Diophantine equations for Littlewood polynomials

by

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Dedicated to the memory of Andrzej Schinzel

1. Introduction. There are many papers in the literature concerning polynomials with coefficients belonging to the set \{−1, 0, 1\}. For a short survey, we refer to the introduction of the paper [4] and the references there. If the coefficients are only ±1, the polynomials are called Littlewood polynomials. In [4], under certain necessary assumptions, an effective bound for \(\max(|x|, |y|, m)\) in the equation

\[ f(x) = y^m \]

is given in case \(f\) is a Littlewood polynomial and \(x, y, m\) are integral unknowns with \(m \geq 2\). In the present paper we give effective upper bounds for the solutions of the more general equation

\[ f(x) = ay^m + b \]

where \(a, b \in \mathbb{Q}\). Further, we describe all cases where a Littlewood polynomial can have infinitely many common values with another polynomial. In particular, we show that for any \(g(x) \in \mathbb{Q}[x]\), the equation

\[ f(x) = g(y) \]

can have only finitely many solutions in integers \(x, y\), except for certain explicitly given cases.

2. The theorems

THEOREM 2.1. Let \(f(x)\) be a Littlewood polynomial of degree \(n\) with \(n \geq 4\) and \(a, b \in \mathbb{Q}\) with \(a \neq 0\). Then all solutions \(x, y, m \in \mathbb{Z}\) of the equation

\[ f(x) = ay^m + b \]

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(1) \[ f(x) = ay^m + b \]

with \( m \geq 2 \) satisfy \( \max(|x|, |y|, m) \leq C_1 \), except when \( m = 2 \) and

(2) \[ f(x) \in \{ f^*(x), f^*(x) - 2f^*(0), xf^*(x) \pm 1 \} \]

with \( b = 0, -2f^*(0), \pm 1 \), respectively, where

\[ f^*(x) = \pm (x^{2\ell+1} + x^{2\ell} + \cdots + x^{\ell+1} - x^{\ell} - \cdots - 1) \]

or

\[ f^*(x) = \pm ((-x)^{2\ell+1} + (-x)^{2\ell} + \cdots + (-x)^{\ell+1} - (-x)^{\ell} + \cdots - 1) \]

with \( \ell = \lfloor (n-1)/2 \rfloor \), and the solutions are given by \( y = Q(x) \) with \( Q(\pm x) = \pm (x^k + \cdots + x + 1) \). Here \( C_1 \) is an effectively computable constant depending only on \( n, a, b \), and we use the convention that \( m \leq 3 \) if \( |y| \leq 1 \).

**Theorem 2.2.** Let \( f(x) \) be a Littlewood polynomial of degree \( n \) with \( n \geq 4 \) and \( g(x) \in \mathbb{Z}[x] \). Then the equation

(3) \[ f(x) = g(y) \]

has only finitely many solutions in integers \( x, y \), except when \( g(y) = f(T(y)) \) with some polynomial \( T(y) \) of degree \( \geq 1 \) having rational coefficients, or if \( f(x) \) is of the shape (2) and \( g(y) = a(cy+d)^2 + b \) for \( a, b \) as in Theorem 2.1 and \( c, d \in \mathbb{Q}, c \neq 0 \).

**Remark 1.** In both theorems the assumption \( \deg(f) \geq 4 \) is necessary. The case \( \deg(f) = 1 \) is trivial. It is easy to construct infinitely many \( f, a, b \) with \( \deg(f) = 2 \), and \( g(y) = ay^2 + b \) such that equation (1) becomes a Pell equation having infinitely many integer solutions \( x, y \). Finally, also for \( \deg(f) = 3 \) there exist cases not fitting in the families described in the theorems. For example, taking

\[ f(x) = x^3 + x^2 - x + 1, \quad a = \frac{1}{27}, \quad b = \frac{22}{27}, \]

in view of

\[ f(x) - b = a(3x + 5)(3x - 1)^2 \]

we see that equation (1) has infinitely many integer solutions \( x, y \).

It is also necessary that \( f(x) \) is not of the shape (2). We demonstrate this only for one case. The other cases can be checked similarly. Take

\[ f(x) = x(x^{2\ell+1} + \cdots + x^{\ell+1} - x^\ell - \cdots - 1) + 1 = x^n + \cdots + x^{n/2+1} - x^{n/2} - \cdots - x + 1. \]

One can readily check that

\[ f(x) - 1 = x(x - 1)(x^{n/2-1} + \cdots + x + 1)^2. \]
As the Pell equation $x(x - 1) = 2y^2$ has infinitely many solutions, equation $[1]$ has infinitely many solutions in integers $x, y$ when taking $m = 2$, $a = 2$, $b = 1$.

Remark 2. Let $f(x)$ be a Littlewood polynomial and write
\[ f(x) = \varepsilon_0 x^n + \varepsilon_1 x^{n-1} + \varepsilon_2 x^{n-2} + \cdots + \varepsilon_{n-1} x + \varepsilon_n \]
with $\varepsilon_i \in \{-1, 1\}$ ($i = 0, 1, \ldots, n$). Applying the transformation $x \mapsto -x$ if necessary, we may assume that $\varepsilon_0 = \varepsilon_1$. Then, taking out a factor $-1$ if necessary, we may suppose that $\varepsilon_0 = \varepsilon_1 = 1$. Since our statements concern the root structure of $f(x)$ and $f'(x)$, and equations involving $f(x)$, we can clearly do this in our arguments without loss of generality. So from this point on, we shall assume that $f(x)$ is of the shape
\[ f(x) = x^n + x^{n-1} + \varepsilon_2 x^{n-2} + \cdots + \varepsilon_{n-1} x + \varepsilon_n. \]

3. Auxiliary results. We present some lemmas which we shall use in the proofs of the theorems. By the height $H(F(x))$ of a polynomial $F(x)$ with integer coefficients we mean the maximum of the absolute values of its coefficients.

Lemma 3.1. Let $F(x) \in \mathbb{Z}[x]$ of degree $D$ and height $H$ have two distinct (complex) roots, and let $B$ be a non-zero rational number. Then the equation
\[ F(x) = By^m \]
with $x, y \in \mathbb{Z}$, $|y| > 1$, implies that $m < C_2$, where $C_2$ is effectively computable and depends only on $B$, $D$ and $H$.

Proof. The statement follows from the Schinzel–Tijdeman theorem [6].

The following lemma is a theorem of Brindza [2]. For any finite set $S$ of primes, write $\mathbb{Q}_S$ for those rationals whose denominators (in their primitive forms) are composed exclusively from the primes in $S$. By the height $h(s)$ of a rational number $s$ we mean the height of its minimal defining polynomial.

Lemma 3.2. Let $F(x) \in \mathbb{Z}[x]$ be of degree $D$ and height $H$, and write
\[ F(x) = A \prod_{i=1}^{\ell} (x - \gamma_i)^{r_i}, \]
where $A$ is the leading coefficient of $F$, and $\gamma_1, \ldots, \gamma_\ell$ are the distinct complex roots of $F(x)$, with multiplicities $r_1, \ldots, r_\ell$, respectively. Further, let $m$ be an integer with $m \geq 2$, and put
\[ q_i = \frac{m}{(m, r_i)} \quad (i = 1, \ldots, \ell). \]
Suppose that $(q_1, \ldots, q_\ell)$ is not a permutation of any of the $\ell$-tuples
\[ (q, 1, \ldots, 1) \quad (q \geq 1), \quad (2, 2, 1, \ldots, 1). \]
Then for any finite set $S$ of primes and non-zero rational $B$, the solutions $x, y \in \mathbb{Q}_S$ of the equation

$$F(x) = By^m$$

satisfy

$$\max(h(x), h(y)) < C_3,$$

where $C_3$ is effectively computable and depends only on $B, m, D, H, S$.

In the proof of Theorem 2.2, the decomposability of polynomials will play an important role. We call $F(x) \in \mathbb{Q}[x]$ decomposable over $\mathbb{Q}$ if there exist $G(x), H(x) \in \mathbb{Q}[x]$ with $\deg(G) > 1, deg(H) > 1$ such that $F = G(H)$, and otherwise indecomposable.

**Lemma 3.3.** Let $F(x) \in \mathbb{Z}[x]$ be of the form

$$F(x) = x^n + u_1x^{n-1} + \cdots + u_{n-1}x + u_n.$$  

If $\gcd(u_1, n) = 1$ then $F(x)$ is indecomposable over $\mathbb{Q}$.

**Proof.** The statement is a simple consequence of [3, Theorems 2 and 3].

We further apply a deep result of Bilu and Tichy. Let $\delta$ be a non-zero rational number and $\mu$ be a positive integer. Then the $\mu$th Dickson polynomial is defined by

$$D_\mu(x, \delta) := \sum_{i=0}^{[\mu/2]} d_{\mu,i}x^{\mu-2i}$$

where $d_{\mu,i} = \frac{\mu}{\mu - i} \binom{\mu - i}{i} (-\delta)^i$.

For properties of Dickson polynomials see e.g. [5]. The polynomials $F, G \in \mathbb{Q}[x]$ form a standard pair over $\mathbb{Q}$ if either $(F(x), G(x))$ or $(G(x), F(x))$ appears in Table 1.

**Table 1.** Standard pairs. Here $\alpha, \beta$ are non-zero rational numbers, $\mu, \nu, q$ are positive integers, $p$ is a non-negative integer, $v(x) \in \mathbb{Q}[x]$ is a non-zero but possibly constant polynomial.

<table>
<thead>
<tr>
<th>Kind</th>
<th>Standard pair (unordered)</th>
<th>Parameter restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>$(x^q, \alpha x^p v(x)^q)$</td>
<td>$0 \leq p &lt; q, (p,q) = 1,$ $p + \deg(v) &gt; 0$</td>
</tr>
<tr>
<td>Second</td>
<td>$(x^2, (\alpha x^2 + \beta)v(x)^2)$</td>
<td>—</td>
</tr>
<tr>
<td>Third</td>
<td>$(D_\mu(x, \alpha^\nu), D_\nu(x, \alpha^\mu))$</td>
<td>$\gcd(\mu, \nu) = 1$</td>
</tr>
<tr>
<td>Fourth</td>
<td>$(\alpha^{-\mu/2}D_\mu(x, \alpha), -\beta^{-\nu/2}D_\nu(x, \beta))$</td>
<td>$\gcd(\mu, \nu) = 2$</td>
</tr>
<tr>
<td>Fifth</td>
<td>$((\alpha x^2 - 1)^3, 3x^4 - 4x^3)$</td>
<td>—</td>
</tr>
</tbody>
</table>

**Lemma 3.4 (Bilu, Tichy [1, Theorem 1.1]).** Let $f(x), g(x) \in \mathbb{Q}[x]$ be non-constant polynomials. Then the following two statements are equivalent:
(I) The equation \( f(x) = g(y) \) has infinitely many rational solutions \( x, y \) with a bounded denominator.

(II) We have \( f = \varphi(F(\kappa)) \) and \( g = \varphi(G(\lambda)) \), where \( \kappa(x), \lambda(x) \in \mathbb{Q}[x] \) are linear polynomials, \( \varphi(x) \in \mathbb{Q}[x] \), and \( F(x), G(x) \) form a standard pair over \( \mathbb{Q} \) such that the equation \( F(x) = G(y) \) has infinitely many rational solutions with a bounded denominator.

Lemma 3.5. Let \( f(x) \) be a Littlewood polynomial and \( b \in \mathbb{Q} \). If \( f(x) - b \) has a root of multiplicity \( \geq 3 \), or has at least two roots of multiplicities \( \geq 2 \), then \( b \in \mathbb{Z} \). Further, in both cases the multiple roots of \( f(x) - b \) are units.

Proof. Let \( f(x) \) be given by (4) as in Remark 2. For any root \( \alpha \) of \( f(x) - b \) let \( v_\alpha(x) \) denote the monic minimal defining polynomial of \( \alpha \) over \( \mathbb{Q} \). If \( \alpha \) is a triple (or higher multiplicity) root of \( f(x) - b \), then let \( v(x) = v_\alpha(x) \). Similarly, if \( \alpha \) is a double root of \( f(x) - b \) with \( \deg(v_\alpha) \geq 2 \), then let \( v(x) = v_\alpha(x) \). Finally, if \( \deg(v_\alpha) = 1 \) in the case of at least two roots of multiplicities \( \geq 2 \), then take any other multiple root \( \beta \) of \( f(x) - b \) and let \( v(x) = v_\alpha(x)v_\beta(x) \).

Observe that in each case, we can write
\[
(5) \quad f(x) - b = g(x)(v(x))^\ell
\]
with a monic \( g \in \mathbb{Q}[x] \) and \( \ell \geq 2 \), and either \( k := \deg(v) \geq 2 \) or \( \ell \geq 3 \). Write \( b = q_1/q_2 \) with coprime integers \( q_1, q_2 (q_2 > 0) \), and \( v(x) = v^*(x)/v_0 \), \( g(x) = g^*(x)/g_0 \) with \( v^*, g^* \in \mathbb{Z}[x] \) primitive polynomials, \( v_0, g_0 \) positive integers. (Since \( v \) and \( g \) are monic, such \( v^*, g^*, v_0, g_0 \) exist.) Rewrite (5) as
\[
(6) \quad q_2 f(x) - q_1 = \frac{q_2}{g_0 v_0^\ell} g^*(x)(v^*(x))^\ell.
\]

Since \( q_2 f(x) - q_1 \) and \( g^*(x)(v^*(x))^\ell \) are primitive polynomials in \( \mathbb{Z}[x] \) (the latter one by the Gauss lemma), we see that \( q_2/g_0 v_0^\ell = 1 \) in (6). Suppose that \( q_2 \neq 1 \). Let \( p \) be any prime with \( p \nmid q_2 \). Then taking (6) modulo \( p \), we see that
\[
(7) \quad v^*(x) \equiv c \pmod{p}
\]
for some integer \( c \) with \( p \nmid c \). Taking now derivatives in (5) we obtain
\[
(8) \quad f'(x) = (v(x))^\ell h(x)
\]
with
\[
h(x) = g'(x)v(x) + \ell g(x)v'(x).
\]

Note that \( \deg(f') = n - 1 \), \( \deg(h) = n - 1 - k(\ell - 1) \). There exist coprime positive integers \( h_0, h_1 \) and a primitive polynomial \( h^*(x) \in \mathbb{Z}[x] \) such that \( h(x) = h_1 h^*(x)/h_0 \). Thus we can rewrite (8) as
\[
(9) \quad f'(x) = \frac{h_1}{v_0^{\ell-1} h_0} v^*(x)^{\ell-1} h^*(x).
\]
Recall Remark 2. Since
\[ f'(x) = nx^{n-1} + (n-1)x^{n-2} + \cdots + 2\varepsilon_{n-2}x + \varepsilon_{n-1} \]
as well as \( v^\ast(x)^{\ell-1}h^\ast(x) \) are primitive polynomials in \( \mathbb{Z}[x] \), we see that \( h_1/v_0^{\ell-1}h_0 = 1 \). Taking (9) modulo \( p \) with the above prime \( p|q_2 \), we deduce by (7) that
\[ \deg(f'(x) \mod p) \leq n - 1 - k(\ell - 1). \]
However, since the coefficients of the first two terms of \( f'(x) \) are \( n \) and \( n-1 \), which are coprime, we see that
\[ \deg(f'(x) \mod p) \geq n - 2. \]
As either \( k, \ell \geq 2 \) or \( \ell \geq 3 \), this is a contradiction. Hence we conclude that \( q_2 = 1 \), hence \( b \in \mathbb{Z} \).

Next we show that under the assumptions of the lemma, the multiple roots of \( f(x) - b \) are units. Let \( \alpha \) be any such root. Then, since \( b \in \mathbb{Z} \), \( \alpha \) is an algebraic integer. Thus \( v_\alpha(x) \in \mathbb{Z}[x] \) and \( (v_\alpha(x))^2 | f(x) - b \) over \( \mathbb{Z} \), whence \( v_\alpha(x) | f'(x) \) over \( \mathbb{Z} \). As \( f'(0) = \pm 1 \), our claim follows.

We shall also apply the following information concerning the roots of shifted Littlewood polynomials.

**Lemma 3.6.** Let \( f(x) \) be a Littlewood polynomial of degree \( n \) and let \( b \in \mathbb{Z} \). Then for any root \( \alpha \) of \( f(x) - b \) with \( |\alpha| > 2 \) we have
\[ \frac{|\alpha| - 2}{|\alpha| - 1}|\alpha|^n < |b|. \]

**Proof.** We have
\[ |\alpha|^n \leq |\alpha|^{n-1} + |\alpha|^{n-2} + \cdots + |\alpha| + 1 + |b| = \frac{|\alpha|^{n-1}}{|\alpha| - 1} + |b| < \frac{|\alpha|^n}{|\alpha| - 1} + |b|. \]
From this the statement follows.

Finally, we shall also use the following result from [4].

**Lemma 3.7.** Let \( Q(x) \in \mathbb{Z}[x] \) be a non-constant polynomial and \( r, t \) be integers with \( 0 \leq r < t \), \( t \geq 2 \). If all the coefficients of the polynomial \( (x-1)^rQ(x)^t \) belong to \( \{-1, 1\} \), then \( t = 2 \), \( r = 1 \) and \( Q(x) \) is of the form
\[ Q(x) = \pm(x^k + \cdots + x + 1) \]
with some \( k \geq 1 \). If all the coefficients of the polynomial \( (x+1)^rQ(x)^t \) belong to \( \{-1, 1\} \), then \( t = 2 \), \( r = 1 \) and \( Q(x) \) is of the form
\[ Q(-x) = \pm(x^k + \cdots + x + 1) \]
with some \( k \geq 1 \).

**Proof.** The first statement is [4, Lemma 3.6]. The second statement follows by the substitution \( x \mapsto -x \).
4. Proofs of the theorems

Proof of Theorem 2.1  
Let \( f(x) \) be given by (4). The bound for \( m \) follows from Lemma 3.1 unless \( f(x) - b \) is of the shape \( f(x) = (x - s)^n \) with \( s \in \mathbb{Q} \). Since Lemma 3.5 implies \( b \in \mathbb{Z} \), we have \( s \in \mathbb{Z} \). However, we get a contradiction with the fact that the coefficient of \( x^{n-1} \) is 1 in \( f(x) - b \).

Thus, by Lemma 3.1 we may assume that \( m \) is fixed. Now our claim follows from Lemma 3.2, except for the following two cases:

(i) \( m \geq 2 \) is arbitrary and \( f(x) - b = (P(x))^r(Q(x))^t \) with \( 0 \leq r < t, t \geq 2 \) and \( P, Q \in \mathbb{Q}[x], \deg(P) \leq 1 \);

(ii) \( m = 2 \) and \( f(x) - b = P(x)(Q(x))^2 \) with \( P, Q \in \mathbb{Q}[x], \deg(P) = 2 \).

Throughout the proof we suppose without loss of generality that \( P, Q \) are both monic.

For \( n = 4 \) a simple computer calculation shows that (i) is impossible, while (ii) can only occur when we have

\[
(f(x), b) = (x^4 + x^3 - x^2 - x \pm 1, \pm 1).
\]

Since this possibility is among the exceptional cases (2), we may assume that \( n \geq 5 \). Lemma 3.5 implies \( b \in \mathbb{Z} \), so we infer that \( P, Q \in \mathbb{Z}[x] \). We consider cases (i) and (ii) in turn.

Assume first that (i) holds. If \( r = 0 \) or \( P(x) \) is constant then, since the coefficient of \( x^{n-1} \) is 1 on the left-hand side, while it is divisible by \( t \) on the right-hand side, we get a contradiction. So \( r \geq 1 \) and \( P(x) \) is linear. We write \( P(x) = x - s \) with \( s \in \mathbb{Z} \). Since either \( t \geq 3 \) or \( \deg(Q) \geq 2 \), and the roots of \( Q \) are multiple roots of \( f(x) - b \), by Lemma 3.5 we obtain \( Q(0) = \pm 1 \). Further, the same lemma implies that for \( r \geq 2 \) we have \( s = \pm 1 \). We apply Lemma 3.7 and obtain a contradiction. We conclude that the statement of Theorem 2.1 holds if \( r \geq 2 \).

So we may assume that \( r = 1 \). Comparing the constant terms, we see that \( b = \varepsilon_n \pm s \). Lemma 3.6 with \( n \geq 5 \), \( |b| \leq 3 \) yields \( |s| < 3 \).

If \( |s| = 1 \) then Lemma 3.7 implies that \( Q(x) \) is of the form (10) or (11). This leads to the first two options of (2).

If \( s = 0 \) then comparing the coefficient of \( x^{n-1} \) on both sides we get a contradiction: it is 1 on the left-hand side, while it is a multiple of \( t \) on the right-hand side.

Hence we are left with \( s = \pm 2 \). Since \( s \) is a root of \( f(x) - b \), we have (recall Remark 2)

\[
f(s) - b = s^n + s^{n-1} + \varepsilon_2 s^{n-2} + \cdots + \varepsilon_{n-1} s + \varepsilon_n - b = 0.
\]

In view of \( |s^n + s^{n-1}| \geq 2^{n-1} \), and as by \( \varepsilon_n - b = \pm 2 \) we have

\[
|\varepsilon_2 s^{n-2} + \cdots + \varepsilon_{n-1} s + \varepsilon_n - b| \leq 2^{n-2} + 2^{n-3} + \cdots + 2^1 + 2 = 2^{n-1},
\]

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the equality (12) is only possible if \( s = -2 \) and all other terms in (12) have signs opposite to that of \( s^n \). Thus we conclude
\[
(13) \quad f(x) - b = x^n + x^{n-1} - x^{n-2} + \cdots + (-1)^{n-2}x + (-1)^{n-1} \cdot 2.
\]

Hence we easily get
\[
(Q(x))^t = x^{n-1} - x^{n-2} + \cdots.
\]

However, this is not possible, since the coefficient of \( x^{n-2} \) is not divisible by \( t \). Thus the theorem is true in case (i).

Suppose that (ii) holds. Write \( P(x) = x^2 + ux + w \). Recall that \( n = \deg(f) \geq 5 \); thus now in fact \( n \geq 6 \). First we clarify the parity of \( u \) and \( w \).

Taking the equation in (ii) modulo 2 we obtain
\[
x^n + x^{n-1} + x^{n-2} + x^{n-3} + \cdots \equiv (x^2 + ux + w)(x^{n-2} + \delta_1 x^{n-4} + \delta_2 x^{n-6} + \cdots) \pmod{2}.
\]

Here a priori \( \delta_1, \delta_2 \in \{0, 1\} \). Comparing the coefficients of \( x^{n-1}, x^{n-3}, x^{n-2} \) (in this order) on both sides, we successively find that \( u \) is odd, \( \delta_1 = 1 \) and \( w \) is even.

Since \( n \geq 6 \), Lemma 3.5 implies (as in the case \( r \geq 2 \) of (i)) that \( Q(0) = \pm 1 \), and consequently \( f(0) - b = w \). Observe that \( P(x) \) has a root \( \alpha \) with \( |\alpha| \geq \sqrt{|w|} \). Since \( b = -w \pm 1 \), for \( n \geq 6 \) Lemma 3.6 yields
\[
\frac{\sqrt{|w|} - 2}{\sqrt{|w|} - 1} |w|^3 < |w| + 1.
\]

This implies \( |w| \leq 4 \). Hence by the parity condition above, we obtain \( w \in \{0, \pm 2, \pm 4\} \). Assume first that \( w = 0 \). Then \( f(0) - b = 0 \), and on taking out a factor \( x \) the equality in (ii) simplifies to
\[
\frac{f(x) - b}{x} = (x + u)(Q(x))^2.
\]

Observe that the polynomial on the left-hand side is a Littlewood polynomial. So \( u = \pm 1 \), and by Lemma 3.7 we obtain (similarly to the case \( r = 1, s = \pm 1 \) of (i)) that \( Q(x) \) is of the form (10) or (11). This yields the third option of (2) and
\[
\frac{f(x) - b}{x} = (x - 1)(x^k + \cdots + x + 1)^2.
\]

From this our claim follows in the case \( w = 0 \). For the remaining values of \( w \), Lemma 3.6 implies
\[
\frac{|\alpha_{1,2}| - 2}{|\alpha_{1,2}| - 1} |\alpha_{1,2}|^6 < |w| + 1 \leq 5
\]
We handle these possibilities in turn.

Then where

Since \( M < 4 \), however, a computer calculation shows that \( |\alpha_1| < 2.1 \). By a further calculation, both roots are below this bound in absolute value only if \( u = 0 \) for \( w = -4 \); \( |u| \leq 4 \) for \( w = 4 \); \( |u| \leq 1 \) for \( w = -2 \); \( |u| \leq 3 \) for \( w = 2 \). Since \( u \) must be odd, we are left with the following polynomials:

\[
P(x) = x^2 \pm 3x + 4, \ x^2 \pm x + 4, \ x^2 \pm x - 2, \ x^2 \pm 3x + 2, \ x^2 \pm x + 2.
\]

We handle these possibilities in turn.

Let \( \alpha \) be a root of any of the polynomials \( P(x) = x^2 \pm 3x + 4, \ x^2 \pm x + 4 \). Then \( |\alpha| = 2 \), and \( \alpha \) is a root of \( f(x) - b \). Since the constant term of \( f(x) - b \) is 4, we obtain

\[
2^n = |\alpha|^n \leq |\alpha|^{n-1} + \cdots + |\alpha|^4 + M = 2^n - 16 + M,
\]

where

\[
M = \max_{\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}} |\varepsilon_3 \alpha^3 + \varepsilon_2 \alpha^2 + \varepsilon_1 \alpha + 4|.
\]

However, a computer calculation shows that \( M < 16 \) for these choices of \( P(x) \). Hence these cases cannot occur.

Consider now the polynomials \( P(x) = x^2 \pm x - 2, \ x^2 \pm 3x + 2 \). Observe that \(-2\) or \(2\) is a root of these polynomials. Further, the constant term of \( f(x) - b \) equals \(\pm 2\) in these cases. Thus we get (similar to (13), recall that \( f(x) \) is of the form (4), and that \( n \) is even)

\[
f(x) - b = x^n + x^{n-1} - x^{n-2} + x^{n-3} - x^{n-4} + \cdots - x^2 + x - 2
\]

with a root \(-2\). This, in view of the signs of the constant terms, rules out the polynomials \( P(x) = x^2 \pm 3x + 2 \). In case \( P(x) = x^2 \pm x - 2 \) we get, since \( f \) is a Littlewood polynomial,

\[
(Q(x))^2 = x^{n-2} + x^{n-4} + \cdots + x^2 + 1.
\]

Then, writing

\[
Q(x) = x^{(n-2)/2} + q_1 x^{(n-4)/2} + q_2 x^{(n-6)/2} + \cdots ,
\]

from the coefficients of \( x^{n-3} \) we see that \( q_1 = 0 \), and then from the coefficients of \( x^{n-4} \) that \( 2q_2 = 1 \). This contradicts \( Q(x) \in \mathbb{Z}[x] \). So these cases are not possible either.

Thus we are left with \( P(x) = x^2 \pm x + 2 \).

\[
Q(x) = x^k + q_1 x^{k-1} + \cdots + q_{k-1} x + q_k
\]

with \( n = 2k + 2 \). Recall that \( q_1, \ldots, q_k \in \mathbb{Z} \) with \( q_k = \pm 1 \). First we argue that the equality

\[
f(x) - b = (x^2 \pm x + 2)(Q(x))^2
\]
implies that $q_1, \ldots, q_k$ are all odd. Indeed, if $i$ is the smallest index with $q_i$ even, then the coefficients of $x^{2i-1}, x^{2i}, x^{2i+1}$ would all be even in $(Q(x))^2$, so the coefficient of $x^{2i+1}$ would be even in $f(x) - b$, a contradiction. Expanding the first few coefficients on the right-hand side of (14) we get

$$x^n + x^{n-1} + \varepsilon_2 x^{n-2} + \cdots = x^n + (2q_1 \pm 1)x^{n-1} + (2q_2 + q_1^2 \pm 2q_1 + 2)x^{n-2} + \cdots.$$ 

Hence, using the fact that $q_1$ and $q_2$ are odd, we obtain successively

$$q_1 = 1, \quad P(x) = x^2 - x + 2, \quad q_2 = -1, \quad \varepsilon_2 = -1.$$ 

Write $\alpha$ for a root of $x^2 - x + 2$. Since $\alpha$ is a root of $f(x) - b$, we obtain

$$|\alpha^n + \alpha^{n-1} - \alpha^{n-2}| \leq |\alpha^{n-3} + \cdots + |\alpha| + |f(0) - b|.$$ 

Note that $|\alpha| = \sqrt{2}$ and $|\alpha^2 + \alpha - 1| > 6$. Since the constant term $f(0) - b$ of $f(x) - b$ is 2, we obtain

$$6 \cdot (\sqrt{2})^{n-2} < \frac{(\sqrt{2})^{n-2} - 1}{\sqrt{2} - 1} + 1.$$ 

This gives a contradiction, which shows that $P(x) = x^2 - x + 2$ is also impossible. Hence the theorem is proved.

**Proof of Theorem 2.2** Let $f(x)$ be of the form (4). Then Lemma 3.3 implies that $f(x)$ is indecomposable over $\mathbb{Q}$. Thus, if equation (3) has infinitely many solutions in integers $x, y$, then by Lemma 3.4 we have only two options. Either $g(x)$ is of the form $g(x) = f(T(x))$ with some $T(x) \in \mathbb{Z}[x]$ (in which case (3) clearly has infinitely many integer solutions indeed) or $f(x)$ is of the shape

$$f(x) = AF(ux + w) + B,$$

with some $A, B, u, w \in \mathbb{Q}$, $Au \neq 0$, where $F$ belongs to a standard pair from Table 1. Only the latter case needs more investigation.

Suppose first that $F(x)$ belongs to case I or II of Table 1. Since a Littlewood polynomial cannot be a perfect power of another polynomial, in these cases $G(x)$ is a perfect power of $x$ and $F(x)$ is the other possibility in Table 1. Therefore $f(x)$ is of the shape occurring as (i) or (ii) in the proof of Theorem 2.1 and $g(x) = P(cx + d)$ for some $c, d \in \mathbb{Q}, c \neq 0$. So the statement follows from Theorem 2.1 in cases I and II.

Now assume that we are in case III or IV of Table 1. Then $F(x)$ is a constant multiple of a Dickson polynomial in (15). Clearly, (15) is equivalent to

$$f \left( \frac{x - w}{u} \right) = AD_n(x, \delta) + B$$
with some non-zero $\delta \in \mathbb{Q}$, where $n$ is the degree of $f$. (Here in case III, $A$ is replaced by another constant.) Recall that $n \geq 4$. Since $D_n$ is either an odd or an even polynomial (depending on the parity of $n$), comparing the coefficients of $x^{n-1}$ and $x^{n-3}$ in (16), we get

$$-nw \frac{u}{u^n} + \frac{1}{u^{n-1}} = 0$$

or

$$-\left(\frac{n}{3}\right) w^3 \frac{u}{u^n} + \left(\frac{n}{2}\right) \frac{w^2}{u^{n-1}} - \varepsilon_2 \frac{nw}{u^{n-2}} + \varepsilon_3 \frac{1}{u^{n-3}} = 0,$$

respectively. These equalities imply

$$(3\varepsilon_3 - 3\varepsilon_2 + 1)n^2 - 1 = 0.$$ 

Hence $n = \pm 1$, which is excluded.

Finally, suppose that $F(x)$ comes from case V of Table 1. The polynomial $(\alpha x^2 - 1)^3$ is an even polynomial of degree 6 and it can be handled and excluded in the same way as the possibilities in cases III and IV. If $F(x) = 3x^4 - 4x^3$, then (15) gives

$$f(x) = A(3(ux + w)^4 - 4(ux + w)^3) + B.$$ 

A simple calculation shows that $f(x)$ cannot be a Littlewood polynomial. Hence the theorem is proved.

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