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Solutions of some quasianalytic equations

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Abstract. In this article we are interested in solving systems of equations $F_j(t,x) = 0$, $1 \le j \le p$, for C^∞ functions F_j in a neighborhood of $(0,0) \in \mathbb{R} \times \mathbb{R}^p$, which are in a given differentiable quasianalytic system $\mathcal{C}$ (for example, a quasianalytic Denjoy–Carleman class or the class of infinitely differentiable functions definable in a polynomially bounded o-minimal structure). We suppose that the implicit function theorem is true in this system, that is, the equations $F_i(t,x) = 0$ have a unique solution $x_j(t)$, $j = 1, \dots, p$, in a neighborhood at $t = 0$ if the Jacobian of the functions F_i with respect to the variables x_j is nonzero at $x_j = t = 0$, $j = 1, \dots, p$. This condition is not necessary for the equations $F_i = 0$ to have a solution in the system. An example is the analytic system. A theorem of Artin can be used to give sufficient conditions under which a system of analytic equations has analytic solutions. In the case of a general quasianalytic system the theorem of Artin is not available. In this article weaker conditions will be given. We show that if $F(t,x) = (F_1(t,x), \dots, F_p(t,x)) = 0$ has a formal power series solution $u(t) = (u_1(t), \dots, u_p(t)) \in (\mathbb{R}[[t]])^p$, and $\det D_x F(t, u(t)) \neq 0$, then for each $j = 1, \dots, p$, $u_j(t)$ is the Taylor expansion at $0 \in \mathbb{R}$ of a function in the system $\mathcal{C}$. We also treat the same problem in the case of more independent variables.

1. Introduction. It is well known that a C^∞ function or a formal power series that satisfies a nonzero real analytic equation is necessarily real analytic. There are several different proofs of this fact in the literature using different methods. For example, a proof is given in [11, Theorem 2]. For another proof using functional analysis methods, see [3, Theorem 6]. Finally, for a proof using a theorem of Malgrange, see [7, Chapter VI, Example 3.11.2].

It is natural to ask whether a similar result holds for a finite number of equations in a quasianalytic local ring (see below). The problem seems difficult to address in full generality. In this paper, we consider the particular case of one independent variable and a finite number of equations.

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Consider an arbitrary C^∞ quasianalytic system $\mathcal{C} = \{\mathcal{C}_n \mid n \in \mathbb{N}\}$ (see Section 2), for example, a quasianalytic Denjoy–Carleman class or the class of infinitely differentiable functions definable in a polynomially-bounded o-minimal structure. The aim of this paper is to solve equations $F(t,x) = 0$, where

$$F(t,x) = F(t,x_1,\dots,x_p) = (F_1(t,x),\dots,F_p(t,x))$$

and $F_i \in \mathcal{C}_{p+1}$ for each $i = 1,\dots,p$.

If the implicit function theorem is true for the quasianalytic system $\mathcal{C}$, we know that if $F(0,0) = 0$, then the equation $F(t,x) = 0$ has a unique solution $x(t) = (x_1(t),\dots,x_p(t)) \in \{\mathcal{C}_1\}^p$ if the Jacobian of the functions F_j, $j = 1,\dots,p$, with respect to the variables x_j, $j = 1,\dots,p$, is nonzero at the point $(0,0) \in \mathbb{R}^{p+1}$. This condition is sufficient, but not necessary for the equations F_j, $j = 1,\dots,p$, to have a solution in the system. For example, if $\mathcal{C}$ is the system of analytic equations, i.e. each F_i is an analytic function in a neighborhood of the origin in $\mathbb{R}^{p+1}$, then a theorem of M. Artin [1] can be used to give a sufficient condition under which a system of analytic equations has analytic solutions.

Recall that M. Artin's approximation theorem ensures that if a finite system of analytic equations has a formal solution, then we can also find a convergent one, even more: we can approximate the formal solution by convergent ones up to a given order. It is proved by induction on the dimension and the inductive step is an application of the Weierstrass division theorem.

Since Weierstrass's division theorem is not true in the setting of a quasianalytic system strictly containing the analytic system [6], we cannot use Artin's theorem. In this article a weaker condition will be given for a general quasianalytic system to have a solution in the system. The proof presented here only involves function-theoretic aspects.

We denote by $\det D_xF(t,u(t))$ the determinant of the matrix $D_xF(t,u(t))$ of partial derivatives of the mapping $(t,x) \mapsto F(t,x)$ with respect to x at the point $(t,u(t))$. The main result of this article is the following theorem:

THEOREM 1.1. *Assume that* $F(t,x) = 0$ *has a formal power series solution* $u(t) = (u_1(t),\dots,u_p(t))$, *i.e.* $F(t,u(t)) = F(t,u_1(t),\dots,u_p(t)) = 0$, *and suppose that* $\det D_xF(t,u(t)) \neq 0$. *Then for all* $j = 1,\dots,p$, $u_j(t)$ *is the Taylor expansion at* $0 \in \mathbb{R}$ *of a quasianalytic function in the same system containing* F.

If $p = 1$, we will see ultimately that the condition $\det D_xF(t,u(t)) \neq 0$ can be relaxed considerably.

At the end, we prove that if a C^∞ real valued function $y = f(x)$ on an open subset U of $\mathbb{R}^n$ satisfies a quasianalytic equation $G(x,y) = 0$, then f is *arc-quasianalytic*, i.e., its restriction to every quasianalytic arc, $\gamma : \mathbb{R} \supset I \to U$, $f \circ \gamma$ is quasianalytic.

2. Differentiable quasianalytic system. For $n \in \mathbb{N}$, we denote by $\mathcal{E}_n$ the ring of smooth germs at the origin of $\mathbb{R}^n$ and by $\mathcal{A}_n \subset \mathcal{E}_n$ the subring of analytic germs.

A *differentiable system* is a collection of rings of smooth germs at the origin of $\mathbb{R}^n$, $\mathcal{C} = \{\mathcal{C}_n \mid n \in \mathbb{N}\}$, such that, for each $n \in \mathbb{N}$, $\mathcal{C}_n \subset \mathcal{E}_n$ is a subalgebra and the following properties hold for all $n \in \mathbb{N}$:

(1) The ring $\mathcal{A}_n$ is contained in $\mathcal{C}_n$.
(2) (Stability by composition) If $f \in \mathcal{C}_n$ and $g_1, \ldots, g_n \in \mathcal{C}_m$ with $g_1(0) = \cdots = g_n(0) = 0$, then $f(g_1, \ldots, g_n) \in \mathcal{C}_m$.
(3) (Stability under monomial division) If $f \in \mathcal{C}_n$ satisfies

$$f(x_1, \ldots, x_{i-1}, 0, x_i, \ldots x_n) = 0,$$

then there exists $g \in \mathcal{C}_n$ such that $f(x) = x_i g(x)$.
(4) (Stability by implicit equation) Suppose that $f = (f_1, \ldots, f_m) \in (\mathcal{C}_{n+m})^m$ with $f(0,0) = 0$. Put $y = (y_1, \ldots, y_m)$ and suppose that

$$\det\left(\frac{\partial f_i}{\partial y_j}(0,0)\right)_{1 \le i,j \le m} \neq 0.$$

Then there is a (unique) $g = (g_1, \ldots, g_m) \in (\mathcal{C}_n)^m$ with $g(0) = 0$ such that $f(x, g(x)) \equiv 0$.

Remark 2.1. It can be easily seen that the above properties imply that the algebras $\mathcal{C}_n$ are closed under partial differentiation [10, p. 423].

The mapping

$$\hat{} : \mathcal{C}_n \to R[[x_1, \ldots, x_n]]$$

which associates to each $f \in \mathcal{C}_n$ its Taylor expansion at the origin is called the *Borel mapping.* We consider the following condition:

(Q) the Borel mapping is an injective homomorphism.

Definition 2.2. A differentiable system is called *quasianalytic* if condition (Q) holds.

Examples 2.3. (1) For each $n \in \mathbb{N}$, we denote by $\mathcal{N}_n$ the ring of germs, at the origin in $\mathbb{R}^n$, of Nash functions, i.e. algebraic on the ring of polynomials $\mathbb{R}[x_1, \ldots, x_n]$. The system $\mathcal{N} = \{\mathcal{N}_n \mid n \in \mathbb{N}\}$ is a quasianalytic system.

(2) The system $\mathcal{A} = \{\mathcal{A}_n \mid n \in \mathbb{N}\}$ is a quasianalytic system.

(3) Let $\mathcal{R}$ be a polynomially bounded o-minimal structure which is an expansion of the ordered field of reals. For more details about an o-minimal structure over the field of reals, we refer the reader to [14]. We denote by $\mathcal{D}_n$ the ring of germs, at the origin in $\mathbb{R}^n$, of C^∞ definable functions in a neighborhood of the origin in $\mathbb{R}^n$. By [8], the system $\mathcal{D} = \{\mathcal{D}_n \mid n \in \mathbb{N}\}$ is a quasianalytic system.

(4) Let $\mathcal{E}(U)$ denote the ring of C^∞ functions on an open set $U \subset \mathbb{R}^n$. As usual, for each n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we set

$$|\alpha| = \alpha_1 + \dots + \alpha_n \quad \text{and} \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} \dots \partial^{\alpha_n}}.$$

Let $M = (M_p)_{p=0}^\infty$ be an increasing sequence of positive real numbers and $\rho > 0$. For each $f \in \mathcal{E}(U)$,

$$\|f\|_{U,\rho,M} = \sup_{x\in U,\, p\in\mathbb{N}} \sup_{|\alpha|=p} \frac{|D^\alpha f(x)|}{\rho^p M_p p!}$$

defines a norm on

$$C_{M,\rho}(U) = \{f \in \mathcal{E}(U) \mid \|f\|_{U,\rho,M} < \infty\},$$

which turns out to be a Banach space.

Definition 2.4. We say that $f \in \mathcal{E}(U)$ is in the *Denjoy–Carleman class* defined by the sequence M ($f \in C_M(U)$ for short) if any point $x \in U$ admits a neighborhood $U_x \subset U$ such that $f \in C_{M,\rho}(U_x)$, for some $\rho > 0$.

Let

$$\Delta_n(r) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid |x_i| < r,\, i = 1, \dots, n\}$$

be the cube in $\mathbb{R}^n$. We denote by $C_M(\mathbb{R}^n, 0)$ the inductive limit of $C_M(\Delta_n(r))$ when $r \to 0$.

In order to get classes of functions with some good structural properties, we require two conditions on the sequence $M = (M_p)_{p=0}^\infty$ (see [4]):

$$M = (M_p)_{p=0}^\infty \text{ is logarithmically convex and } M_0 = 1, \tag{2.1}$$

$$\sup_p \left(\frac{M_{p+1}}{M_p}\right)^{1/p} < \infty \tag{2.2}$$

The assumptions (2.1) and (2.2) imply that the system $C_M = \{C_M(\mathbb{R}^n, 0) \mid n \in \mathbb{N}\}$ is a differentiable system [4].

In order to deal with quasianalytic classes, we require the following condition:

$$\sum_{p\in\mathbb{N}} \frac{M_p}{(p+1)M_{p+1}} = \infty. \tag{2.3}$$

In summary, if the sequence $M = (M_p)_{p=0}^\infty$ satisfies conditions (2.1)–(2.3), then $C_M = \{C_M(\mathbb{R}^n, 0) \mid n \in \mathbb{N}\}$ is a quasianalytic system.

The injectivity of the Borel mapping for a differentiable quasianalytic system $\mathcal{C} = \{\mathcal{C}_n \mid n \in \mathbb{N}\}$ ensures that $\mathcal{C}_n$ can be identified with a subring of the ring of formal power series; this identification is used throughout this paper.

We remark that the conditions on a quasianalytic system imply that, for each $n \in \mathbb{N}$, the ring $\mathcal{C}_n$ is local and the maximal ideal of $\mathcal{C}_n$ is

$$\underline{m_n} := \{f \in \mathcal{C}_n \mid f(0) = 0\} = (x_1, \dots, x_n)\mathcal{C}_n$$

and its completion in the m_n-adic topology is the ring $\mathbb{R}[[x_1, \dots, x_n]]$ of formal power series.

Proposition 2.5. *If* $\mathcal{C} = \{\mathcal{C}_n \mid n \in \mathbb{N}\}$ *is a differentiable quasianalytic system, then* $\mathcal{C}_1$ *is noetherian.*

Proof. The ring $\mathcal{C}_1$ is noetherian since any local ring whose maximal ideal $\underline{m_1}$ is principal and such that $\bigcap_n \underline{m_1^n} = \{0\}$ turns out to be a principal ideal domain [9, Lemma 31.5]. ∎

Remark 2.6. We still do not know if any ring of a differentiable quasianalytic system strictly bigger than the analytic system, apart from the ring of one variable germs, is noetherian or satisfies Artin's approximation theorem [5].

3. Notations and lemma. If $\varphi = \sum_{n=1}^{\infty} \varphi_n t^n \in \mathbb{R}[[t]]$ is a formal power series in one variable t and $m \in \mathbb{N}$, we put

$$T_m\varphi = \sum_{n=1}^{m} \varphi_n t^n \in \mathbb{R}[t] \quad \text{and} \quad T^{[m]}\varphi = \sum_{n=1}^{\infty} \varphi_{m+n} t^n \in \mathbb{R}[[t]].$$

It is clear that

$$\varphi = T_m\varphi + t^m T^{[m]}\varphi.$$

We extend this notation to matrices.

If $M(t) = (u_{i,j}(t))_{1\le i\le s,\,1\le j\le q}$ is an $s\times q$ matrix with coefficients in $\mathbb{R}[[t]]$, we also write

$$T_m M(t) = (T_m u_{i,j}(t))_{1\le i\le s,\,1\le j\le q}, \qquad T^{[m]}M(t) = (T^{[m]}u_{i,j}(t))_{1\le i\le s,\,1\le j\le q}.$$

In the following, we fix a quasianalytic system, $\mathcal{C} = \{\mathcal{C}_n \mid n \in \mathbb{N}\}$, strictly containing the analytic system.

Theorem 1.1 is a direct consequence of an auxiliary result (Lemma 3.1) which will be given in a more general form than necessary for our present purposes. The proof follows the idea of Tougeron's generalization of the implicit function theorem [13, pp. 56–57].

Lemma 3.1. *Let* $F \in (\mathcal{C}_{p+1})^p$,

$$F(t,x) = (F_1(t,x), \dots, F_p(t,x)), \qquad x = (x_1, \dots, x_p).$$

Let

$$u(t) = (u_1(t), \dots, u_p(t)) \in (\mathbb{R}[[t]])^p, \qquad u(0) = (0, \dots, 0),$$

be a formal solution of the equation

$$F(t,x) = 0. \tag{3.1}$$

Assume there exists a $p\times p$ *matrix* $M(t)$ *with coefficients in* $\mathbb{R}[[t]]$ *such that*

$$M(t)D_xF(t,u(t)) = t^\nu I_{(p,p)}, \qquad \nu\in\mathbb{N},$$

where $I_{(p,p)}$ *is the unit* $p\times p$ *matrix and* $D_xF(t,u(t))$ *is the matrix of partial derivatives of the mapping* $(t,x)\mapsto F(t,x)$ *with respect to* x *at* $(t,u(t))$*. Then there exists* $g=(g_1,\dots,g_p)\in(\mathcal{C}_1)^p$ *with the following properties:*

(1) *If* $\xi(t)$ *is a formal solution of the equation*

$$t^\nu\xi(t) = g(t), \tag{3.2}$$

then $x(t)=T_{2\nu}u(t)+t^{2\nu}\xi(t)$ *is a solution of* (3.1).

(2) $T^{[2\nu]}u(t)$ *is a formal solution of* (3.2).

Proof. If $\nu=0$, then the matrix $D_xF(t,u(t))$ is nonsingular. According to the implicit function theorem, there exists a function $g\in(\mathcal{C}_1)^p$ such that (3.1) is equivalent to the equation

$$x = g(t).$$

Now suppose that $\nu\ge 1$. We consider the Taylor expansion of F:

$$F(t,T_{2\nu}u(t)+x) = F(t,T_{2\nu}u(t)) + D_xF(t,T_{2\nu}u(t))x + G(t,x), \tag{3.3}$$

where $G\in(\mathcal{C}_{p+1})^p$ contains only second and higher order terms in x. Putting $N(t)=T_{2\nu}M(t)$, we have

$$N(t)D_xF(t,T_{2\nu}u(t)) = t^\nu(I_{(p,p)}+t^\nu A(t)),$$

where $A(t)$ is a matrix with coefficients in $\mathcal{C}_1$.

Since $\nu>0$, the matrix $I_{(p,p)}+t^\nu A(t)$ is invertible. Therefore, there exists a nonsingular matrix $P(t)$ with coefficients in $\mathcal{C}_1$ such that

$$P(t)D_xF(t,T_{2\nu}u(t)) = t^\nu I_{(p,p)}.$$

By (3.3), we have

$$P(t)F(t,T_{2\nu}u(t)+x) = P(t)F(t,T_{2\nu}u(t)) + t^\nu x + P(t)G(t,x).$$

We substitute $x=t^{2\nu}\overline{x}$ and remark that

$$P(t)G(t,t^{2\nu}\overline{x}) = t^{2\nu}G_1(t,t^\nu\overline{x})$$

for some $G_1\in(\mathcal{C}_{p+1})^p$, since G does not contain linear or constant terms in x. We obtain

$$P(t)F(t,T_{2\nu}u(t)+t^{2\nu}\overline{x}) = P(t)F(t,T_{2\nu}u(t)) + t^{2\nu}(t^\nu\overline{x}+G_1(t,t^\nu\overline{x})).$$

Since

$$F(t,T_{2\nu}u(t)+t^{2\nu}T^{[2\nu]}u(t)) = F(t,u(t)) = 0,$$

we must have

$$P(t)F(t,T_{2\nu}u(t)) = t^{2\nu}h(t)$$

for some $h\in(\mathcal{C}_1)^p$.

We put

$$H(t, t^\nu \overline{x}) = h(t) + G_1(t, t^\nu \overline{x}).$$

Then we have

$$P(t)F(t, T_{2\nu}u(t) + t^{2\nu}\overline{x}) = t^{2\nu}(t^\nu \overline{x} + H(t, t^\nu \overline{x})).$$

We see that the equation

$$F(t, T_{2\nu}u(t) + t^{2\nu}\overline{x}) = 0$$

is equivalent to the equation

$$\Gamma(t, t^\nu \overline{x}) := t^\nu \overline{x} + H(t, t^\nu \overline{x}) = 0.$$

We recall that $H \in (\mathcal{C}_{p+1})^p$ contains only constant second and higher order terms in $t^\nu \overline{x}$. We put $Y = t^\nu \overline{x}$.

The equation

$$\Gamma(t, Y) = Y + H(t, Y) = 0$$

can be solved in a unique way as $Y = g(t)$, with $g \in (\mathcal{C}_1)^p$, thanks to the implicit function theorem. It is easily seen that g has the properties (1) and (2). ■

Proof of Theorem 1.1. Under the assumptions of the theorem, there exists a $p \times p$ matrix $M(t)$ with coefficients in $\mathbb{R}[[t]]$ and $\nu \in \mathbb{N}$ such that

$$M(t)D_x F(t, u(t)) = t^\nu I_{(p,p)},$$

where $I_{(p,p)}$ is the unit $p \times p$ matrix.

By Lemma 3.1(2), $T^{[2\nu]}u(t)$ is a formal solution of an equation of the form

$$t^\nu Y = g(t),$$

where $g \in (\mathcal{C}_1)^p$. Hence

$$u(t) = T_{[2\nu]}u(t) + t^{2\nu}T^{[2\nu]}u(t) = T_{[2\nu]}u(t) + t^\nu g(t) \in (\mathcal{C}_1)^p.$$

This proves the theorem. ■

We now give a corollary of Lemma 3.1 when the solution of equation (3.1) is a C^∞ function.

COROLLARY 3.2. *Let $F \in (\mathcal{C}_{p+1})^p$, $F(t,x) = (F_1(t,x), \ldots, F_p(t,x))$, $x = (x_1, \ldots, x_p)$. Let $v(t) = (v_1(t), \ldots, v_p(t)) \in (\mathcal{E}_1)^p$, $v(0) = (0, \ldots, 0)$, be a C^∞ solution of the equation*

$$F(t, x) = 0. \tag{3.4}$$

If there exists a $p \times p$ matrix $M(t)$ with coefficients in $\mathbb{R}[[t]]$ such that

$$M(t)D_x F(t, \hat{v}(t)) = t^\nu I_{(p,p)}, \qquad \nu \in \mathbb{N},$$

where $\hat{v}(t) = (\hat{v}_1(t), \ldots, \hat{v}_p(t))$ and $\hat{v}_j(t)$ is the image of v_j by the Borel mapping, $j = 1, \ldots, p$, then $v \in (\mathcal{C}_1)^p$.

Proof. Denoting by $T_{2\nu}v(t)$ the 2νth Taylor polynomial of v at the origin, the same computation as in Lemma 3.1 yields

$$P(t)F(t,T_{2\nu}v(t)+t^{2\nu}\overline{x})=t^{2\nu}\Gamma(t,t^{\nu}\overline{x}), \tag{3.5}$$

where $P(t)$ is a square $p\times p$ matrix with coefficients in $\mathcal{C}_1$ such that $\det P(0)\neq 0$, and Γ is an element of $(\mathcal{C}_{p+1})^p$ such that $\Gamma(0,0)=0$ and $D_Y\Gamma(0,0)=I_{(p,p)}$. By the implicit function theorem, the equation $\Gamma(t,Y)=0$ has a unique solution $Y=g(t)$ in $(\mathcal{C}_1)^p$.

By Taylor's formula with integral remainder, the function

$$t\mapsto t^{-2\nu}(v(t)-T_{2\nu}v(t))$$

is an element of $(\mathcal{E}_1)^p$. We put

$$T^{[2\nu]}v(t):=t^{-2\nu}(v(t)-T_{2\nu}v(t)).$$

We have $F(t,T_{2\nu}v(t)+t^{2\nu}T^{[2\nu]}v(t))=0$, hence by (3.5) we obtain

$$\Gamma(t,t^{\nu}T^{[2\nu]}v(t))=0.$$

By considering implicit functions, we can now infer that $t^{\nu}T^{[2\nu]}v(t)=g(t)$ as function germs. Thus, by monomial division, $T^{[2\nu]}v(t)$ actually belongs to $(\mathcal{C}_1)^p$ and so does v. ■

4. Case of one equation. If $p=1$, the condition $\det D_xF(t,u(t))\neq 0$ in Theorem 1.1 can be relaxed.

PROPOSITION 4.1. *Let $F\in\mathcal{C}_2$, $F(0,0)=0$ and let $u(t)\in\mathbb{R}[[t]]$ be such that $F(t,u(t))=0$. If F is not identically zero, then $u\in\mathcal{C}_1$.*

Proof. First, we can suppose that the function $x\mapsto F(0,x)$ is not identically zero. In fact, if $F(0,x)\equiv 0$, then F is divisible by t and $u(t)$ is also a formal solution of the equation

$$F_1(t,x)=0,$$

where $tF_1(t,x)=F(t,x)$ (stability under monomial division).

We may repeat this procedure until we obtain a function $F_s(t,x)$ for which $u(t)$ is a formal solution of $F_s(t,x)=0$ and $F_s(0,x)\neq 0$.

Suppose now that the function $x\mapsto F(0,x)$ is not identically zero. If $\frac{\partial F}{\partial x}(t,u(t))=0$, then $u(t)$ is a formal solution of the equation $\frac{\partial F}{\partial x}(t,x)=0$. If $\frac{\partial^2 F}{\partial x^2}(t,u(t))=0$ then we repeat this procedure. It cannot happen that $\frac{\partial^k F}{\partial x^k}(t,u(t))=0$ for all $k\in\mathbb{N}$, because this will imply that $\frac{\partial^k F}{\partial x^k}(0,0)=0$ for all $k\in\mathbb{N}$, and hence the function $x\mapsto F(0,x)$ is identically zero.

It follows that for some $s\in\mathbb{N}$ we have

$$\frac{\partial^s F}{\partial x^s}(t,u(t))=0 \quad\text{and}\quad \frac{\partial^{s+1} F}{\partial^{s+1} x}(t,u(t))\neq 0.$$

We apply Theorem 1.1 to the equation $G(t,x)=\frac{\partial^s F}{\partial x^s}(t,x)=0$. ■

REMARK 4.2. Proposition 4.1 follows immediately from [2, proof of Theorem 1.1], because there is no need to blow up the domain of a quasianalytic function of one variable in order to get a normal crossing.

Using the same proof as for Proposition 4.1, but applying Corollary 3.3 instead of Theorem 1.1, we get

PROPOSITION 4.3. *Let* $F \in \mathcal{C}_2$ *with* $F(0,0) = 0$ *and let* $v(t) \in \mathcal{E}_1$ *be such that* $F(t, v(t)) \equiv 0$. *If* F *is not identically zero, then* $v \in \mathcal{C}_1$.

REMARK 4.4. The result of Proposition 4.3 is a generalization of a result proved in [12] when the function $F(t,x)$ is a distinguished polynomial in x with coefficients in $\mathcal{C}_1$.

5. Several independent variables. We fix a differentiable quasianalytic system $\mathcal{C} = \{\mathcal{C}_n \mid n \in \mathbb{N}\}$. Solutions of one quasianalytic equation, with one unknown function of several variables, have been studied in [2].

The focus of this section is to find generalizations of that result to the case of systems of quasianalytic equations. More precisely, given a system of real quasianalytic equations

$$G_j(x,y) = 0, \quad 1 \le j \le p,\ x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_p),$$

under what condition can one conclude that a C^∞ or a formal power series solution is necessarily in some quasianalytic system?

DEFINITION 5.1. A $\mathcal{C}$-*quasianalytic arc*

$$\gamma = (\gamma_1, \ldots, \gamma_n) : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$$

is a map such that each γ_j, $1 \le j \le n$, is in $\mathcal{C}_1$.

DEFINITION 5.2. A germ $g \in \mathcal{E}_n$ is called *arc* $\mathcal{C}$-*quasianalytic* if $g \circ \gamma \in \mathcal{C}_1$ for every $\mathcal{C}$-quasianalytic arc $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$.

We put

$$G(x,y) = (G_1(x,y), \ldots, G_p(x,y)), \quad G_j \in \mathcal{C}_{n+p},\ 1 \le j \le p.$$

Let $g(x) = (g_1(x), \ldots, g_p(x))$ with $g(0) = 0$ be a C^∞ solution of the equation

$$G(x,y) = 0.$$

PROPOSITION 5.3. *Let* $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ *be a* $\mathcal{C}$-*quasianalytic arc such that* $D_y G(\gamma(t), \widehat{g_\gamma(t)}) \ne 0$, *where* $g_\gamma(t) = (g_1 \circ \gamma(t), \ldots, g_p \circ \gamma(t))$. *Then* $g_j \circ \gamma \in \mathcal{C}_1$ *for each* $1 \le j \le p$.

Proof. We apply Corollary 3.3 to the function $(t,y) \mapsto F(t,y) = G(\gamma(t), y)$. ∎

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