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Embeddings between Lorentz sequence spaces are strictly but not finitely strictly singular

by

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Abstract. Given $0 < p, q, r < \infty$ and $q < r \leq \infty$ we consider the natural embedding $\ell_{p,q} \hookrightarrow \ell_{p,r}$ between Lorentz sequence spaces. We introduce a new method of proving that this non-compact embedding is always strictly singular but not finitely strictly singular.

1. Introduction. Let $T : X \to Y$ be a linear operator between Banach spaces. We recall that T is strictly singular if there is no infinite-dimensional closed subspace Z of X such that $T : Z \to T(Z)$, the restriction of T to Z, is an isomorphism. Moreover, we say that T is *finitely strictly singular* if, for every $\varepsilon > 0$, there exists $n_\varepsilon \geq 1$ such that for every subspace E of X with dimension greater than n_ε, there exists x in the unit sphere of E such that $\|T(x)\|_Y \leq \varepsilon$.

Strictly singular operators and finitely strictly singular operators, which encompass compact operators, possess similar properties which are habitually connected with compact operators. For example, it is well known that Fredholm operators are invariant when perturbed by strictly singular operators (i.e. if T is Fredholm and S is strictly singular then $T + S$ is Fredholm; see [1, Theorem 4.63]).

Many of the non-compact operators in Analysis are strictly singular or finitely strictly singular. For example, the Fourier transform, which is obviously non-compact when considered as a map from L^p into $L^{p'}$, is finitely strictly singular for $1 < p < 2$ and strictly singular when $p = 1$ (see [6]).

The natural embedding of sequence spaces

$$I : \ell_p \to \ell_q \quad \text{for } p < q,$$

which is non-compact, is finitely strictly singular (see [8]).

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When we consider weighted sequence spaces $d(\omega, p)$, with a weight $\omega = \omega(j)$, equipped with a quasi-norm

$$\|u\|_{d(\omega,p)} = \Big(\sum_{j=1}^{\infty}((u^*)(j))^p\omega(j)\Big)^{1/p},$$

then the natural embedding $d(\omega, p) \hookrightarrow d(\omega', p)$, with $p \geq 1$, is strictly singular if and only if

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n}\omega'(j)}{\sum_{j=1}^{n}\omega(j)} = 0.$$

For the proof and more details about this startling result see the paper by Hernández, Sánchez and Semenov [4, Theorem 3.1].

Also the limiting optimal Sobolev embedding E_d into continuous functions,

$$E_d : W_0^1 L^{d,1}((0,1)^d) \hookrightarrow C((0,1)^d) \tag{1.1}$$

(where $W_0^1 L^{d,1}((0,1)^d)$ denotes the space of all functions u for which $|\nabla u|$ belongs to the Lorentz space $L^{d,1}$ and u has a zero trace), is non-compact but finitely strictly singular (see [5]).

From the above examples, a natural and quite intriguing question arises: are all limiting Sobolev embeddings on a bounded smooth domain strictly singular or finitely strictly singular?

Since the optimal target spaces for Sobolev embeddings are quite often Lorentz spaces, in order to investigate the above question one needs to get more information about strict singularity for natural embeddings between Lorentz sequence spaces:

$$I : \ell_{p,q} \to \ell_{s,r}, \quad 0 < p \leq s < \infty,\ 0 < q, r < \infty.$$

The case $0 < p = s < \infty$ and $0 < q < r < \infty$ is the most interesting as the other cases can be derived from existing results. Note that for $p < s$ the embedding I is obviously finitely strictly singular (see [8]).

The paper is structured as follows. In Sect. 2 we recall the definitions we use, and we collect all the necessary material and technical lemmas. In Sect. 3 we prove, by a new method, that the embedding $\ell_{p,q} \to \ell_{p,r}$ is strictly singular for all range of indices $0 < p < \infty$, $0 < q < r \leq \infty$. In Sect. 4 we apply ideas inspired by the previous sections and show that this embedding is not finitely strictly singular.

The results obtained illustrate that the impact of the second index for embeddings between Lorentz sequence spaces is more pronounced than the impact of the first index.

2. Preliminaries. In this section we recall the definitions, notations and some technical lemmas needed in Sections 3 and 4. We start by the definition of strictly singular and finitely strictly singular operators on quasi-Banach spaces, which satisfy the triangle inequality with a constant. For a sequence $u = (u(1), u(2), \dots)$ of real numbers denote $|u| = (|u(1)|, |u(2)|, \dots)$. We say that $|u| \le |v|$ if $|u(i)| \le |v(i)|$ for each $i \in \mathbb{N}$.

DEFINITION 2.1. Let $\mathcal{S}$ be the set of all sequences of real numbers and $\|\cdot\| : \mathcal{S} \to [0, \infty]$. Assume that for all $u, v \in \mathcal{S}$ and $\alpha \in \mathbb{R}$ the function $\|\cdot\|$ satisfies:

(i) $\|u + v\| \le T(\|u\| + \|v\|)$ for some $T \ge 1$,
(ii) $\|\alpha u\| = |\alpha|\ \|u\|$,
(iii) $\|u\| \ge 0$ and $\|u\| = 0$ if and only if $u = 0$,
(iv) $\|u\| = \||u|\|$,
(v) if $|u| \le |v|$, then $\|u\| \le \|v\|$,
(vi) if $0 \le u_n \nearrow u$, then $\|u_n\| \nearrow \|u\|$,
(vii) if $\#\{i;\ u(i) \ne 0\} < \infty$, then $\|u\| < \infty$.

Define $X := \{u;\ \|u\| < \infty\}$. Then we call X a *quasi-Banach sequence space.*

In an analogous way we could define a quasi-Banach function space of functions on a domain Ω. Note that each quasi-Banach function space is complete (for details see for instance [7, Corollary 3.7]).

DEFINITION 2.2. A bounded operator $T : X \to Y$ between quasi-Banach spaces is said to be *strictly singular* if there is no infinite-dimensional closed subspace Z of X such that $T : Z \to T(Z)$, the restriction of T to Z, is an isomorphism.

See [1, Section 4.5] for more about strictly singular operators in the case X, Y being Banach spaces.

DEFINITION 2.3. An operator T from a quasi-Banach space X into a quasi-Banach space Y is *finitely strictly singular* if, for every $\varepsilon > 0$, there exists $n_\varepsilon \ge 1$ such that for every subspace E of X of dimension greater than n_ε, there exists x in the unit sphere of E such that $\|T(x)\|_Y \le \varepsilon$.

Let $T : X \to Y$ be a linear map between quasi-Banach spaces. Then the nth *Bernstein number* (or *Bernstein width*) is defined by

$$b_n(T) = \sup_{E \subset X,\, \dim(E) = n}\ \inf_{f \in E,\, \|f\|_X = 1} \|T(f)\|_Y.$$

It is not too hard to see that the operator T is finitely strictly singular if and only if $b_n(T) \to 0$ and that we have the following relations:

$$\text{compact} \implies \text{finitely strictly singular} \implies \text{strictly singular.}$$

For a finite set F denote by $\#(F)$ the number of elements of F.

DEFINITION 2.4. Given a sequence $a = (a(1), a(2), \dots) \in c_0$ we set

$$\mu_a(\lambda) = \#\{i;\ |a(i)| > \lambda\} \quad \text{for } \lambda > 0$$

and define $a^* = (a^*(1), a^*(2), \dots)$, the *non-increasing rearrangement* of a, by

$$a^*(j) = \min\{\lambda > 0;\ \mu_a(\lambda) \le j\} \quad \text{for } j \in \mathbb{N}.$$

For a sequence $a = (a(1), a(2), \dots) \in c_0$ denote

$$\operatorname{supp} a = \{j \in \mathbb{N};\ a(j) \neq 0\}.$$

DEFINITION 2.5. Given a sequence $u = (u(1), u(2), \dots) \in c_0$ with $\operatorname{supp} u = \{n_1, \dots, n_k\} \subset \mathbb{N}$ and $n_1 < \dots < n_k$, we define the *non-increasing rearrangement $u^\diamond$ of u with respect to* $\operatorname{supp} u$ by

$$\begin{cases} u^\diamond(n_j) = u^*(j), & j \in \{1, \dots, k\}, \\ u^\diamond(i) = 0, & i \notin \{n_1, \dots, n_k\}. \end{cases}$$

REMARK 2.6. If $\operatorname{supp} u := \{n+1, n+2, \dots, m\}$ then

$$u^\diamond(j) = \begin{cases} u^*(j-n) & \text{if } j \in \{n+1, n+2, \dots, m\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Next we recall the definition of sequence Lorentz spaces.

DEFINITION 2.7. Let $p \in (0, \infty)$ and $q \in (0, \infty]$. For a sequence u define

$$\|u\|_{p,q} = \begin{cases} \Big(\sum_{j=1}^{\infty} j^{q/p-1}(u^*(j))^q\Big)^{1/q} & \text{if } q < \infty, \\ \sup\{j^{1/p}u^*(j);\ j = 1, 2, \dots\} & \text{if } q = \infty. \end{cases}$$

We define the *Lorentz space* $\ell_{p,q}$ as the collection of all sequences u for which the norm $\|u\|_{p,q}$ is finite.

The following is easy to see.

REMARK 2.8. Let $p \in (0, \infty)$, $q \in (0, \infty]$ and $\delta > 0$. If $\|u\|_{p,q} \le \delta$, then

$$|u(s)| \le \delta \quad \text{for all } s.$$

LEMMA 2.9. *Let $0 < p < \infty$ and $0 < q \le \infty$. The space $\ell_{p,q}$ is a quasi-Banach function space.*

Proof. As in [3, (1.16) in Proposition 1.7] we can prove

$$(u+v)^*(i+j) \le u^*(i) + v^*(j).$$

Split the sum

$$\sum_{j=1}^{\infty} j^{q/p-1}((u+v)^*(j))^q$$

into two sums, over the odd numbers and over the even numbers. For both sums we can easily prove the quasi-triangle inequality. The other properties are easy. ■

Given two positive quantities A, B we write $A \lesssim B$ if there is a positive constant C with $A \le CB$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

LEMMA 2.10. *Let $0 < p < \infty$ and $0 < q < \infty$. Then for all $n \in \mathbb{N}$,*

$$\Big(\sum_{j=1}^{n} j^{q/p-1}\Big)^{1/q} \approx n^{1/p}.$$

Proof. For all n we have

$$\sum_{j=1}^{n} j^{q/p-1} \approx \int_0^n t^{q/p-1}\,dt = \frac{p}{q} n^{q/p} \approx n^{q/p}. \ \blacksquare$$

PROPOSITION 2.11. *Let $0 < p < \infty$ and $0 < q < r \le \infty$. Then we have $\ell_{p,q} \hookrightarrow \ell_{p,r}$, i.e.*

$$\|a\|_{p,r} \le D_{q,r}\|a\|_{p,q} \quad \text{for all sequences } a, \tag{2.2}$$

where $D_{q,r}$ is the norm of this embedding.

DEFINITION 2.12. Let X be a quasi-Banach function space of functions defined over Ω. We say that $f \in X$ *has an absolutely continuous norm in* X, written $f \in X_a$, if for every non-increasing sequence of measurable sets $G_n \subset \Omega$ with $|G_n| \searrow 0$ we have $\|f\chi_{G_n}\| \searrow 0$. We say that X *has an absolutely continuous norm* if $X_a = X$.

LEMMA 2.13. *Let $0 < p < \infty$ and $0 < q < \infty$. Then $\ell_{p,q}$ has an absolutely continuous norm.*

Proof. Take $u \in \ell_{p,q}$. Set

$$u_n(j) = \begin{cases} 0, & 1 \le j \le n, \\ u(j), & n+1 \le j. \end{cases}$$

Since $\|u\|_{p,q} \le K < \infty$, by (2.2) we have, for each n,

$$K \ge \Big(\sum_{j=1}^{\infty} j^{q/p-1}(u^*(j))^q\Big)^{1/q} \gtrsim n^{1/p}u^*(n)$$

and so

$$u^*(n) \lesssim n^{-1/p}.$$

This implies for any $j \in \mathbb{N}$ that $\lim_{n\to\infty} u_n^*(j) = 0$ and consequently, due to the Lebesgue dominated convergence theorem we obtain

$$\|u_n\|_{p,q} = \Big(\sum_{j=1}^{\infty} j^{q/p-1}(u_n^*(j))^q\Big)^{1/q} \to 0 \quad \text{as } n \to \infty. \ \blacksquare$$

For a sequence $b = (b(1), b(2), \dots)$ and $m \in \mathbb{N}$ set

$$P_m(b) = (b(1), b(2), \dots, b(m), 0, 0, \dots),$$
$$R_m(b) = b - P_m b = (0, 0, \dots, 0, b(m+1), b(m+2), \dots).$$

Let $X \subset \ell_{p,q}$ be a closed subspace with $\dim X = \infty$. Define $X_m = R_m(X)$. It is easy to see that X_m is a closed subspace with $\dim X_m = \infty$.

Let $0 < p < \infty, 0 < q \leq \infty$. Since $\ell_{p,q}$ is a quasi-Banach sequence space, there exists $T \geq 1$ such that

$$\|u+v\|_{p,q} \leq T(\|u\|_{p,q} + \|v\|_{p,q}).$$

Note that this implies directly

$$u = v + w \implies \|v\|_{p,q} \geq \frac{1}{T}\|u\|_{p,q} - \|w\|_{p,q}, \tag{2.3}$$

$$\Big\|\sum_{j=1}^{n} u_j\Big\|_{p,q} \leq \sum_{j=1}^{n} T^j \|u_j\|_{p,q}. \tag{2.4}$$

LEMMA 2.14. *Let $0 < p < \infty$, $0 < q < \infty$ and $\alpha > 0$. Assume that $v_j \in \ell_{p,q}$, $j = 1, 2, \dots$, have pairwise disjoint supports and $\|v_j\|_{p,q} \geq \alpha$. Then*

$$\lim_{k\to\infty} \Big\|\sum_{j=1}^{k} v_j\Big\|_{p,q} = \infty.$$

Proof. Since the v_j have pairwise disjoint supports we can write

$$\Big\|\sum_{j=1}^{k} v_j\Big\|_{p,q} = \Big\|\sum_{j=1}^{k} |v_j|\Big\|_{p,q}.$$

Assume that there exists a positive constant C independent of k such that

$$C \geq \Big\|\sum_{j=1}^{k} v_j\Big\|_{p,q} = \Big\|\sum_{j=1}^{k} |v_j|\Big\|_{p,q}.$$

Since

$$\Big\|\sum_{j=1}^{k} |v_j|\Big\|_{p,q} \nearrow \Big\|\sum_{j=1}^{\infty} |v_j|\Big\|_{p,q}$$

we have

$$C \geq \Big\|\sum_{j=1}^{\infty} |v_j|\Big\|_{p,q}.$$

By the absolute continuity of $\|\cdot\|_{p,q}$ we obtain

$$\alpha \leq \|v_n\|_{p,q} = \big\||v_n|\big\|_{p,q} \leq \Big\|\sum_{j=n}^{\infty} |v_j|\Big\|_{p,q} \to 0,$$

which is a contradiction. ∎

LEMMA 2.15. *Suppose $0 < p < \infty$ and $0 < q < \infty$. Let $X \subset \ell_{p,q}$ be a closed subspace with* $\dim X = \infty$. *Assume $n, N \in \mathbb{N}$ and $\varepsilon > 0$, $1/T > \delta > 0$. Then there exist $m \in \mathbb{N}$ and $u \in X_n$ such that denoting $v := P_m u$ and*

$w := R_m u$ we have

(2.5) $\|u\|_{p,q} = 1,$

(2.6) $m > 2n, \quad m \geq N,$

(2.7) $\operatorname{supp} v \subset \{n+1, n+2, \ldots, m\},$

(2.8) $|v(j)| \leq \varepsilon \quad$ *for all* $j,$

(2.9) $1/T - \delta \leq \|v\|_{p,q} \leq 1,$

(2.10) $\|w\|_{p,q} \leq \delta.$

Proof. Set $n_0 := n$ and construct by induction sequences $n_0 < n_1 < n_2 < \cdots$ and $u_i \in X$ such that setting $v_i := P_{n_i} u_i$, $w_i := R_{n_i} u_i$ we have

(2.11) $\operatorname{supp} v_i \subset \{n_{i-1}+1, n_{i-1}+2, \ldots, n_i\},$

(2.12) $\dfrac{1}{T} - \dfrac{\delta}{(2T)^i} \leq \|v_i\|_{p,q} \leq 1,$

(2.13) $\|w_i\|_{p,q} \leq \dfrac{\delta}{(2T)^i}.$

Since $\dim X_n = \infty$ we can find $u_1 \in X_n$ with $\|u_1\|_{p,q} = 1$. Take $n_1 > n$ such that $\|R_{n_1} u_1\|_{p,q} \leq \delta/(2T)$. Denote $v_1 := P_{n_1} u_1$ and $w_1 := R_{n_1} u_1$. Clearly, $\operatorname{supp} v_1 \subset \{n+1, n+2, \ldots, n_1\}$ and

$$1 \geq \|v_1\|_{p,q} \overset{(2.3)}{\geq} \frac{1}{T}\|u_1\|_{p,q} - \|w_1\|_{p,q} \geq \frac{1}{T} - \frac{\delta}{2T}.$$

Suppose that we have constructed $n_0 < n_1 < \cdots < n_k$, $u_1, \ldots, u_k \in X$ and appropriate functions $v_1, \ldots, v_k$ satisfying (2.11) and (2.12). Since $\dim X_{n_k} = \infty$ we can find $u_{k+1} \in X_{n_k}$ with $\|u_{k+1}\|_{p,q} = 1$. It is easy to see that we can take $n_{k+1} \geq n_k$ such that $\|R_{n_{k+1}} u_{k+1}\|_{p,q} \leq \delta/(2T)^{k+1}$. Set $w_{k+1} = R_{n_{k+1}} u_{k+1}$ and $v_{k+1} = P_{n_{k+1}} u_{k+1}$. Consequently,

$$1 \geq \|v_{k+1}\|_{p,q} \overset{(2.3)}{\geq} \frac{1}{T}\|u_{k+1}\|_{p,q} - \|w_{k+1}\|_{p,q} \geq \frac{1}{T} - \frac{\delta}{(2T)^{k+1}}.$$

Moreover, $\operatorname{supp} v_{k+1} \subset \{n_k+1, n_k+2, \ldots, n_{k+1}\}$.

Consider now the sequences

$$y_k := \sum_{j=1}^{k} u_j, \qquad s_k := \|y_k\|_{p,q}.$$

By (2.3), (2.4) and (2.13) we can write

$$\begin{aligned} s_k &= \Big\|\sum_{j=1}^{k} u_j\Big\|_{p,q} = \Big\|\sum_{j=1}^{k} v_j + \sum_{j=1}^{k} w_j\Big\|_{p,q} \\ &\overset{(2.3)}{\geq} \frac{1}{T}\Big\|\sum_{j=1}^{k} v_j\Big\|_{p,q} - \Big\|\sum_{j=1}^{k} w_j\Big\|_{p,q} \end{aligned}$$

$$\overset{(2.4)}{\geq} \frac{1}{T}\Big\|\sum_{j=1}^{k} v_j\Big\|_{p,q} - \sum_{j=1}^{k} T^j \|w_j\|_{p,q} \overset{(2.13)}{\geq} \frac{1}{T}\Big\|\sum_{j=1}^{k} v_j\Big\|_{p,q} - \sum_{j=1}^{k} T^j \frac{\delta}{(2T)^j}$$
$$\geq \frac{1}{T}\Big\|\sum_{j=1}^{k} v_j\Big\|_{p,q} - \delta.$$

Since by (2.12) we obtain

$$\|v_i\|_{p,q} \geq \frac{1}{T} - \frac{\delta}{(2T)^i} \geq \frac{1}{T} - \frac{\delta}{2T} =: \alpha > 0$$

and the v_j have pairwise disjoint supports by (2.11), and with the above $\alpha > 0$, Lemma 2.14 gives

$$\lim_{k\to\infty} \Big\|\sum_{j=1}^{k} v_j\Big\|_{p,q} = \infty$$

and consequently $s_k \to \infty$.

Then we can find k large enough such that

$$k > 2n, \qquad k \geq N, \qquad \frac{1}{s_k}\left(1 + \frac{\delta}{2T-1}\right) \leq \varepsilon \tag{2.14}$$

and set

$$u = \frac{1}{s_m}\sum_{j=1}^{m} u_j, \qquad m = n_k.$$

Clearly, $m \geq k$ and so $m > 2n$, $m \geq N$, which proves (2.6). It is seen from the definition of s_m that $\|u\|_{p,q} = 1$, which proves (2.5).

By the definition of $v = P_m u$ we obtain directly

$$\operatorname{supp} v \subset \{n_0+1, n_0+2, \dots, m\} = \{n+1, n+2, \dots, m\},$$

which proves (2.7).

Fix now $t \in \mathbb{N}$. If $t > m$ we have $|v(t)| = 0$.

Assume $t \leq m$ and $\|u_k\|_{p,q} = 1$. By Remark 2.8 we have $|u_k(s)| \leq 1$ for each $s \in \mathbb{N}$ and so

$$|v_k(s)| \leq |u_k(s)| \leq 1.$$

Moreover, by (2.13) we have $\|w_k\|_{p,q} \leq \delta/(2T)^k$ and again by Remark 2.8 we obtain

$$|w_k(s)| \leq \frac{\delta}{(2T)^k} \qquad \text{for all } s.$$

Recall that $\operatorname{supp} v_i \subset \{n_{i-1}+1, n_{i-1}+2, \dots, n_i\}$ by (2.11) and so the v_i have pairwise disjoint supports. Thus

$$|v(t)| \le |u(t)| \le \frac{1}{s_m}\sum_{j=1}^m |u_j(t)| \le \frac{1}{s_m}\sum_{j=1}^m |v_j(t)| + \frac{1}{s_m}\sum_{j=1}^m |w_j(t)|$$
$$\le \frac{1}{s_m} + \frac{1}{s_m}\sum_{j=1}^m \frac{\delta}{(2T)^j} \le \frac{1}{s_m} + \frac{1}{s_m}\sum_{j=1}^\infty \frac{\delta}{(2T)^j} \le \frac{1}{s_m}\left(1+\frac{\delta}{2T-1}\right) \overset{(2.14)}{\le} \varepsilon,$$

which proves (2.8).

Next,

$$w = R_m u = R_m\left(\frac{1}{s_m}\sum_{j=1}^m u_j\right) = R_m\left(\frac{1}{s_m}\sum_{j=1}^m v_j + \frac{1}{s_m}\sum_{j=1}^m w_j\right)$$
$$= \frac{1}{s_m}\sum_{j=1}^m R_m(v_j) + \frac{1}{s_m}\sum_{j=1}^m R_m(w_j) = \frac{1}{s_m}\sum_{j=1}^m R_m(w_j).$$

Thus

$$\|w\|_{p,q} \overset{(2.4)}{\le} \frac{1}{s_m}\sum_{j=1}^m T^j \|R_m(w_j)\|_{p,q} \le \frac{1}{s_m}\sum_{j=1}^m T^j\|w_j\|_{p,q}$$
$$\overset{(2.13)}{\le} \frac{1}{s_m}\sum_{j=1}^m T^j\frac{\delta}{(2T)^j} = \frac{1}{s_m}\sum_{j=1}^m \frac{\delta}{2^j} \le \frac{\delta}{s_m} \le \delta,$$

which proves (2.10).

Finally, (2.9) follows directly from

$$1 \ge \|v\|_{p,q} \overset{(2.3)}{\ge} \frac{1}{T}\|u\|_{p,q} - \|w\|_{p,q} \ge \frac{1}{T} - \delta,$$

which finishes the proof. ■

3. The embedding is strictly singular. In this section we prove that the embedding $\ell_{p,q} \hookrightarrow \ell_{p,r}$ is strictly singular for all the range of indices $0 < p < \infty$, $0 < q < r \le \infty$.

THEOREM 3.1. *Let $0 < p, q, r < \infty$ and $q < r$. Then the embedding $\ell_{p,q} \hookrightarrow \ell_{p,r}$ is strictly singular.*

Proof. Let $X \subset \ell_{p,q}$ be a closed subspace with $\dim X = \infty$ and fix a sequence $\widetilde{a} \in \ell_{p,q}$ with $\widetilde{a}(1) \ge \widetilde{a}(2) \ge \cdots > 0$ such that

$$0 < \|\widetilde{a}\|_{p,q} \le 1. \tag{3.1}$$

Having a sequence $0 = n_0 < n_1 < n_2 < \cdots$ and $u_k \in X_{n_{k-1}} = R_{n_{k-1}}(X)$ we denote, for $k \ge 1$,

$$v_k = P_{n_k}u_k,\ w_k = R_{n_k}u_k,$$
$$I_k = \{n_{k-1}+1, n_{k-1}+2, \dots, n_k\},$$
$$b_k = \min\{|v_k(j)|;\ v_k(j) \ne 0,\ j \in I_k\}.$$

Choose $0 < \delta < 1/T$.

We will construct by induction a sequence of integers $0 = n_0 < n_1 < \cdots$, a sequence of positive real numbers $\varepsilon_1, \varepsilon_2, \dots$, a sequence of functions $u_k \in X_{n_{k-1}}$, $k \geq 1$, and a fixed sequence $a(1) \geq a(2) \geq \cdots > 0$ such that if we set

$$c_k = \min\left\{\frac{1}{k n_k j^{q/p-1}};\ j = 1, \dots, n_k\right\}, \tag{3.2}$$

then for $k \geq 1$,

$$\|u_k\|_{p,q} = 1, \tag{3.3}$$

$$2n_{k-1} < n_k, \tag{3.4}$$

$$\operatorname{supp} v_k \subset I_k, \tag{3.5}$$

$$\varepsilon_{k+1} \leq \min\{b_k, c_k^{1/q}, a(n_k)\}, \tag{3.6}$$

$$\varepsilon_{k+1} n_k^{1/p} \leq 1, \tag{3.7}$$

$$|v_k(j)| \leq \varepsilon_k \quad \text{for } j \in \mathbb{N}, \tag{3.8}$$

$$a(j) \leq \widetilde{a}(j) \quad \text{for } j \in \mathbb{N}, \tag{3.9}$$

$$a(n_k + 1) \leq b_k, \tag{3.10}$$

$$1/T - \delta \leq \|v_k\|_{p,q} \leq 1, \tag{3.11}$$

$$\|w_k\|_{p,q} \leq \delta/(2T)^k. \tag{3.12}$$

Consider first $k = 1$. Find $u_1 \in X_0 = X$ with $\|u_1\|_{p,q} = 1$ and set $\varepsilon_1 = 1$. There exists $n_1 > n_0$ such that $\|w_1\|_{p,q} \leq \delta/(2T)$ and set $a(i) = \widetilde{a}(i)$, $i \in I_1$. Clearly,

$$1 = \|u_1\|_{p,q} \geq \|v_1\|_{p,q} \overset{(2.3)}{\geq} \frac{1}{T}\|u_1\|_{p,q} - \|w_1\|_{p,q} \geq \frac{1}{T} - \delta. \tag{3.13}$$

Now, it is easy to verify (3.3)–(3.12).

Suppose that we have constructed $n_0 < n_1 < \cdots < n_k$, ε_i, c_i for $1 \leq i \leq k$, the sequence $a(i)$ for $i \in I_1 \cup \cdots \cup I_k$ and functions $u_1, \dots, u_k \in X$ satisfying the above conditions.

Choose ε_{k+1} such that

$$\varepsilon_{k+1} \leq \min\{b_k, c_k^{1/q}, a(n_k)\} \quad \text{and} \quad \varepsilon_{k+1} n_k^{1/p} \leq 1. \tag{3.14}$$

According to Lemma 2.15 with $\varepsilon := \varepsilon_{k+1}$ and $\delta := \delta/(2T)^{k+1}$ there exist $m > 2n_k$ and $u \in X_{n_k}$ with $\|u\|_{p,q} = 1$ such that (2.5)–(2.10) are satisfied with $v := P_m u$ and $w := R_m u$. Set

$$n_{k+1} := m, \qquad u_{k+1} := u.$$

Then

$$v = v_{k+1} = P_{n_{k+1}} u_{k+1}, \qquad w = w_{k+1} = R_{n_{k+1}} u_{k+1}.$$

Set

$$\lambda_k := \min\left\{\frac{a(n_k)}{\widetilde{a}(n_k+1)}, \frac{b_k}{\widetilde{a}(n_k+1)}, 1\right\}, \tag{3.15}$$

$$a(j) := \lambda_k \widetilde{a}(j), \quad j \in I_{k+1}. \tag{3.16}$$

Now, (3.12) follows from (2.10).

Further

$$1 \geq \|v_{k+1}\|_{p,q} \overset{(2.9)}{\geq} \frac{1}{T} - \frac{\delta}{(2T)^{k+1}} \geq \frac{1}{T} - \delta,$$

which proves (3.11).

Properties (3.6) and (3.7) are immediate consequences of the choice of ε_{k+1} in (3.14).

Property (3.8) follows directly from (2.8). Moreover, by (3.15) and (3.16),

$$a(n_k+1) = \lambda_k \widetilde{a}(n_k+1) \leq b_k,$$

which confirms (3.10).

We now verify that $a(i)$ is non-increasing. If $i, j \in I_{k+1}$ with $i < j$, then

$$a(i) = \lambda_k \widetilde{a}(i) \geq \lambda_k \widetilde{a}(j) = a(j).$$

Moreover,

$$a(n_k+1) = \lambda_k \widetilde{a}(n_k+1) \overset{(3.15)}{\leq} \frac{a(n_k)}{\widetilde{a}(n_k+1)} \widetilde{a}(n_k+1) = a(n_k)$$

and $a(i)$ is indeed non-increasing.

By (3.15) we have $\lambda_k \leq 1$ and so by (3.16) we have (3.9).

Finally, properties (2.7), (2.6) and (2.5) give (3.5), (3.4) and (3.3), which finishes the construction of n_k, ε_k, u_k and a.

By (3.2) and (3.6) we obtain (with the convention $\sum_1^0 = 0$)

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\varepsilon_k^q}{k} \sum_{j=1}^{n_{k-1}} j^{q/p-1} &\overset{\text{L. 2.10}}{\lesssim} \sum_{k=2}^{\infty} \frac{\varepsilon_k^q}{k} n_{k-1}^{q/p} \overset{(3.6)}{\leq} \sum_{k=2}^{\infty} \frac{c_{k-1}}{k} n_{k-1}^{q/p} \\ &\overset{(3.2)}{\leq} \sum_{k=2}^{\infty} \frac{1}{k} \frac{1}{(k-1) n_{k-1}^{q/p}} n_{k-1}^{q/p} = \sum_{k=2}^{\infty} \frac{1}{k(k-1)} =: B < \infty. \end{aligned} \tag{3.17}$$

Due to (3.9), (3.1) and the embedding $\ell_{p,q} \hookrightarrow \ell_{p,r}$ we have

$$\|a\|_{p,r} \leq D_{q,r} \|a\|_{p,q} \leq D_{q,r} \|\widetilde{a}\|_{p,q} \leq D_{q,r}. \tag{3.18}$$

Set

$$z_N = \sum_{k=1}^{N} k^{-1/q} u_k.$$

Then $z_N \in X$. We estimate

$$(3.19)\qquad \|z_N\|_{p,q} = \Big\|\sum_{k=1}^{N} k^{-1/q} u_k\Big\|_{p,q} = \Big\|\sum_{k=1}^{N} k^{-1/q} v_k + \sum_{k=1}^{N} k^{-1/q} w_k\Big\|_{p,q}$$
$$\overset{(2.3)}{\geq} \frac{1}{T}\Big\|\sum_{k=1}^{N} k^{-1/q} v_k\Big\|_{p,q} - \Big\|\sum_{k=1}^{N} k^{-1/q} w_k\Big\|_{p,q}.$$

Clearly we have

$$(3.20)\qquad \Big\|\sum_{k=1}^{N} k^{-1/q} w_k\Big\|_{p,q} \overset{(2.4)}{\leq} \sum_{k=1}^{N} k^{-1/q} T^k \|w_k\|_{p,q} \overset{(3.12)}{\leq} \sum_{k=1}^{N} k^{-1/q} T^k \frac{\delta}{(2T)^k}$$
$$\leq \sum_{k=1}^{N} \frac{\delta}{2^k} \leq \delta.$$

Denote $A_k = \{i \in I_k;\ v_k(i) = 0\}$ and define

$$a_k(j) = (a\chi_{I_k})(j), \qquad j \in \mathbb{N},$$
$$\tilde{v}_k(j) = |v_k(j)| + a_k(j)\chi_{A_k}(j), \qquad j \in \mathbb{N},$$

where a is the fixed constructed sequence.

Fix now $i \in I_k$, $j \in I_{k+1}$. If $v_k(i) \neq 0$ then

$$\widetilde{v}_k(i) = |v_k(i)| \geq b_k \overset{(3.6)}{\geq} \varepsilon_{k+1} \overset{(3.8)}{\geq} |v_{k+1}(j)|$$

and also

$$\widetilde{v}_k(i) \geq b_k \overset{(3.10)}{\geq} a(n_k + 1) \geq a(j).$$

So

$$\widetilde{v}_k(i) \geq |v_{k+1}(j)| + a(j)\chi_{A_{k+1}}(j) = \widetilde{v}_{k+1}(j).$$

If $v_k(i) = 0$, then

$$\widetilde{v}_k(i) = a(i) \geq a(n_k) \overset{(3.6)}{\geq} \varepsilon_{k+1} \overset{(3.8)}{\geq} |v_{k+1}(j)|$$

and also

$$\widetilde{v}_k(i) \geq a(n_k) \geq a(n_k + 1) \geq a(j),$$

which gives again

$$\widetilde{v}_k(i) \geq |v_{k+1}(j)| + a(j)\chi_{A_{k+1}}(j) = \widetilde{v}_{k+1}(j).$$

This implies $\tilde{v}_k(i) \geq \tilde{v}_{k+1}(j)$ for $i \in I_k$ and $j \in I_{k+1}$, which immediately yields

$$(3.21)\qquad k^{-1/q}\tilde{v}_k(i) > (k+1)^{-1/q}\tilde{v}_{k+1}(j), \qquad i \in I_k,\ j \in I_{k+1},$$

and so

$$\left(\sum_{k=1}^{N} k^{-1/q}\tilde{v}_k\right)^* = \sum_{k=1}^{N} k^{-1/q}\tilde{v}_k^{\diamond} = \sum_{k=1}^{N} k^{-1/q} \sum_{j=n_{k-1}+1}^{n_k} \tilde{v}_k^{\diamond}\chi_{\{j\}}. \tag{3.22}$$

Since the supp v_k are pairwise disjoint we have

$$\begin{aligned}\Big\|\sum_{k=1}^{N} k^{-1/q}v_k\Big\|_{p,q} &= \Big\|\sum_{k=1}^{N} k^{-1/q}|v_k|\Big\|_{p,q} \\ &= \Big\|\sum_{k=1}^{N} k^{-1/q}\tilde{v}_k - \sum_{k=1}^{N} k^{-1/q}a_k\chi_{A_k}\Big\|_{p,q} \\ &\overset{(2.3)}{\geq} \frac{1}{T}\Big\|\sum_{k=1}^{N} k^{-1/q}\tilde{v}_k\Big\|_{p,q} - \Big\|\sum_{k=1}^{N} k^{-1/q}a_k\Big\|_{p,q}.\end{aligned}$$

Since $k^{-1/q} \leq 1$ and the supp a_k are pairwise disjoint we have

$$\Big\|\sum_{k=1}^{N} k^{-1/q}a_k\Big\|_{p,q} \leq \|a\|_{p,q},$$

which yields

$$\Big\|\sum_{k=1}^{N} k^{-1/q}v_k\Big\|_{p,q} \geq \frac{1}{T}\Big\|\sum_{k=1}^{N} k^{-1/q}\tilde{v}_k\Big\|_{p,q} - \|a\|_{p,q}. \tag{3.23}$$

Further

$$\begin{aligned}&\Big\|\sum_{k=1}^{N} k^{-1/q}\widetilde{v}_k\Big\|_{p,q}^q \\ &\overset{(3.22)}{=} \sum_{k=1}^{N} \sum_{j=n_{k-1}+1}^{n_k} j^{q/p-1}k^{-1}((\widetilde{v}_k)^{\diamond}(j))^q = \sum_{k=1}^{N} k^{-1} \sum_{j=n_{k-1}+1}^{n_k} j^{q/p-1}((\widetilde{v}_k)^{\diamond}(j))^q \\ &= \sum_{k=1}^{N} k^{-1} \sum_{j=1}^{n_k-n_{k-1}} (j+n_{k-1})^{q/p-1}((\widetilde{v}_k)^{\diamond}(j+n_{k-1}))^q \\ &\overset{(2.1)}{=} \sum_{k=1}^{N} k^{-1} \sum_{j=1}^{n_k-n_{k-1}} (j+n_{k-1})^{q/p-1}((\widetilde{v}_k)^*(j))^q \\ &\geq \sum_{k=1}^{N} k^{-1} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} (j+n_{k-1})^{q/p-1}((\widetilde{v}_k)^*(j))^q.\end{aligned}$$

Since $\tilde{v}_k^*(j) \geq v_k^*(j)$ we obtain

$$\Big\|\sum_{k=1}^{N} k^{-1/q}\tilde{v}_k\Big\|_{p,q}^{q} \geq \sum_{k=1}^{N} k^{-1} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} (j+n_{k-1})^{q/p-1}(v_k^*(j))^q \tag{3.24}$$
$$= \sum_{k=1}^{N} k^{-1} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} \left(\frac{j+n_{k-1}}{j}\right)^{q/p-1} j^{q/p-1}(v_k^*(j))^q.$$

Since

$$1 \leq \frac{j+n_{k-1}}{j} \leq 2 \quad \text{for } n_{k-1}+1 \leq j$$

we have

$$\min\{1, 2^{q/p-1}\} \leq \left(\frac{j+n_{k-1}}{j}\right)^{q/p-1} \leq \max\{1, 2^{q/p-1}\} \tag{3.25}$$

which yields, with (3.24),

$$\Big\|\sum_{k=1}^{N} k^{-1/q}\tilde{v}_k\Big\|_{p,q}^{q} \gtrsim \sum_{k=1}^{N} k^{-1} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} j^{q/p-1}(v_k^*(j))^q$$
$$\geq \sum_{k=1}^{N} k^{-1} \sum_{j=1}^{n_k-n_{k-1}} j^{q/p-1}(v_k^*(j))^q$$
$$- \sum_{k=1}^{N} k^{-1} \sum_{j=1}^{n_{k-1}} j^{q/p-1}(v_k^*(j))^q.$$

Clearly,

$$\sum_{j=1}^{n_k-n_{k-1}} j^{q/p-1}(v_k^*(j))^q = \|v_k\|_{p,q}^q,$$

and so

$$\Big\|\sum_{k=1}^{N} k^{-1/q}\tilde{v}_k\Big\|_{p,q}^{q} \overset{(3.8)}{\gtrsim} \sum_{k=1}^{N} k^{-1}\|v_k\|_{p,q}^q - \sum_{k=1}^{N} k^{-1} \sum_{j=1}^{n_{k-1}} j^{q/p-1}\varepsilon_k^q \tag{3.26}$$
$$\overset{(3.11)}{\geq} \sum_{k=1}^{N} k^{-1}\left(\frac{1}{T}-\delta\right)^q - \sum_{k=1}^{N} k^{-1} \sum_{j=1}^{n_{k-1}} j^{q/p-1}\varepsilon_k^q$$
$$\overset{(3.17)}{\gtrsim} \left(\frac{1}{T}-\delta\right)^q \sum_{k=1}^{N} k^{-1} - B \gtrsim A\ln N - B.$$

Now,

$$\begin{aligned}
\|z_N\|_{p,q} &\overset{(3.19)}{\geq} \frac{1}{T}\Big\|\sum_{k=1}^{N} k^{-1/q}\ v_k\Big\|_{p,q} - \Big\|\sum_{k=1}^{N} k^{-1/q}\ w_k\Big\|_{p,q} \\
&\overset{(3.20),\,(3.23)}{\geq} \frac{1}{T}\Big(\frac{1}{T}\Big\|\sum_{k=1}^{N} k^{-1/q}\ \tilde{v}_k\Big\|_{p,q} - \|a\|_{p,q}\Big) - \delta \\
&\overset{(3.26)}{\gtrsim} \frac{1}{T^2}(A\ln N - B)^{1/q} - \frac{\|a\|_{p,q}}{T} - \delta.
\end{aligned}$$

This implies

$$\|z_N\|_{p,q} \to \infty \quad \text{as } N \to \infty. \tag{3.27}$$

It remains to estimate $\|z_N\|_{p,r}$. Clearly

$$\begin{aligned}
\|z_N\|_{p,r} &= \Big\|\sum_{k=1}^{N} k^{-1/q} u_k\Big\|_{p,r} \\
&= \Big\|\sum_{k=1}^{N} k^{-1/q} v_k + \sum_{k=1}^{N} k^{-1/q} w_k\Big\|_{p,r} \\
&\leq T\Big(\Big\|\sum_{k=1}^{N} k^{-1/q} v_k\Big\|_{p,r} + \Big\|\sum_{k=1}^{N} k^{-1/q} w_k\Big\|_{p,r}\Big) \\
&\overset{(3.20)}{\leq} T\Big(\Big\|\sum_{k=1}^{N} k^{-1/q} v_k\Big\|_{p,r} + D_{q,r}\delta\Big).
\end{aligned} \tag{3.28}$$

Further,

$$\begin{aligned}
\Big\|\sum_{k=1}^{N} k^{-1/q} v_k\Big\|_{p,r}^{r} &\leq \Big\|\sum_{k=1}^{N} k^{-1/q} \tilde{v}_k\Big\|_{p,r}^{r} = \Big\|\Big(\sum_{k=1}^{N} k^{-1/q} \tilde{v}_k\Big)^{*}\Big\|_{p,r}^{r} \\
&\overset{(3.22)}{=} \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{n_k} j^{r/p-1} (\tilde{v}_k^{\diamond}(j))^r \\
&= \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1} (\tilde{v}_k^{\diamond}(j))^r \\
&\quad + \sum_{k=1}^{N} k^{-r/q} \sum_{j=2n_{k-1}+1}^{n_k} j^{r/p-1} (\tilde{v}_k^{\diamond}(j))^r.
\end{aligned} \tag{3.29}$$

First we estimate

$$
\begin{aligned}
(3.30)\qquad \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}(\tilde{v}_k^{\diamond}(j))^r
&= \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}((|v_k| + a_k\chi_{A_k})^{\diamond}(j))^r \\
&\overset{(3.8)}{\leq} \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}((\varepsilon_k + a_k\chi_{A_k})^{\diamond}(j))^r.
\end{aligned}
$$

Clearly,

$$
(3.31)\qquad \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1} \lesssim \int_{n_{k-1}}^{2n_{k-1}} x^{r/p-1}\,dx \lesssim n_{k-1}^{r/p}.
$$

Since a is a non-increasing sequence and ε_k is constant on I_k we have $(\varepsilon_k + a_k)^{\diamond}(j) = \varepsilon_k + a_k(j)$, which implies

$$
\begin{aligned}
(3.32)\qquad & \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}((\varepsilon_k + a_k\chi_{A_k})^{\diamond}(j))^r \\
&\leq \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}(\varepsilon_k + a_k(j))^r \\
&\lesssim \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}\varepsilon_k^r + \sum_{k=1}^{N} k^{-r/q} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}a_k^r(j) \\
&\overset{(3.31)}{\lesssim} \sum_{k=1}^{N} k^{-r/q} n_{k-1}^{r/p}\varepsilon_k^r + \sum_{k=1}^{N} \sum_{j=n_{k-1}+1}^{2n_{k-1}} j^{r/p-1}a_k^r(j) \\
&\overset{(3.7)}{\lesssim} \sum_{k=1}^{\infty} k^{-r/q} + \|a\|_{p,r}^r \overset{(3.18)}{=} K_1 < \infty.
\end{aligned}
$$

We estimate

$$
\begin{aligned}
\sum_{j=2n_{k-1}+1}^{n_k} j^{r/p-1}(\tilde{v}_k^{\diamond}(j))^r &\overset{(2.1)}{=} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} (j + n_{k-1})^{r/p-1}(\tilde{v}_k^{*}(j))^r \\
&\overset{(3.25)}{\lesssim} \sum_{j=n_{k-1}+1}^{n_k-n_{k-1}} j^{r/p-1}(\tilde{v}_k^{*}(j))^r
\end{aligned}
$$

$$\leq \sum_{j=1}^{n_k-n_{k-1}} j^{r/p-1}(\tilde{v}_k^*(j))^r = \|\tilde{v}_k\|_{p,r}^r$$
$$\overset{(2.2)}{\lesssim} D_{q,r}^r \|\tilde{v}_k\|_{p,q}^r \lesssim \|v_k + a_k\chi_{A_k}\|_{p,q}^r \lesssim \|v_k\|_{p,q}^r + \|a_k\|_{p,q}^r$$
$$\overset{(3.11)}{\lesssim} 1 + \|a\|_{p,q}^r := K_2 < \infty.$$

Thus

$$\sum_{k=1}^{N} k^{-r/q} \sum_{j=2n_{k-1}+1}^{n_k} j^{r/p-1}(\tilde{v}_k^\diamond(j))^r \leq K_2 \sum_{k=1}^{N} k^{-r/q} < \infty. \tag{3.33}$$

Now, (3.28), (3.29), (3.30), (3.32) and (3.33) show that

$$\|z_N\|_{p,r} \leq K < \infty \quad \text{for all } N$$

and this with (3.27) finishes the proof. ∎

Now we will investigate the case $q = \infty$.

Lemma 3.2. *Let X, Y, Z be quasi-Banach spaces and let $T : X \to Y$, $S : Y \to Z$ be bounded linear mappings. Assume that T is strictly singular. Then the composition $S \circ T : X \to Z$ is strictly singular.*

Proof. Assume $S \circ T$ is not strictly singular. Then there are an infinite-dimensional subspace $P \subset X$ and positive constants c_1, c_2 such that for all $u \in P$ we have

$$c_1\|S(T(u))\|_Z \leq \|u\|_X \leq c_2\|S(T(u))\|_Z.$$

Due to the boundedness of T we obtain, for each $u \in P$,

$$\frac{1}{\|T\|}\|T(u)\|_Y \leq \|u\|_X \leq c_2\|S(T(u))\|_Z \leq c_2\|S\|\,\|T(u)\|_Y$$

and so P and $T(P)$ are isomorphic. But this contradicts the assumption and so $S \circ T$ is strictly singular. ∎

Theorem 3.3. *Let $0 < p < \infty$ and $0 < q < r \leq \infty$. Then the embedding $\ell_{p,q} \hookrightarrow \ell_{p,r}$ is strictly singular.*

Proof. The case $r < \infty$ is proved in Theorem 3.1. Assume now $r = \infty$. Choose s, $q < s < \infty$. Then

$$\ell_{p,q} \hookrightarrow \ell_{p,s} \hookrightarrow \ell_{p,\infty}.$$

Since $\ell_{p,q} \hookrightarrow \ell_{p,s}$ is strictly singular by Theorem 3.1, we infer by Lemma 3.2 that the embedding $\ell_{p,q} \hookrightarrow \ell_{p,\infty}$ considered as a composition of two embeddings is strictly singular. ∎

4. The embedding is not finitely strictly singular. We prove in this section that the embedding $\ell_{p,q} \hookrightarrow \ell_{p,r}$ is not finitely strictly singular.

DEFINITION 4.1 (Rademacher system of functions). Given $n \in \mathbb{N}$ we define a function on the interval $[0,1]$ by

$$R_n(t) = \operatorname{sign} \sin 2^n \pi t.$$

We can describe R_n in a more natural way. Set $I_i = \left(\frac{i-1}{2^n}, \frac{i}{2^n}\right)$, $i = 1, 2, \ldots, 2^n$. Then we have

$$R_n(t) = \sum_{i=1}^{2^n} (-1)^{i+1} \chi_{I_i}(t).$$

Let us recall the well-known Khinchin's inequality. A proof can be found in [2, Theorem 1.4].

THEOREM 4.2 (Khinchin's inequality). *Let $0 < p < \infty$. Then there are positive constants A_p, B_p such that for all $N \in \mathbb{N}$ and $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ we have*

$$A_p \Big\| \sum_{i=1}^{N} a_i R_i \Big\|_{L^p(0,1)} \le \|a\|_{\ell_2} := \Big(\sum_{i=1}^{N} a_i^2 \Big)^{1/2} \le B_p \Big\| \sum_{i=1}^{N} a_i R_i \Big\|_{L^p(0,1)}.$$

To each $a = (a_1, \ldots, a_{2^n}) \in \mathbb{R}^{2^n}$ we can assign a function $A(\cdot)$ defined on $[0,1]$ by

$$A(t) = \sum_{i=1}^{2^n} a_i \chi_{I_i}(t).$$

For $0 < p, q < \infty$ we set

$$a_{p,q} = \min\{1, p/q, 2^{1-q/p}\}, \qquad b_{p,q} = \max\{1, p/q, 2^{1-q/p}\}.$$

LEMMA 4.3. *Let $0 < p, q < \infty$. Then*

$$a_{p,q} 2^{-nq/p}\, i^{q/p-1} \le \int_{I_i} t^{q/p-1}\, dt \le b_{p,q} 2^{-nq/p}\, i^{q/p-1} \tag{4.1}$$

for all $i = 1, \ldots, 2^n$.

Proof. Set $J(i) = \int_{I_i} t^{q/p-1}\, dt$. If $q \ge p$ we have

$$J(i) \le \frac{1}{2^n} \left(\frac{i}{2^n} \right)^{q/p-1} = 2^{-nq/p} i^{q/p-1}.$$

If $q \le p$ and $i \ge 2$ we obtain

$$\begin{aligned} J(i) &\le \frac{1}{2^n} \left(\frac{i-1}{2^n} \right)^{q/p-1} = \frac{1}{2^n} \left(\frac{i-1}{i} \right)^{q/p-1} \left(\frac{i}{2^n} \right)^{q/p-1} \\ &\le 2^{1-q/p} \frac{1}{2^n} \left(\frac{i}{2^n} \right)^{q/p-1} = 2^{1-q/p} 2^{-nq/p} i^{q/p-1}. \end{aligned}$$

Assume now $q \le p$ and $i = 1$. Then

$$J(i) = J(1) = \int_0^{1/2^n} t^{q/p-1}\,dt = \frac{p}{q}\Big(\frac{1}{2^n}\Big)^{q/p} = \frac{p}{q}2^{-nq/p} = \frac{p}{q}2^{-nq/p}i^{q/p-1}.$$

Altogether we have

$$J(i) \le \max\{1, p/q, 2^{1-q/p}\}2^{-nq/p}i^{q/p-1} = b_{p,q}2^{-nq/p}i^{q/p-1}.$$

Let us prove the second part of (4.1). Assume first $q \ge p$ and $i \ge 2$. Then

$$\begin{aligned} J(i) &\ge \frac{1}{2^n}\Big(\frac{i-1}{2^n}\Big)^{q/p-1} = \frac{1}{2^n}\Big(\frac{i-1}{i}\Big)^{q/p-1}\Big(\frac{i}{2^n}\Big)^{q/p-1} \\ &\ge 2^{1-q/p}\frac{1}{2^n}\Big(\frac{i}{2^n}\Big)^{q/p-1} = 2^{1-q/p}2^{-nq/p}i^{q/p-1}. \end{aligned}$$

If $q \ge p$ and $i = 1$ we have

$$J(i) = J(1) = \int_0^{1/2^n} t^{q/p-1}\,dt = \frac{p}{q}\Big(\frac{1}{2^n}\Big)^{q/p} = \frac{p}{q}\,2^{-nq/p} = \frac{p}{q}\,2^{-nq/p}i^{q/p-1}.$$

Finally, let $q \le p$. Then

$$J(i) \ge \frac{1}{2^n}\Big(\frac{i}{2^n}\Big)^{q/p-1} = 2^{-nq/p}i^{q/p-1}.$$

We can conclude that

$$J(i) \ge \min\{1, p/q, 2^{1-q/p}\}\,2^{-nq/p}i^{q/p-1} = a_{p,q}2^{-nq/p}i^{q/p-1}$$

and (4.1) is proved. ∎

Lemma 4.4. *Let $0 < p, q < \infty$. Then for all $a = (a_1, \dots, a_{2^n}) \in \mathbb{R}^{2^n}$ we have*

$$a_{p,q}^{1/q}\|a\|_{\ell_{p,q}} \le 2^{n/p}\|A\|_{L^{p,q}(0,1)} \le b_{p,q}^{1/q}\|a\|_{\ell_{p,q}}.$$

Proof. Fix $a = (a_1, \dots, a_{2^n}) \in \mathbb{R}^{2^n}$. The function A satisfies

$$\begin{aligned} \|A\|_{L^{p,q}(0,1)} &= \Big\|\sum_{i=1}^{2^n} a_i\chi_{I_i}(t)\Big\|_{L^{p,q}(0,1)} = \Big\|\sum_{i=1}^{2^n} a_i^*\chi_{I_i}(t)\Big\|_{L^{p,q}(0,1)} \\ &= \Big(\int_0^1 t^{q/p-1}\Big(\sum_{i=1}^{2^n} a_i^*\chi_{I_i}(t)\Big)^q\,dt\Big)^{1/q} = \Big(\sum_{i=1}^{2^n}\int_{I_i} t^{q/p-1}(a_i^*)^q\,dt\Big)^{1/q} \\ &= \Big(\sum_{i=1}^{2^n}(a_i^*)^q \int_{(i-1)/2^n}^{i/2^n} t^{q/p-1}\,dt\Big)^{1/q}. \end{aligned}$$

Clearly, by (4.1) we obtain

$$\begin{aligned} a_{p,q}^{1/q}2^{-n/p}\|a\|_{\ell^{p,q}} &= a_{p,q}^{1/q}2^{-n/p}\Big(\sum_{i=1}^{2^n}(a_i^*)^q i^{q/p-1}\Big)^{1/q} \\ &= \Big(\sum_{i=1}^{2^n}(a_i^*)^q a_{p,q}2^{-nq/p}i^{q/p-1}\Big)^{1/q} \le \|A\|_{L^{p,q}(0,1)} \\ &\le \Big(\sum_{i=1}^{2^n}(a_i^*)^q b_{p,q}2^{-nq/p}i^{q/p-1}\Big)^{1/q} \\ &= b_{p,q}^{1/q}2^{-n/p}\Big(\sum_{i=1}^{2^n}(a_i^*)^q i^{q/p-1}\Big)^{1/q} = b_{p,q}^{1/q}2^{-n/p}\|a\|_{\ell_{p,q}}, \end{aligned}$$

which proves the lemma. ■

LEMMA 4.5. *Let $0 < p < \infty$. Then for all $a = (a_1, \dots, a_{2^n}) \in \mathbb{R}^{2^n}$ we have*

$$\|a\|_{\ell_{p,\infty}} = 2^{n/p}\|A\|_{L^{p,\infty}(0,1)}.$$

Proof. Fix $a = (a_1, \dots, a_{2^n}) \in \mathbb{R}^{2^n}$. Then

$$\begin{aligned} \|A\|_{L^{p,\infty}(0,1)} &= \Big\|\sum_{i=1}^{2^n} a_i\chi_{I_i}(t)\Big\|_{L^{p,\infty}(0,1)} = \Big\|\sum_{i=1}^{2^n} a_i^*\chi_{I_i}(t)\Big\|_{L^{p,\infty}(0,1)} \\ &= \sup_{t\in(0,1)} t^{1/p}\sum_{i=1}^{2^n} a_i^*\chi_{I_i}(t) = \max_{i=1,\dots,2^n}\Big(\frac{i}{2^n}\Big)^{1/p} a_i^* \\ &= 2^{-n/p}\max_{i=1,\dots,2^n} i^{1/p}a_i^* = 2^{-n/p}\|a\|_{\ell_{p,\infty}}. \ ■ \end{aligned}$$

The next lemma is an easy modification of Khinchin's inequality for Lorentz spaces.

LEMMA 4.6. *Let $0 < p < \infty$ and $0 < q \le \infty$. Then there is a constant $C_{p,q}$ such that for all $N \in \mathbb{N}$ and $a = (a_1, \dots, a_N) \in \mathbb{R}^N$ we have*

$$C_{p,q}^{-1}\Big\|\sum_{i=1}^{N} a_iR_i\Big\|_{L^{p,q}(0,1)} \le \|a\|_{\ell_2} \le C_{p,q}\Big\|\sum_{i=1}^{N} a_iR_i\Big\|_{L^{p,q}(0,1)}.$$

Proof. Take $p_1 < p < p_2$. Then $L^{p_2}(0,1) \hookrightarrow L^{p,q}(0,1) \hookrightarrow L^{p_1}(0,1)$, so there is $K > 0$ such that for all u we have

$$K^{-1}\|u\|_{L^{p_1}(0,1)} \le \|u\|_{L^{p,q}(0,1)} \le K\|u\|_{L^{p_2}(0,1)}.$$

Consider the function $\sum_{i=1}^N a_i R_i$. By Theorem 4.2 we have

$$\begin{aligned}\frac{A_{p_2}}{K}\Big\|\sum_{i=1}^N a_i R_i\Big\|_{L^{p,q}(0,1)} &\le A_{p_2}\Big\|\sum_{i=1}^N a_i R_i\Big\|_{L^{p_2}(0,1)} \le \|a\|_{\ell_2}\\ &\le B_{p_1}\Big\|\sum_{i=1}^N a_i R_i\Big\|_{L^{p_1}(0,1)}\\ &\le KB_{p_1}\Big\|\sum_{i=1}^N a_i R_i\Big\|_{L^{p,q}(0,1)}. \ \blacksquare\end{aligned}$$

Consider an embedding $\ell_{p,q}\to\ell_{p,r}$ with $q<r$. Given $L_n \subset\subset \ell_{p,q}$ with $\dim L_n = n$ set

$$b_n(L_n) = \inf_{u\in L_n}\frac{\|u\|_{\ell_{p,r}}}{\|u\|_{\ell_{p,q}}}.$$

Say that $M\subset\mathbb{N}$ is an *interval* if for all $i<j<k$ we have $j\in M$ provided $i,k\in M$. Let $M_1,M_2\subset\mathbb{N}$. Say that $M_1\prec M_2$ if for all $i\in M_1, j\in M_2$ we have $i<j$.

Let $n\in\mathbb{N}$ and let $1\le i\le n$. Split the set $\{1,2,\dots,2^n\}$ into 2^i pairwise disjoint intervals M_k, $k=1,\dots,2^i$, and such that $\#M_k = 2^{n-i}$, and $M_k\prec M_l$ provided $k<l$. Define now sequences $r_{i,n}$ by

$$r_{i,n}(j) = \begin{cases}(-1)^{k+1}, & j\in M_k,\\ 0, & j>2^n.\end{cases}$$

For given n we can see

$$\begin{aligned}
r_{n,n} &= \overbrace{1,-1,\ \ 1,-1,\ \ 1,-1,\ \ 1,-1,\dots,-1,}^{2^{n-1}}\ \overbrace{1,\dots,\ \ 1,-1,\ \ 1,-1,\ \ 1,-1}^{2^{n-1}},0,0,\dots\\
r_{n-1,n} &= 1,\ \ 1,-1,-1,\ \ 1,\ \ 1,-1,-1,\dots,-1,\ \ 1,\dots,-1,-1,\ \ 1,\ \ 1,-1,-1,0,0,\dots\\
r_{n-2,n} &= 1,\ \ 1,\ \ 1,\ \ 1,-1,-1,-1,-1,\dots,-1,\ \ 1,\dots,\ \ 1,\ \ 1,-1,-1,-1,-1,0,0,\dots\\
&\ \ \vdots\\
r_{1,n} &= 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\dots,\ \ \ 1,-1,\dots,-1,-1,-1,-1,-1,-1,0,0,\dots.
\end{aligned}$$

It is a system analogous to Rademacher functions. We can also write

$$r_{i,n}(j) = \begin{cases}\operatorname{sign}\sin 2^{i-n}\pi j, & 1\le j\le 2^n,\\ 0, & j>2^n.\end{cases}$$

Note that the appropriate function to $r_{i,n}$ is R_i.

THEOREM 4.7. *Let $0<p<\infty$ and $0<q<r\le\infty$. Then there is $\alpha>0$ such that for all $n\in\mathbb{N}$,*

$$b_n := b_n(\mathrm{id}:\ell_{p,q}\hookrightarrow\ell_{p,r})\ge\alpha.$$

Proof. Consider $\mathcal{R}_n = \operatorname{span}(r_{1,n}, r_{2,n}, \dots, r_{n,n})$. Then $\dim(\mathcal{R}_n) = n$. Moreover, the appropriate function to a linear combination $\sum_{i=1}^n a_i r_{i,n}$ is a function $A = \sum_{i=1}^n a_i R_i(\cdot)$. Clearly, by Lemmas 4.5 and 4.4,

$$\begin{aligned} b_n &= \sup_{L_n} \inf_{a\in L_n} \frac{\|a\|_{\ell_{p,r}}}{\|a\|_{\ell_{p,q}}} \ge \inf_{a\in\mathcal{R}_n} \frac{\|a\|_{\ell_{p,r}}}{\|a\|_{\ell_{p,q}}} \\ &\ge \inf_{a\in\mathcal{R}_n} \frac{a_{p,q}^{1/q}}{b_{p,r}^{1/r}} \frac{2^{n/p}\|A\|_{L^{p,r}(0,1)}}{2^{n/p}\|A\|_{L^{p,q}(0,1)}}. \end{aligned}$$

Using Lemma 4.6 we obtain

$$\begin{aligned} b_n &\ge C \inf_{a\in\mathcal{R}_n} \frac{\|A\|_{L^{p,r}(0,1)}}{\|A\|_{L^{p,q}(0,1)}} \\ &= C \inf_{a\in\mathcal{R}_n} \frac{\|\sum_{i=1}^n a_i R_i\|_{L^{p,r}(0,1)}}{\|\sum_{i=1}^n a_i R_i\|_{L^{p,q}(0,1)}} \\ &\ge C \frac{C_{p,r}^{-1}\|a\|_{\ell_2}}{C_{p,q}\|a\|_{\ell_2}} = \frac{C}{C_{p,r}C_{p,q}} := \alpha. \ \blacksquare \end{aligned}$$

As an immediate consequence we obtain the following theorem.

THEOREM 4.8. *Let $0 < p < \infty$ and $0 < q < r \le \infty$. Then the embedding $\ell_{p,q} \hookrightarrow \ell_{p,r}$ is not finitely strictly singular.*

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