

## A continuous-parameter Katznelson–Tzafriri theorem for algebras of analytic functions

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**Abstract.** We prove a continuous-parameter version of a theorem of Katznelson–Tzafriri type for power-bounded operators which have a bounded calculus for analytic Besov functions. We also show that the result can be extended to some operators which have functional calculi with respect to some larger algebras.

**1. Introduction.** In 1986, Katznelson and Tzafriri [13] proved a theorem concerning asymptotics of the discrete semigroup  $(T^n)_{n \geq 0}$  for a power-bounded operator  $T$  on a complex Banach space  $X$ . They showed that  $\lim_{n \rightarrow \infty} \|T^n(I - T)\| = 0$  if  $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$ . More generally, they considered functions in the Wiener algebra  $W^+(\mathbb{D})$  of the form  $f(z) = \sum_{k=0}^{\infty} a_k z^k$ , where  $\sum_{k=0}^{\infty} |a_k| < \infty$  and  $|z| \leq 1$ . Let  $f(T) = \sum_{k=0}^{\infty} a_k T^k$ . Let  $W(\mathbb{T})$  be the space of all functions on  $\mathbb{T}$  of the form  $g(z) = \sum_{k=-\infty}^{\infty} b_k z^k$ , where  $\|g\|_{W(\mathbb{T})} := \sum_{k=-\infty}^{\infty} |b_k| < \infty$ . It was shown in [13] that  $\lim_{n \rightarrow \infty} \|T^n f(T)\| = 0$  if  $f \in W^+(\mathbb{D})$  and  $f$  is of spectral synthesis in  $W(\mathbb{T})$  with respect to  $\sigma(T) \cap \mathbb{T}$ . This assumption means that there exist functions  $(g_k)_{k \geq 1}$  in  $W(\mathbb{T})$  such that each  $g_k$  vanishes on a neighbourhood  $U_k$  of  $\sigma(T) \cap \mathbb{T}$  in  $\mathbb{T}$  and  $\lim_{k \rightarrow \infty} \|g_k - f\|_{W(\mathbb{T})} = 0$ .

These theorems have had a variety of applications. For a selection of them, see Section 4 of the recent survey article [6]. One drawback of the more general theorem is the assumption of spectral synthesis, which is stronger than simply assuming that  $f$  vanishes on  $\sigma(T) \cap \mathbb{T}$ . The theorem is not true in general if the weaker assumption is used, but the two assumptions are equivalent if  $\sigma(T) \cap \mathbb{T}$  is countable, for example. If  $X$  is a Hilbert space, then the weaker assumption is sufficient [14]. The survey article [6] covers these and other variants of the theorem, mainly in the discrete case.

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The following theorem is an analogue of the original theorem for bounded  $C_0$ -semigroups, and it was proved in [11] and [19]. Those proofs were quite different from the proofs in [13] and from each other. For a discussion of other proofs, see [6, Section 3.1]. In this paper we shall closely follow the approach used in [19].

**THEOREM 1.1.** *Let  $-A$  be the generator of a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Banach space  $X$ . Let  $f$  be the Laplace transform of a function in  $L^1(\mathbb{R}_+)$ , and assume that  $f$  is of spectral synthesis in  $L^1(\mathbb{R})$  with respect to  $\sigma(A) \cap i\mathbb{R}$ . Then  $\lim_{t \rightarrow \infty} \|T(t)f(A)\| = 0$ .*

Some variants of this continuous-parameter version of the Katznelson–Tzafriri theorem have been obtained. If  $X$  is a Hilbert space, the result is true with the weaker assumption that  $f$  vanishes on  $\sigma(A) \cap i\mathbb{R}$  [18]. Several extensions to more general semigroups of operators have appeared in [18] and [21]. Some papers have considered Banach algebras other than the Laplace transforms of integrable functions. An example in the discrete case is a Banach algebra  $\mathcal{B}(\mathbb{D})$  of analytic Besov functions on the unit disc  $\mathbb{D}$ , and Peller showed in [17] that any power-bounded operator on a Hilbert space has a bounded  $\mathcal{B}(\mathbb{D})$ -calculus. In the continuous-parameter case, the theory of functional calculus for the corresponding algebra  $\mathcal{B}$  of analytic Besov functions on the right half-plane  $\mathbb{C}_+$  and bounded  $C_0$ -semigroups has recently been developed in [3] and [4].

In [7], we proved a version of the Katznelson–Tzafriri theorem in the discrete case where  $T$  is assumed to have a bounded functional calculus with respect to the Banach algebra  $\mathcal{B}(\mathbb{D})$ . It applies to functions  $f \in \mathcal{B}(\mathbb{D})$  which vanish on  $\sigma(T) \cap \mathbb{T}$ , so it includes not only some functions outside  $W^+(\mathbb{D})$ , but also functions in  $W^+(\mathbb{D})$  which are not of spectral synthesis with respect to  $\sigma(T) \cap \mathbb{T}$ . When we wrote [7], we did not see how to obtain a corresponding result in the continuous-parameter case. Subsequently we have been able to prove the result stated in Theorem 1.2 below, by using an approximation argument based on Arveson’s theory of spectral subspaces for  $C_0$ -groups of isometries; see [18] for a related argument. We also rely crucially on results by Cojuhari and Gomilko [8] concerning a certain integral resolvent condition which characterises those operators that admit a bounded  $\mathcal{B}$ -calculus.

We now state the main result of our paper for operators which have a bounded  $\mathcal{B}$ -calculus. The assumption that  $f$  vanishes at infinity is natural in the result because infinity can be thought of as being an invisible part of  $\sigma(A)$  when  $A$  is unbounded; indeed without this assumption the theorem would be false if  $\sigma(A) \cap i\mathbb{R}$  is empty,  $(T(t))_{t \geq 0}$  is not exponentially stable, and  $f$  is a constant function. In the statement, the operator  $f(A)$  is defined by the  $\mathcal{B}$ -calculus for  $A$ .

**THEOREM 1.2.** *Let  $-A$  be the generator of a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  and assume that  $A$  has a bounded  $\mathcal{B}$ -calculus. Let  $f \in \mathcal{B}$ , and assume that  $f$  vanishes on  $\sigma(A) \cap i\mathbb{R}$  and  $\lim_{z \in \mathbb{C}_+, |z| \rightarrow \infty} f(z) = 0$ . Then*

$$\lim_{t \rightarrow \infty} \|T(t)f(A)\| = 0.$$

Since Laplace transforms of functions in  $L^1(\mathbb{R}_+)$  lie in  $\mathcal{B}$  and vanish at infinity, this result extends Theorem 1.1 in the case when  $A$  admits a bounded  $\mathcal{B}$ -calculus, both by enlarging the class of functions to which it applies and by weakening the spectral synthesis condition to the condition that  $f$  vanishes on  $\sigma(A) \cap i\mathbb{R}$  (which is necessary for the conclusion of Theorem 1.2). Furthermore, as the negative generator of every bounded  $C_0$ -semigroup on a Hilbert space admits a bounded  $\mathcal{B}$ -calculus (see [3, Section 4]), we obtain the following immediate consequence of Theorem 1.2.

**COROLLARY 1.3.** *Let  $-A$  be the generator of a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Hilbert space. Let  $f \in \mathcal{B}$ , and assume that  $f$  vanishes on  $\sigma(A) \cap i\mathbb{R}$  and  $\lim_{z \in \mathbb{C}_+, |z| \rightarrow \infty} f(z) = 0$ . Then  $\lim_{t \rightarrow \infty} \|T(t)f(A)\| = 0$ .*

This extends the implication (i) $\Rightarrow$ (iii) of [18, Theorem 3.1] in the case when  $S = \mathbb{R}_+$  to a larger class of functions  $f$ ; see also [14].

In the next section, we recall relevant facts for understanding the statement and proof of Theorem 1.2. We originally considered only the algebra  $\mathcal{B}$ , but an anonymous referee suggested that we might be able to prove Theorem 1.2 for algebras other than  $\mathcal{B}$  by similar methods. After some time, we proved a version of the result for some algebras larger than  $\mathcal{B}$ , and that version is set out in Theorem 2.1. The proof is given in Section 3.

The papers [1] and [2] have identified two function algebras which are larger than  $\mathcal{B}$ , and to which Theorem 2.1 can be extended in a slightly modified form. These algebras are briefly described in Section 4.

**2. The setting and the main result.** Let  $\mathcal{B}$  be the space of all holomorphic functions  $f$  on  $\mathbb{C}_+$  such that

$$\|f\|_{\mathcal{B}_0} := \int_0^\infty \sup_{\beta \in \mathbb{R}} |f'(\alpha + i\beta)| d\alpha < \infty.$$

These functions are bounded and uniformly continuous on  $\mathbb{C}_+$ , and  $\mathcal{B}$  is a Banach algebra in the norm

$$\|f\|_{\mathcal{B}} := \|f\|_{\mathcal{B}_0} + \|f\|_\infty,$$

where  $\|\cdot\|_\infty$  is the supremum norm on  $H^\infty(\mathbb{C}_+)$ . We will consider any function  $f \in \mathcal{B}$  to be defined and uniformly continuous on  $\overline{\mathbb{C}_+}$ . For details about  $\mathcal{B}$ , see [3].

We will also consider the following subalgebras of  $\mathcal{B}$ , adopting the notation used in [3–5]:

$$\begin{aligned} \mathcal{L}\mathcal{L}^1 &:= \{\mathcal{L}g : g \in L^1(\mathbb{R}_+)\}, \quad \text{where } \mathcal{L} \text{ is the Laplace transform,} \\ \mathcal{B}_{00} &:= \left\{ f \in \mathcal{B} : \lim_{|\beta| \rightarrow \infty} f(i\beta) = 0 \right\} = \left\{ f \in \mathcal{B} : \lim_{z \in \mathbb{C}_+, |z| \rightarrow \infty} f(z) = 0 \right\}. \end{aligned}$$

The algebra  $\mathcal{L}\mathcal{L}^1$  is dense in  $\mathcal{B}_{00}$  in the  $\mathcal{B}$ -norm [4, Theorem 4.4], and  $\mathcal{B}_{00}$  is a closed subalgebra of  $\mathcal{B}$ , so  $\mathcal{B}_{00}$  is the closure of  $\mathcal{L}\mathcal{L}^1$  in  $\mathcal{B}$ . The norms  $\|\cdot\|_{\mathcal{B}_0}$  and  $\|\cdot\|_{\mathcal{B}}$  are equivalent on  $\mathcal{B}_{00}$ , but  $\|\cdot\|_{\mathcal{B}_0}$  is not submultiplicative. Note that the space  $\mathcal{B}_{00}$  is denoted by  $\mathcal{B} \cap C_0(\overline{\mathbb{C}_+})$  in [4].

Let  $-A$  be the generator of a bounded  $C_0$ -semigroup on a Banach space  $X$ , so that  $\sigma(A) \subseteq \overline{\mathbb{C}_+}$ . We say that  $A$  satisfies the (GSF) condition if, for all  $x \in X$  and  $x^* \in X^*$ ,

$$(2.1) \quad \sup_{\alpha > 0} \int_{\mathbb{R}} | \langle (\alpha + i\beta + A)^{-2} x, x^* \rangle | d\beta < \infty.$$

The Closed Graph Theorem then implies that there exists a finite constant  $\gamma_A$  such that the supremum above is bounded by  $\gamma_A \|x\| \|x^*\|$  for all  $x \in X$  and  $x^* \in X^*$ .

Let  $L(X)$  be the Banach algebra of bounded linear operators on  $X$ . For  $\lambda \in \mathbb{C}$ , we will let  $r_\lambda$  denote the function  $r_\lambda(z) := (\lambda + z)^{-1}$  with appropriate domain in  $\mathbb{C}$ . Let  $\mathcal{A}$  be a Banach algebra such that  $\mathcal{B}$  is continuously included in  $\mathcal{A}$  and  $\mathcal{A}$  is continuously included in  $H^\infty(\mathbb{C}_+)$ . We say that  $A$  has a bounded  $\mathcal{A}$ -calculus if there is a bounded algebra homomorphism  $\Phi : \mathcal{A} \rightarrow L(X)$  such that  $\Phi(r_\lambda) = (\lambda + A)^{-1}$  for some (or equivalently, all)  $\lambda \in \mathbb{C}_+$ .

It is shown in [3, Theorem 4.4] and [4, Theorem 6.1] that  $A$  satisfies the (GSF) condition if and only if  $A$  has a bounded  $\mathcal{B}$ -calculus. Moreover,  $A$  satisfies the (GSF) condition if  $\Phi$  is defined and bounded only on  $\mathcal{B}_{00}$ , as all the functions  $G_{\alpha, \varphi}$  in the proof of [4, Theorem 6.1] belong to  $\mathcal{B}_{00}$ . In addition, any bounded  $\mathcal{B}$ -calculus for  $A$  is unique [4, Theorem 6.2], and we will denote it by  $\Phi_{\mathcal{B}}^A$ .

In the case when  $\mathcal{A} = \mathcal{B}$ , the following result coincides with Theorem 1.2, since  $\mathcal{B}_{00}$  is the closure of  $\mathcal{L}\mathcal{L}^1$  in  $\mathcal{B}$ . In general, the closure of  $\mathcal{L}\mathcal{L}^1$  in  $\mathcal{A}$  is contained in  $\{f \in \mathcal{A} : \lim_{z \in \mathbb{C}_+, |z| \rightarrow \infty} f(z) = 0\}$ , but equality may not hold.

**THEOREM 2.1.** *Let  $\mathcal{A}$  be a Banach algebra such that  $\mathcal{B}$  is continuously included in  $\mathcal{A}$ , and  $\mathcal{A}$  is continuously included in  $H^\infty(\mathbb{C}_+)$ . Let  $-A$  be the generator of a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  and assume that  $A$  has a bounded  $\mathcal{A}$ -calculus. Let  $f$  be in the closure of  $\mathcal{L}\mathcal{L}^1$  in  $\mathcal{A}$ , and assume that  $f$  vanishes on  $\sigma(A) \cap i\mathbb{R}$ . Then  $\lim_{t \rightarrow \infty} \|T(t)f(A)\| = 0$ .*

To prove Theorem 2.1, we will need to consider the case of a skew-hermitian operator  $Z$  on a Banach space  $Y$ , so that  $-Z$  generates a  $C_0$ -group of isometries on  $Y$ . Then  $\sigma(Z) \subseteq i\mathbb{R}$ . If  $Z$  has a bounded  $\mathcal{B}$ -calculus, then

$Z$  also has a bounded  $C_0(i\mathbb{R})$ -calculus, that is, a bounded algebra homomorphism from  $C_0(i\mathbb{R})$  to  $L(X)$  mapping  $r_\lambda$  to  $(\lambda + Z)^{-1}$  for some (or all)  $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ . The converse also holds. These statements can be seen in [4, Theorem 6.5], or by using a combination of [4, Theorem 6.1], [8, Lemma 3.4] and [10, Theorem 3.6]. The density of the algebra generated by the resolvent functions  $\{r_\lambda : \lambda \in \mathbb{C} \setminus i\mathbb{R}\}$  in  $C_0(i\mathbb{R})$  shows that any bounded  $C_0(i\mathbb{R})$ -calculus for  $Z$  is unique, and we denote it by  $\Phi_{C_0}^Z$ . If  $f \in \mathcal{B}_{00}$ , then  $\Phi_{\mathcal{B}}^Z(f)$  and  $\Phi_{C_0}^Z(f)$  are both defined (here we use  $f$  to denote the boundary function of  $f$  on  $i\mathbb{R}$ ). The proof of [4, Theorem 6.5] did not show explicitly that the two definitions coincide. This compatibility can be deduced from the paragraph following the proof of Theorem 6.5 and equation (6.4) in [4]. They establish a multiplier formula for  $\Phi_{\mathcal{B}}^Z$ , namely

$$(2.2) \quad \Phi_{\mathcal{B}}^Z(f)\Phi_{C_0}^Z(g) = \lim_{n \rightarrow \infty} \Phi_{C_0}^Z(fe_n)\Phi_{C_0}^Z(g)$$

in the strong operator topology, where  $g \in C_0(i\mathbb{R})$  and  $(e_n)_{n \geq 1}$  is a bounded approximate unit in  $C_0(i\mathbb{R})$ . The same formula holds when  $\Phi_{\mathcal{B}}^Z(f)$  is replaced by  $\Phi_{C_0}^Z(f)$ . The compatibility of the two calculi follows; see also Remark 3.3.

If  $Z$  satisfies the (GSF) condition, then  $-Z$  also satisfies the (GSF) condition. This can be seen from [8, Lemma 3.4]. It corresponds to the fact that if  $Z$  has a bounded  $C_0(i\mathbb{R})$ -calculus, then so does  $-Z$ , with  $\Phi_{C_0}^{-Z}(f) = \Phi_{C_0}^Z(\tilde{f})$ , where  $\tilde{f}(i\beta) = f(-i\beta)$  for  $f \in C_0(i\mathbb{R})$  and  $\beta \in \mathbb{R}$ .

**3. The proof of Theorem 2.1.** The first step in the proof is the observation that it suffices to prove the result under the additional assumption that  $\mathcal{B}$  is dense in  $\mathcal{A}$ . We make this assumption without any loss of generality, as  $\mathcal{A}$  can be replaced by the closure of  $\mathcal{B}$  in  $\mathcal{A}$ . This assumption implies that all functions in  $\mathcal{A}$  are bounded and uniformly continuous on  $\overline{\mathbb{C}_+}$  and any bounded  $\mathcal{A}$ -calculus for an operator  $A$  is unique, by the uniqueness of any  $\mathcal{B}$ -calculus and the density of  $\mathcal{B}$  in  $\mathcal{A}$ . Throughout this section, we take  $\mathcal{A}$  to be a fixed Banach algebra of this form. It may be advantageous to take  $\mathcal{A} = \mathcal{B}$  on first reading of the proof.

Next we consider the case of a skew-hermitian operator  $Z$  which has a bounded  $\mathcal{B}$ -calculus. Then  $Z$  also has a bounded  $\mathcal{A}$ -calculus  $\Phi_{\mathcal{A}}^Z$ , given by the multiplier formula (2.2) with  $\mathcal{B}$  replaced by  $\mathcal{A}$ . This calculus is an extension of the  $\mathcal{B}$ -calculus for  $Z$ , and it is compatible with the  $C_0(i\mathbb{R})$ -calculus for  $Z$ .

Now assume that  $Z$  is a bounded skew-hermitian operator with a bounded  $\mathcal{B}$ -calculus, and let  $K$  be a compact subset of  $i\mathbb{R}$  such that  $\sigma(Z)$  is contained in the interior of  $K$  in  $i\mathbb{R}$ . The following lemma shows that there is a bounded  $C(K)$ -calculus for  $Z$  with properties similar to those of a single polynomially bounded isometry; see [15, Lemma 1.1]. A *bounded  $C(K)$ -calculus* for  $Z$  is a bounded algebra homomorphism mapping the constant function 1 to the

identity operator  $I$  and the identity function to  $Z$ , or equivalently mapping  $r_\lambda$  to  $(\lambda + Z)^{-1}$  for all  $\lambda \in \mathbb{C} \setminus K$ . Such a calculus is necessarily unique by the density of the polynomials in  $C(K)$ .

LEMMA 3.1. *Let  $Z$  be a bounded skew-hermitian operator on a Banach space  $Y$ , and assume that  $Z$  has a bounded  $\mathcal{B}$ -calculus. Let  $\mathcal{A}$  be a Banach algebra as described above, and let  $K$  be a compact subset of  $i\mathbb{R}$  such that  $\sigma(Z)$  is contained in the interior of  $K$  in  $i\mathbb{R}$ . There is a unique bounded  $C(K)$ -calculus  $\Psi$  for  $Z$ . It has the following properties:*

- (a) *If  $f \in C_0(i\mathbb{R})$  and  $g$  is the restriction of  $f$  to  $K$ , then  $\Psi(g) = \Phi_{C_0}^Z(f)$ .*
- (b) *If  $f \in \mathcal{A}$  and  $g$  is the restriction of  $f$  to  $K$ , then  $\Psi(g) = \Phi_{\mathcal{A}}^Z(f)$ .*
- (c) *If  $g \in C(K)$ , then  $\sigma(\Psi(g)) = g(\sigma(Z))$ .*
- (d) *If  $g \in C(K)$  and  $g$  vanishes on  $\sigma(Z)$ , then  $\Psi(g) = 0$ .*

*Proof.* Since  $Z$  has a bounded  $\mathcal{B}$ -calculus, the (GSF) condition holds for  $Z$  and for  $-Z$ , so, for all  $\alpha > 0$ ,  $y \in Y$  and  $y^* \in Y^*$ ,

$$\alpha \int_{\mathbb{R}} |\langle (\alpha + i\beta - Z)^{-2} y, y^* \rangle| d\beta \leq \gamma_{-Z} \|y\| \|y^*\|.$$

For  $\alpha \in (0, 1]$  and  $\beta \in \mathbb{R}$ , let

$$\begin{aligned} \Delta(\alpha, \beta, -Z) &:= (\alpha + i\beta - Z)^{-1} - (-\alpha + i\beta - Z)^{-1} \\ &= -2\alpha(\alpha + i\beta - Z)^{-1}(-\alpha + i\beta - Z)^{-1}. \end{aligned}$$

Fix  $k > \|Z\|$ , and let  $K' = \{\beta \in \mathbb{R} : i\beta \in K\}$ . Then

$$\begin{aligned} \|\Delta(\alpha, \beta, -Z)\| &\leq \frac{2\alpha}{(|\beta| - \|Z\|)^2}, & \alpha \in (0, 1], |\beta| > k, \\ \|\Delta(\alpha, \beta, -Z)\| &\leq c, & \alpha \in (0, 1], |\beta| \leq k, \beta \in \mathbb{R} \setminus K', \\ \lim_{\alpha \rightarrow 0^+} \|\Delta(\alpha, \beta, -Z)\| &= 0, & i\beta \notin \sigma(Z), \end{aligned}$$

for some constant  $c$ . The Dominated Convergence Theorem and these estimates imply that

$$(3.1) \quad \lim_{\alpha \rightarrow 0^+} \int_{\mathbb{R} \setminus K'} \|\Delta(\alpha, \beta, -Z)\| d\beta = 0.$$

Let  $f \in C_0(i\mathbb{R})$ . It follows from [10, Theorem 3.6] and (3.1) that the  $C_0(i\mathbb{R})$ -calculus for  $Z$  is given by

$$\begin{aligned} \langle \Phi_{C_0}^Z(f)y, y^* \rangle &= \frac{1}{2\pi} \lim_{\alpha \rightarrow 0^+} \int_{\mathbb{R}} f(i\beta) \langle \Delta(\alpha, \beta, -Z)y, y^* \rangle d\beta \\ &= \frac{1}{2\pi} \lim_{\alpha \rightarrow 0^+} \int_{K'} f(i\beta) \langle \Delta(\alpha, \beta, -Z)y, y^* \rangle d\beta. \end{aligned}$$

From [8, Lemma 3.4] applied to  $-iZ$ , there is a constant  $C$  such that

$$\int_{\mathbb{R}} |\langle \Delta(\alpha, \beta, -Z)y, y^* \rangle| d\beta \leq C \|y\| \|y^*\|, \quad \alpha \in (0, 1], y \in Y, y^* \in Y^*.$$

Then

$$\begin{aligned} |\langle \Phi_{C_0}^Z(f)y, y^* \rangle| &\leq \frac{1}{2\pi} \int_{K'} \|f\|_{C(K)} |\langle \Delta(\alpha, \beta, -Z)y, y^* \rangle| d\beta \\ &\leq \frac{C}{2\pi} \|f\|_{C(K)} \|y\| \|y^*\|. \end{aligned}$$

This implies that

$$(3.2) \quad \|\Phi_{C_0}^Z(f)\| \leq \frac{C}{2\pi} \|f\|_{C(K)}.$$

We can now define a bounded  $C(K)$ -calculus for  $Z$ , as follows. Let  $g \in C(K)$ , and let  $f \in C_0(i\mathbb{R})$  be any extension of  $g$ . Define  $\Psi(g) := \Phi_{C_0}^Z(f)$ . It follows from (3.2) that this definition is independent of the choice of  $f$ , and  $\Psi$  is a bounded algebra homomorphism from  $C(K)$  to  $L(Y)$  mapping  $r_\lambda$  to  $(\lambda + Z)^{-1}$  for  $\lambda \in \mathbb{C} \setminus K$ . Thus it is a bounded  $C(K)$ -calculus for  $Z$ . Property (a) in Lemma 3.1 is immediate from the definition of  $\Psi$ .

The map from  $\mathcal{A}$  to  $L(Y)$  given by  $f \mapsto \Psi(f|_K)$  is a bounded algebra homomorphism, and it is a bounded  $\mathcal{A}$ -calculus for  $Z$ . Now (b) follows from the uniqueness of the  $\mathcal{A}$ -calculus for  $Z$ ; see the first paragraph in this section.

The proofs of (c) and (d) are very similar to the proofs of [15, Lemma 1.1]. For (c), we may take any commutative Banach subalgebra  $\mathcal{C}$  of  $L(Y)$  containing  $I$ ,  $Z$ ,  $\Psi(g)$  and all their resolvents. Then the spectra of  $Z$  and  $\Psi(g)$  in  $L(Y)$  coincide with their spectra in  $\mathcal{C}$ , so it suffices to show that  $\chi(\Psi(g)) = g(\chi(Z))$  for every character  $\chi$  of  $\mathcal{C}$ . We may approximate  $g$  uniformly on  $K$  by a sequence  $(p_n)_{n \geq 1}$  of polynomials, so  $\Psi(g) = \lim_{n \rightarrow \infty} p_n(Z)$  in operator norm. Then

$$\chi(\Psi(g)) = \lim_{n \rightarrow \infty} \chi(p_n(Z)) = \lim_{n \rightarrow \infty} p_n(\chi(Z)) = g(\chi(Z)).$$

This establishes (c).

For (d), first assume that  $g \in C(K)$  vanishes on a neighbourhood of  $\sigma(Z)$  in  $K$ . Choose  $h \in C(K)$  such that  $h(\lambda) = 1$  for all  $\lambda \in \sigma(Z)$  and  $gh = 0$ . By (c),  $\sigma(\Psi(h)) = \{1\}$ , so  $\Psi(h)$  is invertible. Since  $\Psi(g)\Psi(h) = 0$  it follows that  $\Psi(g) = 0$ .

Now assume only that  $g \in C(K)$  vanishes on  $\sigma(Z)$ . Then there is a sequence  $(g_n)_{n \geq 1}$  in  $C(K)$  where each  $g_n$  vanishes on a neighbourhood  $U_n$  of  $\sigma(Z)$  in  $K$ , and  $\|g_n - g\|_{C(K)} \rightarrow 0$ . Statement (d) follows from (3.2) and the paragraph above. ■

Properties (a) and (b) in Lemma 3.1 establish that the functional calculus  $\Psi$  for a bounded skew-hermitian operator  $Z$  is compatible with the  $C_0(i\mathbb{R})$ -

calculus and the  $\mathcal{A}$ -calculus for functions  $f \in \mathcal{A}$ . We will use the simple notation  $f(Z)$  for the resulting operators in any of these calculi.

We now return to the case when  $Z$  is an unbounded skew-hermitian operator with a bounded  $\mathcal{B}$ -calculus.

**LEMMA 3.2.** *Let  $Z$  be a skew-hermitian operator on a Banach space  $Y$ , and assume that  $Z$  has a bounded  $\mathcal{B}$ -calculus. Let  $f \in \mathcal{A}$ , and assume that  $f$  vanishes on  $\sigma(Z)$ . Then  $f(Z) = 0$ .*

*Proof.* We will apply the Arveson spectral theory for  $C_0$ -groups of isometries to the  $C_0$ -group  $(V(t))_{t \in \mathbb{R}}$  generated by  $-Z$  on  $Y$ ; see [9, Section 8] for an account of the theory in this context.

For  $k \in \mathbb{N}$ , let  $Y_k$  be the spectral subspace corresponding to the interval  $[-k, k]$ . Then  $Y_k$  is a closed subspace of  $Y$  which is invariant under the operators  $V(t)$ , and the restrictions  $V_k(t)$  of  $V(t)$  to  $Y_k$  form a norm-continuous group of isometries on  $Y_k$ . Moreover, the negative generator  $Z_k$  is a bounded operator on  $Y_k$ , so  $Y_k$  is contained in the domain of  $Z$ , and  $Zy = Z_k y$  for all  $y \in Y_k$ , and  $\sigma(Z_k) \subseteq \sigma(Z) \cap i[-k, k]$ ; see [9, Theorems 8.19 and 8.27]. By restricting the operators  $f(Z)$  to the subspace  $Y_k$  for  $f \in C_0(i\mathbb{R})$ , we obtain a bounded  $C_0(i\mathbb{R})$ -calculus for  $Z_k$ .

Now let  $f \in \mathcal{A}$ , and assume that  $f$  vanishes on  $\sigma(Z)$ . For each  $k$ ,  $f$  vanishes on  $\sigma(Z_k)$ . Taking  $K = i[-(k+1), k+1]$ , Lemma 3.1 shows that  $f(Z_k) = 0$ . This implies that  $f(Z)y = 0$  for all  $y \in Y_k$ , since  $f(Z_k)$  is the restriction of  $f(Z)$  to  $Y_k$ . Since  $\bigcup_{k \in \mathbb{N}} Y_k$  is dense in  $Y$  (see [9, Lemma 8.12]), this implies that  $f(Z) = 0$ . ■

**REMARK 3.3.** Let  $Z$  be a skew-hermitian operator on  $Y$ , with a bounded  $\mathcal{B}$ -calculus, and let  $Y_k$  and  $Z_k$  be as in the proof of Lemma 3.2. Also let  $f \in \mathcal{A} \cap C_0(i\mathbb{R})$ . Parts (a) and (b) of Lemma 3.1 show that  $\Phi_{\mathcal{A}}^{Z_k}(f) = \Phi_{C_0}^{Z_k}(f)$ . Hence  $\Phi_{\mathcal{A}}^Z(f)$  and  $\Phi_{C_0}^Z(f)$  coincide on  $\bigcup_{k \in \mathbb{N}} Y_k$ , and then by continuity they coincide on  $Y$ . This is an alternative proof that the two calculi are compatible.

We now prove Theorem 2.1. The structure of our argument is the same as that in [19], but we need the lemmas above to justify the crucial stage in the argument which enables us to cover functions outside  $\mathcal{L}L^1$  and to avoid assumptions of spectral synthesis.

*Proof of Theorem 2.1.* If  $i\mathbb{R} \subseteq \sigma(A)$ ,  $f \in \mathcal{A}$ , and  $f$  vanishes on  $i\mathbb{R}$ , then  $f$  vanishes on  $\mathbb{C}_+$ . So  $f(A) = 0$ , and the result holds trivially. Thus we may assume that  $i\mathbb{R}$  is not contained in  $\sigma(A)$ . We will initially show that  $\lim_{t \rightarrow \infty} T(t)f(A) = 0$  in the strong operator topology.

We use the limit isometric semigroup, as in [19, Theorem 3.2] or [20, Proposition 3.1]. There is a Banach space  $Y$  (which may be  $\{0\}$ ), a bounded map  $\pi : X \rightarrow Y$  with dense range such that  $\|\pi(x)\|_Y = \limsup_{t \rightarrow \infty} \|T(t)x\|$

for  $x \in X$ , and a  $C_0$ -semigroup  $(V(t))_{t \geq 0}$  of (not necessarily invertible) isometries on  $Y$  such that  $V(t)\pi = \pi T(t)$  and the negative generator  $Z$  of  $(V(t))_{t \geq 0}$  satisfies  $\sigma(Z) \subseteq \sigma(A) \cap i\mathbb{R}$ .

By a property of semigroups of isometries (see [16, Lemma, p. 38] or [12, pp. 419, 420]), the inequality  $\|Zy - \lambda y\| \geq \operatorname{Re} \lambda \|y\|$  is valid for all  $\lambda \in \mathbb{C}_+$  and  $y \in Y$ . Since  $i\mathbb{R}$  is not contained in  $\sigma(Z)$  and  $Z$  has no approximate eigenvalues in  $\mathbb{C}_+$ , it follows that  $\sigma(Z) \cap \mathbb{C}_+$  is empty, and  $\|(\lambda - Z)^{-1}\| \leq (\operatorname{Re} \lambda)^{-1}$  for  $\lambda \in \mathbb{C}_+$ . By the Hille–Yosida theorem,  $Z$  generates a  $C_0$ -semigroup, and hence  $-Z$  generates a  $C_0$ -group of invertible isometries  $(V(t))_{t \in \mathbb{R}}$  on  $Y$ .

For  $g \in \mathcal{B}$ , the operator  $g(A) \in L(X)$  commutes with  $T(t)$  for all  $t \geq 0$ , and so there is a unique operator  $\mathcal{Y}(f) \in L(Y)$  such that  $\mathcal{Y}(f)\pi = \pi f(A)$ . Then  $\mathcal{Y}$  is a bounded algebra homomorphism of  $\mathcal{B}$  into  $L(Y)$  mapping  $r_\lambda$  to  $(\lambda + Z)^{-1}$  for  $\lambda \in \mathbb{C}_+$ , so it is a bounded  $\mathcal{B}$ -calculus for  $Z$ .

Now assume that  $f$  is in the closure of  $\mathcal{L}L^1$  in  $\mathcal{A}$  and  $f$  vanishes on  $\sigma(A) \cap i\mathbb{R}$ . Then  $f$  vanishes on  $\sigma(Z)$ , and it follows from Lemma 3.2 that  $\mathcal{Y}(f) = 0$ . Then  $\pi f(A) = 0$ , and hence  $\lim_{t \rightarrow \infty} T(t)f(A) = 0$  in the strong operator topology.

It remains to lift this conclusion to the operator-norm topology. This can be achieved by using the method of [19, Theorem 3.2] and [20, Theorem 3.9]. Consider the induced bounded  $C_0$ -semigroup of left multiplications by  $T(t)$  on the Banach space  $\mathcal{X}$  of operators  $S \in L(X)$  such that  $t \mapsto T(t)S$  is norm-continuous on  $\mathbb{R}_+$ . Let  $H$  be the negative generator of this  $C_0$ -semigroup on  $\mathcal{X}$ . It is easily seen that  $\sigma(H) \subseteq \sigma(A)$ , and that the operators of left multiplication by  $f(A)$ , for  $f \in \mathcal{A}$ , form an  $\mathcal{A}$ -calculus for  $H$ . It is well known and elementary that  $g(A) \in \mathcal{X}$  for all  $g \in \mathcal{L}L^1$ . Applying the result established in the paragraph above to the  $C_0$ -semigroup on  $\mathcal{X}$  shows that  $\lim_{t \rightarrow \infty} \|T(t)f(A)g(A)\| = 0$  for all  $g \in \mathcal{L}L^1$ .

Let  $(e_n)_{n \geq 1}$  be a bounded approximate unit for  $L^1(\mathbb{R}_+)$ , and let  $g_n = \mathcal{L}e_n$  for  $n \geq 1$ . Then  $(g_n)_{n \geq 1}$  is a bounded approximate unit for  $\mathcal{L}L^1$  and hence for the closure of  $\mathcal{L}L^1$  in  $\mathcal{A}$ ; see Section 2. Thus  $\lim_{n \rightarrow \infty} \|fg_n - f\|_{\mathcal{B}} = 0$ , and so  $\lim_{n \rightarrow \infty} \|f(A)g_n(A) - f(A)\| = 0$ . Since the semigroup  $(T(t))_{t \geq 0}$  is bounded and  $\lim_{t \rightarrow \infty} \|T(t)f(A)g_n(A)\| = 0$  for all  $n \geq 1$ , it follows that  $\lim_{t \rightarrow \infty} \|T(t)f(A)\| = 0$ . ■

REMARKS 3.4. (a) The above proof, without the final paragraph, shows that  $\lim_{t \rightarrow \infty} T(t)f(A) = 0$  in the strong operator topology and

$$\lim_{t \rightarrow \infty} \|T(t)f(A)g(A)\| = 0$$

for all  $g \in \mathcal{L}L^1$  even without the assumption that  $f$  vanishes at infinity.

(b) If  $\sigma(A) \cap i\mathbb{R}$  is bounded then the application of Lemma 3.2 in the proof of Theorem 2.1 may be replaced by an application of Lemma 3.1.

(c) In Theorem 1.2, where  $\mathcal{A} = \mathcal{B}$  and  $f \in \mathcal{B}$ , the condition that  $f$  vanishes on  $\sigma(A) \cap i\mathbb{R}$  is necessary for the conclusion. Indeed, let  $-A$  be the generator of a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  and assume that  $A$  has a bounded  $\mathcal{B}$ -calculus. Let  $f \in \mathcal{B}$ , and suppose that  $\lim_{t \rightarrow \infty} \|T(t)f(A)\| = 0$ . Let  $z \in \sigma(A) \cap i\mathbb{R}$ . Then  $e^{tz}f(z) \in \sigma(T(t)f(A))$  for all  $t \geq 0$  by the spectral inclusion theorem for the  $\mathcal{B}$ -calculus [3, Theorem 4.17], and hence  $|f(z)| \leq \|T(t)f(A)\| \rightarrow 0$  as  $t \rightarrow \infty$ . It follows that  $f(z) = 0$ , as required.

(d) In Theorem 2.1, we have assumed that  $\mathcal{A}$  contains  $\mathcal{B}$ , and so contains the constant functions. It would suffice to assume that  $\mathcal{A}$  contains the subalgebra  $\mathcal{B}_0 := \{f \in \mathcal{B} : \lim_{\operatorname{Re} z \rightarrow \infty} f(z) = 0\}$ , as one may pass to the algebra obtained by adjoining the constant functions.

**4. Examples.** Here we briefly describe two Banach algebras introduced by Arnold and Le Merdy [1], to which Theorem 2.1 can be applied. Arnold and Le Merdy introduce a Banach algebra  $\mathcal{A}(\mathbb{C}_+)$  which is continuously included in  $H^\infty(\mathbb{C}_+)$ , they identify a Banach subalgebra  $\mathcal{A}_0(\mathbb{C}_+)$  and they show that it is the closure of the algebra  $\mathcal{L}L^1$  in  $\mathcal{A}(\mathbb{C}_+)$  [1, Lemma 3.14]. They also show that  $\mathcal{B}_0$  is properly and continuously included in  $\mathcal{A}(\mathbb{C}_+)$  [1, Proposition 5.2, Theorem 5.3]. (Readers should be aware that our spaces  $\mathcal{B}_0$  and  $\mathcal{B}_{00}$  are denoted by  $\mathcal{B}(\mathbb{C}_+)$  and  $\mathcal{B}_0(\mathbb{C}_+)$ , respectively, in [1].) In finding the Banach algebra  $\mathcal{A}(\mathbb{C}_+)$  with these properties, the authors were guided by Peller's work in the discrete case [17]. It is not easy to identify specific functions which are in  $\mathcal{A}(\mathbb{C}_+)$  but not in  $\mathcal{B}$ .

In a further paper Arnold, Le Merdy and Zadeh [2] introduce a Banach algebra  $\mathcal{A}_S(\mathbb{C}_+)$  such that

$$\mathcal{A}(\mathbb{C}_+) \subsetneq \mathcal{A}_S(\mathbb{C}_+) \subsetneq H^\infty(\mathbb{C}_+),$$

with continuous inclusions. The closure of  $\mathcal{L}L^1$  in  $\mathcal{A}_S(\mathbb{C}_+)$  is an identified subalgebra  $\mathcal{A}_{0,S}(\mathbb{C}_+)$ . Again the definitions are complex, and finding explicit functions is not easy.

Let  $-A$  be the generator of a bounded  $C_0$ -semigroup on a Hilbert space. In [1] it is shown that  $A$  has a (unique) bounded  $\mathcal{A}(\mathbb{C}_+)$ -calculus, and in [2] that  $A$  has a bounded  $\mathcal{A}_S(\mathbb{C}_+)$ -calculus. Hence Corollary 1.3 can be extended to the following.

**COROLLARY 4.1.** *Let  $-A$  be the generator of a bounded  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on a Hilbert space. Let  $f \in \mathcal{A}_{0,S}(\mathbb{C}_+)$ , and assume that  $f$  vanishes on  $\sigma(A) \cap i\mathbb{R}$ . Then  $\lim_{t \rightarrow \infty} \|T(t)f(A)\| = 0$ .*

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