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APPLICATION OF PERRON TREES TO GEOMETRIC MAXIMAL OPERATORS

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Abstract. We characterize the $L^p(\mathbb{R}^2)$ boundedness of the geometric maximal operator $M_{a,b}$ associated to the basis $\mathcal{B}_{a,b}$ $(a, b > 0)$ which is composed of rectangles R whose eccentricity and orientation are of the form

$$(e_R, \omega_R) = \left(\frac{1}{n^a}, \frac{\pi}{4n^b}\right)$$

for some $n \in \mathbb{N}^*$. The proof involves *generalized Perron trees*, as constructed by Hare and Röning [J. Fourier Anal. Appl. 4 (1998)].

1. Introduction. In [5], Bateman has conducted the study of *directional maximal operators in the plane*. In this text, we study *geometric maximal operators*, which are a natural generalization of the directional operators. However, their study requires a precise understanding of the correlation between the *eccentricity* and the *orientation* of families of rectangles. In this text, we prove sharp results concerning the $L^p(\mathbb{R}^2)$ boundedness of geometric maximal operators whose parameters vary in a polynomial way.

1.1. Definitions. We work in the euclidean plane $\mathbb{R}^2$; if A is a measurable subset of $\mathbb{R}^2$ we denote by $|A|$ its two-dimensional Lebesgue measure. We denote by $A \sqcup B$ the union of A and B when $|A \cap B| = 0$.

Denote by $\mathcal{R}$ the collection of all rectangles of $\mathbb{R}^2$; for $R \in \mathcal{R}$ we define its *orientation* as the angle $\omega_R \in [0, \pi)$ that its longest side makes with the Ox-axis, and its *eccentricity* as the ratio $e_R \in (0, 1]$ of its shortest side to its longest side.

For an arbitrary non-empty family $\mathcal{B}$ contained in $\mathcal{R}$, we define the associated *differentiation basis* $\mathcal{B}^*$ by

$$\mathcal{B}^* = \{\vec{t} + hR : \vec{t} \in \mathbb{R}^2,\ h > 0,\ R \in \mathcal{B}\}.$$

The differentiation basis $\mathcal{B}^*$ is simply the smallest collection which is invari-

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ant by dilation and translation and contains $\mathcal{B}$. Without loss of generality, we identify the differentiation basis $\mathcal{B}^*$ and any of its generators $\mathcal{B}$.

Our object of interest will be the *geometric maximal operator* $M_{\mathcal{B}}$ *generated by* $\mathcal{B}$, defined as

$$M_{\mathcal{B}} f(x) := \sup_{x \in R \in \mathcal{B}^*} \frac{1}{|R|} \int_R |f|$$

for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$. Observe that the upper bound is taken over elements of $\mathcal{B}^*$ that contain the point x. The definitions of $\mathcal{B}^*$ and $M_{\mathcal{B}}$ remain valid when $\mathcal{B}$ is an arbitrary family of open bounded convex sets. For example in this note, for technical reasons, we will sometimes work with triangles instead of rectangles.

For $p \in (1, \infty]$ we define as usual the operator norm $\|M_{\mathcal{B}}\|_p$ of $M_{\mathcal{B}}$ by

$$\|M_{\mathcal{B}}\|_p = \sup_{\|f\|_p = 1} \|M_{\mathcal{B}} f\|_p.$$

If $\|M_{\mathcal{B}}\|_p < \infty$, we say that $M_{\mathcal{B}}$ is *bounded* on $L^p(\mathbb{R}^2)$. The boundedness of $M_{\mathcal{B}}$ is related to the geometry of the family $\mathcal{B}$.

Definition 1.1. We will say that $M_{\mathcal{B}}$ is *good* when it is bounded on $L^p(\mathbb{R}^2)$ for any $p > 1$. On the other hand, it is *bad* when it is unbounded on $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$.

On the $L^p(\mathbb{R}^2)$ scale, to be able to say that an operator $M_{\mathcal{B}}$ is good or bad is an optimal result. One can also be interested in the behavior near the endpoints ($p = 1$ and $p = \infty$) but we will not consider this question here; the reader might consult D'Aniello, Moonens and Rosenblatt [3], D'Aniello and Moonens [2]–[4] or Stokolos [15].

1.2. Directional maximal operators. Research has been done in the case where $\mathcal{B}$ is equal to

$$\mathcal{R}_{\Omega} := \{R \in \mathcal{R} : \omega_R \in \Omega\}$$

with Ω an arbitrary set of directions in $[0, \pi)$. In other words, $\mathcal{R}_{\Omega}$ is the set of *all* rectangles whose orientation belongs to Ω. We say that $\mathcal{R}_{\Omega}$ is a *directional basis* and to simplify notation we write

$$M_{\mathcal{R}_{\Omega}} := M_{\Omega}.$$

In the literature, the operator M_{Ω} is said to be a *directional maximal operator*. The study of those operators goes back at least to Strömberg [16] or Córdoba and Fefferman [8] who use geometric techniques to show that if $\Omega = \{\pi/2^k\}_{k \geq 1}$ then M_{Ω} has weak type $(2, 2)$. A year later, using Fourier analysis techniques, Nagel, Stein and Wainger [13] proved that M_{Ω} is actually bounded on $L^p(\mathbb{R}^2)$ for any $p > 1$. In [14], Sjögren and Sjölin proved that if Ω is a *lacunary set of finite order* then M_{Ω} is bounded on $L^p(\mathbb{R}^2)$

for any $p > 1$; this result was next proved in a more general context by Alfonseca [1]. Finally, Bateman [5] proved the converse and so characterized the $L^p(\mathbb{R}^2)$ boundedness of directional operators. Precisely he proved the following theorem.

THEOREM 1.2 (Bateman's Theorem). *Fix an arbitrary set $\Omega \subset [0, \pi)$ of directions. The directional maximal operator M_Ω is either good or bad.*

Hence we know that any set Ω of directions always yields a directional operator M_Ω that is either good or bad. Merging the vocabulary, we use the following definition.

DEFINITION 1.3. We say that a set Ω of directions is *good* when M_Ω is good, and *bad* when M_Ω is bad.

This notion of good/bad is perfectly understood. To say it bluntly, Ω is a good set of directions if and only if it can be included in a finite union of lacunary sets of finite order. If this is not possible, then Ω is bad; see [5]. We now turn our attention to maximal operators which are not directional.

1.3. Geometric maximal operators. *In this text, we will focus on geometric maximal operators which are not directional.* We recall two results in the direction of Bateman's Theorem for an arbitrary basis $\mathcal{B}$ included in $\mathcal{R}$. The first one is a result of Hagelstein and Stokolos [11].

THEOREM 1.4. *Fix an arbitrary basis $\mathcal{B}$ in $\mathcal{R}$ and suppose that there exist constants $t_0 \in (0, 1)$ and $C_0 > 1$ such that for any bounded measurable set $E \subset \mathbb{R}^2$ one has*

$$|\{M_{\mathcal{B}} \mathbb{1}_E > t_0\}| \leq C_0 |E|.$$

Then there exists p_0 depending on (t_0, C_0) such that for any $p > p_0$ we have $\|M_{\mathcal{B}}\|_p < \infty$.

In [9], we have shown that one can associate to any basis $\mathcal{B}$ included in $\mathcal{R}$ a geometric quantity denoted by $\lambda_{[\mathcal{B}]} \in \mathbb{N} \cup \{\infty\}$ that we call the *analytic split* of $\mathcal{B}$. We stress that the analytic split *is not* defined by abstract but really concrete means; in certain settings one can easily compute it. The analytic split of a basis allows us to control the p-norm of the associated geometric maximal operator.

THEOREM 1.5. *For any basis $\mathcal{B}$ in $\mathcal{R}$ and any $1 < p < \infty$ we have*

$$\log(\lambda_{[\mathcal{B}]}) \lesssim_p \|M_{\mathcal{B}}\|_p^p.$$

This theorem implies that any basis $\mathcal{B}$ whose analytic split is infinite yields a bad maximal operator $M_{\mathcal{B}}$. Moreover, it is easy to exhibit a lot of bases $\mathcal{B}$ whose analytic split is infinite.

THEOREM 1.6. *If $\lambda_{[\mathcal{B}]} = \infty$ then $M_{\mathcal{B}}$ is bad.*

1.4. Results. *As said earlier, we consider a family of geometric maximal operators which are* not *directional. Moreover, we will always work with bases $\mathcal{B}$ such that the associated set of directions*

$$\Omega_{\mathcal{B}} := \{\omega_R : R \in \mathcal{B}\}$$

is bad. Indeed, if $\Omega_{\mathcal{B}}$ is good then using the trivial estimate $M_{\mathcal{B}} \leq M_{\Omega_{\mathcal{B}}}$ we know that $M_{\mathcal{B}}$ is also a good operator.

Fix real numbers $a, b > 0$ and denote by $\mathcal{B}_{a,b}$ the basis of rectangles R whose eccentricity and orientation are of the form

$$(e_R, \omega_R) = \left(\frac{1}{n^a}, \frac{\pi}{4n^b}\right)$$

for some $n \in \mathbb{N}^*$. We denote by $M_{a,b}$ the geometric maximal operator associated to the basis $\mathcal{B}_{a,b}$. We prove the following theorem.

THEOREM 1.7. *If $a < b$ then $M_{a,b}$ is a good operator. If not then $M_{a,b}$ is bad.*

In the proof we will use Theorems 1.8 and 1.9 below. Denote by $\boldsymbol{t} = \{t_k\}_{k\geq 1} \subset [0, \pi/4]$ a sequence decreasing to 0 and by $\boldsymbol{e} = \{e_k\}_{k\geq 1} \subset (0, 1]$ any positive sequence. One should view $\boldsymbol{t}$ as a sequence of angles (or tangents of angles) that forms a bad set of directions, whereas $\boldsymbol{e}$ stands for an arbitrary sequence of eccentricities. For $k \geq 1$ consider the rectangle

$$R_k := R_k(\boldsymbol{e}, \boldsymbol{t})$$

whose orientation and eccentricity are $(e_{R_k}, \omega_{R_k}) = (e_k, t_k)$. Define

$$\mathcal{B} = \mathcal{B}(\boldsymbol{t}, \boldsymbol{e})$$

to be the basis generated by the rectangles $\{R_k\}_{k\geq 1}$. Our first result reads as follows.

THEOREM 1.8. *Suppose there is a constant $C > 0$ such that $t_k \leq Ce_k$ for any $k \geq 1$. Then $M_{\mathcal{B}}$ is a good operator.*

We now define the following quantity associated to the sequence $\boldsymbol{t}$:

$$\tau_{\boldsymbol{t}} := \sup_{k\geq 0, l\leq k} \left(\frac{t_{k+2l} - t_{k+l}}{t_{k+l} - t_k} + \frac{t_{k+l} - t_k}{t_{k+2l} - t_{k+l}}\right) \in (0, \infty].$$

This quantity yields information on the goodness/badness of the set $\{t_k\}_{k\geq 1}$ seen as a set of directions. Indeed, if $\tau_{\boldsymbol{t}}$ is finite then $\Omega = \{t_k\}_{k\geq 1}$ forms a bad set of directions. In some sense, this quantity indicates to what extent the sequence $\boldsymbol{t}$ is *uniformly distributed near* 0. For example, $\boldsymbol{t} = \{1/k\}$ looks like a uniform distribution near 0 and we have $\tau_{\boldsymbol{t}} < \infty$. On the other hand, $\boldsymbol{t} = \{1/2^k\}_{k\geq 1}$ converges rapidly to 0 and we have $\tau_{\boldsymbol{t}} = \infty$. Our second result reads as follows.

THEOREM 1.9. *Suppose that $\tau_{\boldsymbol{t}} < \infty$ and also that there is a constant $\mu_0 > 0$ such that for any $k \geq 1$ we have $e_k < \mu_0 |t_k - t_{k+1}|$. Then $M_{\mathcal{B}}$ is a bad operator.*

Before going into the proofs, let us make some general remarks about geometric maximal operators.

1.5. How can we prove that $M_{\mathcal{B}}$ is bad? To prove that $M_{\mathcal{B}}$ is bad, the idea is to create an exceptional geometric set adapted to the basis $\mathcal{B}$; more precisely, one can try to find a small fixed value $0 < \eta_0 < 1$ such that for any $\epsilon > 0$ there is a subset X in $\mathbb{R}^2$ satisfying

$$|X| \leq \epsilon |\{M_{\mathcal{B}} \mathbb{1}_X > \eta_0\}|.$$

If this holds then for any $p > 1$ we have

$$\int (M_{\mathcal{B}} \mathbb{1}_X)^p \geq \eta_0^p |\{M_{\mathcal{B}} \mathbb{1}_X > \eta_0\}| \geq \eta_0^p \frac{\|\mathbb{1}_X\|_p^p}{\epsilon}$$

since $|X|^{1/p} = \|\mathbb{1}_X\|_p$. Hence for any $\epsilon > 0$ we have $\|M_{\mathcal{B}}\|_p \geq \eta_0^p \epsilon^{-1/p}$ and $\|M_{\mathcal{B}}\|_p^p = \infty$ for any $1 < p < \infty$. The question remains how one can find/-construct such an X. Of course this depends on the basis $\mathcal{B}$. For example, consider $\mathcal{B} := \mathcal{R}$, biggest possible. The following property is a consequence of Proposition 3.3: for any large constant $A > 1$ there exists a finite family $\{R_i\}_{i \leq m}$ of rectangles in $\mathcal{R}$ satisfying

$$\Big|\bigcup_{i \leq m} 2R_i\Big| \geq A \Big|\bigcup_{i \leq m} R_i\Big|.$$

Considering then the set $X = \bigcup_{i \leq m} R_i$ it is easy to see that

$$|X| \leq \frac{1}{A} |\{M_{\mathcal{R}} \mathbb{1}_X > 1/4\}|,$$

which implies that $M_{\mathcal{R}}$ is a bad operator. A *Perron tree* (or *generalized Perron tree*) formed with a basis $\mathcal{B}$ of rectangles is a concrete construction of such a set X (or more precisely a sequence of sets) for any $\epsilon > 0$ and a fixed value η_0.

1.6. From rectangles to triangles. Without loss of generality, we will work at some point with triangles instead of rectangles. For any $k \geq 1$ define

$$T_k = T_k(\boldsymbol{e}, \boldsymbol{t}) := OA_kE_k$$

where $O = (0, 0)$, $A_k = (1, t_k)$ and $E_k = (1, t_k + e_k)$. Loosely speaking, T_k is a triangle which is *oriented* along the direction t_k and of *eccentricity* e_k. Denoting by $\mathcal{B}'$ the basis generated by the triangles T_k, one can observe the following property: for any $R \in \mathcal{B}$ there exists $T \in \mathcal{B}'$ satisfying, for some vector $\vec{t} \in \mathbb{R}^2$,

$$\vec{t} + \tfrac{1}{16} T \subset R \subset T,$$

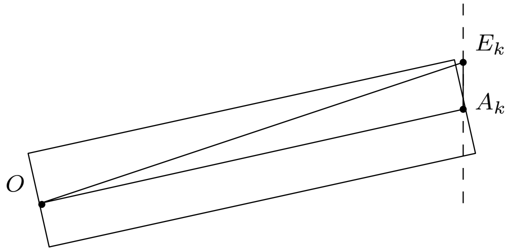

Fig. 1. A rectangle R_k and a triangle T_k, both oriented along $\simeq t_k$ and with eccentricity $\simeq e_k$.

and conversely, for any $T \in \mathcal{B}'$ there exists $R \in \mathcal{B}$ satisfying, for some $\vec{t} \in \mathbb{R}^2$,

$$\vec{t} + \tfrac{1}{16}R \subset T \subset R.$$

This implies that for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ we have

$$M_{\mathcal{B}}f(x) \simeq M_{\mathcal{B}'}f(x).$$

Hence it is equivalent to work with $\mathcal{B}$ or with $\mathcal{B}'$ and we will denote both bases by $\mathcal{B}$.

2. Proof of Theorem 1.8. It is well known that the operator $M_{\{0\}}$ associated to the basis $\mathcal{R}_{\{0\}} = \{R \in \mathcal{R} : \omega_R = 0\}$ is good. Now by easy geometric observations and using the property that $t_k < Ce_k$, one can prove that for any $R \in \mathcal{B}$ there exists a rectangle $P \in \mathcal{R}_{\{0\}}$ such that

$$R \subset P \quad \text{and} \quad |P| \leq 8(1+C)|R|.$$

This allows us to use the operator $M_{\{0\}}$ in order to dominate $M_{\mathcal{B}}$ pointwise. Fix any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and any $R \in \mathcal{B}$ and the associated rectangle $P \in \mathcal{R}_{\{0\}}$; then

$$\frac{1}{|R|}\int_R |f| \leq \frac{8(1+C)}{|P|}\int_P |f|,$$

which shows that for any $x \in \mathbb{R}^2$ we have

$$M_{\mathcal{B}}f(x) \leq 8(1+C)M_{\{0\}}f(x).$$

The conclusion comes from the fact that the strong maximal operator $M_{\{0\}}$ is good.

3. Proof of Theorem 1.9. The proof of Theorem 1.9 relies on geometric estimates and the construction of generalized Perron trees.

3.1. Geometric estimates. We start by establishing two geometric estimates. Fix an arbitrary open triangle $\Delta = ABC$ and consider the triangle $\Delta_2 := \vec{B} + \frac{1}{2}(\Delta - \vec{A})$.

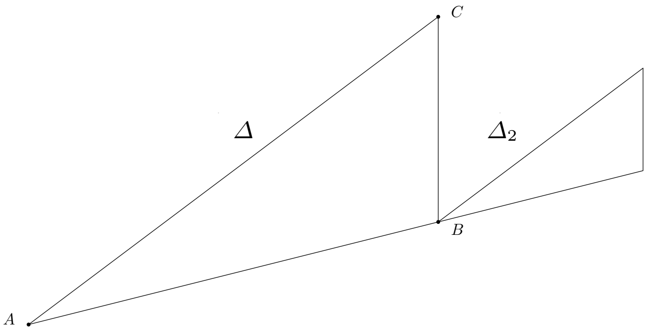

Fig. 2. The triangles $\varDelta$ and $\varDelta_2$ will usually be in this position.

Lemma 3.1 (Geometric estimate I). *The following inclusion holds:*

$$\varDelta_2 \subset \{M_{\{\varDelta\}} \mathbb{1}_{\varDelta} \geq 1/4\}.$$

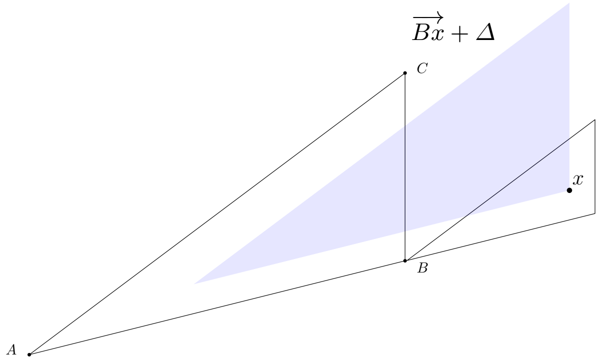

Fig. 3. The proof of Lemma 3.1 relies on the fact that $|\varDelta \cap (Bx + \varDelta)| \geq \frac{1}{4}|\varDelta|$.

Proof. Fix $x \in \vec{B} + \frac{1}{2}(\varDelta - \vec{A})$. It suffices to observe that $x \in Bx + \varDelta$ and

$$|\varDelta \cap (Bx + \varDelta)| \geq \tfrac{1}{4}|Bx + \varDelta|.$$

Hence $x \in \{M_{\{\varDelta\}} \mathbb{1}_{\varDelta} \geq 1/4\}$. ■

Actually, we need a more general version of the previous estimate. For $e \in \mathbb{R}_+$ and $\varDelta = ABC$ as before, define the triangle

$$T = T(e, \varDelta) := AB(B + eBC).$$

Lemma 3.2 (Geometric estimate II). *For any pair* $(\varDelta, T)$ *as defined above,*

$$\varDelta_2 \subset \{M_{\{T\}} \mathbb{1}_{\varDelta} \geq \eta(e)\} \quad \textit{where} \quad \eta(e) = \min\left\{\tfrac{1}{4}, \tfrac{1}{4e}\right\}.$$

Proof. The proof is akin to that of Lemma 3.1 and we invite the reader

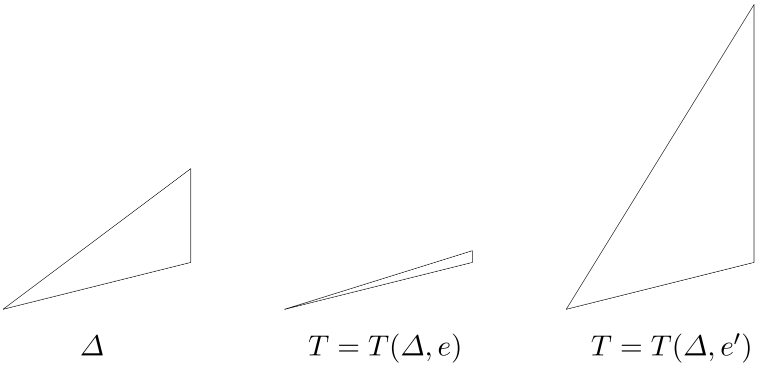

Fig. 4. A representation of Δ and $T = T(\Delta, e)$ for $e \ll 1$ and $e' > 1$.

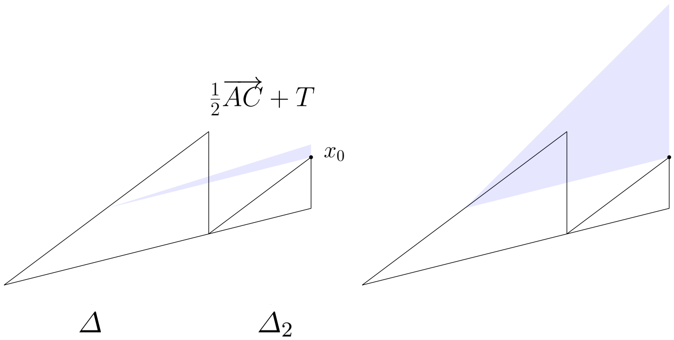

Fig. 5. An illustration of the argument of Lemma 3.2; $0 < e \leq 1$ (left) and $e > 1$ (right); the shaded triangle represents $\frac{1}{2}AC + T$.

to look at Figure 5 for a geometric representation. It is enough to check

$$x_0 \in \{M_{\{T\}} \mathbb{1}_\Delta > \eta(e)\}$$

where $x_0 = B + \frac{1}{2}AC$, because this is the worst case. To begin, observe that $x_0 \in \frac{1}{2}AC + T$. We distinguish two situations; if

$$0 < e \leq 1,$$

we are in the situation corresponding to the left part of Figure 5, that is,

$$\left|\Delta \cap \left(\tfrac{1}{2}AC + T\right)\right| = \tfrac{1}{4}|T|$$

and hence also

$$x_0 \in \{M_{\{T\}} \mathbb{1}_\Delta \geq 1/4\}.$$

The second situation corresponds to $1 < e$; in this case (see Figure 5 again),

$$\left|\Delta \cap \left(\tfrac{1}{2}AC + T\right)\right| \geq \tfrac{1}{4}|\Delta| \geq \tfrac{1}{4e}|T|.$$

This shows that $x_0 \in \{M_{\{T\}} \mathbb{1}_\Delta > 1/(4e)\}$ and concludes the proof. ∎

3.2. Generalized Perron trees. Denote by Δ_k the triangle with vertices $O, A_k = (1, t_k)$ and $A_{k+1} = (1, t_{k+1})$. Recall that we have supposed $\tau_{\boldsymbol{t}} < \infty$. We now give a slightly improved version of the construction of generalized Perron trees as defined in [12].

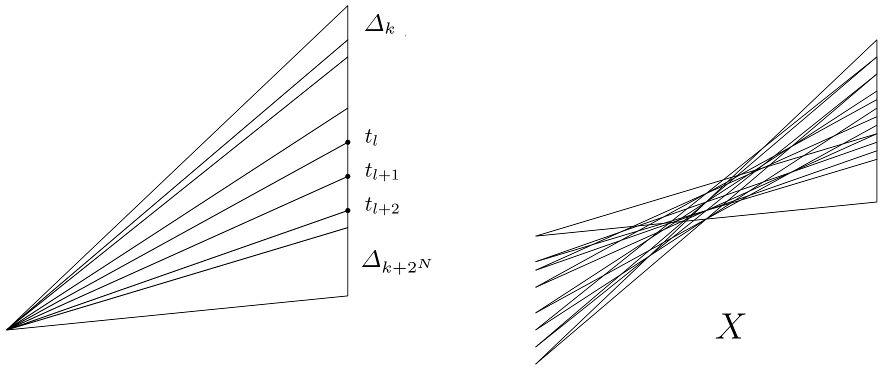

Fig. 6. A representation of some Δ_k and (on the left) a Perron tree X generated with those triangles. The idea is that for large n one has $|X| \ll |\Delta_k \sqcup \cdots \sqcup \Delta_{k+2^N}|$, plus the second property of Proposition 3.3.

Proposition 3.3 (Generalized Perron tree). *For any positive ratio α close to 1 and any integer $n \geq 1$, there exists an integer $N \gg 1$ and 2^n vectors $\vec{s}_k := (0, s_k)$ such that defining*

$$X = \bigcup_{N+1 \leq k \leq N+2^n} (\vec{s}_k + \Delta_k)$$

we have the following properties:

- $|X| \leq (\alpha^{2n} + \tau_{\boldsymbol{t}}(1-\alpha))|\Delta_{N+1} \sqcup \cdots \sqcup \Delta_{N+2^n}|$;
- *for any $k \neq l$ the triangles $(\vec{A}_k + \vec{s}_k) + \frac{1}{2}\Delta_k$ and $(\vec{A}_l + \vec{s}_l) + \frac{1}{2}\Delta_l$ are disjoint.*

We say that X is a generalized Perron tree of scale (α, n) *and we denote it by $X_{\alpha,n}(\boldsymbol{t})$.*

Note that the fact that the triangles $(\vec{A}_k + \vec{s}_k) + \frac{1}{2}\Delta_k$ and $(\vec{A}_l + \vec{s}_l) + \frac{1}{2}\Delta_l$ are disjoint is not proved in [12] yet it is easy to check. Observe that for any $\epsilon > 0$, one can first choose α close to 1 and then n large enough in order to have

$$|X_{\alpha,n}(\boldsymbol{t})| \leq \epsilon |\Delta_{N+1} \sqcup \cdots \sqcup \Delta_{N+2^n}|$$

for some large N. To obtain such an inequality, we need to ensure that the thin triangles Δ_k are comparable in some sense. Indeed, suppose that for any $k \geq 1$ the triangle Δ_k has vertices O, $G_k = (1, 1/2^k)$ and $G_{k+1} = (1, 1/2^{k+1})$.

Then for any $I \subset \mathbb{N}$ and any sequence of vectors $\{\vec{s}_i\}_{i\in I} \subset \mathbb{R}^2$ the set

$$X_I = \bigcup_{i\in I} (\vec{s}_i + \Delta_i)$$

satisfies

$$|X_I| \geq |\Delta_{i_0}| \geq \frac{1}{2}\Big|\bigcup_{i\in I} \Delta_i\Big|$$

where $i_0 := \min I$. Hence we cannot hope to stack up the triangles Δ_k into a set X that has a small area compared to the sum of the areas of the Δ_k. Hopefully this example sheds some light on the condition imposed on $\boldsymbol{t}$,

$$\tau_{\boldsymbol{t}} := \sup_{k\geq 0,\, l\leq k} \left(\frac{t_{k+2l} - t_{k+l}}{t_{k+l} - t_k} + \frac{t_{k+l} - t_k}{t_{k+2l} - t_{k+l}} \right) < \infty.$$

This ensures that the triangles Δ_k are comparable in some sense and that we can construct generalized Perron trees from them.

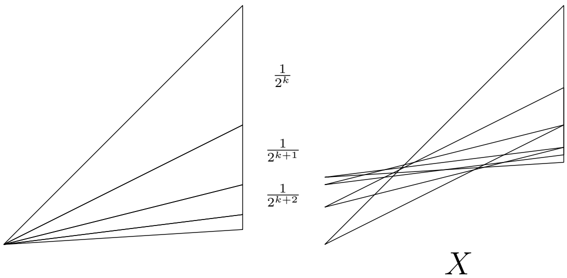

Fig. 7. It is quite difficult to construct a Perron tree; one needs to ensure that the triangles Δ_k are comparable. In this figure, the Δ_k differ too much in volume and one will always have $|X| \simeq |\bigcup \Delta_i|$ as explained.

3.3. Proof of Theorem 1.9. Recall that we suppose there is a constant $\mu_0 > 0$ such that $e_k < \mu_0 |t_k - t_{k+1}|$ for any $k \geq 1$. We will first construct a Perron tree $X_{\alpha,n}(\boldsymbol{t})$ from the triangles $\{\Delta_k\}_{k\geq 1}$. Then we will exploit this tree to show that $M_{\mathcal{B}}$ with $\mathcal{B} = \{T_k\}_{k\geq 1}$ is a bad operator. More precisely, we prove the following:

Claim. *For any α close to 1 and any $n \in \mathbb{N}$, the Perron tree $X := X_{\alpha,n}(\boldsymbol{t})$ satisfies*

$$|X| \leq \epsilon |\{M_{\mathcal{B}} \mathbb{1}_X > \eta(\mu_0)\}| \quad \textit{where} \quad \epsilon = \alpha^{2n} + \tau_{\boldsymbol{t}}(1-\alpha).$$

Proof. Fix α close to 1 and $n \in \mathbb{N}$ and consider the Perron tree

$$X = X_{\alpha,n}(\boldsymbol{t}) := \bigcup_{N+1\leq k\leq N+2^n} (\vec{s}_k + \Delta_k)$$

of scale (α, n) where N is given by Proposition 3.3. Fix any $k \in \{N+1, \ldots, N+2^n\}$ and consider the pair of triangles

$$(\vec{s}_k + \Delta_k, \vec{s}_k + T_k)$$

or more simply the pair (Δ_k, T_k), which is the same up to a translation. We can apply Lemma 3.2 to this pair, which yields

$$(\vec{A}_{k+1} + \vec{s}_k) + \tfrac{1}{2}\Delta_k \subset \{M_{\{T_k\}}\mathbb{1}_{\vec{s}_k+\Delta_k} > \eta(\mu_0)\}.$$

Since $M_{T_k} \leq M_{\mathcal{B}}$, we also have

$$(\vec{A}_{k+1} + \vec{s}_k) + \tfrac{1}{2}\Delta_k \subset \{M_{\mathcal{B}}\mathbb{1}_{\vec{s}_k+\Delta_k} > \eta(\mu_0)\}.$$

This inclusion yields

$$\bigsqcup_{k=N}^{N+2^n} (\vec{A}_{k+1} + \vec{s}_k) + \tfrac{1}{2}\Delta_k \subset \{M_{\mathcal{B}}\mathbb{1}_X > \eta(\mu_0)\}.$$

Here the union is disjoint by Proposition 3.3. Hence, in terms of Lebesgue measure,

$$\sum_{N+1\leq k\leq N+2^n} \frac{1}{4}|\Delta_k| \leq |\{M_{\mathcal{B}}\mathbb{1}_X > \eta(\mu_0)\}|.$$

fromUsing the fact that X is a Perron tree constructed with the triangles Δ_k we have

$$|X| \leq (\alpha^{2n} + \tau_{\boldsymbol{t}}(1-\alpha))|\Delta_{N+1} \sqcup \cdots \sqcup \Delta_{N+2^n}|.$$

In other words,

$$|X| \leq 4(\alpha^{2n} + \tau_{\boldsymbol{t}}(1-\alpha))|\{M_{\mathcal{B}}\mathbb{1}_X > \eta(\mu_0)\}|. \ \blacksquare$$

Observe finally that the Claim implies that for any $p > 1$ we have

$$\|M_{\mathcal{B}}\|_p \geq \eta(\mu_0)(4\alpha^{2n} + 4\tau_{\boldsymbol{t}}(1-\alpha))^{-1/p}$$

for any α close to 1 and any $n \in \mathbb{N}$. The fact that the constant $\eta(\mu_0)$ is independent of the scale (α, n) concludes the proof: we have $\|M_{\mathcal{B}}\|_p = \infty$ for any $p > 1$, i.e. $M_{\mathcal{B}}$ is a bad operator.

4. Proof of Theorem 1.7. Fix $a, b > 0$ and recall that $\mathcal{B}_{a,b}$ is the basis generated by a sequence $\{R_n\}_{n\geq 1}$ of rectangles satisfying

$$(e_{R_n}, \omega_{R_n}) = \left(\frac{1}{n^a}, \frac{\pi}{4n^b}\right).$$

Recall also that, for any $b > 0$, letting $\boldsymbol{\omega} = \{\pi/(4n^b)\}$ we have $\tau_{\boldsymbol{\omega}} < \infty$.

Case $a \leq b$. In this case, for $n \geq 1$ we have

$$\frac{4}{\pi n^b} \lesssim \frac{1}{n^a},$$

and so applying Theorem 1.8 we deduce that $M_{\mathcal{B}}$ is a good operator.

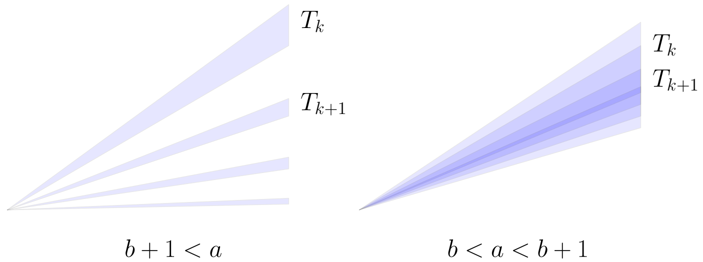

Fig. 8. Left: the regime $a > b+1$, the triangles T_k do not overlap at all for large k (actually the gap gets bigger with k). Right: in the regime $b+1 > a > b$, the triangles T_k tend to completely overlap.

Case $a \geq b+1$. In this case, for $n \geq 1$ we have

$$\left|\frac{\pi}{n^b} - \frac{\pi}{(n+1)^b}\right| \simeq \frac{1}{n^{b+1}}$$

and so

$$\frac{1}{n^a} \lesssim \left|\frac{\pi}{n^b} - \frac{\pi}{(n+1)^b}\right|.$$

Theorem 1.9 implies that $M_{\mathcal{B}}$ is a bad operator.

Case $b < a < b+1$. Now for any $\ell \in \mathbb{N}^*$, we have $\mathcal{B}_{\ell a, \ell b} \subset \mathcal{B}_{a,b}$. Hence, trivially,

$$M_{\ell a, \ell b} \leq M_{a,b}.$$

Since $a > b$, for $\ell_0 \gg 1$ we have $a > b + 1/\ell_0$, that is,

$$\ell_0 a > \ell_0 b + 1.$$

Applying the previous case, we conclude that $M_{\ell_0 a, \ell_0 b}$ is a bad operator, and so is $M_{a,b}$.

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