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Explicit rank-1 constructions for irrational rotations

by

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Abstract. Let $\theta \in (0,1)$ be an irrational number and let $\lambda := e^{2\pi i\theta}$. For each *well approximable* irrational θ, we provide an explicit rank-1 construction of the λ-rotation R_λ on the circle $\mathbb{T}$. This solves "almost surely" a problem by del Junco. For *every* irrational θ, we construct explicitly a rank-1 transformation with an eigenvalue λ. For every irrational θ, two infinite σ-finite invariant measures μ_λ and μ'_λ on $\mathbb{T}$ are constructed explicitly such that $(\mathbb{T},\mu_\lambda,R_\lambda)$ is *rigid* and of rank 1 and $(\mathbb{T},\mu'_\lambda,R_\lambda)$ is of *zero type* and of rank 1. The centralizer of the latter system consists of just the powers of R_λ. Some versions of the aforementioned results are proved under an extra condition on boundedness of the sequence of cuts in the rank-1 construction.

1. Introduction. By a *dynamical system* we mean a quadruple $(X,\mathfrak{B},\mu,T)$, where $(X,\mathfrak{B})$ is a standard Borel space, μ is a σ-finite measure on $\mathfrak{B}$, and T is an invertible μ-preserving transformation of X. The dynamical system (or just T) is called *of rank* 1 if there is a sequence of finite T-Rokhlin towers that approximates the subring of subsets of finite measure in $\mathfrak{B}$. There is an alternative (explicit) definition of a rank-1 system via an inductive construction process of cutting-and-stacking with a single tower at each step. It is completely determined by two underlying sequences of *cuts* and *spacer mappings*. For details and for the equivalence of various definitions of rank-1 we refer to [Fe].

Let $\theta \in (0,1)$ be an irrational number and let $\lambda := e^{2\pi i\theta}$. Denote by R_λ the λ-rotation on the circle $\mathbb{T}$. It was shown by Del Junco [dJ2] that R_λ is of rank 1 if $\mathbb{T}$ is furnished with the Haar measure. He also raised a related (more subtle) problem in [dJ1]:

PROBLEM I. *Given θ, provide an explicit construction* (*i.e. find sequences of cuts and spacer mappings*) *of a rank-*1 *transformation which is isomorphic to R_λ.*

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A solution of Problem I for an uncountable subset of well approximable irrationals of zero Lebesgue measure was found recently in [Dr–Si].

We also consider a weak version of Problem I.

PROBLEM II. *Given θ, provide an explicit construction of a rank-1 probability preserving transformation T which has an eigenvalue λ.*

We recall that a number $\lambda \in \mathbb{T}$ is an eigenvalue of $(X, \mathfrak{B}, \mu, T)$ if there is a Borel function $f : X \to \mathbb{T}$ such that $f \circ T = \lambda f$ almost everywhere. This implies that R_λ is a factor of T. Hence R_λ is isomorphic to T if and only if f is one-to-one. Del Junco solved Problem II for a.e. $\theta \in (0,1)$ in [dJ1].

We now state the main results of the first part (related to the probability preserving systems) of the present paper.

MAIN RESULT A.

- *Problem II is solved for every θ.*
- *Problem I is solved for each well approximable θ.*
- *For almost all $\theta \in (0,1)$, including the badly approximable reals and the algebraic numbers, we solve Problem II in the subclass of rank-1 transformations with only two cuts at every step of their inductive construction.*

As the subset of well approximable reals from $(0,1)$ is of Lebesgue measure 1, Problem I is solved "almost surely".

In connection with the third point of Main Result A, we note that Problem II (and hence Problem I) cannot be solved for any irrational θ in the subclass of rank-1 transformations *with bounded parameters* (1), as the eigenvalues of every such transformation are of finite order (see [El–Ru, Theorem 3] or [Da5, Theorem M]). We also provide a short alternative proof of this fact.

In the second part of the paper we consider Problem I within the class of infinite measure preserving dynamical systems. The main difference from the probability preserving case is that for each irrational θ, there exist *uncountably many* mutually disjoint R_λ-invariant infinite σ-finite measures on $\mathbb{T}$ (see, e.g., [Sc]). Infinite measure preserving *rank*-1 rotations on $\mathbb{T}$ were under study in a recent paper [Dr–Si]. Construction of the rank-1 systems there is based on the same cutting-and-stacking algorithm as in [dJ1] but without the spacer growth restriction (to obtain infinite measure). For a.e. $\theta \in (0,1)$, an infinite measure m_θ on $\mathbb{T}$ was constructed in [Dr–Si] such that the system $(\mathbb{T}, m_\theta, R_\lambda)$ is of rank 1 with explicit cutting-and-stacking parameters. Moreover, it was shown that the system is rigid if and only if θ is well approximable. We generalize and sharpen those results in the following two theorems.

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(1) This means that the number of cuts and the total number of spacers added at the nth step of the construction are both uniformly bounded in n.

Theorem B. *Let $\lambda \in \mathbb{T}$ be of infinite order. Then there is an infinite σ-finite R_λ-invariant non-atomic Borel measure μ_λ on $\mathbb{T}$ such that*

- *the dynamical system $(\mathbb{T}, \mu_\lambda, R_\lambda)$ is of rank* 1*, the parameters of the underlying cutting-and-stacking construction are explicitly described,*
- *the number of cuts at each step of the construction is* 2*,*
- *$(\mathbb{T}, \mu_\lambda, R_\lambda)$ is rigid, hence the centralizer $C(R_\lambda)$ of R_λ is uncountable.*

Theorem C. *For each element $\lambda \in \mathbb{T}$ of infinite order, there is an infinite σ-finite R_λ-invariant non-atomic Borel measure μ'_λ on $\mathbb{T}$ such that*

- *the dynamical system $(\mathbb{T}, \mu'_\lambda, R_\lambda)$ is of rank* 1*, the parameters of the underlying cutting-and-stacking construction are explicitly described,*
- *$(\mathbb{T}, \mu'_\lambda, R_\lambda)$ is totally ergodic and of zero type,*
- *$C(R_\lambda) = \{R_\lambda^n \mid n \in \mathbb{Z}\}$,*
- *$\mu'_\lambda \circ R_\beta \perp \mu'_\lambda$ whenever $\beta \notin \{\lambda^n \mid n \in \mathbb{Z}\}$,*
- *if an element $\omega \in \mathbb{T}$ is of infinite order with $\omega \notin \{\lambda, \lambda^{-1}\}$ then $\mu'_\omega \perp \mu'_\lambda$.*

As far as we know, Theorem C provides the first examples of *spectrally mixing* (2) ergodic infinite invariant measures for irrational rotations.

Everywhere below in this paper we construct the rank-1 systems via the (C, F)-construction. It was introduced in [dJ3] and [Da1] (in a different form). Various kinds of the (C, F)-construction, interrelationship among them and the classical cutting-and-stacking are discussed in detail in [Da3].

The outline of the present paper is as follows. Preliminary information from the theory of continuous fractions, dynamical systems and (C, F)-construction is collected in §2. In §3 we study eigenvalues and eigenfunctions of the (C, F)-systems. Main Result A is proved in §4 and §5 is devoted to the proof of Theorems B and C.

2. Preliminaries

Continued fractions. We recall some basic facts from the theory of continued fractions. Every irrational number θ can be represented as an infinite continued fraction $[a_0; a_1, a_2, \ldots]$ with $a_j \in \mathbb{N}$ for each $j > 0$. The rational numbers $p_k/q_k := [a_0; a_1, a_2, \ldots, a_k]$ with $(p_k, q_k) = 1$ are called the *convergents* for θ. For each $k \geq 2$,

$$p_k = a_{k-1} q_{k-1} + p_{k-2}, \quad p_0 = 1, \quad p_1 = a_0,$$
$$q_k = a_{k-1} q_{k-1} + q_{k-2}, \quad q_0 = 0, \quad q_1 = 1.$$

(2) An infinite measure preserving transformation S is of zero type if and only if the measure of maximal spectral type of S is Rajchman, i.e. the Koopman operator associated with S is mixing.

Definition 2.1. An irrational number θ is called *badly approximable* if the sequence $(a_k)_{k=1}^\infty$ is bounded. The irrational numbers that are not badly approximable are called *well approximable.*

We will utilize the following well known results from the theory of continuous fractions (see [Kh] for the proof).

Fact 2.2.

(i) $(q_n\theta - p_n)(q_{n+1}\theta - p_{n+1}) < 0$ *for each* $n > 0$.

(ii) $\dfrac{1}{q_n + q_{n+1}} < |\theta q_n - p_n| < \dfrac{1}{q_{n+1}}$ *for each* $n > 0$.

(iii) $|q_n\theta - p_n| < \min\{|b\theta - a| \mid \mathbb{N} \ni b \leq q_n,\ a \in \mathbb{Z},\ a/b \neq p_n/q_n\}$ *for each* $n > 0$.

(iv) *If there exist* $p, q \in \mathbb{N}$ *such that*

$$|q\theta - p| < \min\left\{|b\theta - a| \,\middle|\, \mathbb{N} \ni b \leq q,\ a \in \mathbb{Z},\ \frac{a}{b} \neq \frac{p}{q}\right\}$$

then $p/q = p_n/q_n$ *for some* $n \in \mathbb{N}$.

(v) θ *is badly approximable if and only if there exists a real* $\delta > 0$ *such that* $\min_{p\in\mathbb{Z}} |q\theta - p| > \delta/q$ *for each* $q \in \mathbb{N}$.

(vi) *If* θ *is algebraic of power* n *then there exists a real* $\delta > 0$ *such that* $\min_{p\in\mathbb{Z}} |q\theta - p| > \delta/q^n$ *for each* $q \in \mathbb{N}$.

(vii) *The set of badly approximable numbers has Lebesgue measure zero.*

Dynamical systems. We recall that a dynamical system $(X, \mathfrak{B}, \mu, T)$ (or just T) is called

- *ergodic* if each T-invariant subset is either μ-null or μ-conull;
- *totally ergodic* if T^p is ergodic for each $p \in \mathbb{N}$;
- *rigid* if there is a sequence $n_1 < n_2 < \cdots$ such that
$$\mu(T^{n_k}A \cap B) \to \mu(A \cap B) \quad \text{as } k \to \infty$$
for all subsets $A, B \in \mathfrak{B}$ with $\mu(A) < \infty$ and $\mu(B) < \infty$;
- *of zero-type* if
$$\mu(T^n A \cap B) \to 0 \quad \text{as } n \to \infty$$
for all subsets $A, B \in \mathfrak{B}$ with $\mu(A) < \infty$ and $\mu(B) < \infty$;
- *of rank* 1 if there are subsets $B_1, B_2, \dots$ of X with $\mu(B_k) < \infty$ for each $k \in \mathbb{N}$ and a sequence of positive integers $n_1 < n_2 < \cdots$ such that $T^l B_k \cap T^m B_k = \emptyset$ whenever $0 \leq l < m < n_k$ and $k \in \mathbb{N}$ and, for each subset $A \in \mathfrak{B}$ of finite measure,
$$\lim_{k\to\infty} \min_{J\subset\{0,\dots,n_k-1\}} \mu\Big(A \triangle \bigsqcup_{j\in J} T^j B_k\Big) = 0.$$

Of course, if T is of zero-type then $\mu(X) = \infty$. If T is of rank 1 then T is ergodic.

Definition 2.3. Suppose that T is ergodic. A number $\lambda \in \mathbb{T}$ is called an *eigenvalue* of T if there is a measurable function $f : X \to \mathbb{T}$ such that $f \circ T = \lambda f$. The function f is called a λ-*eigenfunction* of T. It is defined up to a multiplicative constant from $\mathbb{T}$. The set of all eigenvalues is called the L^∞-*spectrum* of T and denoted by $e(T)$.

Of course, $e(T)$ is a subgroup of $\mathbb{T}$. If $\mu(X) < \infty$ then $e(T)$ is countable. It is straightforward to verify that T is totally ergodic if and only if $e(T)$ is torsion free.

For the λ-rotation R_λ on $\mathbb{T}$ endowed with the Haar measure, $e(R_\lambda) = \{\lambda^n \mid n \in \mathbb{Z}\}$.

The *centralizer* $C(T)$ of T is the set of invertible μ-preserving transformations that commute with T. Of course, $C(T)$ is a group.

(C,F)-dynamical systems. For a detailed exposition of the (C,F)-construction of (funny) rank-1 actions we refer to [Da1] and [Da3]. Let $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ be two sequences of finite subsets in $\mathbb{Z}$ such that for each $n > 0$,

$$F_0 = \{0\}, \quad \#C_n > 1, \tag{2.1}$$

$$F_n + C_{n+1} \subset F_{n+1}, \tag{2.2}$$

$$(F_n + c) \cap (F_n + c') = \emptyset \quad \text{if } c, c' \in C_{n+1} \text{ and } c \neq c'. \tag{2.3}$$

We let $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$ and endow this set with the infinite product topology. Then X_n is a compact Cantor space. The mapping

$$X_n \ni (f_n, c_{n+1}, c_{n+2} \ldots) \mapsto (f_n + c_{n+1}, c_{n+2}, \ldots) \in X_{n+1}$$

is a topological embedding of X_n into X_{n+1}. Therefore the inductive limit X of the sequence $(X_n)_{n\geq 0}$ furnished with these embeddings is well defined. Moreover, X is a locally compact Cantor space. Given a subset $A \subset F_n$, we let

$$[A]_n := \{x = (f_n, c_{n+1}, \ldots) \in X_n \mid f_n \in A\}$$

and call this set an n-*cylinder* in X. It is open and compact in X. For brevity, we will write $[f]_n$ for $[\{f\}]_n$, $f \in F_n$.

Also, we will write $\mathbf{0}$ for $(0, 0, \ldots) \in X_0 \subset X$. There exists a unique σ-finite Borel measure μ on X such that $\mu(X_0) = 1$ and

$$\mu([f]_n) = \mu([f']_n) \quad \text{for all } f, f' \in F_n,\ n \geq 0.$$

It is easy to verify that

$$\mu([A]_n) = \frac{\#A}{\#C_1 \cdots \#C_n} \quad \text{for each subset } A \subset F_n,\ n > 0.$$

Returning to the construction of the (C,F)-system, we also note that $\mu(X)<\infty$ if and only if

$$\sum_{n=1}^{\infty}\frac{\#F_{n+1}-\#F_n\#C_{n+1}}{\#F_{n+1}}<\infty. \tag{2.4}$$

From now on,

$$F_n=\{0,1,\dots,h_n-1\}\quad\text{for some } h_n>0 \text{ and every } n\in\mathbb{N}. \tag{2.5}$$

In particular, $h_n=\#F_n$. Then we can define a transformation T on X. We first note that for μ-a.e. $x\in X$, there is $n>0$ such that $x=(f_n,c_{n+1},\dots)\in X_n$ and $1+f_n\in F_n$. We now let

$$Tx:=(1+f_n,c_{n+1},\dots)\in X_n\subset X.$$

Then T is a well defined μ-preserving transformation of X. We call (X,μ,T) the (C,F)-*dynamical system associated with the sequence* $(C_n,F_{n-1})_{n\ge 0}$. This system is of rank 1.

Remark 2.4. We recall the connection between the (C,F)-construction and the classical cutting-and-stacking construction of rank-1 transformations. A *tower* $\mathcal{A}$ is an ordered finite collection of pairwise disjoint intervals (called the *levels* of $\mathcal{A}$) in $\mathbb{R}$, each of the same Lebesgue measure. We think of the levels in a tower as being stacked on top of each other, so that the $(j+1)$st level is directly above the jth level. Every tower $\mathcal{A}=(I_j)_j$ is associated with a natural tower map $T_{\mathcal{A}}$ sending each point in a level I_j to the point directly above it in I_{j+1}. A *rank*-1 *cutting-and-stacking construction* of a measure preserving transformation T consists of a sequence $(\mathcal{A}_n)_{n\ge 0}$ of towers such that $\mathcal{A}_0$ is a single interval $[0,1)$, each tower $\mathcal{A}_{n+1}$ is obtained from $\mathcal{A}_n$ by cutting $\mathcal{A}_n$ into $r_n\ge 2$ subtowers of equal width, adding some number $\sigma_n(k)$ of new levels (called *spacers*) above the kth subcolumn, $k=1,\dots,r_n$, and stacking every subtower over the subtower to its left. We note the spacers are intervals drawn from $\mathbb{R}$ that are disjoint from the levels of $\mathcal{A}_n$ and the other spacers added to it. They are of the same length as the levels of the subcolumns of $\mathcal{A}_n$. It is easy to see that the restriction of $T_{\mathcal{A}_{n+1}}$ to $\mathcal{A}_n$ coincides with $T_{\mathcal{A}_n}$ for each n. We now set $X:=\bigcup_{n\ge 0}\bigsqcup_{I\in\mathcal{A}_n}I$, endow X with the Lebesgue measure and define T to be the pointwise limit of $T_{\mathcal{A}_n}$ as $n\to\infty$. Then T is a measure preserving invertible transformation of X. Note that T is completely defined by the sequence $(r_n)_{n\ge 0}$ of "cuts" and the sequence $(\sigma_n)_{n\ge 0}$ of "spacer" maps $\sigma_n:\{1,\dots,r_n\}\ni k\mapsto\sigma_n(k)\in\mathbb{Z}_+$. We show that T is isomorphic to the (C,F)-dynamical system in such a way that $\mathcal{A}_n=\{[f_n]_n\mid f_n\in F_n\}$ and $r_n=\#C_{n+1}$ for each $n\ge 0$. Thus, $\#F_n$ is the height and $\#C_{n+1}$ is the number of cuts of the nth tower. For that, we define the sequence $(F_n,C_{n+1})_{n=0}^{\infty}$ recurrently by setting $h_0:=0$ and, for each $n \geq 0$,

$$
(2.6)\qquad
\begin{aligned}
h_{n+1} &:= r_n h_n + \sum_{k=1}^{r_n} \sigma_n(k),\\
C_{n+1} &:= \Big\{ j h_n + \sum_{k=1}^{j} \sigma_n(k) \;\Big|\; j = 0, \dots, r_n - 1 \Big\}.
\end{aligned}
$$

It is straightforward to check that T is isomorphic to the (C,F)-system associated with $(F_n, C_{n+1})_{n=0}^{\infty}$. Conversely, given a (C,F)-system associated with a sequence $(F_n, C_{n+1})_{n=0}^{\infty}$ such that $F_n = \{0, \dots, h_n - 1\}$, we can "recover" the sequences $(r_n)_{n\geq 1}$ of cuts and $(\sigma_n)_{n\geq 1}$ of spacer maps such that (2.6) holds. We leave the details to the reader.

We will need the following two facts about T.

FACT 2.5. *If there is an infinite sequence $n_1 < n_2 < \cdots$ such that $\#C_{n_k} \to \infty$ as $k \to \infty$ and C_{n_k} is an arithmetic progression then T is rigid.*

FACT 2.6. *If for each $n > 0$,*

- $F_n + F_n + C_{n+1} \subset F_{n+1}$,
- *the sets $F_n - F_n + c - c'$, $c \neq c' \in C_{n+1}$, and $F_n - F_n$ are all mutually disjoint,*
- $\#C_n \to \infty$ *as* $n \to \infty$,

then T is of zero type.

Fact 2.5 is well known (see, for instance, [Da4, proof of Theorem 0.1]). A proof of Fact 2.6 can be obtained as a slight modification of [Da2, proof of Theorem 6.1].

We also consider an equivalence relation $\mathcal{R}$ on X:

$$
(2.7)\qquad (x,y) \in \mathcal{R} \iff \exists n > 0 \text{ such that } x = (f_n, c_{n+1}, \dots) \in X_n,\quad y = (f'_n, c'_{n+1}, \dots) \in X_n, \text{ and } c_j = c'_j \text{ for each } j > n.
$$

We call $\mathcal{R}$ the (C,F)-*equivalence relation* on X. The T-orbit equivalence relation coincides with $\mathcal{R}$ reduced to a μ-conull subset.

3. Eigenfunctions of (C,F)-equivalence relations. Let $\mathcal{R}$ be the (C,F)-equivalence relation on X defined by (2.7). We now define a Borel mapping $d : \mathcal{R} \to \mathbb{Z}$ by setting

$$
d(x,y) := f_n - f'_n \qquad \text{for each } (x,y) \in \mathcal{R}.
$$

It is straightforward to verify that

$$
d(x,y) + d(y,z) = d(x,z) \qquad \text{for all } (x,y), (y,z) \in \mathcal{R}.
$$

In other words, d is a Borel cocycle of $\mathcal{R}$ with values in $\mathbb{Z}$.

Definition 3.1. We call a complex number $\lambda \in \mathbb{T}$ an *eigenvalue* of $\mathcal{R}$ if the cocycle $\mathcal{R} \ni (x,y) \mapsto \lambda^{d(x,y)} \in \mathbb{T}$ is a coboundary, i.e. there exists a Borel map $\varphi : X \to \mathbb{T}$ and a subset $A \subset X$ such that $\mu(A) = 0$ and

$$\lambda^{d(x,y)} = \varphi(x)\varphi(y)^{-1} \quad \text{for all } (x,y) \in \mathcal{R} \text{ with } x, y \notin A.$$

We call φ a λ-*eigenfunction* of $\mathcal{R}$. It is defined up to a multiplicative constant. We denote by $e(\mathcal{R})$ the set of all eigenvalues of $\mathcal{R}$.

Remark 3.2. Let $(X, \mathfrak{B}, \mu, T)$ be a (C,F)-dynamical system and let $\mathcal{R}$ be the (C,F)-equivalence relation on X. Then it is straightforward to verify that $e(T) = e(\mathcal{R})$.

Continuous eigenfunctions. In this subsection we study only continuous eigenfunctions of $\mathcal{R}$ and the corresponding eigenvalues.

Proposition 3.3. *Let $\lambda \in e(\mathcal{R})$. If a λ-eigenfunction φ is continuous at a point, then it is continuous everywhere on X and for each sequence $(c_k)_{k=1}^\infty \in C_1 \times C_2 \times \cdots$, the series $\prod_{k=1}^\infty \lambda^{c_k}$ converges. Moreover, for each $x = (f_n, c_{n+1}, c_{n+2}, \dots) \in X_n \subset X$, we have*

$$\varphi(x) = \varphi(\mathbf{0})\, \lambda^{f_n} \prod_{k=n+1}^{\infty} \lambda^{c_k}.$$

Proof. We will only consider the case where φ is continuous at $\mathbf{0}$; the other cases are considered in a similar way. Take a point $x \in X$. Then there is $n > 0$ such that $x \in X_n$ and $x = (f_n, c_{n+1}, c_{n+2}, \dots)$ for some $f_n \in F_n$ and $c_k \in C_k$ if $k > n$. We now set

$$p_k(x) := (0, \dots, 0, c_{k+1}, c_{k+2}, \dots) \in X_n$$

for each $k \geq n$. Then $p_k(x) \to \mathbf{0}$ as $k \to \infty$. So $\varphi(p_k(x)) \to \varphi(\mathbf{0})$ as $k \to \infty$. On the other hand, $(x, p_k(x)) \in \mathcal{R}$ and $d(x, p_k(x)) = f_n + c_{n+1} + \cdots + c_k$. We now have

$$\lambda^{f_n + c_{n+1} + \cdots + c_k} = \lambda^{d(x, p_k(x))} = \varphi(x)\varphi(p_k(x))^{-1}.$$

The right-hand side of this formula tends to $\varphi(x)\varphi(\mathbf{0})^{-1}$ as $k \to \infty$. Hence the limit of the left-hand side exists. Of course, this limit equals $\lambda^{f_n} \prod_{k>n} \lambda^{c_k}$, as desired.

It remains to prove that φ is continuous at x. Given $z \in \mathbb{T}$, we write $\operatorname{Arg} z = \tau$ if $z = e^{i\tau}$ and $-\pi < \tau \leq \pi$. For each $k \in \mathbb{N}$, we select $a_k, b_k \in C_k$ so that

$$\max_{c \in C_k} \operatorname{Arg} \lambda^c = \operatorname{Arg} \lambda^{a_k} \quad \text{and} \quad \min_{c \in C_k} \operatorname{Arg} \lambda^c = \operatorname{Arg} \lambda^{b_k}.$$

Since the series $\prod_{k=1}^\infty \lambda^{a_k}$ and $\prod_{k=1}^\infty \lambda^{b_k}$ converge, we can find, for each $\epsilon > 0$, a number $N > 0$ such that

$$\left|\operatorname{Arg}\prod_{k=l}^{\infty}\lambda^{a_k}\right| < \epsilon \quad \text{and} \quad \left|\operatorname{Arg}\prod_{k=l}^{\infty}\lambda^{b_k}\right| < \epsilon \quad \text{whenever } l > N.$$

Therefore, if $(y_l, y_{l+1}, \dots) \in C_l \times C_{l+1} \times \cdots$ then

$$-\epsilon < \operatorname{Arg}\prod_{k=l}^{\infty}\lambda^{b_k} \le \operatorname{Arg}\prod_{k=l}^{\infty}\lambda^{y_k} \le \operatorname{Arg}\prod_{k=l}^{\infty}\lambda^{a_k} < \epsilon.$$

Let $f_l := f_n + c_{n+1} + \cdots + c_l$. Then $x \in [f_l]_l$ and $[f_l]_l$ is a compact open neighborhood of x. Hence, for each $y = (f_l, y_{l+1}, y_{l+2}, \dots) \in [f]_l$,

$$\begin{aligned}|\varphi(x) - \varphi(y)| &= \left|\varphi(0)\lambda^{f_l}\prod_{k>l}\lambda^{c_k} - \varphi(0)\lambda^{f_l}\prod_{k>l}\lambda^{y_k}\right| = \left|\prod_{k>l}\lambda^{c_k} - \prod_{k>l}\lambda^{y_k}\right| \\ &\le \left|\operatorname{Arg}\prod_{k>l}\lambda^{b_k} - \operatorname{Arg}\prod_{k>l}\lambda^{a_k}\right| < 2\epsilon. \ \blacksquare\end{aligned}$$

We note that the reasoning above also proves the converse to Proposition 3.3.

Proposition 3.4. *If $\lambda \in \mathbb{T}$ and the series $\prod_{k=1}^{\infty}\lambda^{c_k}$ converges for each sequence $(c_k)_{k=1}^{\infty}$ with $c_k \in C_k$ for every $k > 0$ then a function $\varphi : X \to \mathbb{T}$ is well defined by the formula*

$$X \supset X_n \ni x = (f_n, c_{n+1}, c_{n+2}, \dots) \mapsto \varphi(x) := \lambda^{f_n}\prod_{k=n+1}^{\infty}\lambda^{c_k}. \tag{3.1}$$

This function is continuous on X. Moreover, $\lambda \in e(\mathcal{R})$ and φ is a λ-eigenfunction of $\mathcal{R}$.

Corollary 3.5. *If $\sum_{n=1}^{\infty}\max_{c\in C_n}|1 - \lambda^c| < \infty$ then $\lambda \in e(\mathcal{R})$ and the function φ defined by* (3.1) *is a continuous λ-eigenfunction of $\mathcal{R}$.*

Proof. We note that

$$|1 - \lambda^{\sum_{j=1}^{n} a_n}| \le \sum_{j=1}^{n}|1 - \lambda^{a_j}|$$

for arbitrary $a_1, \dots, a_n \in \mathbb{Z}$. Hence the condition of the corollary and the Cauchy criterion of convergence imply that the series $\prod_{k=1}^{\infty}\lambda^{c_k}$ converges for each sequence $(c_k)_{k=1}^{\infty}$ with $c_k \in C_k$ for every $k > 0$. It remains to apply Proposition 3.4. ■

We now provide a sufficient condition for the existence of one-to-one eigenfunctions of $\mathcal{R}$.

Proposition 3.6. *Let $\lambda \in \mathbb{T}$ be of infinite order. Suppose that for each $n \geq 0$,*

$$\min_{f \neq f' \in F'_n} |1 - \lambda^{f-f'}| > \sum_{k>n} \max_{c,c' \in C_k} |1 - \lambda^{c-c'}|.$$

Then $\lambda \in e(\mathcal{R})$ and each λ-eigenfunction is continuous and one-to-one.

Proof. It follows from the condition of the proposition that

$$\sum_{k=1}^{\infty} \max_{c \in C_k} |1 - \lambda^c| < \infty.$$

Hence $\lambda \in e(\mathcal{R})$ by Corollary 3.5. Let φ be the λ-eigenfunction of $\mathcal{R}$ defined by (3.1). It is continuous by Corollary 3.5. Suppose that $\varphi(x) = \varphi(y)$ for some $x, y \in X$ and $x \neq y$. Then there is $n > 0$ such that $x, y \in X_n$, $x = (f_n, c_{n+1}, c_{n+2}, \dots)$, $y = (f'_n, c'_{n+1}, c'_{n+2}, \dots)$ and $f_n \neq f'_n \in F_n$. It follows from (3.1) that

$$\lambda^{f_n} \prod_{k>n} \lambda^{c_k} = \lambda^{f'_n} \prod_{k>n} \lambda^{c'_k}, \text{ i.e. } \lambda^{f'_n - f_n} = \prod_{k>n} \lambda^{c_k - c'_k}.$$

Hence

$$|1 - \lambda^{f'_n - f_n}| = \Big|1 - \prod_{k>n} \lambda^{c_k - c'_k}\Big| \leq \sum_{k>n} |1 - \lambda^{c_k - c'_k}|.$$

This contradicts the condition of the proposition. Thus, φ is one-to-one. An arbitrary λ-eigenfunction is also one-to-one because it is proportional to φ. ∎

We will need one more sufficient condition for the existence of one-to-one eigenfunctions. It is a close analogue of Proposition 3.6. Since it is proved in a very similar way to Proposition 3.6, we state it without proof ([3]).

Proposition 3.7. *Let $\lambda \in \mathbb{T}$ be of infinite order. If for each $n > 0$,*

$$\min_{f \neq f' \in F_n} |\operatorname{Arg} \lambda^{f'-f}| > \sum_{k>n} \max_{c,c' \in C_k} |\operatorname{Arg} \lambda^{c-c'}|,$$

then $\lambda \in e(\mathcal{R})$ and each λ-eigenfunction is one-to-one.

Unfortunately, we were unable to figure out if Proposition 3.7 is equivalent to Proposition 3.6.

Measurable eigenfunctions. We now consider the general case, provide an eigenvalue criterion and describe the structure of arbitrary measurable eigenfunctions of $\mathcal{R}$.

Proposition 3.8. *Let $\lambda \in \mathbb{T}$. Then $\lambda \in e(\mathcal{R})$ if and only if for each $\epsilon > 0$, there is $n > 0$ such that for every $m \geq n$, there exists a subset $E_{n,m} \subset C_n + \cdots + C_m$ satisfying the following:*

([3]) We only note that $|\operatorname{Arg} \lambda^s| \geq |1 - \lambda^s|$ for each $s \in \mathbb{Z}$.

(i) $\dfrac{\#E_{n,m}}{\#C_n\cdots\#C_m} > 1-\epsilon$,

(ii) $\max_{c,c'\in E_{n,m}} |1-\lambda^{c-c'}| < \epsilon$.

Proof. Let $\lambda \in e(\mathcal{R})$ and let φ be a λ-eigenfunction. Given $\epsilon > 0$, there are $w \in \mathbb{T}$ and a subset $A \subset X$ of positive measure such that

$$|\varphi(x) - w| < \epsilon/2 \quad \text{for each } x \in A. \tag{3.2}$$

Then we can find $n > 0$ and $f \in F_{n-1}$ such that

$$\mu(A \cap [f]_{n-1}) > (1-\epsilon^2)\mu([f]_{n-1}).$$

Let $E_{n,m} := \{c \in C_n + \cdots + C_m \mid \mu(A \cap [f+c]_m) > (1-\epsilon)\mu([f+c]_m)\}$. Since $[f]_{n-1} = \bigsqcup_{c\in C_n+\cdots+C_m}[f+c]_m$, it follows that $\frac{\#E_{n,m}}{\#C_n\cdots\#C_m} > 1-\epsilon$. Without loss of generality we may assume that $\epsilon < 0.1$. For each pair $c, c' \in E_{n,m}$, we let $B := T^{c-c'}(A \cap [f+c]_m) \cap (A \cap [f+c']_m)$. Then $\mu(B) > 0$, $B \subset A$ and $T^{c'-c}B \subset A$. Select $x \in B$ such that $\varphi(T^{c'-c}x) = \lambda^{c'-c}\varphi(x)$. Since $T^{c'-c}x \in A$, from (3.2) we have

$$|\varphi(T^{c'-c}x) - w| < \epsilon/2.$$

This equality and (3.2) yield

$$\epsilon > |\varphi(x) - \varphi(T^{c'-c}x)| = |\varphi(x) - \lambda^{c'-c}\varphi(x)| = |1-\lambda^{c-c'}|,$$

as desired.

Conversely, suppose that (i) and (ii) are satisfied. Then we can construct an infinite sequence $n_1 < n_2 < \cdots$ such that for each $k > 0$,

(iii) $\dfrac{\#E_{n_k,n_{k+1}-1}}{\#C_{n_k}\cdots\#C_{n_{k+1}-1}} > 1 - \dfrac{1}{2^k}$,

(iv) $\displaystyle\max_{c,c'\in E_{n_k,n_{k+1}-1}} |1-\lambda^{c-c'}| < \frac{1}{2^k}$.

We note that the mapping

$$(c_{n_1}, c_{n_1+1}, \dots) \mapsto ((c_{n_1} + \cdots + c_{n_2-1}), (c_{n_2} + \cdots + c_{n_3-1}), \dots)$$

is a measure preserving isomorphism of the probability space

$$\left([0]_{n_1-1}, \frac{\mu \restriction [0]_{n_1-1}}{\mu([0]_{n_1-1})}\right)$$

onto the infinite product space $\bigotimes_{k\geq 1}(C_{n_k} + \cdots + C_{n_{k+1}-1}, \tau_k)$, where τ_k is the equidistribution on $C_{n_k} + \cdots + C_{n_{k+1}-1}$. Hence, the Borel–Cantelli lemma and (iii) imply that there is a μ-null subset $Y_0 \subset [0]_{n_1-1}$ such that, for each

$$x = (0, c_{n_1}, c_{n_1+1}, \dots) \in [0]_{n_1-1} \setminus Y_0, \tag{3.3}$$

there is $K = K(x) \in \mathbb{N}$ with

$$c_{n_k} + \cdots + c_{n_{k+1}-1} \in E_{n_k,n_{k+1}-1} \quad \text{for each } k > K.$$

Fix an element $(0, c'_{n_1}, c'_{n_1+1}, \dots) \in [0]_{n_1-1}$ with $c'_{n_k} + \dots + c'_{n_{k+1}-1} \in E_{n_k, n_{k+1}-1}$ for each $k > 0$. We now define a map $\phi : [0]_{n_1-1} \setminus Y_0 \to \mathbb{T}$ by setting

$$\phi(x) = \prod_{k=1}^{\infty} \lambda^{(c_{n_k} - c'_{n_k}) + \dots + (c_{n_{k+1}-1} - c'_{n_{k+1}-1})}$$

for each x from (3.3). This is well defined in view of (iv). It is straightforward to verify that if $(x, y) \in \mathcal{R}$ and $x, y \in [0]_{n_1-1} \setminus Y_0$ then

$$\phi(x) = \lambda^{d(x,y)} \phi(y). \tag{3.4}$$

We recall that the cocycle d is defined at the beginning of this section. Thus, λ is an eigenvalue for the restriction of $\mathcal{R}$ to $[0]_{n_1-1}$. Let Y denote the smallest $\mathcal{R}$-invariant subset that includes Y_0. Of course, $\mu(Y) = 0$. Then ϕ extends to the entire X in such a way that (3.4) holds for all $(x, y) \in \mathcal{R} \cap ((X \setminus Y) \times (X \setminus Y))$. Indeed, given $y \in X \setminus Y$, there is (non-unique!) $x \in [0]_{n_1-1} \setminus Y_0$ such that $(x, y) \in \mathcal{R}$. We now define ϕ at x as $\lambda^{d(x,y)} \phi(y)$. It is a routine to verify that ϕ is well defined and (3.4) holds for ϕ on the entire space $X \setminus Y$. Hence $\lambda \in e(\mathcal{R})$. ∎

Corollary 3.9. *Suppose the sequence* $(\#C_n)_{n=1}^{\infty}$ *is bounded. If* $\lambda \in e(\mathcal{R})$ *then* $\max_{c \in C_n} |1 - \lambda^c| \to 0$ *as* $n \to \infty$.

We now consider the case where λ is of finite order.

Corollary 3.10. *Let* $\lambda \in e(\mathcal{R})$ *be of finite order* p. *Then for each* $\epsilon > 0$, *there exists* $n > 0$ *such that for every* $m \geq n$, *there is a subset* $C_m^0 \subset C_m$ *such that* $\frac{\#C_m^0}{\#C_m} > 1 - \epsilon$ *and* p *divides* $c - c'$ *for all* $c, c' \in C_m^0$.

Proof. Fix an arbitrary $\epsilon > 0$. We note that $\lambda^p = 1$. Therefore, for each m and $c, c' \in C_m$, we have $\lambda^{c-c'} = \lambda^{\widetilde{c} - \widetilde{c'}}$, where $0 \leq \widetilde{c} < p$, $0 \leq \widetilde{c'} < p$ and the differences $c - \widetilde{c}$ and $c' - \widetilde{c'}$ are divisible by p. Therefore, if $\epsilon > 0$ then the equality $\lambda^{c-c'} = 1 \pm \epsilon$ implies $\lambda^{\widetilde{c} - \widetilde{c'}} = 1 \pm \epsilon$. As the set $\{\lambda^{\widetilde{c} - \widetilde{c'}} \mid 0 \leq \widetilde{c}, c' < p\}$ is finite and ϵ is arbitrarily small, we obtain $\lambda^{\widetilde{c} - \widetilde{c'}} = 1$, i.e. $p \mid (\widetilde{c} - \widetilde{c'})$ and hence $p \mid (c - c')$ if ϵ is small enough. It remains to apply Proposition 3.8. ∎

4. (C, F)**-systems with finite invariant measure and their eigenfunctions.** In this section we consider rank-1 dynamical systems with finite invariant measure.

Solution of Problem II. Let $\epsilon > 0$. We say that a finite subset $P \subset \mathbb{T}$ is an ϵ-*net* if

$$\max_{z \in \mathbb{T}} \min_{p \in P} |z - p| < \epsilon.$$

Theorem 4.1. *Given an element* $\lambda \in \mathbb{T}$ *of infinite order, there is an explicit* (C, F)*-construction of a rank*-1 *transformation* T *with finite invariant measure and* $\lambda \in e(T)$.

Proof. It is well known (and easy to verify) that for each $n \in \mathbb{N}$, there is a number $j_n \in \{1, \dots, n\}$ such that $\delta_n := |1 - \lambda^{j_n}| < 2\pi/n$. Hence the subset

$$P(\lambda, n) := \left\{ \lambda^{j_n k} \;\middle|\; 1 \le k \le \frac{2\pi}{\delta_n} \right\} \subset \mathbb{T}$$

is a $(2\pi/n)$-net.

We now construct inductively a sequence $(C_n, F_{n-1})_{n=1}^{\infty}$ of subsets in $\mathbb{Z}$. Suppose we have already determined $(C_k, F_k)_{k=1}^{n-1}$ for some $n > 1$. Suppose, in addition, that an auxiliary condition

$$h_{n-1} > \frac{n^4}{\delta_{n^2}} \tag{4.1}$$

is satisfied. Our purpose is to define C_n and F_n (or equivalently h_n). Since $\lambda^{h_{n-1}} \cdot P(\lambda, n^2)$ is a $(2\pi/n^2)$-net, there is a positive integer $k_1 \le 2\pi/\delta_{n^2}$ such that

$$|1 - \lambda^{h_{n-1} + k_1 j_{n^2}}| < \frac{2\pi}{n^2}.$$

We now set $a(1) := h_{n-1} + k_1 j_{n^2}$. In a similar way, there is a positive integer $k_2 \le 2\pi/\delta_{n^2}$ such that

$$|1 - \lambda^{a(1) + h_{n-1} + k_2 j_{n^2}}| < \frac{2\pi}{n^2}.$$

We now set $a(2) := a(1) + h_{n-1} + k_2 j_{n^2}$. Continuing this process infinitely many times we obtain an infinite sequence $a(1), a(2), \dots$ such that

$$a(l) = a(l-1) + h_{n-1} + k_l j_{n^2} \quad \text{for some positive } k_l \le \frac{2\pi}{\delta_{n^2}}, \tag{4.2}$$

$$|1 - \lambda^{a(l)}| < \frac{2\pi}{n^2} \tag{4.3}$$

for each $l > 0$. Let r_n be the smallest l with $a(r_n - 1) + h_{n-1} > \frac{(n+1)^4}{\delta_{(n+1)^2}}$. Then we set

$$C_n := \{0, a(1), a(2), \dots, a(r_n - 1)\}, \qquad h_n := a(r_n - 1) + h_{n-1}.$$

Thus, we have defined C_n and F_n. Moreover, (4.1) holds if we replace $n-1$ with n. Continuing this process infinitely many times, we obtain the entire sequence $(C_n, F_n)_{n=1}^{\infty}$. It is straightforward to check that (2.1)–(2.3) are satisfied for this sequence. Hence the associated (C, F)-dynamical system $(X, \mathfrak{B}, \mu, T)$ is well defined. It is of rank 1.

We now verify that $\mu(X) < \infty$. Indeed, applying (4.1) and (4.2) and using the fact that $j_{n^2} \le n^2$ we obtain

$$\frac{h_n - h_{n-1} r_n}{h_n} = \frac{(k_1 + \dots + k_{r_n - 1}) j_{n^2}}{h_n} \le \frac{2\pi r_n n^2}{\delta_{n^2} h_{n-1} r_n} < \frac{2\pi}{n^2}.$$

Hence $\sum_{n=1}^{\infty} \frac{h_n - h_{n-1} r_n}{h_n} < \infty$. By (2.4), $\mu(X) < \infty$, as desired.

It remains to check that $\lambda \in e(T)$. It follows from (4.3) that

$$\sum_{n=1}^{\infty} \max_{c \in C_n} |1-\lambda^c| < \sum_{n=1}^{\infty} \frac{2\pi}{n^2} < \infty.$$

Hence, Corollary 3.5 and Remark 3.2 show that $\lambda \in e(T)$. ∎

Thus, Problem II is solved.

Boundedness of the number of cuts. In this subsection we refine Theorem 4.1 for a.e. $\lambda \in \mathbb{T}$.

Given $\lambda \in \mathbb{T}$ and $n \in \mathbb{N}$, we let

$$\delta_n(\lambda) := \min_{j \in \{1,\dots,n\}} |1-\lambda^j|, \qquad E := \bigcup_{N \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \left\{\lambda \in \mathbb{T} \;\middle|\; \frac{n^4}{N2^{n-1}} < \delta_{n^2}(\lambda)\right\}.$$

Of course, if $\lambda \in E$ then λ is of infinite order in $\mathbb{T}$.

THEOREM 4.2. *Given $\lambda \in E$, there is an explicit (C, F)-construction of a rank-1 finite measure preserving transformation T such that $\lambda \in e(T)$ and $\#C_n = 2$ for each $n \in \mathbb{N}$.*

Proof. Let $\lambda \in \bigcap_{n \in \mathbb{N}} \{\lambda \in \mathbb{T} \mid \frac{n^4}{N2^{n-1}} < \delta_{n^2}(\lambda)\}$ for some $N \in \mathbb{N}$. We then repeat the construction in the proof of Theorem 4.1 almost verbatim but replace the "stopping time" condition (4.1) with the following one:

$$h_{n-1} > N2^n. \tag{4.4}$$

Below, we use the same notation as in the proof of Theorem 4.1. Since $a(1) = h_{n-1} + k_1 j_{n^2}$, it follows that $a(1) + h_{n-1} > 2h_{n-1} > N2^{n+1}$. Hence $r_n = 2$. We recall that $\#C_n = r_n$. Utilizing (4.4) and the definition of λ we obtain

$$\frac{h_n - h_{n-1}r_n}{h_n} < \frac{k_1 n^2}{2h_{n-1}} < \frac{\pi n^2}{\delta_{n^2(\lambda)} h_{n-1}} < \frac{\pi}{2n^2}.$$

Hence $\sum_{n=1}^{\infty} \frac{h_n - h_{n-1}r_n}{h_n} < \infty$. Therefore, if (X, μ, T) stands for the associated (C, F)-dynamical system then $\mu(X) < \infty$. We also have $|1-\lambda^{a(1)}| < 2\pi/n^2$. Therefore

$$\sum_{n=1}^{\infty} \max_{c \in C_n} |1-\lambda^c| = \sum_{n=1}^{\infty} |1-\lambda^{c_n}| < \sum_{n=1}^{\infty} \frac{2\pi}{n^2} < \infty,$$

where c_n is determined by the equality $C_n = \{1, c_n\}$. Hence $\lambda \in e(T)$. Thus, the theorem is proved completely. ∎

Our next purpose is to show that E is "large". Denote by τ the Haar measure on $\mathbb{T}$.

PROPOSITION 4.3. $\tau(E) = 1$.

Proof. Let

$$D_n := \left\{ z \in \mathbb{T} \;\middle|\; \exists k \in \mathbb{N},\, k \le n^2 \text{ with } |z^k - 1| < \frac{n^4}{2^{n-1}} \right\}.$$

Since the map $\mathbb{T} \ni z \mapsto z^k \in \mathbb{T}$ preserves τ, we obtain

$$\tau(D_n) \le \sum_{k=1}^{n^2} \tau\left(\left\{ z \in \mathbb{T} \;\middle|\; |z^k - 1| < \frac{n^4}{2^{n-1}} \right\} \right) \le 2\pi \sum_{k=1}^{n^2} \frac{n^4}{2^n} < 2\pi \frac{n^6}{2^n}.$$

Hence $\sum_{n=1}^{\infty} \tau(D_n) < \infty$. The Borel–Cantelli lemma implies that for τ-a.e. $z \in \mathbb{T}$ there is $M > 0$ such that $z \notin D_n$ for each $n > M$. Thus, $n^4/2^{n-1} < \delta_{n^2}(z)$ for each $n > M$. Of course, $\delta_{n^2}(z) > 0$ for each $n \in \mathbb{N}$ if z is of infinite order in $\mathbb{T}$. Hence there exists $N > 0$ such that $\frac{n^4}{N 2^{n-1}} < \delta_{n^2}(z)$ for each $n \in \mathbb{N}$, i.e. $z \in E$. ∎

PROPOSITION 4.4. *If an irrational $\theta \in (0,1)$ is badly approximable or algebraic then $e^{2\pi i\theta} \in E$.*

Proof. It follows from Fact 2.2(v) that if θ is badly approximable then there is a real $d > 0$ such that $|e^{2\pi i\theta q} - 1| > d/q$ for each $q \in \mathbb{N}$. Hence $\delta_{n^2}(e^{2\pi i\theta}) > d/n^2$. This implies that $e^{2\pi i\theta} \in E$, as desired.

A similar argument "works" for algebraic θ if one refers to Fact 2.2(vi). ∎

The (C,F)-parameters $(C_n, F_{n-1})_{n=1}^{\infty}$ are called *bounded* if the sequences $(\#C_n)_{n=1}^{\infty}$ and $(\#F_n - \#F_{n-1}\#C_n)_{n=1}^{\infty}$ are both bounded (see [El–Ru], [Ry], [Da5]). It follows from the structural theorems [El–Ru, Theorem 3] or [Da5, Theorem M] that if T is a (C,F)-transformation with bounded parameters then every $\lambda \in e(T)$ is of finite order in $\mathbb{T}$. We now provide a short direct proof of this fact.

PROPOSITION 4.5. *If T is a (C,F)-transformation with bounded parameters and $\lambda \in e(T)$ then λ is of finite order.*

Proof. We prove this by contraposition. Suppose that $\lambda \in e(T)$ is of infinite order and

$$M := \sup_{n \in \mathbb{N}} \max(\#C_n, \#F_n - \#F_{n-1}\#C_n) < \infty.$$

For each $n \in \mathbb{N}$, we denote by c_n the least positive element of C_n. Then, of course, $0 \le c_{n+1} - \#F_n \le M$. Therefore,

$$\begin{aligned}|c_{n+1} - c_n\#C_n| &\le |c_{n+1} - \#F_n| + |\#F_n - \#F_{n-1}\#C_n| + \#C_n|\#F_{n-1} - c_n| \\ &\le M + M + M^2.\end{aligned}$$

Let $\delta := \min\{|\lambda^k - 1| \mid k = 1, \ldots, 2M + M^2\}$. Then $\delta > 0$ and for each $n \in \mathbb{N}$,

$$|\lambda^{c_{n+1}} - \lambda^{c_n\#C_n}| = |\lambda^{c_{n+1} - c_n\#C_n} - 1| \ge \delta. \tag{4.5}$$

On the other hand, $\lim_{n\to\infty}|\lambda^{c_n}-1|=0$ by Corollary 3.9. Passing to the limit in (4.5), we obtain $|\lambda^{c_{n+1}}-\lambda^{c_n\#C_n}|\to 0$, a contradiction. ∎

Solution of Problem I for the well approximable irrationals. This subsection is devoted entirely to the proof of the following theorem.

THEOREM 4.6. *Let $\lambda=e^{2\pi i\theta}$ for a well approximable irrational $\theta\in(0,1)$. There is an explicit (C,F)-construction of a rank-1 probability preserving dynamical system $(X,\mathfrak{B},\mu,T)$ such that $\lambda\in e(T)$ and the corresponding λ-eigenfunctions are one-to-one continuous maps from X to $\mathbb{T}$.*

Proof. We first outline the roadmap of the proof. The proof is based on Proposition 3.7. According to this proposition, we have to construct the (C,F)-sequence $(C_n,F_{n-1})_{n=1}^\infty$ in such a way that $\max_{c\in C_n}\operatorname{Arg}\lambda^c$ decreases rapidly as $n\to\infty$. In addition, as we want the corresponding invariant measure to be finite, we have to satisfy (2.4). Roughly speaking, this means that for any two neighboring elements $c<d$ of C_{n+1}, the ratio $\frac{c+\#F_n}{d}$ is close to 1 and that the ratio $\frac{\max C_{n+1}+\#F_n}{\#F_{n+1}}$ is also close to 1. To this end, we first define the entire sequence $(F_n)_{n=0}^\infty$. It is technically more difficult to define the sequence $(C_n)_{n=1}^\infty$. First, for each n, we construct inductively an auxiliary increasing sequence $(b(k))_{k=0}^\infty$ (it depends also on n) such that $\frac{b(k+1)-b(k)}{\#F_n}\approx 1$ and $\operatorname{Arg}\lambda^{b_k}\approx 1$ for each k. Then we find a "stopping time" r_n such that $\frac{b(r_n-1)+\#F_n}{\#F_{n+1}}$ is close to 1 and set $C_{n+1}:=\{b(k)\mid k=0,\dots,r_n-1\}$. If the above approximations are chosen sufficiently sharp, then the assertion of the theorem follows.

We now pass to a detailed exposition of the construction. Let $\theta=[a_0;a_1,\dots]$ stand for the expansion of θ into a continued fraction. Let $(p_k/q_k)_{k=1}^\infty$ be the sequence of convergents for θ. It follows from Fact 2.2(ii) that

$$\frac{|\theta q_n-p_n|}{|\theta q_{n+1}-p_{n+1}|}>\frac{q_{n+2}}{q_n+q_{n+1}}=\frac{a_{n+1}q_{n+1}+q_n}{q_{n+1}+q_n}\ge\max\left(\frac{a_{n+1}}{2},1\right)\tag{4.6}$$

for each $n\in\mathbb{N}$. Since

$$|\theta q_n-p_n|\to 0\quad\text{as } n\to\infty,\tag{4.7}$$

we obtain

$$|\operatorname{Arg}\lambda^{q_n}|=|\operatorname{Arg}e^{2\pi i(\theta q_n-p_n)}|=2\pi|\theta q_n-p_n|.\tag{4.8}$$

As θ is well approximable, the sequence $(a_n)_{n=1}^\infty$ is unbounded. Hence, in view of (4.6)–(4.8), there exists an infinite sequence $m_1<m_2<\cdots$ of positive integers such that

$$\frac{|\operatorname{Arg}\lambda^{q_{m_k-1}}|}{|\operatorname{Arg}\lambda^{q_{m_k}}|} > \max\{2k^2, 4\}, \tag{4.9}$$

$$|\operatorname{Arg}\lambda^{q_{m_{k-1}}}| > |\operatorname{Arg}\lambda^{q_{m_k-1}}|. \tag{4.10}$$

More precisely, (4.9) follows from (4.6) and (4.8), while (4.10) follows from (4.7) and (4.8). We have to construct a sequence $(C_n, F_{n-1})_{k=1}^{\infty}$ satisfying (2.1)–(2.3). We let $h_n := q_{m_n}$ for each $n \in \mathbb{N}$. Thus, the sequence $(F_n)_{n=0}^{\infty}$ is determined completely.

It remains to construct $(C_n)_{n=1}^{\infty}$. Fix n. To specify C_n, we first construct an auxiliary sequence $(b(j))_{j=0}^{\infty}$ of positive integers. Note that, strictly speaking, $(b(j))_{j=0}^{\infty}$ will depend also on n. However, we do not reflect this fact in our notation as no confusion can arise. Let $b(0) := 0$. The other terms of this sequence will be specified in an inductive way. Suppose that we already have $b(1), \dots, b(N)$ for some $N > 0$. If

$$|\operatorname{Arg}\lambda^{b(N)}| < \tfrac{1}{2}|\operatorname{Arg}\lambda^{q_{m_{n-1}-1}}|$$

then we call N *good for* b and set $b(N+1) := b(N) + h_{n-1}$. If

$$|\operatorname{Arg}\lambda^{b(N)}| \geq \tfrac{1}{2}|\operatorname{Arg}\lambda^{q_{m_{n-1}-1}}|$$

then we call N *bad for* b and set $b(N+1) := b(N) + q_{m_{n-1}-1} + 2h_{n-1}$. Thus, we determine inductively the entire sequence $(b(j))_{j=0}^{\infty}$. Of course, $b(0) < b(1) < \cdots$ and hence $b(j) \to \infty$ as $j \to \infty$. Let r_n be the greatest $j > 0$ such that $b(j-1) + h_{n-1} < h_n$. Then we set

$$C_n := \{b(0), \dots, b(r_n - 1)\}.$$

Thus, we obtain the infinite sequence $(C_k)_{k=1}^{\infty}$. It follows from the construction that (2.1)–(2.3) are satisfied for $(C_k, F_{k-1})_{k=1}^{\infty}$. Hence the (C, F)-dynamical system $(X, \mathfrak{B}, \mu, T)$ associated with this sequence is well defined. It remains to show that

(i) $\mu(X) < \infty$,
(ii) $\lambda \in e(T)$,
(iii) the λ-eigenfunctions of T are continuous and one-to-one.

We will use (2.4) to prove (i), and Proposition 3.7 to prove (ii) and (iii).

Prior to this, we fix $n \in \mathbb{N}$ and analyze some properties of C_n and F_n. For brevity, we will write m for m_{n-1}. To be specific, we assume that $\operatorname{Arg}\lambda^{q_m} > 0$. (The other case is considered in a similar way.) Then, in view of Fact 2.2(i), $\operatorname{Arg}\lambda^{q_{m-1}} < 0$. Denote by $M_1 < M_2 < \cdots$ the sequence of natural numbers that are bad for b. We also set $M_0 := -1$. Then $(\operatorname{Arg}\lambda^{b(j)})_{j=M_l+1}^{M_{l+1}}$ is an arithmetic progression with common difference $\operatorname{Arg}\lambda^{q_m}$ for each $l \geq 0$.

According to our construction,

$$\max_{0\le j<M_1}\operatorname{Arg}\lambda^{b(j)}<-\tfrac12\operatorname{Arg}\lambda^{q_{m-1}},$$

$$-\tfrac12\operatorname{Arg}\lambda^{q_{m-1}}\le\operatorname{Arg}\lambda^{b(M_1)}<-\tfrac12\operatorname{Arg}\lambda^{q_{m-1}}+\operatorname{Arg}\lambda^{q_m},$$

$$\tfrac12\operatorname{Arg}\lambda^{q_{m-1}}+2\operatorname{Arg}\lambda^{q_m}\le\operatorname{Arg}\lambda^{b(M_1+1)}<\tfrac12\operatorname{Arg}\lambda^{q_{m-1}}+3\operatorname{Arg}\lambda^{q_m}$$

and so on. It follows that for each $j\ge 0$,

$$\operatorname{Arg}\lambda^{b(j)}\in\big[\tfrac12\operatorname{Arg}\lambda^{q_{m-1}}+2\operatorname{Arg}\lambda^{q_m},-\tfrac12\operatorname{Arg}\lambda^{q_{m-1}}+\operatorname{Arg}\lambda^{q_m}\big).$$

We illustrate this by Figure 1 below.

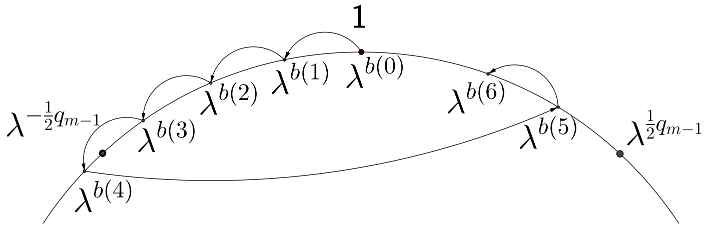

Fig. 1. $i=1,2,3,5,6$ are good for b, and $i=4$ is bad for b.

Hence for all $c,c'\in C_n$,

$$|\operatorname{Arg}\lambda^{c-c'}|=|\operatorname{Arg}\lambda^{c}-\operatorname{Arg}\lambda^{c'}|<|\operatorname{Arg}\lambda^{q_{m-1}}|-|\operatorname{Arg}\lambda^{q_m}|. \tag{4.11}$$

Applying (4.9), we also obtain that for each $j>0$,

$$M_{j+1}-M_j\ge\frac{\big(-\frac12\operatorname{Arg}\lambda^{q_{m-1}}\big)-\big(\frac12\operatorname{Arg}\lambda^{q_{m-1}}+3\operatorname{Arg}\lambda^{q_m}\big)}{\operatorname{Arg}\lambda^{q_m}}$$
$$=\frac{-\operatorname{Arg}\lambda^{q_{m-1}}}{\operatorname{Arg}\lambda^{q_m}}-3=\frac{|\operatorname{Arg}\lambda^{q_{m-1}}|}{|\operatorname{Arg}\lambda^{q_m}|}-3>2(n-1)^2-3>(n-1)^2$$

if $n\ge 3$. In a similar way,

$$M_1\ge\frac{-\frac12\operatorname{Arg}\lambda^{q_{m-1}}}{\operatorname{Arg}\lambda^{q_m}}=\frac{|\operatorname{Arg}\lambda^{q_{m-1}}|}{2|\operatorname{Arg}\lambda^{q_m}|}>(n-1)^2.$$

Thus, there are at least $(n-1)^2$ good integers (for b) between any two subsequent bad ones, and there are at least $(n-1)^2$ good integers before the first bad one. Hence

$$h_n\ge Bq_m(n-1)^2, \tag{4.12}$$

where B is the number of bad integers that are less than r_n. On the other hand,

$$h_n-h_{n-1}r_n=(q_m+q_{m-1})B+(h_n-b(r_n-1)-h_{n-1}).$$

By the definition of r_n,

$$b(r_n-1)+h_{n-1}<h_n\le b(r_n)+h_{n-1}.$$

Hence $h_n - b(r_n-1) - h_{n-1} \le b(r_n) - b(r_n-1) \le q_{m-1} + 2q_m$. Therefore

$$h_n - h_{n-1}r_n \le (q_m + q_{m-1})B + q_{m-1} + 2q_m.$$

This inequality and (4.12) imply that

$$\text{(4.13)} \qquad \frac{h_n - h_{n-1}r_n}{h_n} \le \frac{(q_m+q_{m-1})B + q_{m-1} + 2q_m}{B(n-1)^2 q_m} \le \frac{2Bq_m + 3Bq_m}{B(n-1)^2 q_m} = \frac{5}{(n-1)^2}.$$

If $f, f' \in F_n$ then $-q_{m_n} < f'_n - f_n < q_{m_n}$ and

$$\min_{f \neq f' \in F_n} |\operatorname{Arg} \lambda^{f-f'}| = \min_{0<k<q_{m_n}} |\operatorname{Arg} e^{2\pi i k\theta}|.$$

For each positive $k < q_{m_n}$, there is $l_k \ge 0$ such that $|\operatorname{Arg} e^{2\pi i k\theta}| = 2\pi|k\theta - l_k|$. We claim that $|k\theta - l_k| \ge |q_{m_n-1}\theta - p_{m_n-1}|$. Indeed, if this inequality does not hold then there is $k \in \{1, \dots, q_{m_n}-1\}$ such that

$$|k\theta - l_k| < |q_{m_n-1}\theta - p_{m_n-1}|.$$

It follows from Fact 2.2(iii) that $k > q_{m_n-1}$. We now select

$$q \in \{q_{m_n-1}+1, \dots, q_{m_n}-1\}$$

such that

$$\text{(4.14)} \qquad |q\theta - l_q| = \min_{q_{m_n-1}+1<k<q_{m_n}-1} |k\theta - l_k| < |q_{m_n-1}\theta - p_{m_n-1}|.$$

Consider two cases. If $q_{m_{n-1}} < a \le q$ and $b \in \mathbb{Z}$ then

$$|a\theta - b| \ge |a\theta - l_a| \ge |q\theta - l_q|.$$

Moreover, $|a\theta - b| = |q\theta - l_q|$ if and only if $a = q$ and $b = l_q$ and hence $b/a = l_q/q$.

If $q_{m_{n-1}} \ge a \ge 1$ and $b \in \mathbb{Z}$ then, in view of Fact 2.2(iii) and (4.14),

$$|a\theta - b| \ge |q_{m_n-1}\theta - p_{m_n-1}| > |q\theta - l_q|.$$

Therefore we deduce from Fact 2.2(iv) that l_q/q is a convergent for θ. This contradicts the fact that $q_{m_n-1} < q < q_{m_n}$. Thus, we have proved that

$$|k\theta - l_k| \ge |q_{m_n-1}\theta - p_{m_n-1}|.$$

Therefore,

$$2\pi|k\theta - l_k| \ge 2\pi|q_{m_{n-1}}\theta - p_{m_n-1}| = |\operatorname{Arg} \lambda^{q_{m_n-1}}|.$$

Hence,

$$\text{(4.15)} \qquad \min_{f \neq f' \in F_n} |\operatorname{Arg} \lambda^{f-f'}| \ge |\operatorname{Arg} \lambda^{q_{m_n-1}}|.$$

We are now ready to verify (i)–(iii). It follows from (4.13) that

$$\sum_{n=1}^{\infty} \frac{h_n - h_{n-1} r_n}{h_n} < \infty.$$

Hence $\mu(X) < \infty$ by (2.4). Next, for each $n \in \mathbb{N}$, we utilize (4.11) infinitely many times to obtain

$$\sum_{k>n} \max_{c,c' \in C_k} |\mathrm{Arg}\, \lambda^{c-c'}| \le \sum_{k>n} (|\mathrm{Arg}\, \lambda^{q_{m_{k-1}-1}}| - |\mathrm{Arg}\, \lambda^{q_{m_{k-1}}}|). \tag{4.16}$$

Since (4.10) holds, it follows that

$$\xi := \sum_{k \ge n} (|\mathrm{Arg}\, \lambda^{q_{m_k}}| - |\mathrm{Arg}\, \lambda^{q_{m_{k+1}-1}}|) > 0.$$

Therefore

$$\sum_{k \ge n} (|\mathrm{Arg}\, \lambda^{q_{m_k}-1}| - |\mathrm{Arg}\, \lambda^{q_{m_k}}|) = |\mathrm{Arg}\, \lambda^{q_{m_n}-1}| - \xi < |\mathrm{Arg}\, \lambda^{q_{m_n}-1}|.$$

This inequality, (4.16) and (4.15) yield

$$\min_{f \ne f' \in F_n} |\mathrm{Arg}\, \lambda^{f'-f}| > \sum_{k>n} \max_{c,c' \in C_k} |\mathrm{Arg}\, \lambda^{c-c'}|.$$

Hence, by Proposition 3.7, $\lambda \in e(T)$ and each λ-eigenfunction is continuous and one-to-one. ∎

5. (C, F)-systems with infinite invariant measure and irrational rotations

Proof of Theorem B. We define a sequence $(C_k, F_{k-1})_{k \ge 1}$ inductively. Suppose that we have already constructed $(C_k)_{k=1}^{n-1}$ and $(h_k)_{k=1}^{n-1}$ for some $n > 1$. Our purpose is to define C_n and h_n. Since $\lambda^{h_{n-1}}$ is of infinite order in $\mathbb{T}$, we can select $q_n > 1$ so that

$$\min_{1 \le k \le n} \min_{f \ne f' \in F_k} |\lambda^{f-f'} - 1| > 2^n \max_{|j| < 2^n} |1 - \lambda^{j q_n h_{n-1}}|. \tag{5.1}$$

We now set

$$C_n := \{0, q_n h_{n-1}, 2 q_n h_{n-1}, \ldots, (2^n - 1) q_n h_{n-1}\}, \qquad h_n := 2^n q_n h_{n-1}.$$

Thus we define inductively the entire sequences $(h_k)_{k \ge 0}$ (and hence $(F_k)_{k \ge 0}$) and $(C_k)_{k \ge 1}$. It is straightforward to verify that (2.1)–(2.3) are satisfied for these sequences. Hence the associated (C, F)-system $(X, \mathfrak{B}, \mu, T)$ is well defined. As $h_n > 2 h_{n-1} \# C_n$, it follows from (2.4) that $\mu(X) = \infty$. Since C_n is an arithmetic progression and $\# C_n \to \infty$, we deduce from Fact 2.5 that T is rigid. For each $n \in \mathbb{N}$, it follows from (5.1) that

$$\sum_{k>n} \max_{c,c' \in C_k} |1 - \lambda^{c-c'}| < \sum_{k>n} \frac{1}{2^k} \min_{f \ne f' \in F_n} |\lambda^{f-f'} - 1| < \min_{f \ne f' \in F_n} |\lambda^{f-f'} - 1|.$$

Then Proposition 3.6 shows that $\lambda \in e(T)$ and each λ-eigenfunction f of T is one-to-one. We now set $\mu_\lambda := \mu \circ f^{-1}$. Then f is a measure preserving isomorphism of (X, μ, T) onto $(\mathbb{T}, \mu_\lambda, R_\lambda)$.

It remains to choose another (C, F)-sequence $(C'_k, F'_{k-1})_{k\geq 1}$ with $\#C'_k = 2$ for each $k \in \mathbb{N}$ in such a way that the associated (C, F)-system is isomorphic to (X, μ, T). To this end, for each $n \in \mathbb{N}$ and $k \in \{0, \dots, n-1\}$, we let

$$C_{n,k} := \{0, 2^k q_n h_{n-1}\}.$$

It is easy to see that for each $n \in \mathbb{N}$,

$$C_n = C_{n,0} + \cdots + C_{n,n-1}.$$

We now define the sequences $(C'_k)_{k=1}^\infty$ and $(h'_k)_{k=0}^\infty$ by listing their terms as follows:

$$C_1, C_{2,0}, C_{2,1}, C_{3,0}, C_{3,1}, C_{3,2}, C_{4,0}, \dots,$$
$$h_0, 2q_1h_0, 2q_2h_1, 2^2q_2h_1, 2q_3h_2, 2^2q_3h_2, 2^3q_3h_2, \dots$$

respectively. We also let $F'_k := \{0, 1, \dots, h'_k - 1\}$. It is routine to check that (2.1)–(2.3) are satisfied for $(C'_k, F'_{k-1})_{k\geq 1}$. Of course, the associated (C, F)-system (X', μ', T') is canonically isomorphic to (X, μ, T). ([4]) Thus, Theorem B is proved completely. ∎

Proof of Theorem C. Fix a partition of $\mathbb{N}$ into infinitely many infinite subsets $\mathcal{N}_p$, $p \geq 2$. Utilizing an inductive argument, we can construct a sequence $(C_n, F_{n-1})_{n=1}^\infty$ such that for each $n > 0$,

(i) $F_n = \{0, 1, \dots, h_n - 1\}$ for some $h_n > 0$, and $\#C_n > 1$,
(ii) $F_n + F_n + C_{n+1} \subset F_{n+1}$,
(iii) the sets $F_n - F_n + c - c'$, $c \neq c' \in C_{n+1}$, and $F_n - F_n$ are all mutually disjoint,
(iv) $\#C_n \to \infty$ as $n \to \infty$,
(v) $\min_{1\leq k\leq n} \min_{f\neq f'\in F_k} |\lambda^{f-f'} - 1| > 2^{n+1} \max_{c\in C_n} |1 - \lambda^c|$,
(vi) if $n \in \mathcal{N}_p$ for some $p \geq 2$ then there are two subsets $C_n^{(1)}$ and $C_n^{(2)}$ of C_n such that $\#C_n^{(1)} > 0.3\#C_n$, $\#C_n^{(2)} > 0.3\#C_n$, every element of $C_n^{(1)}$ is divisible by p, and $c \equiv 1 \pmod p$ for each $c \in C_n^{(2)}$.

Note that (i)–(iv) are needed to construct a (C, F)-system of zero type. Then (v) yields an isomorphism between this (C, F)-system and R_λ. Finally, (vi) implies that $e(R_\lambda)$ is torsion free.

This sequence is constructed in an inductive way. Suppose that we have already constructed a finite sequence $(C_k, F_k)_{k=1}^n$ satisfying (i)–(vi) for some $n \geq 1$. Our purpose is to define C_{n+1} and h_{n+1}. Fix $p > 2$ such that $n + 1$ is

([4]) This follows from the fact that $(C_k, F_{k-1})_{k\geq 1}$ is a *telescoping* of $(C'_k, F'_{k-1})_{k\geq 1}$. See [Da5, §1.4] for details.

in $\mathcal{N}_p$. We now define inductively an auxiliary increasing sequence $(a(j))_{j=0}^{\infty}$ of non-negative integers. Let $a(0) := 0$. Suppose that there is $l > 0$ such that the elements $(a(j))_{j=1}^{l}$ have been defined and

(a) the sets $F_n - F_n + a(i) - a(j)$, $i \neq j \in \{0, \dots, l\}$ and $F_n - F_n$ are mutually disjoint,
(b) $\min_{1\le k\le n} \min_{f\neq f'\in F_k} |\lambda^{f-f'} - 1| > 2^{n+1} \max_{1\le j\le l} |1 - \lambda^{a(j)}|$.

Consider separately two cases. Suppose first that $l + 1$ is even. Since the set $\{\lambda^{pq} \mid q \in \mathbb{N}\}$ is dense in $\mathbb{T}$, we can find $Q \in \mathbb{N}$ such that

- $pQ > 3(h_n + a(l))$,
- $\min_{1\le k\le n} \min_{f\neq f'\in F_k} |\lambda^{f-f'} - 1| > 2^{n+1}|1 - \lambda^{pQ}|$.

We then put $a(l + 1) := pQ$.

Suppose now that l is even. Since the set $\{\lambda^{pq+1} \mid q \in \mathbb{N}\}$ is dense in $\mathbb{T}$, we can find $Q \in \mathbb{N}$ such that

- $pQ + 1 > 3(h_n + a(l))$,
- $\min_{1\le k\le n} \min_{f\neq f'\in F_k} |\lambda^{f-f'} - 1| > 2^{n+1}|1 - \lambda^{pQ+1}|$.

In this case put $a(l + 1) := pQ + 1$.

We note that in both cases, (a) and (b) hold with $l + 1$ in place of l.

Hence, we define inductively an infinite sequence $(a(j)_{j=0}^{\infty}$. We now set $C_{n+1} := \{a(0), a(1), \dots, a(n + 1)\}$. Choose h_{n+1} large enough so that (ii) is satisfied. Thus, we have defined C_{n+1} and F_{n+1} in a such a way that (i)–(v) are satisfied. It remains to check that (vi) holds. Let $C_{n+1}^{(1)} = \{a(j) \in C_{n+1} \mid j \text{ is even}\}$ and $C_{n+1}^{(2)} = \{a(j) \in C_{n+1} \mid j \text{ is odd}\}$. Then, of course, $\#C_{n+1}^{(1)} > 0.3\#C_{n+1}$ and $\#C_{n+1}^{(2)} > 0.3\#C_{n+1}$, every element of $C_{n+1}^{(1)}$ is divisible by p, and $c \equiv 1 \pmod p$ for each $c \in C_{n+1}^{(2)}$, i.e. (vi) holds. Thus, we have defined the entire sequence $(C_n, F_{n-1})_{n=1}^{\infty}$ and (i)–(vi) hold for each $n \in \mathbb{N}$.

It follows from (i)–(iii) that (2.1)–(2.3) are satisfied. Therefore the associated (C, F)-system (X, μ, T) is well defined. It follows from (ii) that $h_{n+1} > 2h_n\#C_{n+1}$ for each $n > 0$. Hence $\mu(X) = \infty$ by (2.4). We deduce from (ii)–(iv) that T is of zero type in view of Fact 2.6. Finally, (v) implies that the condition of Proposition 3.6 is satisfied. Hence Proposition 3.6 shows that $\lambda \in e(T)$ and every λ-eigenfunction $f : X \to \mathbb{T}$ of T is one-to-one. Let $\mu'_\lambda := \mu \circ f^{-1}$. Then the dynamical system $(\mathbb{T}, \mu'_\lambda, R_\lambda)$ is of rank 1 (with explicitly determined cutting-and-stacking parameters) and of zero type. Hence, by the main result of [RyTh], $C(R_\lambda) = \{R_\lambda^n \mid n \in \mathbb{Z}\}$. On the other hand, it is well known (and easy to verify) that

$$C(R_\lambda) = \{R_\xi \text{ for } \xi \in \mathbb{T} \mid \mu'_\lambda \circ R_\xi = \mu'_\lambda\}.$$

Moreover, since R_λ is ergodic, it follows that for each $\xi \in \mathbb{T}$, we have either

$\mu'_\lambda \circ R_\xi \sim \mu'_\lambda$ or $\mu'_\lambda \circ R_\xi \perp \mu'_\lambda$ [Na, 9.22]. We thus obtain

$$\mathbb{T} \setminus \{\lambda^n \mid n \in \mathbb{Z}\} = \{\xi \in \mathbb{T} \mid \mu'_\lambda \circ R_\xi \perp \mu'_\lambda\}.$$

Suppose that $e(T)$ contains a torsion λ of order $p \geq 2$. Then by Corollary 3.10, there exist $n > 0$ and subsets $C^0_m \subset C_m$ for each $m \geq n$ such that $\#C^0_m > 0.9\#C_m$ and $c \equiv c' \pmod p$ for all $c, c' \in C^0_m$. This contradicts (vi). Hence $e(T)$ is torsion free. Therefore T is totally ergodic.

It remains to prove the final claim of Theorem C. Let $\omega \in \mathbb{T} \setminus \{\lambda, \lambda^{-1}\}$ be of infinite order. Suppose first that there are $n, m \in \mathbb{Z}$ such that $\lambda^n = \omega^m$. Then μ'_ω and μ'_λ are two measures on $\mathbb{T}$ that are invariant under the same transformation $R^n_\lambda = R^m_\omega$. Moreover, this transformation is ergodic with respect to each of these two measures because the systems $(\mathbb{T}, \mu'_\lambda, R_\lambda)$ and $(\mathbb{T}, \mu'_\omega, R_\omega)$ are totally ergodic. Hence either $\mu'_\omega \perp \mu'_\lambda$ or $\mu'_\omega \sim \mu'_\lambda$. If the latter holds then $\{z \in \mathbb{T} \mid \mu'_\lambda \circ R_z \sim \mu'_\lambda\} = \{z \in \mathbb{T} \mid \mu'_\omega \circ R_z \sim \mu'_\omega\}$. Hence $\{\lambda^l \mid l \in \mathbb{Z}\} = \{\omega^l \mid l \in \mathbb{Z}\}$. This is only possible if either $\omega = \lambda$ or $\omega = \lambda^{-1}$. Thus, we obtain a contradiction. Hence $\mu'_\omega \perp \mu'_\lambda$, as claimed.

Suppose now that ω and λ are independent and $\mu'_\omega \not\perp \mu'_\lambda$. Then there is a Borel subset $A \subset \mathbb{T}$ such that $\mu'_\lambda(A) > 0$ and $1_A \cdot \mu'_\lambda \prec \mu'_\omega$. Then for each $n \in \mathbb{N}$,

$$(1_A \cdot \mu'_\lambda) \circ R^n_\lambda \prec \mu'_\omega \circ R^n_\lambda \quad \text{and} \quad \mu'_\omega \circ R^n_\lambda \perp \mu'_\omega.$$

Hence $(1_A \cdot \mu'_\lambda) \circ R^n_\lambda \perp \mu'_\omega$, i.e. $1_{R^{-n}_\lambda A} \cdot \mu'_\lambda \perp \mu'_\omega$. It follows that

$$1_{\bigcup_{n>0} R^{-n}_\lambda A} \cdot \mu'_\lambda \perp \mu'_\omega.$$

Since μ'_λ is non-atomic and ergodic with respect to R_λ, we deduce that $\bigcup_{n>0} R^{-n}_\lambda A = \mathbb{T} \pmod{\mu'_\lambda}$. Hence $\mu'_\omega \perp \mu'_\lambda$. ∎

A subset $\Lambda \subset \mathbb{T}$ is called *independent* if whenever $\lambda_1^{k_1} \ldots \lambda_n^{k_n} = 1$ for some $k_1, \ldots, k_n \in \mathbb{Z}$, $\lambda_1, \ldots, \lambda_n \in \Lambda$ and $n \in \mathbb{N}$ then $k_1 = \cdots = k_n = 0$.

Remark 5.1. Theorems B and C can be strengthened in the following way. Let Λ be a countable independent subset of $\mathbb{T}$. Then there exists

- a rigid (C, F)-dynamical system (X, μ, T) with explicitly defined sequence $(C_n, F_{n-1})_{n=1}^\infty$ such that $\mu(X) = \infty$, $\#C_n = 2$ for each $n \in \mathbb{N}$, $\Lambda \subset e(T)$ and every λ-eigenfunction is one-to-one for each $\lambda \in \Lambda$;
- a zero type (C, F)-dynamical system (X, μ, T) with explicitly defined sequence $(C_n, F_{n-1})_{n=1}^\infty$ such that $\mu(X) = \infty$, $\Lambda \subset e(T)$ and every λ-eigenfunction is one-to-one for each $\lambda \in \Lambda$.

The proof of these statements is only a slight modification of the proof of Theorems B and C respectively, based on the well known fact that for each $\epsilon > 0$, there is $n \in \mathbb{N}$ such that $\sup_{\lambda \in \Lambda} |1 - \lambda^n| < \epsilon$. We leave details to the reader.

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