)$ as $n \to \infty$ provided that $p^+ < \infty$.*

We finish this section with the definition of once-differentiable Sobolev space with variable exponent.

Definition 2.6. The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is defined to consist of all $u \in L^{p(\cdot)}(\Omega)$ whose distributional derivative $\partial_j u$ is in $L^{p(\cdot)}(\Omega)$ for all $j \in \{1,\dots,d\}$.

The space $W^{1,p(\cdot)}(\Omega)$ is a Banach space under the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \sum_{j=1}^{d} \|\partial_j u\|_{L^{p(\cdot)}(\Omega)}.$$

We also write $W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ for the space of all measurable functions u on Ω such that $u|_{\Omega'} \in W^{1,p(\cdot)}(\Omega')$ for all $\Omega' \Subset \Omega$.

3. Kato's inequality. In this section, let $\Omega \subset \mathbb{R}^d$ be open. We emphasize that Ω need not be bounded. We assume throughout that $p(\cdot) \in C^1(\Omega)$ satisfies

$$1 < p^- < p^+ < \infty$$

and $\nabla p = 0$ on $\mathcal{B}$, where $\mathcal{B}$ is given by (1.4). Also fix a constant γ_1 such that

$$\gamma_1 \in (1, 2 \wedge p^-). \tag{3.1}$$

Hereafter we always set

$$u_\epsilon := (u^2 + \epsilon^2)^{1/2}$$

for each measurable $u : \Omega \to \mathbb{R}$ and $\epsilon \in (0,1)$.

If $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ then

$$\nabla u_\epsilon = \frac{u}{u_\epsilon} \nabla u,$$

whence

$$|\nabla u_\epsilon| \le |\nabla u|.$$

We will make use of this estimate frequently and implicitly in what follows.

In order to deal with the log-term in (1.5), we will often employ the following logarithmic inequality. For all $\alpha \in (0,1)$ there exists a constant $C = C(\alpha) > 0$ such that

$$|\log t| \le C(t^{-\alpha} + t^\alpha) \quad \text{for all } t > 0. \tag{3.2}$$

PROPOSITION 3.1. *The integrals in* (1.5) *are well-defined.*

Proof. Let $u \in \mathcal{A}_{p(\cdot)}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$. Then

$$|(\operatorname{sgn} u)(\widetilde{\Delta}_{p(\cdot)} u)\varphi\, \mathbb{1}_{[\nabla u \neq 0]}| \le |\widetilde{\Delta}_{p(\cdot)} u|\, \|\varphi\|_{L^\infty(\Omega)} \in L^1(\operatorname{supp}\varphi)$$

and

$$\begin{aligned} \Big| \big|\nabla |u|\big|^{p(\cdot)-2} \nabla |u| \cdot \nabla\varphi\, \mathbb{1}_{[\nabla u \neq 0]} \Big| &\le \big|\nabla |u|\big|^{p(\cdot)-1} \|\nabla \varphi\|_{L^\infty(\Omega)} \\ &= |\nabla u|^{p(\cdot)-1} \|\nabla\varphi\|_{L^\infty(\Omega)} \in L^1(\operatorname{supp}\varphi). \end{aligned}$$

Concerning the integral with the logarithmic term, we have

$$\begin{aligned} &\Big| \big|\nabla |u|\big|^{p(\cdot)-2} \log\big(\big|\nabla|u|\big|\big) \nabla |u| \cdot \nabla p\, \varphi\, \mathbb{1}_{[\nabla u \neq 0]} \Big| \\ &\qquad \le C \big|\nabla |u|\big|^{p(\cdot)-1} \big(\big|\nabla|u|\big|^{-\alpha} + \big|\nabla |u|\big|^{\alpha} \big) \|\nabla p\, \varphi\|_{L^\infty(\Omega)}\, \mathbb{1}_{[\nabla u \neq 0]} \\ &\qquad = C \big(|\nabla u|^{p(\cdot)-1-\alpha} + |\nabla u|^{p(\cdot)-1+\alpha} \big) \|\nabla p\, \varphi\|_{L^\infty(\Omega)}\, \mathbb{1}_{[\nabla u \neq 0]} \\ &\qquad \le 2C \big(|\nabla u|^{p(\cdot)} + 1 \big) \|\nabla p\, \varphi\|_{L^\infty(\Omega)} \in L^1(\operatorname{supp}\varphi) \end{aligned}$$

for all $\alpha \in (0, \gamma_1 - 1)$, where $C = C(\alpha) > 0$ and we have used (3.2) in the second step as well as Lemma 2.3 in the last step. ∎

Now we proceed to the proof of Theorem 1.4. The main idea in proving Kato's inequality for the strong $p(\cdot)$-Laplacian is to perform various approximating and limiting procedures. We note that the expressions in (1.5) contain singularities on the set $[\nabla u = 0] \cap [p(\cdot) < 2]$. These singularities do not behave well in limiting procedures. Therefore, it is natural to introduce the following approximate operator.

For each $\eta > 0$ define formally

$$\begin{aligned}\widetilde{\Delta}_{p(\cdot),\eta} u := (\eta^2\, \mathbb{1}_{[p(\cdot)<2]} + |\nabla u|^2)^{\frac{p(\cdot)-4}{2}} \\ \times [(p(\cdot)-2)\Delta_\infty u + (\eta^2\, \mathbb{1}_{[p(\cdot)<2]} + |\nabla u|^2)\Delta u].\end{aligned}$$

If $u \in C^2(\Omega)$ then we may also write

$$\begin{aligned}&\widetilde{\Delta}_{p(\cdot),\eta} u \\ &\quad = \begin{cases} \operatorname{div}\big((\eta^2 + |\nabla u|^2)^{\frac{p(\cdot)-2}{2}} \nabla u\big) \\ \qquad -\frac{1}{2}\big(\eta^2 + |\nabla u|^2\big)^{\frac{p(\cdot)-2}{2}} \log(\eta^2 + |\nabla u|^2)\nabla u \cdot \nabla p & \text{on } [p(\cdot) < 2], \\ \operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u) \\ \qquad -\frac{1}{2}|\nabla u|^{p(\cdot)-2} \log(|\nabla u|^2)\nabla u \cdot \nabla p & \text{on } [p(\cdot) \geq 2] \setminus \mathcal{B}, \end{cases}\end{aligned}$$

where $\mathcal{B}$ is the boundary set given by (1.4). Since $p \in C^1(\Omega)$ satisfies $\nabla p = 0$ on $\mathcal{B}$ by hypothesis, it follows that

$$(\eta^2 + |\nabla u|^2)^{\frac{p(\cdot)-2}{2}} \nabla u \in (C^1(\Omega))^d \tag{3.3}$$

and

$$\begin{aligned}&\lim_{\substack{x\to x_0 \\ p(x)<2}} \tfrac{1}{2} \log(\eta^2 + |\nabla u(x)|^2)\nabla u(x) \cdot \nabla p(x) \\ &\qquad\qquad = \tfrac{1}{2} \log(|\nabla u(x_0)|^2) \cdot \nabla u(x_0) \cdot \nabla p(x_0) = 0\end{aligned}$$

for all $x_0 \in \mathcal{B}$ and $u \in C^2(\Omega)$. Hence we also have

$$\widetilde{\Delta}_{p(\cdot),\eta} u \in C(\Omega) \quad \text{for all } u \in C^2(\Omega).$$

The next definition is fundamental to our study.

Definition 3.2. Let $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ and $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$. For each $\eta > 0$ we write

$$\lim_{n\to\infty} \widetilde{\Delta}_{p(\cdot),\eta} u_n = \widetilde{\Delta}_{p(\cdot),\eta} u \quad \text{in } \mathcal{D}'(\Omega)$$

to mean that

$$\lim_{n\to\infty}\Big(\int_{\Omega}(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+|\nabla u_n|^2)^{\frac{p(\cdot)-2}{2}}\nabla u_n\cdot\nabla\varphi$$
$$+\frac{1}{2}\int_{\Omega}(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+|\nabla u_n|^2)^{\frac{p(\cdot)-2}{2}}\log(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+|\nabla u_n|^2)\nabla u_n\cdot\nabla p\,\varphi\Big)$$
$$=\int_{\Omega}(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+|\nabla u|^2)^{\frac{p(\cdot)-2}{2}}\nabla u\cdot\nabla\varphi$$
$$+\frac{1}{2}\int_{\Omega}(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+|\nabla u|^2)^{\frac{p(\cdot)-2}{2}}\log(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+|\nabla u|^2)\nabla u\cdot\nabla p\,\varphi$$

for all $\varphi\in C_c^\infty(\Omega)$.

We also write

$$\lim_{\eta\to 0^+}\widetilde{\Delta}_{p(\cdot),\eta}u=\widetilde{\Delta}_{p(\cdot)}u \quad \text{in } \mathcal{D}'(\Omega)$$

to mean that

$$\lim_{\eta\to 0^+}\Big(\int_{[\nabla u\neq 0]}(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+|\nabla u|^2)^{\frac{p(\cdot)-2}{2}}\nabla u\cdot\nabla\varphi$$
$$+\frac{1}{2}\int_{[\nabla u\neq 0]}(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+|\nabla u|^2)^{\frac{p(\cdot)-2}{2}}\log(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+|\nabla u|^2)\nabla u\cdot\nabla p\,\varphi\Big)$$
$$=\int_{[\nabla u\neq 0]}|\nabla u|^{p(\cdot)-2}\nabla u\cdot\nabla\varphi+\int_{[\nabla u\neq 0]}|\nabla u|^{p(\cdot)-2}\log(|\nabla u|)\nabla u\cdot\nabla p\,\varphi$$

for all $\varphi\in C_c^\infty(\Omega)$.

As preparatory steps, we now provide two technical lemmas to be used in the proof of Theorem 1.4.

LEMMA 3.3. *Let $\eta\in(0,1)$ and $u\in\mathcal{A}_{p(\cdot)}(\Omega)$. Then the following statements hold:*

(i) $\widetilde{\Delta}_{p(\cdot),\eta}u\in L^1_{\mathrm{loc}}(\Omega)$ *and*

$$\lim_{\epsilon\to 0^+}\int_{\Omega}\left|\frac{u}{u_\epsilon}\right|^{p(\cdot)-2}\frac{u}{u_\epsilon}\widetilde{\Delta}_{p(\cdot),\eta}u\varphi=\int_{\Omega}(\operatorname{sgn}u)\widetilde{\Delta}_{p(\cdot),\eta}u\,\varphi \qquad \textit{for all } \varphi\in C_c^\infty(\Omega).$$

(ii) $\lim_{\epsilon\to 0^+}\widetilde{\Delta}_{p(\cdot),\eta}u_\epsilon=\widetilde{\Delta}_{p(\cdot),\eta}|u|$ *in* $\mathcal{D}'(\Omega)$.

Proof. (i) Let $\varphi \in C_c^\infty(\Omega)$. Then

$$\begin{aligned}\left| \left|\frac{u}{u_\epsilon}\right|^{p(\cdot)-2} \frac{u}{u_\epsilon} \widetilde{\Delta}_{p(\cdot),\eta} u\varphi \right| &\le |\widetilde{\Delta}_{p(\cdot),\eta} u| \, \|\varphi\|_{L^\infty(\Omega)} \\ &\le (|p(\cdot)-2|+1)(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla u|^2)^{\frac{p(\cdot)-2}{2}} |\nabla^2 u| \, \|\varphi\|_{L^\infty(\Omega)} \\ &\le [(p^+-1)\vee 2] |\nabla u|^{p(\cdot)-2} |\nabla^2 u| \, \|\varphi\|_{L^\infty(\Omega)} \in L^1(\operatorname{supp}\varphi),\end{aligned}$$

where we have used the fact that $u \in \mathcal{A}_{p(\cdot)}(\Omega)$ in the last step. Clearly

$$\left|\frac{u}{u_\epsilon}\right|^{p(\cdot)-2} \frac{u}{u_\epsilon} \widetilde{\Delta}_{p(\cdot),\eta} u \to (\operatorname{sgn} u) \widetilde{\Delta}_{p(\cdot),\eta} u \qquad \text{a.e. in } \Omega$$

when $\epsilon \to 0^+$. Hence

$$\lim_{\epsilon\to 0^+} \int_\Omega \left|\frac{u}{u_\epsilon}\right|^{p(\cdot)-2} \frac{u}{u_\epsilon} \widetilde{\Delta}_{p(\cdot),\eta} u \,\varphi = \int_\Omega (\operatorname{sgn} u) \widetilde{\Delta}_{p(\cdot),\eta} u \,\varphi$$

by the Lebesgue dominated convergence theorem.

(ii) Let $\varphi \in C_c^\infty(\Omega)$. We consider the limits when $\epsilon \to 0^+$ of the two terms

$$\int_\Omega (\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \nabla u_\epsilon \cdot \nabla\varphi$$

and

$$\int_\Omega (\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \log(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2) \nabla u_\epsilon \cdot \nabla p \,\varphi.$$

Note that

$$(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \nabla u_\epsilon \cdot \nabla\varphi \le |\nabla u|^{p(\cdot)-1} \|\nabla\varphi\|_{L^\infty(\Omega)} \in L^1(\operatorname{supp}\varphi),$$

where the last step follows from Lemma 2.3. Also for all $\alpha \in (0, \gamma_1 - 1)$ one has

$$\begin{aligned}&(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \log(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2) \nabla u_\epsilon \cdot \nabla p\,\varphi \\ &\quad \le \big((\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla u|^2)^{\frac{p(\cdot)-1-\alpha}{2}} + (\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla u|^2)^{\frac{p(\cdot)-1+\alpha}{2}}\big) \\ &\qquad \times \|\nabla p\|_{L^\infty(\operatorname{supp}\varphi)} \|\varphi\|_{L^\infty(\Omega)} \\ &\quad \le 2((1+|\nabla u|)^{p(\cdot)} + 1) \|\nabla p\|_{L^\infty(\operatorname{supp}\varphi)} \|\varphi\|_{L^\infty(\Omega)} \\ &\quad \le 8^{p^+} (|\nabla u|^{p(\cdot)} + 1) \|\nabla p\|_{L^\infty(\operatorname{supp}\varphi)} \|\varphi\|_{L^\infty(\Omega)} \in L^1(\operatorname{supp}\varphi),\end{aligned}$$

where we have used (3.2) and Lemma 2.3 in the second and third steps respectively.

Since $u_\epsilon \to |u|$ a.e. in Ω, we deduce that

$$\lim_{\epsilon\to 0^+} \int_\Omega (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \nabla u_\epsilon \cdot \nabla\varphi = \int_\Omega \left(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + \big|\nabla|u|\big|^2\right)^{\frac{p(\cdot)-2}{2}} \nabla|u| \cdot \nabla\varphi$$

and

$$\lim_{\epsilon\to 0^+} \int_\Omega |\nabla u_\epsilon|^{p(\cdot)-2} \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2) \nabla u_\epsilon \cdot \nabla p\,\varphi = \int_\Omega \left(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + \big|\nabla|u|\big|^2\right)^{\frac{p(\cdot)-2}{2}} \log\left(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + \big|\nabla|u|\big|^2\right) \nabla|u| \cdot \nabla p\,\varphi$$

by the Lebesgue dominated convergence theorem. It follows that

$$\lim_{\epsilon\to 0^+} \widetilde{\Delta}_{p(\cdot),\eta} u_\epsilon = \widetilde{\Delta}_{p(\cdot),\eta} |u| \quad \text{in } \mathcal{D}'(\Omega),$$

which justifies the claim. ■

Lemma 3.4. *Let $u \in \mathcal{A}_{p(\cdot)}(\Omega)$. Then the following statements hold:*

(i) $\displaystyle\lim_{\eta\to 0^+} \int_{[\nabla u\neq 0]} (\operatorname{sgn} u)(\widetilde{\Delta}_{p(\cdot),\eta} u)\varphi = \int_{[\nabla u\neq 0]} (\operatorname{sgn} u)(\widetilde{\Delta}_{p(\cdot)} u)\varphi$ *for all* $\varphi \in C_c^\infty(\Omega)$.

(ii) $\lim_{\eta\to 0^+} \widetilde{\Delta}_{p(\cdot),\eta}|u| = \widetilde{\Delta}_{p(\cdot)}|u|$ *in* $\mathcal{D}'(\Omega)$.

Proof. (i) Observe that

$$\begin{aligned} &|\widetilde{\Delta}_{p(\cdot),\eta} u\,\varphi| \\ &\quad\le \Bigg|(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla u|^2)^{\frac{p(\cdot)-2}{2}} \\ &\qquad\qquad \times \left(\Delta u + \frac{p(\cdot)-2}{\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla u|^2} \sum_{i,j=1}^d (\partial_i u)(\partial_j u)\partial_{ij}^2 u\right)\Bigg| \|\varphi\|_{L^\infty(\Omega)} \\ &\quad\le [(p^+ - 1) \vee 2] |\nabla u|^{p(\cdot)-2} \Big(\sum_{i,j=1}^d |\partial_{ij}^2 u|\Big) \|\varphi\|_{L^\infty(\Omega)} \in L^1_{\mathrm{loc}}(\Omega) \end{aligned}$$

for all $\eta \in (0,1)$, where the last step follows from the fact that $u \in \mathcal{A}_{p(\cdot)}(\Omega)$.

At the same time, $\widetilde{\Delta}_{p(\cdot),\eta} u \to \widetilde{\Delta}_{p(\cdot)} u$ pointwise in $[\nabla u \neq 0]$ as $\eta \to 0^+$, whence (i) follows from the Lebesgue dominated convergence theorem.

(ii) We consider

$$\lim_{\eta\to 0^+} \int_\Omega \left(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + \big|\nabla|u|\big|^2\right)^{\frac{p(\cdot)-2}{2}} \nabla|u| \cdot \nabla\varphi$$

and

$$\lim_{\eta\to 0^+}\int_{\Omega}\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+\big|\nabla|u|\big|^2\big)^{\frac{p(\cdot)-2}{2}}\log\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+\big|\nabla|u|\big|^2\big)\nabla|u|\cdot\nabla p\,\varphi.$$

Let $\eta\in(0,1)$. We have

$$\begin{aligned}\Big|\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+\big|\nabla|u|\big|^2\big)^{\frac{p(\cdot)-2}{2}}\nabla|u|\cdot\nabla\varphi\Big|&\\ &\le\|\nabla\varphi\|_{L^\infty(\operatorname{supp}\varphi)}\big|\nabla|u|\big|^{p(\cdot)-1}\\ &=\|\nabla\varphi\|_{L^\infty(\operatorname{supp}\varphi)}|\nabla u|^{p(\cdot)-1}\in L^1(\operatorname{supp}\varphi)\end{aligned}$$

by Lemma 2.3. Also for all $\alpha\in(0,\gamma_1-1)$ one has

$$\begin{aligned}&\Big|\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+\big|\nabla|u|\big|^2\big)^{\frac{p(\cdot)-2}{2}}\log\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+\big|\nabla|u|\big|^2\big)\nabla|u|\cdot\nabla p\,\varphi\Big|\\ &\quad\le\Big[\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+\big|\nabla|u|\big|^2\big)^{\frac{p(\cdot)-1-\alpha}{2}}+\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+\big|\nabla|u|\big|^2\big)^{\frac{p(\cdot)-1+\alpha}{2}}\Big]\\ &\qquad\times\|\nabla p\|_{L^\infty(\operatorname{supp}\varphi)}\|\varphi\|_{L^\infty(\Omega)}\\ &\quad=\Big[\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+|\nabla u|^2\big)^{\frac{p(\cdot)-1-\alpha}{2}}+\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+|\nabla u|^2\big)^{\frac{p(\cdot)-1+\alpha}{2}}\Big]\\ &\qquad\times\|\nabla p\|_{L^\infty(\operatorname{supp}\varphi)}\|\varphi\|_{L^\infty(\Omega)}\\ &\quad\le 2\big((1+|\nabla u|)^{p(\cdot)}+1\big)\|\nabla p\|_{L^\infty(\operatorname{supp}\varphi)}\|\varphi\|_{L^\infty(\Omega)}\\ &\quad\le 8^{p^+}(|\nabla u|^{p(\cdot)}+1)\|\nabla p\|_{L^\infty(\operatorname{supp}\varphi)}\|\varphi\|_{L^\infty(\Omega)}\in L^1(\operatorname{supp}\varphi),\end{aligned}$$

where we have used (3.2) in the first step and Lemma 2.3 in the fourth step.

To finish we mention the obvious pointwise convergences and then pass to weak limits to get

$$\begin{aligned}&\lim_{\eta\to 0^+}\Big(\int_{[\nabla u\neq 0]}\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+\big|\nabla|u|\big|^2\big)^{\frac{p(\cdot)-2}{2}}\nabla|u|\cdot\nabla\varphi\\ &+\int_{[\nabla u\neq 0]}\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+\big|\nabla|u|\big|^2\big)^{\frac{p(\cdot)-2}{2}}\log\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]}+\big|\nabla|u|\big|^2\big)\nabla|u|\cdot\nabla p\,\varphi\Big)\\ &\qquad=\int_{[\nabla u\neq 0]}\big|\nabla|u|\big|^{p(\cdot)-2}\nabla|u|\cdot\nabla\varphi\\ &\qquad\quad+\int_{[\nabla u\neq 0]}\big|\nabla|u|\big|^{p(\cdot)-2}\log\big(\big|\nabla|u|\big|\big)\nabla|u|\cdot\nabla p\,\varphi.\end{aligned}$$

The proof is complete. ∎

We are now ready to prove the main theorem.

Proof of Theorem 1.4. Since the inequality is of local nature, we may assume without loss of generality that $\Omega=\mathbb{R}^d$.

Let $0 \le \varphi \in C_c^\infty(\mathbb{R}^d)$ and K be open such that $\operatorname{supp}\varphi \Subset K \Subset \mathbb{R}^d$. Let $\chi, \zeta \in C_c^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp}\chi \Subset K \Subset \operatorname{supp}\zeta$ and $\chi|_{\operatorname{supp}\varphi} = \zeta|_K = 1$.

Since $p \in C^1(\mathbb{R}^d)$ with $p^- > 1$, we deduce that $p\zeta \in \mathcal{P}^{\log}(\mathbb{R}^d)$. Hence we may apply Lemma 2.5 in what follows. In particular, for each $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ define

$$u_n(x) = ((u\chi) * J_n)(x) = \int_{\mathbb{R}^d} u(x-y)\chi(x-y)J_n(y)\,dy,$$

where J_n is given by (2.1). Then $u_n \in C_c^\infty(K)$ for all $n \in \mathbb{N}$ and we have

$$\begin{cases} u_n \to u\chi, \\ \partial_j u_n \to \partial_j(u\chi), \\ \partial_{ij}^2 u_n \to \partial_{ij}^2(u\chi), \end{cases} \tag{3.4}$$

a.e. in $\mathbb{R}^d$, in $L^{p(\cdot)}(\mathbb{R}^d)$ and in $L^1(\mathbb{R}^d)$ as $n \to \infty$ for all $i, j \in \{1, \ldots, d\}$ by Lemma 2.5(ii, iii). In addition,

$$\begin{cases} \|u_n\|_{L^{p(\cdot)}(\Omega)} \le \|u\chi\|_{L^{p(\cdot)}(\Omega)}, \\ \|\partial_j u_n\|_{L^{p(\cdot)}(\Omega)} \le \|\partial_j(u\chi)\|_{L^{p(\cdot)}(\Omega)}, \\ \|\partial_{ij}^2 u_n\| \le \|\partial_{ij}^2(u\chi)\|_{L^{p(\cdot)}(\Omega)}, \end{cases} \tag{3.5}$$

by Lemma 2.5(i). Interpolation shows that (3.4) and (3.5) also hold in $L^{r(\cdot)}(\mathbb{R}^d)$ for all $1 \le r(\cdot) \le p(\cdot)$. This fact will be used frequently in the rest of the proof.

Let $\eta \in (0,1)$. Direct calculations give

$$\partial_{ij}^2 (u_n)_\epsilon = \frac{u_n}{(u_n)_\epsilon}\partial_{ij}^2 u_n + \frac{1}{(u_n)_\epsilon}\left[1 - \left(\frac{u_n}{(u_n)_\epsilon}\right)^2\right](\partial_i u_n)(\partial_j u_n)$$

for all $i, j \in \{1, \ldots, d\}$ and

$$\begin{aligned} &\widetilde{\Delta}_{p(\cdot),\eta}(u_n)_\epsilon \\ &\quad = \frac{u_n}{(u_n)_\epsilon}(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \\ &\qquad \times \left(\Delta u_n + \frac{p(\cdot)-2}{\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2}\left(\frac{u_n}{(u_n)_\epsilon}\right)^2 \sum_{i,j=1}^d (\partial_i u_n)(\partial_j u_n)\partial_{ij}^2 u_n\right) \\ &\quad + \frac{1}{(u_n)_\epsilon}\left[1 - \left(\frac{u_n}{(u_n)_\epsilon}\right)^2\right](\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}}|\nabla u_n|^2 \\ &\qquad \times \left[1 + \frac{p(\cdot)-2}{\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2}\left(\frac{u_n}{(u_n)_\epsilon}\right)^2 |\nabla u_n|^2\right]. \end{aligned}$$

Therefore,

$$\widetilde{\Delta}_{p(\cdot),\eta}(u_n)_\epsilon \geq \frac{u_n}{(u_n)_\epsilon}\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2\big)^{\frac{p(\cdot)-2}{2}} \times \Bigg(\Delta u_n + \frac{p(\cdot)-2}{\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2}\bigg(\frac{u_n}{(u_n)_\epsilon}\bigg)^2 \sum_{i,j=1}^{d} (\partial_i u_n)(\partial_j u_n)\partial_{ij}^2 u_n\Bigg). \tag{3.6}$$

We aim to take various limits on both sides of the above inequality to arrive at the result. First observe that

$$\begin{aligned} \int_{\mathbb{R}^d} \widetilde{\Delta}_{p(\cdot),\eta}(u_n)_\epsilon\,\varphi &= -\int_{\mathbb{R}^d} \big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2\big)^{\frac{p(\cdot)-2}{2}} \nabla(u_n)_\epsilon\cdot\nabla\varphi \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^d} \big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2\big)^{\frac{p(\cdot)-2}{2}} \\ &\qquad\qquad \times \log\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2\big)\nabla(u_n)_\epsilon\cdot\nabla p\,\varphi \\ &=: I + II \end{aligned} \tag{3.7}$$

by virtue of (3.3) and the divergence theorem. Now we estimate each term on the right hand side of (3.7) separately.

Term I: By assumption $p^+ < \infty$. So we may choose a constant $s \in (1,\infty)$ such that $p^+ < 1 + \frac{1}{s-1}$, whence

$$1 < s(p(\cdot)-1) < p(\cdot). \tag{3.8}$$

Then

$$\begin{aligned} &\int_K \Big|\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2\big)^{\frac{p(\cdot)-2}{2}} \nabla(u_n)_\epsilon\Big|^s \\ &\qquad \leq \int_K |\nabla(u_n)_\epsilon|^{s(p(\cdot)-1)} \leq \int_K |\nabla u_n|^{s(p(\cdot)-1)} \leq \int_K |\nabla(u\chi)|^{s(p(\cdot)-1)} < \infty. \end{aligned}$$

Moreover,

$$\begin{aligned} &\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2\big)^{\frac{p(\cdot)-2}{2}} \nabla(u_n)_\epsilon \\ &\qquad\qquad \to \big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u\chi)_\epsilon|^2\big)^{\frac{p(\cdot)-2}{2}} \nabla(u\chi)_\epsilon \end{aligned}$$

pointwise in $\mathbb{R}^d$ as $n \to \infty$. As a consequence,

$$\begin{aligned} &\big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2\big)^{\frac{p(\cdot)-2}{2}} \nabla(u_n)_\epsilon \\ &\qquad\qquad \to \big(\eta^2\,\mathbb{1}_{[p(\cdot)<2]} + |\nabla(u\chi)_\epsilon|^2\big)^{\frac{p(\cdot)-2}{2}} \nabla(u\chi)_\epsilon \end{aligned}$$

weakly in $L^s(\mathbb{R}^d)$ as $n \to \infty$. In particular,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \nabla(u_n)_\epsilon \cdot \nabla\varphi$$
$$= \int_{\mathbb{R}^d} (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u\chi)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \nabla(u\chi)_\epsilon \cdot \nabla\varphi$$
$$= \int_{\mathbb{R}^d} (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \nabla u_\epsilon \cdot \nabla\varphi.$$

Term II: Set

$$A := \{x \in K : \eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2 \le 1\}.$$

We write

$$\begin{aligned} II = &-\frac{1}{2}\int_A (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \\ &\qquad \times \log\big(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2\big) \nabla(u_n)_\epsilon \cdot \nabla p\, \varphi \\ &-\frac{1}{2}\int_{A^C} (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \\ &\qquad \times \log\big(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2\big) \nabla(u_n)_\epsilon \cdot \nabla p\, \varphi \\ =:&\ IIa + IIb. \end{aligned}$$

For IIa, observe that

$$\log(|\nabla(u_n)_\epsilon|^2)\, \mathbb{1}_A \le \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)\, \mathbb{1}_A \le 0,$$

from which we infer that

(3.9)
$$\begin{aligned} &\big|(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2) \nabla(u_n)_\epsilon \cdot (\nabla p)\, \mathbb{1}_A\big| \\ &\quad \le \|\nabla p\|_{L^\infty(K)} |\nabla(u_n)_\epsilon|^{p(\cdot)-1} |\log(|\nabla(u_n)_\epsilon|^2)|\, \mathbb{1}_A \\ &\quad \le C(\alpha)\|\nabla p\|_{L^\infty(K)} |\nabla(u_n)_\epsilon|^{p(\cdot)-1} (|\nabla(u_n)_\epsilon|^{-\alpha} + |\nabla(u_n)_\epsilon|^{\alpha})\, \mathbb{1}_A \\ &\quad = C(\alpha)\|\nabla p\|_{L^\infty(K)} (|\nabla(u_n)_\epsilon|^{p(\cdot)-1-\alpha} + |\nabla(u_n)_\epsilon|^{p(\cdot)-1+\alpha})\, \mathbb{1}_A \\ &\quad \le C(\alpha)\|\nabla p\|_{L^\infty(K)} (|\nabla u_n|^{p(\cdot)-1-\alpha} + |\nabla u_n|^{p(\cdot)-1+\alpha})\, \mathbb{1}_A \\ &\quad \le C(\alpha)\|\nabla p\|_{L^\infty(K)} (|\nabla u_n|^{p(\cdot)-1} + |\nabla u_n|^{p(\cdot)-1+\alpha} + 1)\, \mathbb{1}_A \end{aligned}$$

for all $\alpha \in (0, \gamma_1 - 1)$.

Now fix $\alpha \in (0, \gamma_1 - 1)$. As for s in (3.8), we may choose a constant $r \in (1, \infty)$ such that

$$1 < r(p(\cdot) - 1) < r(p(\cdot) - 1 + \alpha) < p(\cdot)$$

and (3.9) gives

$$\int_K \left|(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \times \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)\nabla(u_n)_\epsilon \cdot (\nabla p)\,\mathbb{1}_A\right|^r$$
$$\leq C(\alpha)4^r\|\nabla p\|^r_{L^\infty(K)} \int_K (|\nabla u_n|^{r(p(\cdot)-1)} + |\nabla u_n|^{r(p(\cdot)-1+\alpha)} + 1)$$
$$\leq C(\alpha)4^r\|\nabla p\|^r_{L^\infty(K)} \int_K (|\nabla (u\chi)|^{r(p(\cdot)-1)} + |\nabla (u\chi)|^{r(p(\cdot)-1+\alpha)} + 1) < \infty.$$

We conclude that the sequence

$$\left\{(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)\nabla(u_n)_\epsilon \cdot (\nabla p)\mathbb{1}_A\right\}_{n\in\mathbb{N}}$$

is uniformly bounded in $L^r(K)$.

For *IIb*, let $\beta \in (0, \frac{\gamma_1-1}{2})$. Then (3.2) implies

$$0 < \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)\,\mathbb{1}_{A^C} < \log(1 + |\nabla(u_n)_\epsilon|^2)\,\mathbb{1}_{A^C}$$
$$\leq 1 + (1 + |\nabla(u_n)_\epsilon|^2)^\beta\,\mathbb{1}_{A^C} \leq 2 + |\nabla u_n|^{2\beta}\,\mathbb{1}_{A^C}.$$

As a consequence,

$$\left|(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \log(\eta^2 + |\nabla(u_n)_\epsilon|^2)\nabla(u_n)_\epsilon \cdot (\nabla p)\,\mathbb{1}_{A^C}\right|$$
$$\leq 2\|\nabla p\|_{L^\infty(K)}|\nabla(u_n)_\epsilon|^{p(\cdot)-1}|\log(\eta^2 + |\nabla(u_n)_\epsilon|^2)|\,\mathbb{1}_{A^C}$$
$$\leq 2C(\beta)\|\nabla p\|_{L^\infty(K)}|\nabla u_n|^{p(\cdot)-1}(1 + |\nabla u_n|^{2\beta})\,\mathbb{1}_{A^C}$$
$$= 2C(\beta)\|\nabla p\|_{L^\infty(K)}(|\nabla u_n|^{p(\cdot)-1} + |\nabla u_n|^{p(\cdot)-1+2\beta})\,\mathbb{1}_{A^C}.$$

Note that this is similar to (3.9). Let $t \in (1, \infty)$ be given by

$$1 < t(p(\cdot) - 1) < t(p(\cdot) - 1 + 2\beta) < p(\cdot).$$

Then

$$\int_K \left|(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \times \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)\nabla(u_n)_\epsilon \cdot (\nabla p)\,\mathbb{1}_{A^C}\right|^t$$
$$\leq C(\beta)4^t\|\nabla p\|^t_{L^\infty(K)} \int_K (|\nabla u_n|^{t(p(\cdot)-1)} + |\nabla u_n|^{t(p(\cdot)-1+2\beta)})$$
$$\leq C(\beta)4^t\|\nabla p\|^t_{L^\infty(K)} \int_K (|\nabla (u\chi)|^{t(p(\cdot)-1)} + |\nabla (u\chi)|^{t(p(\cdot)-1+2\beta)}) < \infty.$$

Hence the sequence

$$\left\{(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla(u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \log(\eta^2 + |\nabla(u_n)_\epsilon|^2)\nabla(u_n)_\epsilon \cdot \nabla p\,\mathbb{1}_{A^C}\right\}_{n\in\mathbb{N}}$$

is uniformly bounded in $L^t(K)$.

With the uniform boundedness results above, we continue as follows. Clearly,

$$(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u_n)_\epsilon|^2) \nabla (u_n)_\epsilon \cdot \nabla p$$
$$\to (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u\chi)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u\chi)_\epsilon|^2) \nabla (u\chi)_\epsilon \cdot \nabla p$$

a.e. in $\mathbb{R}^d$ as $n \to \infty$. Hence the weak convergence in $L^w(\mathbb{R}^d)$, where $w \in \{r, t\}$, applies, yielding

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^d} (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \\
&\qquad\qquad \times \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u_n)_\epsilon|^2) \nabla (u_n)_\epsilon \cdot \nabla p\, \varphi \\
&= \int_{\mathbb{R}^d} (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u\chi)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \\
&\qquad\qquad \times \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u\chi)_\epsilon|^2) \nabla (u\chi)_\epsilon \cdot \nabla p\, \varphi \\
&= \int_{\mathbb{R}^d} (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2) \nabla u_\epsilon \cdot \nabla p\, \varphi.
\end{aligned}$$

In sum we obtain

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}^d} \widetilde{\Delta}_{p(\cdot),\eta} (u_n)_\epsilon \varphi &= - \int_{\mathbb{R}^d} (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \nabla u_\epsilon \cdot \nabla \varphi \\
&\quad - \int_{\mathbb{R}^d} (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \log(\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla u_\epsilon|^2) \nabla u_\epsilon \cdot \nabla p\, \varphi.
\end{aligned}$$

Next we consider the right hand side of (3.6). One has

$$\begin{aligned}
&\Bigg| \frac{u_n}{(u_n)_\epsilon} (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \\
&\times \left(\Delta u_n + \frac{p(\cdot)-2}{\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u_n)_\epsilon|^2} \left(\frac{u_n}{(u_n)_\epsilon} \right)^2 \sum_{i,j=1}^d (\partial_i u_n)(\partial_j u_n) \partial^2_{ij} u_n \right) \Bigg| \\
&\qquad \le [(p^+ - 1) \vee 2] (\eta^2 \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} |\nabla^2 u_n| \\
&\qquad \le [(p^+ - 1) \vee 2] \big(\eta^{p^- -2} \mathbb{1}_{[p(\cdot)<2]} + |\nabla (u_n)_\epsilon|^{p(\cdot)-2} \mathbb{1}_{[p(\cdot)\ge 2]} \big) |\nabla^2 u_n| \\
&\qquad \le [(p^+ - 1) \vee 2] \big(\eta^{p^- -2} \mathbb{1}_{[p(\cdot)<2]} + |\nabla u_n|^{p(\cdot)-2} \mathbb{1}_{[p(\cdot)\ge 2]} \big) |\nabla^2 u_n|.
\end{aligned}$$

From this we deduce that

$$\begin{aligned}
&\int_K \Bigg| \frac{u_n}{(u_n)_\epsilon} (\eta^2 + |\nabla (u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \\
&\quad \times \left(\Delta u_n + \frac{p(\cdot)-2}{\eta^2 + |\nabla (u_n)_\epsilon|^2} \left(\frac{u_n}{(u_n)_\epsilon} \right)^2 \sum_{i,j=1}^d (\partial_i u_n)(\partial_j u_n) \partial^2_{ij} u_n \right) \mathbb{1}_{[p(\cdot)<2]} \Bigg|^{\gamma_1}
\end{aligned}$$

$$\leq [(p^+ - 1) \vee 2]^{\gamma_1} \eta^{\gamma_1(p^- - 2)} \int_K |\nabla^2 u_n|^{\gamma_1} \mathbb{1}_{[p(\cdot)<2]}$$
$$\leq [(p^+ - 1) \vee 2]^{\gamma_1} \eta^{\gamma_1(p^- - 2)} \int_K |\nabla^2 (u\chi)|^{\gamma_1} < \infty,$$

where γ_1 is given by (3.1).

Meanwhile, let s be given by (3.8). Then

$$\int_K \left| \frac{u_n}{(u_n)_\epsilon} (\eta^2 + |\nabla (u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \right.$$
$$\left. \times \left(\Delta u_n + \frac{p(\cdot)-2}{\eta^2 + |\nabla (u_n)_\epsilon|^2} \left(\frac{u_n}{(u_n)_\epsilon} \right)^2 \sum_{i,j=1}^d (\partial_i u_n)(\partial_j u_n) \partial_{ij}^2 u_n \right) \mathbb{1}_{[p(\cdot)\geq 2]} \right|^s$$
$$\leq [(p^+ - 1) \vee 2]^s \int_K |\nabla u_n|^{s(p(\cdot)-2)} |\nabla^2 u_n|^s \mathbb{1}_{[p(\cdot)\geq 2]}$$
$$\leq [(p^+ - 1) \vee 2]^s \big\| |\nabla u_n|^{p(\cdot)-2} \mathbb{1}_{[p(\cdot)\geq 2]} \big\|^s_{L^{\frac{s(p(\cdot)-1)}{p(\cdot)-2}}(K)} \times \|\nabla^2 u_n\|^s_{L^{s(p(\cdot)-1)}(K)}$$
$$\leq [(p^+ - 1) \vee 2]^s \big(\big\| |\nabla u_n| \big\|^{s(p(\cdot)-2)^-}_{L^{s(p(\cdot)-1)}(K)} \vee \big\| |\nabla u_n| \big\|^{s(p(\cdot)-2)^+}_{L^{s(p(\cdot)-1)}(K)} \big)$$
$$\times \mathbb{1}_{[p(\cdot)\geq 2]} \|\nabla^2 u_n\|^s_{L_{s(p(\cdot)-1)}(K)}$$
$$\leq [(p^+ - 1) \vee 2]^s \big(\big\| |\nabla (u\chi)| \big\|^{s(p(\cdot)-2)^-}_{L^{s(p(\cdot)-1)}(K)} \vee \big\| |\nabla (u\chi)| \big\|^{s(p(\cdot)-2)^+}_{L^{s(p(\cdot)-1)}(K)} \big)$$
$$\times \mathbb{1}_{[p(\cdot)\geq 2]} \|\nabla^2 u_n\|^s_{L_{s(p(\cdot)-1)}(K)}$$
$$\leq [(p^+ - 1) \vee 2]^s \big(1 + \big\| |\nabla (u\chi)| \big\|^{sp^+}_{L^{s(p(\cdot)-1)}(K)} \big) \|\nabla^2 u\|^s_{L_{s(p(\cdot)-1)}(K)} < \infty,$$

where we have used Lemma 2.2 in the third step, (3.5) in the fourth step and the fact that $u \in W^{2,p(\cdot)}_{\text{loc}}(\mathbb{R}^d)$ in the last step.

Hence as before, using the obvious pointwise convergence and then passing to a weak limit in $L^w(K)$, where $w \in \{\gamma_1, s\}$, we arrive at

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \frac{u_n}{(u_n)_\epsilon} (\eta^2 + |\nabla (u_n)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}}$$
$$\times \left(\Delta u_n + \frac{p(\cdot)-2}{\eta^2 + |\nabla (u_n)_\epsilon|^2} \left(\frac{u_n}{(u_n)_\epsilon} \right)^2 \sum_{i,j=1}^d (\partial_i u_n)(\partial_j u_n) \partial_{ij}^2 u_n \right) \varphi$$
$$= \int_{\mathbb{R}^d} \frac{u\chi}{(u\chi)_\epsilon} (\eta^2 + |\nabla (u\chi)_\epsilon|^2)^{\frac{p(\cdot)-2}{2}}$$
$$\times \left(\Delta (u\chi) + \frac{p(\cdot)-2}{\eta^2 + |\nabla (u\chi)_\epsilon|^2} \left(\frac{u\chi}{(u\chi)_\epsilon} \right)^2 \sum_{i,j=1}^d \partial_i (u\chi) \partial_j (u\chi) \partial_{ij}^2 (u\chi) \right) \varphi$$

$$= \int_{\mathbb{R}^d} \frac{u}{u_\epsilon}(\eta^2 + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \times \left(\Delta u + \frac{p(\cdot)-2}{\eta^2 + |\nabla u_\epsilon|^2} \left(\frac{u}{u_\epsilon}\right)^2 \sum_{i,j=1}^{d} (\partial_i u)(\partial_j u)\partial_{ij}^2 u \right) \varphi.$$

Therefore by multiplying both sides of (3.6) with φ, integrating over $\mathbb{R}^d$ and then letting $n \to \infty$ we obtain

$$\begin{aligned} - \int_{\mathbb{R}^d} (\eta^2 + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \nabla u_\epsilon \cdot \nabla \varphi &- \int_{\mathbb{R}^d} (\eta^2 + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \log(\eta^2 + |\nabla u_\epsilon|^2) \nabla u_\epsilon \cdot \nabla p \, \varphi \\ &\geq \int_{\mathbb{R}^d} \frac{u}{u_\epsilon}(\eta^2 + |\nabla u_\epsilon|^2)^{\frac{p(\cdot)-2}{2}} \\ &\quad \times \left(\Delta u + \frac{p(\cdot)-2}{\eta^2 + |\nabla u_\epsilon|^2} \left(\frac{u}{u_\epsilon}\right)^2 \sum_{i,j=1}^{d} (\partial_i u)(\partial_j u)\partial_{ij}^2 u \right) \varphi. \end{aligned}$$

With this last display in mind we take the limits on both sides of the above inequality when $\epsilon \to 0^+$ and refer to Lemma 3.3 to deduce that

$$\begin{aligned} - \int_{\mathbb{R}^d} \left(\eta^2 + \big|\nabla |u|\big|^2\right)^{\frac{p(\cdot)-2}{2}} \nabla |u| \cdot \nabla \varphi &- \int_{\mathbb{R}^d} \left(\eta^2 + \big|\nabla |u|\big|^2\right)^{\frac{p(\cdot)-2}{2}} \log\left(\eta^2 + \big|\nabla |u|\big|^2\right) \nabla |u| \cdot \nabla p \, \varphi \\ &\geq \int_{\mathbb{R}^d} (\operatorname{sgn} u)(\eta^2 + |\nabla u|^2)^{\frac{p(\cdot)-2}{2}} \\ &\quad \times \left(\Delta u + \frac{p(\cdot)-2}{\eta^2 + |\nabla u|^2} \sum_{i,j=1}^{d} (\partial_i u)(\partial_j u)\partial_{ij}^2 u \right) \varphi \\ &= \int_{\mathbb{R}^d} (\operatorname{sgn} u)(\widetilde{\Delta}_{p(\cdot),\eta} u)\varphi. \end{aligned}$$

Lastly, we let $\eta \to 0^+$ and use Lemma 3.4 to derive

$$- \int_{[\nabla u \neq 0]} \big|\nabla |u|\big|^{p(\cdot)-2} \nabla |u| \cdot \nabla \varphi - \int_{[\nabla u \neq 0]} \big|\nabla |u|\big|^{p(\cdot)-2} \log\left(\big|\nabla |u|\big|\right) \nabla |u| \cdot \nabla p \, \varphi \geq \int_{[\nabla u \neq 0]} (\operatorname{sgn} u)(\widetilde{\Delta}_{p(\cdot)} u)\varphi.$$

That is,

$$\widetilde{\Delta}_{p(\cdot)} |u| \geq (\operatorname{sgn} u)(\widetilde{\Delta}_{p(\cdot)} u) \quad \text{in } \mathcal{D}'(\mathbb{R}^d)$$

as claimed. This completes the proof. ∎

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