The cubic Pell equation $L$-function

by

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Dedicated to the memory of Andrzej Schinzel

1. Introduction. Fix the quadratic number field $K = \mathbb{Q}(\sqrt{-3})$ with ring of integers $\mathcal{O}_K$. We consider the family of cubic Pell equations

$$mx^3 - dny^3 = 1,$$

where $d > 1$ is a fixed cubefree rational integer, $x, y \in \mathcal{O}_K$ are variables, and $m, n \in \mathcal{O}_K$ with $m, n$ squarefree.

Set $\lambda = \sqrt{-3}$. In Definition 5.1, the Pell equation $L$-function

$$L_d(s) := \sum_{\nu \in \lambda^{-3}\mathcal{O}_K} \tau(\nu)\tau(1 + d\nu)|\nu(1 + d\nu)|^{-s} \quad (s \in \mathbb{C}, \text{Re}(s) > 1)$$

is introduced where $\tau(\nu)$ is the Fourier coefficient of the cubic theta function (see Proposition 4.2). It will be shown that the coefficient $\tau(\nu)\tau(1 + d\nu)$ vanishes unless

$$\nu = mx^3, \quad 1 + d\nu = ny^3,$$

where $m, n, x, y \in \mathcal{O}_K$ with $m, n$ squarefree, i.e., the coefficient of the Dirichlet series for $L_d(s)$ vanishes unless it comes from a solution of the Pell equation.

The main result of this paper is the meromorphic continuation of the expression $F(s, (d+1)^2/2d) \cdot L_d(s)$ to $\text{Re}(s) > \frac{1}{2}$, where

$$F(s, x) := \int_{0}^{1} \left( t^{s-4/3} + t^{s-2/3} \right) \left( x \cdot t + \frac{(t - 1)^2}{2} \right)^{-s} dt$$

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for $s \in \mathbb{C}$ with $\text{Re}(s) > \frac{1}{3}$, and $x > 0$ is a special case of Émile Picard’s integral representation [17] of the Appell hypergeometric function.

**Theorem 1.1.** Let $d > 1$ be a cubefree rational integer. The function $\mathcal{F}(s, \frac{(d+1)^2}{2d}) \cdot L_d(s)$ has a meromorphic continuation to $\text{Re}(s) > \frac{1}{2}$ with at most a simple pole at $s = \frac{2}{3}$.

Fix $\varepsilon > 0$. In the region $\text{Re}(s) > \frac{1}{2} + \varepsilon$ and $|s - \frac{2}{3}| > \varepsilon$ we have the bound

$$|\mathcal{F}(s, \frac{(d+1)^2}{2d}) \cdot L_d(s)| \ll_{d, \varepsilon} |s|^3.$$

**Remark 1.2.** The proof of Theorem 1.1 is given in §10 while the residue of the possible pole at $s = \frac{2}{3}$ is determined in Step 8 in §10. Theorem 10.1 introduces a modification of $L_d(s)$ (denoted $L_d^\#(s)$) and it is proved that $L_d^\#(s)$ has meromorphic continuation to $\text{Re}(s) > \frac{1}{3}$ with a possible pole at $s = \frac{2}{3}$ and infinitely many poles on the line $\text{Re}(s) = \frac{1}{2}$ coming from Maass cusp forms including a double pole at $s = \frac{1}{2}$.

With the method of steepest descent (see Lemma 11.2) we show that the Picard integral satisfies

$$\left|\mathcal{F}\left(s, \frac{(d+1)^2}{2d}\right)\right| \sim C_d \cdot |s|^{-1/2}$$

for $\text{Re}(s) > \frac{1}{3}$ fixed and $|\text{Im}(s)| \to \infty$, where $C_d > 0$ is a fixed constant depending at most on $d$. This allows us to obtain the meromorphic continuation and growth of $L_d(s)$ away from poles.

**Theorem 1.3.** Fix $\varepsilon > 0$. Let $d > 1$ be a cubefree rational integer. The function $L_d(s)$ has a meromorphic continuation to $\text{Re}(s) > \frac{1}{2}$ with at most a simple pole at $s = \frac{2}{3}$ and possible poles at the zeros of $\mathcal{F}(s, \frac{(d+1)^2}{2d})$ with $\frac{1}{2} < \text{Re}(s) \leq 1$. In the region $\text{Re}(s) > \frac{1}{2} + \varepsilon$ and $|s - \rho| > \varepsilon$ (for any pole $\rho$ of $L_d(s)$) we have the bound $L_d(s) \ll_{d, \varepsilon} |s|^{7/2}$.

Numerical computations suggest that $\mathcal{F}(s, \frac{(d+1)^2}{2d})$ is nonvanishing on the relevant region, so we formulate the following conjecture.

**Conjecture 1.4.** Fix $\varepsilon > 0$. Let $d > 1$ be a cubefree rational integer. The function $L_d(s)$ has a meromorphic continuation to $\text{Re}(s) > \frac{1}{2}$ with at most a simple pole at $s = \frac{2}{3}$. In the region $\text{Re}(s) > \frac{1}{2} + \varepsilon$ and $|s - \frac{2}{3}| > \varepsilon$ we have the bound $L_d(s) \ll_{d, \varepsilon} |s|^{7/2}$.

In the recent paper of Hoffstein–Jung–Lee [7] it is proved that

$$\sum_{\nu \in \lambda^{-3} \mathcal{O}_K} |\tau(\nu)|^2 |\nu|^{-2s} = 2 \cdot 3^{5+3s} \frac{(1+3^{1-2s})(1-3^{-s})\zeta_K(3s-1)\zeta_K(s)}{(1-3^{-2s})\zeta_K(2s)}$$
where $\zeta_K(s)$ is the zeta function of $K = \mathbb{Q}(\sqrt{-3})$. This result is obtained by considering the inner product of an Eisenstein series with the square of the absolute value of the cubic theta function. The proof of Theorem 1.1 follows a similar approach except, like the Takhtajan–Vinogradov trace formula [20], we use a Poincaré series instead of the Eisenstein series, and unlike both of those, we replace $|\theta|^2$ with $\overline{\theta d}$, where $\theta_d(z) = \theta(dz)$. That is, we consider the inner product $\langle P_1(*, s), \theta \theta_d \rangle$, where $P_1$ is a Poincaré series, $\theta$ is the cubic theta function for $K$, and $\theta_d(z) = \theta(dz)$.

The spectral side of the trace formula is presented in §7 and is evaluated by standard methods, which gives the meromorphic continuation of the inner product of the Poincaré series with $\theta \theta_d$ as well as the explicit computation of the poles and their residues. The possible pole at $s = \frac{2}{3}$ comes from the continuous spectrum (see Theorem 8.17). The poles at the eigenvalues of the Laplacian and the double pole at $s = \frac{1}{2}$ come from the discrete spectrum (see Theorem 8.1). The growth of the inner product away from the poles is obtained in §8.

The main difficulty in proving Theorem 1.1 comes from the geometric side of the trace formula in §9, which involves the function $S_d(s)$ defined in Theorem 9.1. The function $S_d(s)$ is essentially the inner product under consideration with a single term involving multiple gamma functions removed; it is $S_d(s)$ that eventually gives rise to the $L$-functions, so that the spectral analysis of the inner product and knowledge of the gamma function term yields the results for those $L$-functions. Although the coefficients $\tau(\nu)\tau(1 + d\nu)$ coming from the cubic theta function appear in $S_d(s)$, they are twisted by Appell hypergeometric functions, so it is not at all clear how to extract the meromorphic continuation of $L_d(s)$ from $S_d(s)$. What arises naturally is the Dirichlet series $L^\#_d(s)$ (defined in §5), which can be thought of as a version of $L_d(s)$ twisted by Appell hypergeometric functions.

The key idea for extracting $L_d(s)$ by itself is to first express $S_d(s)$ as an integral involving a ratio of gamma functions times the Appell hypergeometric function (with the Appell hypergeometric function coming from an integral involving the product of two Bessel functions that appears when directly taking the inner product) and then shift the line of integration, picking up residues at the poles of the gamma functions. We then replace the Appell hypergeometric function appearing in the shifted integral and the residues with the Émile Picard integral, which enormously simplifies all subsequent computations. In particular, we obtain an integral of the form $\mathcal{F}(s, x)$, allowing us to use the binomial expansion of $\mathcal{F}(s, x)$ about $x = x_0$. This ultimately yields the same expression in the main term for each of the summands, which we can thus pull out of the sum to obtain the $\mathcal{F}(s, \frac{(d+1)^2}{2d})$ term in $L^\#_d(s)$. Specifically, the $k = 0$ and $k = 1$ terms of the binomial
expansion yield expressions involving $L^\#_d(s)$ and $L^\#_d(s + \frac{1}{2})$, while the later terms are much smaller.

The complete proof of the meromorphic continuation of $L_d(s)$ and $L^\#_d(s)$ as well as their growth properties is presented in eight separate steps in §10.

We believe that the techniques used in this paper could be applied to higher-degree theta functions. However, those cases have additional issues to be dealt with that do not arise in the cubic case.

2. Basic notation. The following notation will be used consistently throughout this paper.

- $d \neq 1$ is a cubefree positive rational integer;
- $K := \mathbb{Q}(\lambda)$ with $\lambda = \sqrt{-3}$;
- $O_K := \mathbb{Z}[e^{2\pi i/3}]$;
- $(\frac{a}{b})_3$ (with $a, b \in O_K$) is the cubic residue symbol for $K$.

**Definition 2.1** (Upper half-plane $\mathfrak{h}^2$). We define the classical upper half-plane $\mathfrak{h}^2 := \{x + iy \mid x \in \mathbb{R}, y > 0\}$.

**Definition 2.2** (Quaternionic upper half-space $\mathfrak{h}^3$). The quaternions are expressions of the form $a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = ijk = -1$. Further, $\mathbb{C}$ is identified with the set of quaternions with $j = k = 0$. We define the quaternionic upper half-space to be $\mathfrak{h}^3 := \{x + jy \mid x \in \mathbb{C}, y > 0\}$.

**Definition 2.3** (Trace and exponential function on $\mathfrak{h}^3$). Let $z = x + jy \in \mathfrak{h}^3$. Define the trace $\text{tr}(z) := 2\text{Re}(x) + 2iy$ and the exponential function $e(z) := e^{2\pi i \text{tr}(z)} = e^{-4\pi y}e^{4\pi i \text{Re}(x)}$.

**Definition 2.4** (Action of $\text{SL}(2, \mathbb{C})$ on $\mathfrak{h}^3$). The $\text{SL}(2, \mathbb{C})$ action on $\mathfrak{h}^3$ is given by

$$
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix} z = (az + b)(cz + d)^{-1} \quad (z \in \mathfrak{h}^3, a, b, c, d \in \mathbb{C}, ad - bc = 1).
$$

**Definition 2.5** (The subgroups $\Gamma(N)$ and $\Gamma_\infty(N)$). Let $N \in O_K \setminus \{0\}$. We define the congruence subgroup

$$
\Gamma(N) := \left\{ \gamma \in \text{SL}(2, O_K) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}
$$

and

$$
\Gamma_\infty(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, O_K) \mid c \equiv 0 \pmod{N} \right\}.
$$
3. Cubic Gauss sums

**Definition 3.1** (Cubic Gauss sum). Fix $\lambda = \sqrt{-3}$. For any $\mu \in \lambda^{-3}O_K$ and $a \in O_K$ satisfying $a \equiv 1 \pmod{3}$, the cubic Gauss sum is

$$g(\mu, a) := \sum_{\beta \in O_K \pmod{a}} \left( \frac{3\beta}{a} \right) e\left( \frac{3\mu\beta}{a} \right).$$

For $\mu \in O_K$ this simplifies to

$$g(\mu, a) = \sum_{\beta \in O_K \pmod{a}} \left( \frac{\beta}{a} \right) e\left( \frac{\mu\beta}{a} \right).$$

**Definition 3.2** (The function $\tau$). We define $\tau$ according to the following formulae, where in all of the following expressions, $a, b \in O_K$ with $a, b \equiv 1 \pmod{3}$, and $a$ is squarefree:

$$\tau(\mu) := \begin{cases} 
   g(\lambda^2, a) |b/a| 3^{n/2 + 2} & \text{if } \mu = \pm \lambda^{3n-4}ab^3, n \in \mathbb{Z}_{\geq 1}, \\
   e^{-2\pi i/9} g(\omega\lambda^2, a) |b/a| 3^{n/2 + 2} & \text{if } \mu = \pm \omega\lambda^{3n-4}ab^3, n \in \mathbb{Z}_{\geq 1}, \\
   e^{2\pi i/9} g(\omega^2\lambda^2, a) |b/a| 3^{n/2 + 2} & \text{if } \mu = \pm \omega^2\lambda^{3n-4}ab^3, n \in \mathbb{Z}_{\geq 1}, \\
   g(1, a) |b/a| 3^{n + 5/2} & \text{if } \mu = \pm \lambda^{3n-3}ab^3, n \in \mathbb{Z}_{\geq 0}, \\
   0 & \text{otherwise}. 
\end{cases}$$

**Remark 3.3.** From unique factorization in $K$, any $\mu \in \lambda^{-3}O_K$ has at most one of the above forms, and if it does have one of those forms, the values of $a, b, \text{ and } n$ are unique, so the above definition is correct and we may denote by $a(\mu), b(\mu), \text{ and } n(\mu)$ the values of $a, b, \text{ and } n$ in the decomposition of $\mu$.

4. Cubic theta function. Patterson [15], following Kubota [11], gave a detailed explicit study of the simplest cubic theta function for the field $\mathbb{Q}(\sqrt{-3})$, which we briefly review. Let $\Gamma$ be any congruence subgroup of $SL(2, O_K)$ whose level is divisible by 9. Then for $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma$ define

$$\kappa(\gamma) := \begin{cases} 
   \left( \frac{c}{d} \right) & \text{if } c \neq 0, \\
   1 & \text{if } c = 0, 
\end{cases}$$

to be the Kubota symbol on $\Gamma$. This allows us to construct a metaplectic Eisenstein series.

**Definition 4.1** (Cubic metaplectic Eisenstein series for $\mathbb{Q}(\sqrt{-3})$). Let $s \in \mathbb{C}$ with $\text{Re}(s) > 1$. For $z = x + jy \in \mathbb{H}$ and $I_s(z) := y^s$, we define the cubic metaplectic Eisenstein series by

$$E^{(3)}(z, s) := \sum_{\gamma \in \Gamma_\infty(9)} \kappa(\gamma) I_s(\gamma z)^{2s}.$$
The Eisenstein series \( E^{(3)} \) satisfies the automorphic relation
\[
E^{(3)}(s, \gamma z) = \kappa(\gamma) E^{(3)}(s, z)
\]
for all \( \gamma \in \Gamma(9) \) and has a simple pole at \( s = \frac{2}{3} \) with residue the cubic theta function defined by
\[
\theta(z) = 2 \operatorname{Res}_{s=\frac{2}{3}} E^{(3)}(z, s).
\]

**Proposition 4.2 (Fourier expansion of \( \theta(z) \)).** Let \( z = x + jy \in \mathfrak{h}^3 \). The Fourier expansion of the cubic theta function for \( Q(\sqrt{-3}) \) is given by
\[
\theta(z) := \sigma y^{2/3} + \sum_{\mu \in \lambda^{-3} \mathcal{O}_K, \mu \neq 0} \tau(\mu) y K_{1/3}(4\pi|\mu|y) e(\mu z),
\]
where \( \sigma = \frac{9\sqrt{3}}{2} \) and \( K_{1/3} \) is the modified Bessel function of the second kind with order \( \frac{1}{3} \).

**Proof.** This was proved by Patterson \cite{15, 16}. \( \blacksquare \)

**5. The cubic Pell equation \( L \)-functions \( L_d(s), L_d^*(s), L_d^+(s) \)**

**Definition 5.1 (The cubic Pell equation \( L \)-function \( L_d(s) \)).** Let \( d \neq 1 \) be a cubefree positive integer. Then for \( s \in \mathbb{C} \) with \( \Re(s) \) sufficiently large we define the \( L \)-function
\[
L_d(s) := \sum_{\nu \in \lambda^{-3} \mathcal{O}_K} \tau(\nu) \overline{\tau(1+d\nu)} |\nu(1+d\nu)|^{-s}.
\]

**Proposition 5.2.** The Dirichlet series \( L_d(s) \) in Definition 5.1 converges absolutely for \( \Re(s) > 1 \).

**Proof.** Each of the Gauss sums that may occur in the definition of \( \tau \) has absolute value less than or equal to \( |a| \) (see \cite{14}), so
\[
|\tau(\mu)| \ll 3^{n/2}|b|
\]
for \( \mu \) of the general form \( \mu = \pm \omega^j \lambda^{3n-k} ab^3 \) \((0 \neq a, b \in \mathcal{O}_K, n \in \mathbb{Z}_{\geq 0}) \) with \( 0 \leq j \leq 2 \) and \( k \in \{3, 4\} \). By Cauchy’s inequality, we have
\[
\sum_{\nu} \frac{\tau(\nu) \overline{\tau(1+d\nu)}}{|\nu(1+d\nu)|^{1+\epsilon}} \leq \left( \sum_{\nu} \frac{|\tau(\nu)|^2}{|\nu|^{2+2\epsilon}} \right)^{1/2} \cdot \left( \sum_{\nu} \frac{|\tau(1+d\nu)|^2}{|(1+d\nu)|^{2+2\epsilon}} \right)^{1/2} \ll \left( \sum_{a \in \mathcal{O}_K} \sum_{b \in \mathcal{O}_K} \sum_{n=1}^{\infty} \frac{3^{n}|b|^{2}}{|a|^2 3^{3n}|b|^{6}|1+\epsilon|} \right).
\]
Since the Dirichlet series \( \sum_{a \in \mathcal{O}_K, a \neq 0} |a|^{-s} \) converges absolutely for \( \Re(s) > 2 \), and the sums over \( n, b \) also converge for \( \Re(s) > 2 \), this completes the proof. \( \blacksquare \)
Definition 5.3 (The cubic Pell equation $L^*_d(s)$). Let $d \neq 1$ be a cubefree positive integer. Then for $s \in \mathbb{C}$ with $\text{Re}(s)$ sufficiently large we define the $L$-function

$$L^*_d(s) := \sum_{\nu \in \lambda^{-3} \mathcal{O}_K} \tau(\nu) \tau(1 + d\nu)|\nu(1 + d\nu)|^{-s} \left( a_d(\nu) - \frac{d^2 + 1}{2d} \right),$$

where

$$a_d(\nu) := \frac{|\nu|^2 + |1 + d\nu|^2 - 1}{2|\nu| \cdot |1 + d\nu|} = \frac{d^2 + 1}{2d} \left( 1 + \mathcal{O}_d(|\nu(1 + d\nu)|^{-1/2}) \right).$$

Remark 5.4. It follows from Proposition 5.2 that the Dirichlet series for $L^*_d(s)$ converges absolutely for $\text{Re}(s) > \frac{1}{2}$.

Definition 5.5 (The cubic Pell equation $L^\#_d(s)$). Let $d \neq 1$ be a cubefree positive integer. Then for $s \in \mathbb{C}$ with $\text{Re}(s)$ sufficiently large we define the $L$-function

$$L^\#_d(s) = \mathcal{F} \left( s, \frac{(d + 1)^2}{2d} \right) L_d(s) - s \cdot \mathcal{F} \left( s + 1, \frac{(d + 1)^2}{2d} \right) L^*_d(s).$$

6. Spectral decomposition of $L^2(\Gamma(9d^2) \setminus \mathfrak{h}^3)$. Let $z = x + iy \in \mathfrak{h}^3$ where $x = x_0 + i x_1 \in \mathbb{C}$ with $x_1, x_2 \in \mathbb{R}$. The Laplace–Beltrami differential operator on $\mathfrak{h}^3$ is given by

$$\Delta := y^2 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial}{\partial y}.$$

Recall that $d \neq 1$ is a cubefree positive rational integer. The Hilbert space $L^2(\Gamma(9d^2) \setminus \mathfrak{h}^3)$ decomposes into eigenspaces of $\Delta$ given by Maass cusp forms, Eisenstein series, and residues of Eisenstein series. The Maass cusp forms $u_j(z)$ ($j = 1, 2, \ldots$) have Laplace–Beltrami eigenvalues $\lambda_j = 2s_j(2 - 2s_j)$, where $s_j = \frac{1}{2} + it_j$. The Selberg eigenvalue conjecture predicts that all $t_j$ are in $\mathbb{R}$, and it is known that there can only be finitely many $t_j \in i \cdot \mathbb{R}$. The Maass cusp forms satisfy the automorphic relation $u_j(\gamma z) = u_j(z)$ for all $z \in \mathfrak{h}^3$ and $\gamma \in \Gamma(9d^2)$, and their Fourier expansions are given by

$$u_j(z) = \sum_{m \in \lambda^{-3} \mathcal{O}_K \atop m \neq 0} c_j(m)yK_{2s_j-1}(4\pi|m|y)e(mx) \quad (c_j(m) \in \mathbb{C}),$$

where for $v \in \mathbb{C}$,

$$K_v(y) := \frac{1}{2} \int_0^\infty e^{-\frac{1}{2}y(u+u^{-1})}u^v \frac{du}{u}$$

is the modified Bessel function of the second kind. The Maass forms are normalized so that $\langle u_j, u_j \rangle = 1$. 

The cubic Pell equation $L^*_d(s)$. Let $d \neq 1$ be a cubefree positive integer. Then for $s \in \mathbb{C}$ with $\text{Re}(s)$ sufficiently large we define the $L$-function

$$L^*_d(s) := \sum_{\nu \in \lambda^{-3} \mathcal{O}_K} \tau(\nu) \tau(1 + d\nu)|\nu(1 + d\nu)|^{-s} \left( a_d(\nu) - \frac{d^2 + 1}{2d} \right),$$

where

$$a_d(\nu) := \frac{|\nu|^2 + |1 + d\nu|^2 - 1}{2|\nu| \cdot |1 + d\nu|} = \frac{d^2 + 1}{2d} \left( 1 + \mathcal{O}_d(|\nu(1 + d\nu)|^{-1/2}) \right).$$

Remark 5.4. It follows from Proposition 5.2 that the Dirichlet series for $L^*_d(s)$ converges absolutely for $\text{Re}(s) > \frac{1}{2}$.

Definition 5.5 (The cubic Pell equation $L^\#_d(s)$). Let $d \neq 1$ be a cubefree positive integer. Then for $s \in \mathbb{C}$ with $\text{Re}(s)$ sufficiently large we define the $L$-function

$$L^\#_d(s) = \mathcal{F} \left( s, \frac{(d + 1)^2}{2d} \right) L_d(s) - s \cdot \mathcal{F} \left( s + 1, \frac{(d + 1)^2}{2d} \right) L^*_d(s).$$

6. Spectral decomposition of $L^2(\Gamma(9d^2) \setminus \mathfrak{h}^3)$. Let $z = x + iy \in \mathfrak{h}^3$ where $x = x_0 + i x_1 \in \mathbb{C}$ with $x_1, x_2 \in \mathbb{R}$. The Laplace–Beltrami differential operator on $\mathfrak{h}^3$ is given by

$$\Delta := y^2 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial}{\partial y}.$$
Let $\kappa_1 = \infty$ and $\kappa_2, \ldots, \kappa_r \in \mathbb{C}$ denote the cusps of $\Gamma(9d^2)$. The continuous spectrum consists of the Eisenstein series $E_{\kappa_\ell} (\ell = 1, \ldots, r)$ where the Eisenstein series corresponding to the cusp at infinity is defined by

$$E_{\infty}(z, s) = \sum_{\gamma \in \Gamma_{\infty}(9d^2) \setminus \Gamma(9d^2)} I_s(\gamma z)$$

and the Eisenstein series corresponding to another cusp $\kappa_\ell$ is defined by

$$E_{\kappa_\ell}(z, s) = E_{\infty}(\alpha z, s),$$

where $\alpha \kappa_\ell = \infty$. For a cusp $\kappa_\ell$, the Fourier expansion of $E_{\kappa_\ell}(z, s)$ is

$$E_{\kappa_\ell}(z, s) = \delta_{\kappa_\ell, \infty} y^{2s} + c_{\kappa_\ell}(0, s) y^{2-2s} + \sum_{m \in \lambda^{-3}O_K, \ m \neq 0} c_{\kappa_\ell}(m, s) y K_{2s-1}(4\pi|m|y) \cdot e(mx),$$

where the Fourier coefficients $c_{\kappa_\ell}(m, s)$ are complex numbers.

**Definition 6.3 (Petersson inner product).** For $F, G \in L^2(\Gamma(9d^2) \setminus \mathfrak{h}^3)$ we define the inner product

$$\langle F, G \rangle := \int_{\Gamma(9d^2) \setminus \mathfrak{h}^3} F(z) \overline{G(z)} \frac{dx \, dy}{y^3}.$$

**Proposition 6.4 (Spectral decomposition of $L^2(\Gamma(9d^2) \setminus \mathfrak{h}^3)$).** Suppose that $u_j$ ($j = 1, 2, \ldots$) is an orthonormal basis of Maass cusp forms for $L^2(\Gamma(9d^2) \setminus \mathfrak{h}^3)$ and let $E_{\kappa_\ell}$ ($\ell = 1, \ldots, r$) be the Eisenstein series in (6.2). Define $u_0(z)$ to be the constant function $\equiv \text{Vol}(\Gamma(9d^2) \setminus \mathfrak{h}^3)^{-1/2}$.

Let $F \in L^2(\Gamma(9d^2) \setminus \mathfrak{h}^3)$. Then

$$F(z) = \sum_{j=0}^\infty \langle F, u_j \rangle + \sum_{\ell=1}^r \int_{-\infty}^\infty \langle F, E_{\kappa_\ell}(\ast, \frac{1}{2} + iu) \rangle \cdot E_{\kappa_\ell}(z, \frac{1}{2} + iu) \, du.$$

**Proof.** See [18]. \end{proof}

**7. Spectral side of the cubic Takhtajan–Vinogradov trace formula.** We begin with the definition of the Poincaré series which plays a crucial role in the evaluation of the cubic Takhtajan–Vinogradov trace formula.

**Definition 7.1 (Poincaré series).** Let $d > 1$ be a rational cubefree integer and let $n \in \mathbb{Z}_{>0}$. Then for $z = x + jy \in \mathfrak{h}^3$ and $I_s(z) = y^s$ for $s \in \mathbb{C}$ we define the Poincaré series

$$P_n(z, s) := \sum_{\gamma \in \Gamma_{\infty}(9d^2) \setminus \Gamma(9d^2)} I_s(\gamma z) e(n \gamma z),$$
The cubic Pell equation $L$-function

which converges absolutely and uniformly on compact subsets of $s \in \mathbb{C}$ with $\text{Re}(s) > 1$.

We also define $\theta_d(z) := \theta(dz)$. The cubic Takhtajan–Vinogradov trace formula is an identity that is obtained by computing the inner product $\langle P_1(\ast, s), \theta \overline{\theta_d} \rangle$ in two different ways. The first way is by replacing the Poincaré series $P_1$ with its spectral expansion into Maass cusp forms $u_j$ ($j = 1, 2, \ldots$) and integrals of Eisenstein series $E_{\kappa_\ell} (\ell = 1, \ldots r)$, where $\kappa_1, \ldots, \kappa_r$ are the cusps of $\Gamma(9d^2)$.

**Proposition 7.2 (Spectral decomposition of $P_1$).** Let $\text{Re}(s) > 1$. Then we have the spectral expansion

$$P_1(z, s) = \sum_{j=1}^{\infty} \langle P_1(\ast, s), u_j \rangle \cdot u_j(z)$$

$$+ \frac{1}{4\pi} \sum_{\ell=1}^{r} \int_{-\infty}^{\infty} \langle P_1(\ast, s), E_{\kappa_\ell}(\ast, \frac{1}{2} + iu) \rangle E_{\kappa_\ell}(z, \frac{1}{2} + iu) \, du.$$

**Proof.** This follows immediately from Proposition 6.4.

**Remark 7.3.** The Poincaré series is orthogonal to the residual spectrum which consists only of the constant function $u_0(z)$, so this term does not appear in the spectral expansion.

**Theorem 7.4 (Spectral side of the cubic Takhtajan–Vinogradov trace formula).** Let $\text{Re}(s) > 1$. Then

$$\langle P_1(\ast, s), \theta \overline{\theta_d} \rangle = C(s) + E(s),$$

where

$$C(s) := \sum_{j=1}^{\infty} \langle P_1(\ast, s), u_j \rangle \cdot \langle u_j, \theta \overline{\theta_d} \rangle$$

is the cuspidal contribution and

$$E(s) = \frac{1}{4\pi} \sum_{\ell=1}^{r} \int_{-\infty}^{\infty} \langle P_1(\ast, s), E_{\kappa_\ell}(\ast, \frac{1}{2} + iu) \rangle \cdot \langle E_{\kappa_\ell}(\ast, \frac{1}{2} + iu), \theta \overline{\theta_d} \rangle \, du$$

is the Eisenstein contribution.

**Proof.** This follows immediately after taking the inner product with $\theta \overline{\theta_d}$ of the spectral decomposition of $P_1$ given in Proposition 7.2.

8. **Bounding the spectral side.** In this section we will obtain the meromorphic continuation and bounds (away from poles) for both the cuspidal and Eisenstein contributions to the spectral side of the cubic Takhtajan–Vinogradov trace formula given in Theorem 7.4.
Theorem 8.1 (Bound for the spectral side). The spectral contribution $\mathcal{C}(s) + \mathcal{E}(s)$ given in Theorem 7.4 has a meromorphic continuation to $\text{Re}(s) > 0$ whose set of poles $\mathcal{P}$ in this region consists of a double pole at $s = \frac{1}{2}$ and possible simple poles at $s = \frac{2}{3}$ and $s = s_j, 1 - s_j$ (for $j = 1, 2, \ldots$) where $\lambda_j = 2s_j(2 - 2s_j) \neq 1$ is the Laplace–Beltrami eigenvalue of a Maass cusp form $u_j$. The poles at $s = s_j, 1 - s_j$ occur if and only if $\langle u_j, \theta \overline{\theta_d} \rangle \neq 0$.

Fix $\varepsilon > 0$ (sufficiently small) we define the region (away from poles)
\[ \mathcal{R}_\varepsilon := \{ s \in \mathbb{C} \mid \text{Re}(s) > \varepsilon, |s - \rho| > \varepsilon \text{ for all } \rho \in \mathcal{P} \}. \]

For $s \in \mathcal{R}_\varepsilon$, we have the bound
\[ \langle P_1(*, s), \theta \overline{\theta_d} \rangle \ll_{\varepsilon} |s|^{\max(2 \text{Re}(s)+5/6,4/3)+\varepsilon} e^{-\pi|s|}. \]

Proof. The existence of the double pole at $s = \frac{1}{2}$ is proved in Proposition 8.16. The existence of possible simple poles at the eigenvalues of the Laplacian is a consequence of the fact that
\[ \mathcal{C}(s) := \sum_{j=1}^{\infty} \langle P_1(*, s), u_j \rangle \cdot \langle u_j, \theta \overline{\theta_d} \rangle \]
together with the identity (8.5) which represents $\langle P_1(*, s), u_j \rangle$ in terms of Gamma factors with poles at the eigenvalues of the Laplacian. The possible simple pole at $s = \frac{2}{3}$ comes from the continuous spectrum (see Theorem 8.17).

In Theorems 8.2 and 8.17 we prove the bounds
\[ \mathcal{C}(s) \ll_{\varepsilon} |s|^{\max(2 \text{Re}(s)+5/6,4/3)+\varepsilon} e^{-\pi|s|}, \]
\[ \mathcal{E}(s) \ll_{\varepsilon} |s|^{2 \text{Re}(s)-1/2} e^{-\pi|s|}. \]

By applying Theorem 7.4 and noting that the first of these two upper bounds is always larger, Theorem 8.1 immediately follows. ■

Theorem 8.2 (Bounding $\mathcal{C}(s)$). The cuspidal contribution $\mathcal{C}(s)$ to the spectral side has a meromorphic continuation to $\text{Re}(s) > 0$ whose set of poles $\mathcal{P}'$ in this region consists of a double pole at $s = \frac{1}{2}$ and possible simple poles at $s = s_j, 1 - s_j$ where $\lambda_j = 2s_j(2 - 2s_j) \neq 1$ is the Laplace–Beltrami eigenvalue of a Maass cusp form $u_j$. The aforementioned simple poles occur if and only if $\langle u_j, \theta \overline{\theta_d} \rangle \neq 0$. Define
\[ \mathcal{R}'_\varepsilon := \{ s \in \mathbb{C} \mid \text{Re}(s) > \varepsilon, |s - \rho| > \varepsilon \text{ for all } \rho \in \mathcal{P}' \}. \]

Then for $s \in \mathcal{R}'_\varepsilon$ we have the bound $\mathcal{C}(s) \ll_{\varepsilon} |s|^{\max(2 \text{Re}(s)+5/6,4/3)+\varepsilon} e^{-\pi|s|}$.

We first wish to show that the sum given by the cuspidal contribution $\mathcal{C}(s)$ in Theorem 7.4 converges for $s \in \mathcal{R}_\varepsilon$ and $0 < \text{Re}(s) \leq 1$, that is, for $s$ away from the poles of $\mathcal{C}(s)$. This will require a number of preliminary propositions after which we restate Theorem 8.2 as Proposition 8.14 and give its proof.
The cubic Pell equation L-function

Proposition 8.3 (Bounding the first coefficient of a Maass form). Let $u_j$ $(j = 1, 2, \ldots)$ be an orthonormal basis of Maass cusp forms for the space $L^2(\Gamma(9d^2) \setminus \mathfrak{h}^3)$ with Laplace–Beltrami eigenvalues $2s_j(2 - 2s_j)$ and Fourier coefficients $c_j(m)$ as in (6.1). Then for $s_j = \frac{1}{2} + it_j$ we have the bound

$$|c_j(1)| \ll \frac{|s_j|^\varepsilon}{|\Gamma(s_j)\Gamma(1 - s_j)|^{1/2}} \ll (1 + |t_j|)^{\varepsilon} e^{\frac{\pi}{2}|t_j|}.$$\hspace{1cm}

Proof. By applying the bound from [6] and the results from [12], we have

$$|c_j(1)|^2 \ll \frac{1}{L(1, \text{Ad}(u_j))|\Gamma(\frac{1}{2} + it_j)\Gamma(\frac{1}{2} - it_j)|}.$$\hspace{1cm}

We also have $\frac{1}{X_j} \ll L(1, \text{Ad}(u_j))$ for any fixed $\varepsilon > 0$. This was shown in [9] for Maass forms over $\mathbb{Q}$; an analogous argument holds in this case. Stirling’s bound for the Gamma function implies the proposition. ■

Proposition 8.4 (Inner product of $P_1$ with Maass forms and Eisenstein series). Let $u_j$ $(j = 1, 2, \ldots)$ be an orthonormal basis of Maass cusp forms for $L^2(\Gamma(9d^2) \setminus \mathfrak{h}^3)$ with Fourier coefficients $c_j(m)$ as in (6.1). Let $E_{\kappa_\ell}$ $(\ell = 1, \ldots, r)$ be the Eisenstein series in (6.2) with the Fourier coefficients $c_{\kappa_\ell}(m, \ast)$. We have the inner products

\begin{equation}
\langle P_1(\ast, s), u_j \rangle = \frac{c_j(1) \cdot \text{Vol}(\mathbb{C}/(9d^2\mathcal{O}_K))}{2^{6s-3\pi}2^{s-3/2}} \cdot \frac{\Gamma(2s - 1 + 2it_j)\Gamma(2s - 1 - 2it_j)}{\Gamma(2s - \frac{1}{2})},
\end{equation}

\begin{equation}
\langle P_1(\ast, s), E_{\kappa_\ell}(\ast, \frac{1}{2} + iu) \rangle = \frac{c_{\kappa_\ell}(1, \frac{1}{2} + iu) \cdot \text{Vol}(\mathbb{C}/(9d^2\mathcal{O}_K))}{2^{6s-3\pi}2^{s-3/2}} \cdot \frac{\Gamma(2s - 1 + 2iu)\Gamma(2s - 1 - 2iu)}{\Gamma(2s - \frac{1}{2})}.
\end{equation}

Proof. We explicitly write out the computation for the first of the above inner products; the second one is done by an analogous argument. We have

$$\langle P_1(\ast, s), u_j \rangle = \int_{\Gamma(9d^2)\setminus \mathfrak{h}^3} \sum_{\gamma \in \Gamma_\infty(9d^2) \setminus \Gamma(9d^2)} I_s(\gamma z) e(\gamma z) \cdot \sum_{m \in \lambda^{-3}\mathcal{O}_K \atop m \neq 0} c_j(m)yK_{2s_j - 1}(4\pi|m|y)e(mx) \frac{dx \, dy}{y^3}.$$\hspace{1cm}

$$= \int_{\Gamma_\infty(9d^2)} y^{2s}e^{-4\pi y}e^{4\pi i \text{Re}(x)} \sum_{m \in \lambda^{-3}\mathcal{O}_K \atop m \neq 0} c_j(m)yK_{2it_j}(4\pi|m|y)e^{-4\pi i \text{Re}(mx)} \frac{dx \, dy}{y^3}.$$\hspace{1cm}
Then the proposition immediately follows from Stirling’s bound $\rho_m = \langle P_j \rangle \leq C(1) \log | \lambda_j |$. Let $\epsilon > 0$. It follows from \text{(8.5)} and Proposition \text{8.3} that

\[
\langle P_1(*, s), u_j \rangle \ll \epsilon \left(1 + |t|\right)^{-2\sigma-1} \left(1 + |t_j|\right)^{\epsilon} (1 + |t - t_j|)(1 + |t + t_j|)^{2\sigma-3/2} \cdot e^{-\pi (|t - t_j| + |t + t_j| - |t| - \frac{1}{2}|t_j|)}.
\]

\textbf{Proof.} It follows from \text{(8.5)} and Proposition \text{8.3} that

\[
\langle P_1(*, s), u_j \rangle \ll |c_j(1)| \cdot \left| \frac{\Gamma(2s - 1 + 2it_j)\Gamma(2s - 1 - 2it_j)}{\Gamma(2s - 1/2)} \right| \cdot \frac{|s_j|^\epsilon}{|\Gamma(s_j)|^{1/2} |\Gamma(1 - s_j)|^{1/2}} \cdot \frac{1}{\Gamma(2s - 1 + 2it_j)\Gamma(2s - 1 - 2it_j)}.
\]

The proposition immediately follows from Stirling’s bound

\[
(8.8) \quad |\Gamma(\sigma + it)| \ll (1 + |t|)^{-1/2} e^{-\frac{\pi}{2}|t|}.
\]

\textbf{Lemma 8.9} (Sharp bounds for $K$-Bessel functions). Let $t \in \mathbb{R}$ and $y > 0$. Then

\[
|K_{2it}(4\pi y)| \leq e^{-\pi |t|} f(|t|, y),
\]

where for $t, y > 0$ the function $f(t, y)$ satisfies the following bounds:

\[
f(t, y) \ll \begin{cases} 
1 & \text{if } \frac{1}{4\pi} \leq y \leq \frac{t}{2\pi} - \frac{t^{1/3}}{2^{8/3}\pi}, \\
t^{-1/3} & \text{if } \frac{t}{2\pi} - \frac{t^{1/3}}{2^{8/3}\pi} \leq y \leq \frac{t}{2\pi}, \\
\frac{1}{(4\pi^2 y^2 - t^2)^{1/4}} & \text{if } y \geq \frac{t}{2\pi}.
\end{cases}
\]

\textbf{Proof.} It follows from \text{[2]} that if $4\pi y \geq 2t > 0$, then

\[
f(t, y) = e^{-\sqrt{16\pi^2 y^2 - 4t^2} + 2t \arccos(\frac{2t}{4\pi y})} \min\left(\frac{\sqrt{t}}{16\pi^2 y^2 - 4t^2}^{1/4} \cdot \frac{\Gamma(\frac{1}{3})}{2^{2/3}3^{1/6}} (2t)^{-1/3}\right).
\]
The cubic Pell equation $L$-function

The exponent of the exponential above is always less than zero. It follows that

$$f(t, y) \ll \min \left( \frac{1}{(4\pi^2y^2 - t^2)^{1/4}}, t^{-1/3} \right) \leq \frac{1}{(4\pi^2y^2 - t^2)^{1/4}}.$$

If $1 \leq 4\pi y \leq 2t - \frac{1}{2}(2t)^{1/3}$, then

$$f(t, y) = \frac{5}{(4t^2 - 16\pi^2y^2)^{1/4}} \ll \frac{1}{(t^2 - 4\pi^2y^2)^{1/4}}.$$

If $1 \leq 4\pi y < 2t$ and $4\pi y \geq 2t - \frac{1}{2}(2t)^{1/3}$, then

$$f(t, y) = 4(2t)^{-1/3} \ll t^{-1/3}.$$

**Proposition 8.10 (Bounding the inner product $\langle u_j, \theta \theta_d \rangle$).** Let $u_j$ be a Maass cusp form with Laplace–Beltrami eigenvalue $2s_j(2 - 2s_j)$, where $s_j = \frac{1}{2} + it_j$. Then

$$\langle u_j, \theta \theta_d \rangle \ll (1 + |t_j|)^{-1/6+\epsilon} e^{-\frac{\pi}{2}|t_j|}.$$

**Proof.** We have

$$\langle u_j, \theta \theta_d \rangle = \int_{\Gamma(9d^2)\setminus \mathfrak{h}^3} u_j(z)\overline{\theta(z)}\theta(dz) \frac{dx dy}{y^3}.$$

Using the Fourier expansions of $u_j$ and $\theta$ yields

$$\langle u_j, \theta \theta_d \rangle = \int_{\Gamma(9d^2)\setminus \mathfrak{h}^3} u_j(z)\overline{\theta(z)}\theta(dz) \frac{dx dy}{y^3}$$

$$= \int_{\Gamma(9d^2)\setminus \mathfrak{h}^3} \left( \sum_{n \in \mathfrak{a}^{-3}\mathcal{O}_K} c_j(n)yK_{2it_j}(4\pi|n|y)e^{4\pi i \text{Re}(nx)} \right)$$

$$\cdot \left( \sigma y^{2/3} + \sum_{\mu \in \lambda^{-3}\mathcal{O}_K} \tau(\mu)yK_{1/3}(4\pi|\mu|y)e^{-4\pi i \text{Re}(\mu x)} \right)$$

$$\cdot \left( \sigma y^{2/3} + \sum_{\nu \in \lambda^{-3}\mathcal{O}_K} \tau(\nu)yK_{1/3}(4\pi|\nu|y)e^{4\pi i \text{Re}(\nu x)} \right) dx dy \frac{dx dy}{y^3},$$

where $\sum'_n$ signifies that $n = 0$ is excluded from the sum.

The integral on the right side of (8.11) is an integral of a product of three infinite sums which have very rapid convergence. After interchanging the sums with the integral, the estimation of the infinite sum of integrals will have a dominant integral term given by

$$\sigma^2 c_j(1) \int_{\Gamma(9d^2)\setminus \mathfrak{h}^3} y^{7/3}K_{2it_j}(4\pi y)e^{4\pi i \text{Re}(x)} \frac{dx dy}{y^3}.$$

It follows from Proposition 8.3 that

$$|c_j(1)| \ll (1 + |t_j|)^{\epsilon} e^{-\frac{\pi}{2}|t_j|}.$$
We will use the above bound for $c_j(1)$ and the bounds for the $K$-Bessel function given in Lemma 8.9 to obtain the following bound for the dominant term (8.12) of the inner product $\langle u_j, \theta \bar{\theta}_d \rangle$:

\begin{equation}
(8.13) \quad \sigma^2 c_j(1) \left| \int_{\Gamma(9d^2) \setminus \delta^3} K_{2it_j} (4\pi y) y^{7/3} e^{4\pi i \text{Re}(x)} \frac{dx \, dy}{y^3} \right| \lesssim (1 + |t_j|)^{-1/6+\varepsilon} e^{-\frac{\pi}{2}|t_j|}.
\end{equation}

Following Sarnak [18], we first assume that there is only one cusp; if there are multiple cusps, we evaluate the sum over the cusps, with the computation at each cusp being the same. We evaluate the integral over the Siegel set, which in this case is $F_{O_K} \times (a_d, \infty)$, where $F_{O_K}$ is a fundamental domain for $O_K$ and $a_d$ is a positive constant. Thus the absolute value of the dominant integral term (8.12) is less than a constant times

\[ (1 + |t_j|)^{-1/6+\varepsilon} e^{-\frac{\pi}{2}|t_j|} \lesssim \int_{a_d}^{\infty} f(|t_j|, y) y^{-2/3} dy =: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \]

where

\[ \mathcal{I}_1 = (1 + |t_j|)^{\varepsilon} e^{-\frac{\pi}{2}|t_j|} \int \limits_{a_d}^{\infty} f(|t_j|, y) y^{-2/3} dy \]

\[ \lesssim (1 + |t_j|)^{-1/6+\varepsilon} e^{-\frac{\pi}{2}|t_j|} \int \frac{1}{u^{2/3}(1 - u^2)^{1/4}} du \lesssim (1 + |t_j|)^{-1/6+\varepsilon} e^{-\frac{\pi}{2}|t_j|} \]

and

\[ \mathcal{I}_2 = (1 + |t_j|)^{\varepsilon} e^{-\frac{\pi}{2}|t_j|} \int \frac{|t_j|}{2\pi} f(|t_j|, y) y^{-2/3} dy \]

\[ \lesssim (1 + |t_j|)^{-1/3+\varepsilon} e^{-\frac{\pi}{2}|t_j|} \int \frac{|t_j|}{2\pi} y^{-2/3} dy \lesssim (1 + |t_j|)^{-2/3+\varepsilon} e^{-\frac{\pi}{2}|t_j|}, \]

while

\[ \mathcal{I}_3 = (1 + |t_j|)^{\varepsilon} e^{-\frac{\pi}{2}|t_j|} \int \frac{|t_j|}{2\pi} f(|t_j|, y) y^{-2/3} dy \]

\[ \lesssim (1 + |t_j|)^{-1/6+\varepsilon} e^{-\frac{\pi}{2}|t_j|} \int \frac{1}{u^{2/3}(u^2 - 1)^{1/4}} du \lesssim (1 + |t_j|)^{-1/6+\varepsilon} e^{-\frac{\pi}{2}|t_j|}. \]
The dominant integral term (8.12) in the expression for the inner product $\langle u_j, \theta \overline{d} \rangle$ has the largest value. Any other integral term in the infinite sum of integrals on the right side of (8.11) includes at least one $K$-Bessel function of the form $K_{1/3}(4\pi |\mu| y)$, where $0 \neq \mu \in \lambda^{-3}O_K$. Because $4\pi |\mu| > 1$, the Bessel function $K_{1/3}(4\pi |\mu| y)$ decays exponentially as $y \to \infty$. Thus for all integral terms other than (8.12), the integrand is multiplied by an exponentially decaying function, so its integral is much smaller than the dominant integral term (8.12). This completes the proof of Proposition 8.10.

PROPOSITION 8.14. Let $u_j (j = 1, 2, \ldots)$ be an orthonormal basis of Maass cusp forms for $L^2(\Gamma(9d^2) \setminus \mathfrak{h}^3)$ with Laplace–Beltrami eigenvalues $2s_j(2-2s_j)$, where $s_j = \frac{1}{2} + it_j$. Fix $\varepsilon > 0$ and define the region (away from poles)

$$\mathcal{R}_\varepsilon' = \{ s \in \mathbb{C} | \Re(s) > \varepsilon, |s-s_j| > \varepsilon, |s-(1-s_j)| > \varepsilon \ (j = 1, 2, \ldots) \}.$$ 

Then for $s \in \mathcal{R}_\varepsilon'$ we have the bound

$$\sum_j |\langle P_1(\ast, s), u_j \rangle \langle u_j, \theta \overline{d} \rangle| \ll |s|^{\max(2 \Re(s)+5/6,4/3)+\varepsilon} e^{-\pi |s|}.$$

Proof. Let $s = \sigma + it$. It follows from Proposition 8.7 together with Stirling’s bound for the Gamma function that

$$\langle P_1(\ast, s), u_j \rangle \ll \varepsilon (1 + |t|)^{-2\sigma+1}(1 + |t_j|)^{\varepsilon}((1 + |t-t_j|)(1 + |t+t_j|))^{2\sigma-3/2} e^{-\pi |t-t_j|+|t+t_j|-|t|-\frac{1}{2}|t_j|}$$

and from Proposition 8.10 that

$$\langle u_j, \theta \overline{d} \rangle \ll (1 + |t_j|)^{-1/6+\varepsilon} e^{-\frac{\pi}{2}|t_j|}.$$ 

Combining these bounds gives

$$|\langle P_1(\ast, s), u_j \rangle \langle u_j, \theta \overline{d} \rangle| \ll \varepsilon (1 + |t|)^{-2\sigma+1}(1 + |t_j|)^{-1/6+\varepsilon} \cdot ((1 + |t-t_j|)(1 + |t+t_j|))^{2\sigma-3/2} e^{-\pi |t-t_j|+|t+t_j|-|t|}.$$ 

The following lemma counts the number of Laplace–Beltrami eigenvalues $2s_j(2-2s_j)$ with $s_j = \frac{1}{2} + it_j$ associated to Maass cusp forms $u_j (j = 1, 2, \ldots)$ for $L^2(\Gamma(9d^2) \setminus \mathfrak{h}^3)$ in a given interval.

LEMMA 8.15 (Asymptotic formula for Laplace–Beltrami eigenvalues). The number of $0 < t_j < t$ is asymptotic to a constant times $t^3$ for $t \to \infty$, and the number of $t < t_j < t+k \log(2+t)$ is $\ll k t^2 \log(2+t)$ for $k \geq 1$ and $t \to \infty$.

Proof. See [19].

We suppose that $t = \text{Im}(s) > 0$; the computations for $t < 0$ are analogous. We split the sum.
\[ \sum_j \left| \langle P_1(*, s), u_j \rangle \langle u_j, \theta d_j \rangle \right| \]

into the cases \( t_j > t \) and \( 0 \leq t_j < t \). (We may assume that \( t_j \geq 0 \) because \( t_j \) and \(-t_j\) are completely interchangeable.) Note that the former sum has infinitely many terms, while the latter has only finitely many terms.

Let \( t_{k,t} \) denote the interval \([t + k \log(2 + t), t + (k + 1) \log(2 + t)]\). First suppose that \( t_j > t \). It follows from Lemma 8.15 that

\[
\sum_{t < t_j} (1 + |t|)^{-2\sigma + 1} (1 + |t_j|)^{-1/6 + \varepsilon} \left( (1 + |t - t_j|)(1 + |t + t_j|) \right)^{2\sigma - 3/2} e^{-\pi(|t - t_j| + |t + t_j| - |t|)}
\]

\[= \sum_{t < t_j} (1 + t)^{-2\sigma + 1} (1 + t_j)^{-1/6 + \varepsilon} \left( (1 + t_j - t)(1 + t + t_j) \right)^{2\sigma - 3/2} e^{-\pi(2t_j - t)}
\]

\[= \sum_{k=0}^{\infty} \sum_{t_j \in t_{k,t}} (1 + t)^{-2\sigma + 1} (1 + t_j)^{-1/6 + \varepsilon} \left( (1 + t_j - t)(1 + t + t_j) \right)^{2\sigma - 3/2} e^{-\pi(2t_j - t)}
\]

\[\ll \sum_{k=0}^{\infty} (1 + k)t^2 \log(2 + t) \cdot (1 + t)^{-2\sigma + 1}
\]

\[\cdot (1 + t + (k + 1) \log(2 + t))^{-1/6 + \varepsilon} \left( (1 + (k + 1) \log(2 + t))^{2\sigma - 3/2} \right)
\]

\[\cdot (1 + 2t + (k + 1) \log(2 + t))^{2\sigma - 3/2} (2 + t)^{-\pi k} e^{-\pi t}
\]

\[\ll (1 + t)^{4/3 + \varepsilon} e^{-\pi t}.
\]

The last estimate above comes from the term \( k = 0 \) since the series converges rapidly and is bounded by a positive constant times the first term.

Now suppose that \( t_j < t \). Then

\[
\sum_{t_j < t} (1 + |t|)^{-2\sigma + 1} (1 + |t_j|)^{-1/6 + \varepsilon} \left( (1 + |t - t_j|)(1 + |t + t_j|) \right)^{2\sigma - 3/2} e^{-\pi(|t - t_j| + |t + t_j| - |t|)}
\]

\[= \sum_{t_j < t} (1 + t)^{-2\sigma + 1} (1 + t_j)^{-1/6 + \varepsilon} \left( (1 + t - t_j)(1 + t + t_j) \right)^{2\sigma - 3/2} e^{-\pi t}
\]

\[< \sum_{t_j < t} (1 + t)^{-2\sigma + 1} (1 + t)^{-1/6 + \varepsilon} \left( (1 + t)(1 + 2t) \right)^{2\sigma - 3/2} e^{-\pi t}
\]

\[\ll \sum_{t_j < t} (1 + t)^{2\sigma - 13/6 + \varepsilon} e^{-\pi t} \ll t^3 (1 + t)^{2\sigma - 13/6 + \varepsilon} e^{-\pi t}
\]

\[\ll (1 + t)^{2\sigma + 5/6 + \varepsilon} e^{-\pi t},
\]

where the penultimate step uses the fact that the number of \( j \) such that \( t_j < t \) is asymptotically a constant times \( t^3 \) as \( t \to \infty \) (see [19]).
Therefore, the overall bound is

\[(1 + |t|)^{\max(2\sigma+5/6,4/3)+\varepsilon} e^{-\pi|t|} \ll \varepsilon |s|^{\max(2\text{Re}(s)+5/6,4/3)+\varepsilon e^{-\pi|s|}},\]

completing the proof of Proposition 8.14. \(\blacksquare\)

Finally, we show that the double pole at \(s = \frac{1}{2}\) always occurs.

**Proposition 8.16.** There exists a Maass form \(u_j\) with \(\lambda_j = 1\), so that the double pole of \(\langle P_1(s, s), u_j \rangle\) at \(s = \frac{1}{2}\) is guaranteed to occur.

**Proof.** In [13], Maass showed that there exists a Maass form with eigenvalue \(\frac{1}{4}\) for the congruence subgroup \(\Gamma_0(9d^2)\) of \(SL(2, \mathbb{Z})\). For any such Maass form, we can do a base change to \(K\) to get a Maass form \(u_j\) with eigenvalue \(\lambda_j = 1\) as desired. It is known that the lift exists and that the \(L\)-series of the lift is the product of the original \(L\)-series and the twist of that \(L\)-series by the quadratic character \(\chi_{-3}\), from which we immediately conclude that the lift has the appropriate eigenvalue. See [5].

Because Maass forms are eigenfunctions of the Hecke operators, their first Fourier coefficient is nonzero. Thus from the inner product computed above it follows that any Maass form with eigenvalue 1 will provide a nonzero double pole contribution at \(s = \frac{1}{2}\). \(\blacksquare\)

This completes the proof of the bound for \(C(s)\), i.e., of Theorem 8.2. \(\blacksquare\)

**Theorem 8.17 (Bounding \(E(s)\)).** The Eisenstein contribution \(E(s)\) to the spectral side is meromorphic on \(\text{Re}(s) > 0\), with a possible simple pole at \(s = \frac{2}{3}\) and no other poles, and for any \(\varepsilon > 0\), for \(\text{Re}(s) > \varepsilon\) and \(|s - \frac{2}{3}| > \varepsilon\) it satisfies the bound \(E(s) \ll \varepsilon |s|^{2\text{Re}(s)-\frac{1}{2}} e^{-\pi|s|}\). The residue of \(E(s)\) at \(s = \frac{2}{3}\) is

\[
\text{Vol}(\mathbb{C}/(9d^2\mathcal{O}_K))\sigma^2 \frac{\Gamma(\frac{2}{3})}{\zeta_K(\frac{4}{3})} \left(-\frac{\pi}{\sqrt{3}}(1, \frac{2}{3}) + \sum_{\ell=1}^{r} \overline{c_{\kappa_\ell}}(1, \frac{1}{3}) \overline{c_{\kappa_\ell}}(0, \frac{2}{3})\right).
\]

**Proof.** Recall from Theorem 7.4 and (8.6) that for \(\text{Re}(s) > 1\),

\[
E(s) = \frac{1}{4\pi} \sum_{\ell=1}^{r} \int_{-\infty}^{\infty} \langle P_1(*, s), E_{\kappa_\ell}(*, \frac{1}{2} + iu) \rangle \cdot \langle E_{\kappa_\ell}(*, \frac{1}{2} + iu), \theta_{\tilde{\theta}_d}\rangle \, du
\]

\[
= 9\sqrt{3} \pi^{1/2} \sum_{\ell=1}^{r} \int_{-\infty}^{\infty} c_{\kappa_\ell}(1, \frac{1}{2} + iu) \frac{\Gamma(2s - 1 + 2i\ell)\Gamma(2s - 1 - 2i\ell)}{\Gamma(2s - \frac{1}{2})} \cdot \langle E_{\kappa_\ell}(*, \frac{1}{2} + iu), \theta_{\tilde{\theta}_d}\rangle \, du.
\]

We first bound

\[
\int_{-\infty}^{\infty} c_{\kappa_\ell}(1, \frac{1}{2} + iu) \frac{\Gamma(2s - 1 + 2i\ell)\Gamma(2s - 1 - 2i\ell)}{\Gamma(2s - \frac{1}{2})} \langle E_{\kappa_\ell}(*, \frac{1}{2} + iu), \theta_{\tilde{\theta}_d}\rangle \, du.
\]
To do this, we use an argument similar to the one used in the proof of Lemma 2.4 in [7], though in our case we must use additional residue terms and analytic continuation.

Define

\[ \mathcal{E}_{\kappa \ell}(z; \mu, s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{c_{\kappa \ell}(\mu, \frac{1}{2} + iu) \Gamma(2s - 1 + 2iu) \Gamma(2s - 1 - 2iu)}{\zeta^*_K(1 + 2iu)} \, du. \]

We know that

\[ c_{\kappa \ell}(\mu, \frac{1}{2} + iu) = \frac{\tilde{c}_{\kappa \ell}(\mu, \frac{1}{2} + iu)}{\zeta^*_K(1 + 2iu)}, \]

where \( \tilde{c}_{\kappa \ell}(\mu, \frac{1}{2} + iu) \) is a Dirichlet polynomial in \( iu \) and

\[ \zeta^*_K(s) = \left( \frac{3}{4\pi^2} \right)^{s/2} \Gamma(s) \zeta_K(s) \]

is the completed zeta function for \( K \). We let \( \overline{\tilde{c}_{\kappa \ell}}(\mu, \frac{1}{2} + iu) \) be a Dirichlet polynomial in \( iu \) such that

\[ \overline{\tilde{c}_{\kappa \ell}}(\mu, \frac{1}{2} - iu) = \overline{\tilde{c}_{\kappa \ell}}(\mu, \frac{1}{2} + iu), \]

so that

\[ c_{\kappa \ell}(\mu, \frac{1}{2} + iu) = \overline{\tilde{c}_{\kappa \ell}}(\mu, \frac{1}{2} + iu), \]

Substituting in the Fourier expansion of the Eisenstein series yields

\[ \mathcal{E}_{\kappa \ell}(z; \mu, s) = \frac{\delta_{\kappa \ell, \infty}}{2\pi i} \int_{\text{Re}(w)=0} \overline{\tilde{c}_{\kappa \ell}}(\mu, \frac{1}{2} - w) \frac{\Gamma(2s - 1 + 2w)}{\zeta^*_K(1 - 2w)} \Gamma(2s - 1 - 2w) \cdot \Gamma(2s - 1 - 2w) y^{1+2w} \, dw \]

\[ + \frac{1}{2\pi i} \int_{\text{Re}(w)=0} \overline{\tilde{c}_{\kappa \ell}}(\mu, \frac{1}{2} - w) c_{\kappa \ell}(0, \frac{1}{2} + w) \frac{\Gamma(2s - 1 + 2w)}{\zeta^*_K(1 - 2w)} \Gamma(2s - 1 - 2w) \cdot \Gamma(2s - 1 - 2w) y^{1-2w} \, dw \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\tilde{c}_{\kappa \ell}}(\mu, \frac{1}{2} + iu) \frac{\Gamma(2s - 1 + 2iu) \Gamma(2s - 1 - 2iu)}{\zeta^*_K(1 - 2iu)} \cdot \sum_{0 \neq \nu \in \lambda^{-3}O_K} c_{\kappa \ell}(\nu, \frac{1}{2} + iu) y K_{2iu}(4\pi|\nu|y)e(\nu x) \, du. \]

We have the functional equation \( \zeta^*_K(1 - 2w) = \zeta^*_K(2w) \), so

\[ \frac{\overline{\tilde{c}_{\kappa \ell}}(\mu, \frac{1}{2} - w) c_{\kappa \ell}(0, \frac{1}{2} + w)}{\zeta^*_K(1 - 2w)} = \frac{\overline{\tilde{c}_{\kappa \ell}}(\mu, \frac{1}{2} - w) \tilde{c}_{\kappa \ell}(0, \frac{1}{2} + w)}{\zeta^*_K(1 + 2w)}. \]
where \( \tilde{c}_{\kappa \ell}(0, s) \) is a Dirichlet polynomial such that
\[
c_{\kappa \ell}(0, s) = \frac{\zeta_K^*(2s - 1)}{\zeta_K^*(2s)} \tilde{c}_{\kappa \ell}(0, s).
\]
The series in the third integral is absolutely convergent, and its sum is \( O(e^{-2\pi y}) \). Move the line of integration for the first two integrals, passing over the simple poles at \( w = -s + \frac{1}{2} \) and \( w = s - \frac{1}{2} \), respectively, and no other poles.

We explicitly analyze the first integral; the argument for the second one is analogous. We have
\[
\frac{1}{2\pi i} \int_{\text{Re}(w) = 0} \frac{\overline{c}_{\infty}(\mu, \frac{1}{2} - w)}{\zeta_K^*(1 - 2w)} \Gamma(2s - 1 + 2w) \Gamma(2s - 1 - 2w) y^{1+2w} dw
\]
\[
= \frac{1}{2\pi i} \int_{\text{Re}(w) = -\sigma + 1/2 + \epsilon} \frac{\overline{c}_{\infty}(\mu, \frac{1}{2} - w)}{\zeta_K^*(1 - 2w)} \Gamma(2s - 1 + 2w) \Gamma(2s - 1 - 2w) y^{1+2w} dw
\]
\[
+ 2 \frac{\overline{c}_{\infty}(\mu, s)}{\zeta_K^*(2s)} \Gamma(4s - 2) y^{2-2s}.
\]
We compute
\[
\left\langle \frac{2 \overline{c}_{\infty}(\mu, s)}{\zeta_K^*(2s)} \Gamma(4s - 2) y^{2-2s}, \theta \overline{\theta} \right\rangle = \int_{\Gamma(9d^2) \backslash \mathbb{H}^3} \frac{2 \overline{c}_{\infty}(\mu, s)}{\zeta_K^*(2s)} \Gamma(4s - 2) y^{2-2s}
\]
\[
\cdot \left( \sigma y^{2/3} + \sum_{\mu \in \lambda^{-3}O_K} \tau(\mu) y K_{1/3}(4\pi |\mu| y) e^{-4\pi i \text{Re}(\mu x)} \right)
\]
\[
\cdot \left( \sigma y^{2/3} + \sum_{\nu \in \lambda^{-3}O_K} \tau(\nu) y K_{1/3}(4\pi |\nu| y) e^{4\pi i \text{Re}(\nu x)} \right) \frac{dx \, dy}{y^{3}}.
\]
Continuing the computation we obtain
\[
(8.18) \quad \left\langle \frac{2 \overline{c}_{\infty}(\mu, s)}{\zeta_K^*(2s)} \Gamma(4s - 2) y^{2-2s}, \theta \overline{\theta} \right\rangle = 2 \frac{\overline{c}_{\infty}(\mu, s)}{\zeta_K^*(2s)} \Gamma(4s - 2)
\]
\[
\cdot \int_{\Gamma(9d^2) \backslash \mathbb{H}^3} \left[ \sigma^2 y^{1/3 - 2s} + \sigma y^{2/3 - 2s} \sum_{\mu \in \lambda^{-3}O_K} \tau(\mu) y K_{1/3}(4\pi |\mu| y) e^{-4\pi i \text{Re}(\mu x)} \right.
\]
\[
+ \sigma y^{2/3 - 2s} \sum_{\nu \in \lambda^{-3}O_K} \tau(\nu) y K_{1/3}(4\pi |\nu| y) e^{4\pi i \text{Re}(\nu x)}
\]
\[
+ y^{1-2s} \left( \sum_{\mu \in \lambda^{-3}O_K} \tau(\mu) y K_{1/3}(4\pi |\mu| y) e^{-4\pi i \text{Re}(\mu x)} \right) \left( \sum_{\nu \in \lambda^{-3}O_K} \tau(\nu) y K_{1/3}(4\pi |\nu| y) e^{4\pi i \text{Re}(\nu x)} \right) \right] \frac{dx \, dy}{y^{3}}.
\]
To bound the above integrals we let

$$\Gamma(9d^2) \setminus B_3 = D(\eta) \cup B(\eta),$$

where for \( \eta > 0 \) (sufficiently large) we define the Siegel domain

$$D(\eta) := \{ x + jy \in B_3 : x \in \mathbb{C}/(9d^2O_K), y > \eta \}$$

and where \( B(\eta) \) is a compact set which can be thought of as the bottom of the fundamental domain. Then it is clear that

$$\int_{\Gamma(9d^2) \setminus B_3} = \int_{D(\eta)} + \int_{B(\eta)}.$$

First of all, the integral in (8.18) restricted to the compact domain \( B(\eta) \) is bounded for \( \varepsilon < \text{Re}(s) < 1 \). Next, the integral in (8.18) restricted to the Siegel domain \( D(\eta) \) is given by

$$\int_{D(\eta)} \left[ \sigma^2 y^{1/3-2s} + \sigma y^{2/3-2s} \sum_{\mu \in \lambda^{-3}O_K} \overline{\tau(\mu)} K_{1/3}(4\pi |\mu|y) e^{-4\pi i \text{Re}(\mu x)} \right.$$

$$+ \sigma y^{2/3-2s} \sum_{\nu \in \lambda^{-3}O_K} \tau(\nu) K_{1/3}(4\pi |\nu| dy) e^{4\pi i \text{Re}(\nu dx)}$$

$$+ y^{1-2s} \left( \sum_{\mu \in \lambda^{-3}O_K} \overline{\tau(\mu)} K_{1/3}(4\pi |\mu|y) e^{-4\pi i \text{Re}(\mu x)} \right)$$

$$\cdot \left( \sum_{\nu \in \lambda^{-3}O_K} \tau(\nu) K_{1/3}(4\pi |\nu|dy) e^{4\pi i \text{Re}(\nu dx)} \right) \right] dx \, dy$$

$$= \text{Vol}(\mathbb{C}/(9d^2O_K))$$

$$\cdot \int_{\eta} \left( \sigma^2 y^{1/3-2s} + y^{1-2s} \sum_{\nu \in \lambda^{-3}O_K} \tau(\nu) \overline{\tau(\nu)} K_{1/3}(4\pi |\nu|y) K_{1/3}(4\pi |\nu|dy) \right) dy.$$

If \( \text{Re}(s) > \frac{2}{3} \), then \( y^{1/3-2s} \) is integrable, and its integral is \( \frac{1}{4/3-2s} \eta^{4/3-2s} \).

This expression can be analytically continued to all \( s \neq \frac{2}{3} \). The summands in the second term each have a product of two Bessel functions that decay exponentially, so as in the proof of Proposition 8.10 their integrals are asymptotically smaller than the first term and so can be disregarded. Thus for any \( \varepsilon > 0 \), for \( \text{Re}(s) > \varepsilon \) and \( |s - \frac{2}{3}| > \varepsilon \), we have the bound

$$\left\langle \frac{c_{\infty}(\mu, s)}{c_K(2s)} \Gamma(4s - 2)y^{2-2s}, \theta \overline{\theta}_d \right \rangle \ll_{\varepsilon} |s|^{4\text{Re}(s)-3/2} e^{-2\pi |s|}.$$

The same bound holds for the inner product of the residue from the second integral with \( \theta \overline{\theta}_d \). By the same argument as in [7], the absolute values of the shifted integrals are less than a constant times \( y^{2-2\text{Re}(s)} \), so their inner product with \( \theta \overline{\theta}_d \) converges whenever \( \text{Re}(s) > 0 \) and is less than a constant times \( \eta^{4/3-2\text{Re}(s)} \).
Therefore, by using the above bounds and Stirling’s formula, the absolute value of each individual summand in the definition of $\mathcal{E}(s)$ is less than a constant times $|s|^{2\text{Re}(s)-1/2}e^{-\pi|s|}$. Because the number of such summands is finite and constant, we thus have

$$\mathcal{E}(s) \ll |s|^{2\text{Re}(s)-1/2}e^{-\pi|s|}.$$

From the above computations, we see that the summand in $\mathcal{E}(s)$ corresponding to the pole $\kappa_\ell$ has a simple pole at $s = \frac{2}{3}$ and no other poles with $\text{Re}(s) > 0$, and its residue at $s = \frac{2}{3}$ is

$$\text{Vol}(\mathbb{C}/(9d^2\mathcal{O}_K))\sigma^2 \frac{\Gamma(\frac{2}{3})}{\zeta_K^*(\frac{4}{3})} \left(\delta_{\kappa_\ell,\infty}(1, \frac{2}{3}) + \bar{c}_{\kappa_\ell}(1, \frac{1}{3})c_{\kappa_\ell}(0, \frac{2}{3})\right).$$

Thus $\mathcal{E}(s)$ has a possible simple pole at $s = \frac{2}{3}$ and no other poles with $\text{Re}(s) > 0$, and its residue at $s = \frac{2}{3}$ is

$$\text{Vol}(\mathbb{C}/(9d^2\mathcal{O}_K))\sigma^2 \frac{\Gamma(\frac{2}{3})}{\zeta_K^*(\frac{4}{3})} \left(-\bar{c}_{\infty}(1, \frac{2}{3}) + \sum_{\ell=1}^r \bar{c}_{\kappa_\ell}(1, \frac{1}{3})c_{\kappa_\ell}(0, \frac{2}{3})\right).$$

9. Geometric side of the trace formula. The geometric side of the trace formula is obtained by computing the inner product $\langle P_1(\ast, s), \overline{\theta d} \rangle$ with the Rankin–Selberg method and then using the Fourier expansions of the theta functions. The geometric side is given in the following theorem.

**Theorem 9.1.** Fix $\varepsilon > 0$ and let $s \in \mathbb{C}$ with $\text{Re}(s) > 1 + \varepsilon$. Then

$$\langle P_1(\ast, s), \overline{\theta d} \rangle = \text{Vol}(\mathbb{C}/(9d^2\mathcal{O}_K)) \left(\frac{3^{11/2}2^{-6s}}{\pi^{2s-5/6}} \frac{\Gamma(2s)(2s-\frac{2}{3})}{\Gamma(2s+\frac{1}{6})} + S_d(s)\right),$$

where

$$S_d(s) = \frac{2^{-6s}\pi^{-2s+1/2}\Gamma(2s)^2}{4\pi i \Gamma(s+\frac{1}{2})\Gamma(s)} \int_{\text{Re}(w)=-1-\varepsilon} \frac{\Gamma(s+w)\Gamma(s+w)\Gamma(-w)}{\Gamma(2s+\frac{1}{2}+w)}$$

$$\cdot \sum_{\nu \in \lambda^{-3}\mathcal{O}_K} \frac{\tau(\nu)\tau(1+d\nu)}{2|\nu| \cdot |1+d\nu|} \cdot \left((1-t)^{-4/3-w} + (1-t)^{-2/3-w}\right) t(w) dt dw.$$
The residue of $S_d(s)$ at $s = \frac{1}{3}$ is $-3^{11/2}2^{-3}\pi^{1/6}F(\frac{2}{3})G(\frac{2}{3})$, while the residue of $S_d(s)$ at each of the poles of $\langle P_1(\ast, s), \theta\theta_d \rangle$ is $\frac{1}{\text{Vol}(C/(6d^2\mathcal{O}_K))}$ times the residue of $\langle P_1(\ast, s), \theta\theta_d \rangle$ at that pole.

**Proof.** The proof will be given in four steps.

**STEP 1: Rankin–Selberg method.** We first prove

\[
S_d(s) = \sum_{\nu \in \lambda^{-3}\mathcal{O}_K} \tau(\nu) \tau(1 + d\nu) \int_0^{\infty} K_{1/3}(4\pi|\nu|y) K_{1/3}(4\pi|1 + d\nu|y) e^{-4\pi y^2 s} \frac{dy}{y^3}.
\]

Unraveling the Poincaré series $P_1(\ast, s)$ with the Rankin–Selberg method we see that

\[
\langle P_1(\ast, s), \theta\theta_d \rangle = \int_{\mathcal{I}(9d^2)\setminus \mathcal{H}^3} P_1(z, s)\theta(z)\theta(dz) \frac{dx \, dy}{y^3}
\]

\[
= \int_{\mathcal{I}_\infty(9d^2)\setminus \mathcal{H}^3} y^{2s+4/3} e^{-4\pi y} e^{4\pi i \text{Re}(x)} \theta(z)\theta(dz) \frac{dx \, dy}{y^3}
\]

\[
= \int_{x \in \mathbb{C}/(9d^2\mathcal{O}_K)} \int_{y=0}^{\infty} y^{2s+4/3} e^{-4\pi y} e^{4\pi i \text{Re}(x)}
\]

\[
\cdot \left( \sigma + \sum_{\mu \in \lambda^{-3}\mathcal{O}_K} \frac{\tau(\mu) y^{1/3} K_{1/3}(4\pi|\mu|y) e^{-4\pi i \text{Re}(\mu x)}}{\mathcal{O}_K} \right)
\]

\[
\cdot \left( \sigma + \sum_{\nu \in \lambda^{-3}\mathcal{O}_K} \tau(\nu) y^{1/3} K_{1/3}(4\pi|\nu|y) e^{4\pi i \text{Re}(\nu x)} \right) \frac{dx \, dy}{y^3}.
\]

Continuing the computation, we obtain

\[
\langle P_1(\ast, s), \theta\theta_d \rangle
\]

\[
= \text{Vol}(\mathbb{C}/(9d^2\mathcal{O}_K)) \left( \tau(1)\sigma \int_{y=0}^{\infty} y^{2s+5/3} e^{-4\pi y} K_{1/3}(4\pi y) \frac{dy}{y^3} + A \right.
\]

\[
+ \sum_{\mu, \nu \in \lambda^{-3}\mathcal{O}_K} \frac{\tau(\mu) \tau(\nu)}{\mu - d\nu = 1} \int_0^{\infty} y^{2s+2} e^{-4\pi y} K_{1/3}(4\pi|\mu|y) K_{1/3}(4\pi|\nu|y) \frac{dy}{y^3} \right)
\]

\[
= \text{Vol}(\mathbb{C}/(9d^2\mathcal{O}_K)) \left( 27\sigma \int_0^{\infty} y^{2s-4/3} e^{-4\pi y} K_{1/3}(4\pi y) dy + A \right.
\]

\[
+ \sum_{\mu, \nu \in \lambda^{-3}\mathcal{O}_K} \frac{\tau(\mu) \tau(\nu)}{\mu - d\nu = 1} \int_0^{\infty} y^{2s-1} e^{-4\pi y} K_{1/3}(4\pi|\mu|y) K_{1/3}(4\pi|\nu|y) dy \right),
\]

\[
\text{where } A = \frac{1}{\text{Vol}(\mathbb{C}/(9d^2\mathcal{O}_K))} \int_0^{\infty} y^{2s} e^{-4\pi y} K_{1/3}(4\pi y) dy.
\]
The cubic Pell equation \( L \)-function

where \( A \) is defined as follows. If \(-\frac{1}{d} \notin \lambda^{-3}O_K\), then we directly have \( A = 0 \) because there are no summands containing a nonzero integral since no \( \nu \) in \( \lambda^{-3}O_K \) satisfies \( 1 + d\nu = 0 \). Since \( d \) is a (cubefree) positive integer not equal to 1, we have \(-\frac{1}{d} \notin \lambda^{-3}O_K\) if and only if \( d = 3 \). Thus if \( d \neq 3 \) then \( A = 0 \), and if \( d = 3 \) then

\[
A = \tau(-1/3)\sigma \int_0^\infty y^{2s-4/3} e^{-4\pi y} K_{1/3}\left(\frac{4\pi y}{3}\right) dy.
\]

Because \(-\frac{1}{3} = \lambda^{-2}\), which is not in any of the forms specified in the definition of \( \tau \), we have \( \tau(-\frac{1}{3}) = 0 \). Thus \( A = 0 \) in this case as well. Therefore, \( A = 0 \) in all cases, so we can drop it from the sum. Thus we have

\[
\langle P_1(*, s), \theta \overline{\theta_d} \rangle
\]

\[
= \text{Vol}(\mathbb{C}/(9d^2O_K)) \left(27\sigma \int_0^\infty y^{2s-4/3} e^{-4\pi y} K_{1/3}(4\pi y) dy + \sum_{\substack{\mu, \nu \in \lambda^{-3}O_K \\mu - d\nu = 1}} \overline{\tau(\mu)} \tau(\nu) \int_0^\infty y^{2s-1} e^{-4\pi y} K_{1/3}(4\pi|\mu|y) K_{1/3}(4\pi|\nu|y) dy\right).
\]

The first integral equals

\[
\int_0^\infty y^{2s-4/3} e^{-4\pi y} K_{1/3}(4\pi y) dy
\]

\[
= \frac{\sqrt{\pi}}{(8\pi)^{2s-1/3}} \frac{\Gamma(2s)\Gamma(2s - \frac{2}{3})}{\Gamma(2s + \frac{1}{6})} \cdot F\left(2s, \frac{5}{6}; 2s + \frac{1}{6}; 0\right)
\]

\[
= 2^{-6s+1} \pi^{-2s+5/6} \frac{\Gamma(2s)\Gamma(2s - \frac{2}{3})}{\Gamma(2s + \frac{1}{6})},
\]

where \( F \) is the Gaussian hypergeometric function; the final expression is holomorphic on \( \{ s \in \mathbb{C} \mid \text{Re}(s) > \frac{1}{3} \} \) and has a simple pole at \( s = \frac{1}{3} \) with residue

\[
2^{-1} \pi^{1/6} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{6}\right)}.
\]

The sum on the right side of (9.3) is \( S_d(s) \). This establishes (9.2).

**Step 2: Recomputing \( S_d(s) \) with hypergeometric functions.** In this step we prove that
\begin{equation}
S_d(s) = \frac{2^{-6s} \pi^{-2s+1/2} \Gamma(2s)}{\Gamma(s + \frac{1}{2}) \Gamma(s)} \sum_{\nu \in \lambda^{-3} \mathcal{O}_K} \frac{\tau(\nu) \tau(1 + d\nu)}{2\pi i} \cdot \int_{0 \text{ Re}(w) = -1 - \varepsilon}^{\infty} \frac{\Gamma(s + \frac{1}{2} + w) \Gamma(s + w) \Gamma(-w)}{\Gamma(2s + \frac{1}{2} + w)} \cdot (|\nu|^2 + |1 + d\nu|^2 - 1 + 2|\nu| |1 + d\nu| \cdot \cosh(u)) \cosh\left(\frac{u}{3}\right) dw \, du.
\end{equation}

**Proof.** In the next proposition we show that the integral of a product of two $K$-Bessel functions given on the right hand side of (9.2) can be expressed as an integral of a Gaussian hypergeometric function.

**Proposition 9.6.** For $m, n \in \mathbb{R}_{>0}$, $a \in \mathbb{C}$, and $s \in \mathbb{C}$ with $\text{Re}(s) > |\text{Re}(a)|$,

\[
\int_{0}^{\infty} K_a(my) K_a(ny) e^{-y^2 s} \frac{dy}{y} = \frac{\sqrt{\pi}}{2^{2s}} \frac{\Gamma(2s)}{\Gamma(2s + \frac{1}{2})} \int_{0}^{\infty} F(s + \frac{1}{2}, s; 2s + \frac{1}{2}; 1 - \alpha(u)^2) \cosh(au) du,
\]

where $\alpha(u) = (m^2 + n^2 + 2mn \cosh(u))^{1/2}$.

**Proof.** This identity follows from an analogous argument to the one in the first part of the proof of [10, Lemma 1] because all of the identities that are used in that argument still hold, as does the way of combining them, with the only potential issue being the convergence of the integral on the right. We now show that the integrals on the left and right sides of the identity in Proposition 9.6 converge absolutely if $\text{Re}(s) > |\text{Re}(a)|$.

**Lemma 9.7.** For $m, n \in \mathbb{R}_{>0}$, $a \in \mathbb{C}$, and $y \in \mathbb{R}_{>0}$,

\[
K_a(my) K_a(ny) = \int_{0}^{\infty} K_0((m^2 + n^2 + 2mn \cosh(u))^{1/2} y) \cosh(au) du.
\]

**Proof.** First, use the identity

\[
K_r(X) K_r(x) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2rT - \frac{Xe}{v} \cosh(2T)} e^{-(\frac{v}{2} - \frac{x^2 + y^2}{2v})} \frac{dv}{v} dT,
\]

which holds for any $r \in \mathbb{C}$ and $X, x \in \mathbb{R}_{>0}$ (see [21, p. 440]), and set $X = my$, $x = ny$, and $r = a$, giving

\[
K_a(my) K_a(ny) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2aT - \frac{my^2 e}{v} \cosh(2T)} e^{-(\frac{v}{2} - \frac{(m^2 + n^2)y^2}{2v})} \frac{dv}{v} dT.
\]
Now use the identity
\[
K_r(z) = \frac{1}{2} \left( \frac{1}{2z} \right)^r \int_0^\infty e^{-\frac{t-\frac{z^2}{4t}}{t^{r+1}}} dt,
\]
which holds for any \( r \in \mathbb{C} \) and \( z \in \mathbb{C} \) with \( \text{Re}(z^2) > 0 \) (see [21, p. 183]), and set \( z = (m^2 + n^2 + 2mn \cosh(u))^{1/2}y \) and \( r = 0 \), giving
\[
K_0((m^2 + n^2 + 2mn \cosh(u))^{1/2}y) = \frac{1}{2} \int_0^\infty e^{-t-(m^2+n^2+2mn \cosh(u))y^2/4t} dt.
\]
Combining those two identities proves the lemma.

It follows from Lemma 9.7 that
\[
(9.8) \quad \int_0^\infty K_a(my)K_a(ny)e^{-y^2s} \frac{dy}{y} = \int_0^\infty \int_0^\infty K_0(\alpha(u)y) \cosh(au)e^{-y^2s} \frac{du}{y} \frac{dy}{y},
\]
provided that the integrals converge.

We now verify that the integral on the left side of (9.8) converges for \( \text{Re}(s) > |\text{Re}(a)| \) as follows. For any \( a \in \mathbb{C} \), \( K_a(y) \) has the following asymptotic bounds (see [8, p. 227]):

\[
K_a(y) \ll \begin{cases} 
  y^{-1/2}e^{-y} & \text{for } y > 1 + |a|^2, \\
  y|\text{Re}(a)| + y^{-|\text{Re}(a)|} & \text{for } y < 1 + |a|^2.
\end{cases}
\]

Without loss of generality, suppose that \( m \geq n \). Then
\[
\int_0^\infty K_a(my)K_a(ny)e^{-y^2s} \frac{dy}{y} \ll m^{\text{Re}(a)}n^{\text{Re}(a)} \int_0^{1+|a|^2/m} e^{-y^2s+2|\text{Re}(a)|} \frac{dy}{y} 
\]
\[
+ (m^{\text{Re}(a)}n^{\text{Re}(a)} + m^{\text{Re}(a)}n^{-|\text{Re}(a)|}) \int_0^{1+|a|^2/m} e^{-y^2s} \frac{dy}{y} 
\]
\[
+ m^{-|\text{Re}(a)|}n^{-|\text{Re}(a)|} \int_0^{1+|a|^2/m} e^{-y^2s-2|\text{Re}(a)|} \frac{dy}{y} 
\]
\[
+ m^{-1/2}n^{\text{Re}(a)} \int_0^{1+|a|^2/m} e^{-(1+m)y^2s-\frac{1}{2}+|\text{Re}(a)|} \frac{dy}{y}.
\]
\[ + m^{-1/2} n^{-1} \sum_{a} e^{-(1+m)y} y^{s-\frac{1}{2} - \text{Re}(a)} \frac{dy}{y} \]
\[ + m^{-1/2} n^{-1/2} \sum_{a} e^{-(1+m+n)y} y^{s-1} dy \]
which all converge if \( \text{Re}(s) \geq |\text{Re}(a)| \).

Next, we make use of the Mellin transform
\[ \int_{0}^{\infty} e^{-ay} K_{\nu}(by) y^{s-1} dy = \frac{\sqrt{\pi}}{2a} \left( \frac{b}{a} \right)^{\nu} \frac{\Gamma(s + \nu) \Gamma(s - \nu)}{\Gamma(s + \frac{1}{2})} \cdot \frac{\Gamma(2s)}{\Gamma(2s + \frac{1}{2})} F\left( \frac{s + \nu + 1}{2}, \frac{s + \nu}{2}; s + \frac{1}{2}; 1 - \left( \frac{b}{a} \right)^2 \right), \]
which holds for any \( \nu, a, b, s \in \mathbb{C} \) such that \( \text{Re}(s) > |\text{Re}(\nu)| \) and \( \text{Re}(a+b) > 0 \) (see \cite{3} p. 331) and set \( a = 1, b = \alpha(u), \nu = 0 \) and substitute \( 2s \) for \( s \).

It follows that
\[ \int_{0}^{\infty} K_{0}(\alpha(u)y) e^{-y} y^{2s} \frac{dy}{y} = \frac{\sqrt{\pi}}{2^{2s}} \frac{\Gamma(2s)^2}{\Gamma(2s + \frac{1}{2})} F\left( \frac{s + 1}{2}, s, 2s + \frac{1}{2}; 1 - \alpha(u)^2 \right). \]
Combining this with \( \text{(9.8)} \) gives the formula in the statement of Proposition \( \text{9.6} \). In particular, if \( a = \frac{1}{3} \), the identity holds for all \( s \in \mathbb{C} \) with \( \text{Re}(s) > \frac{1}{3} \).

We now use the following integral representation of the hypergeometric function (see \cite{1}).

**Proposition 9.9.** Fix \( r > 0 \). Then
\[ F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} \cdot \frac{1}{2\pi i} \int_{\text{Re}(w) = -r} \frac{\Gamma(\alpha + w) \Gamma(\beta + w) \Gamma(-w)}{\Gamma(\gamma + w)} (-z)^w dw, \]
where \( |\text{arg}(-z)| < \pi \) and \( \text{Re}(\alpha), \text{Re}(\beta) > r \).

The formula for \( S_{d}(s) \) given in \( \text{(9.5)} \) immediately follows from Propositions \( \text{9.6} \) and \( \text{9.9} \) for \( \text{Re}(s) > 1 + \varepsilon \) with the choice \( r = 1 + \varepsilon \).

**Step 3:** We rewrite \( S_{d}(s) \) using Picard’s integral representation of the Appell hypergeometric function. In this step we complete the proof of the first part of Theorem \( \text{9.1} \). It is helpful to introduce some additional notation.
Let
\[ Z_{d,\nu}(u) = |\nu|^2 + |1 + d\nu|^2 - 1 + 2|\nu| \cdot |1 + d\nu| \cdot \cosh(u) \]
\[ = 2|\nu| \cdot |1 + d\nu| \cdot (a_d(\nu) + \cosh(u)), \]
where
\[ a_d(\nu) := \frac{|\nu|^2 + |1 + d\nu|^2 - 1}{2|\nu| \cdot |1 + d\nu|} = \frac{d^2 + 1}{2d} \left( 1 + \mathcal{O}_d(|\nu(1 + dv)|^{-1/2}) \right). \]

**Lemma 9.10.** Let \( a > 1 \) and \( w \in \mathbb{C} \) with \( \text{Re}(w) < -\frac{1}{3} \). Then
\[ \int_0^\infty (a + \cosh(u))^w \cdot \cosh \left( \frac{u}{3} \right) du = \frac{(a + 1)^w}{18w^2 - 2} [(3 - 9w)\Phi_1(w, a) - (3 + 9w)\Phi_2(w, a)], \]
where
\[ \Phi_1(w, a) := F_1 \left( 1, -w, -w, -w + \frac{2}{3}; \frac{1}{2} - \frac{1}{2} \sqrt{\frac{a - 1}{a + 1}}, \frac{1}{2} + \frac{1}{2} \sqrt{\frac{a - 1}{a + 1}} \right), \]
\[ \Phi_2(w, a) := F_1 \left( 1, -w, -w, -w + \frac{4}{3}; \frac{1}{2} - \frac{1}{2} \sqrt{\frac{a - 1}{a + 1}}, \frac{1}{2} + \frac{1}{2} \sqrt{\frac{a - 1}{a + 1}} \right). \]

**Proof.** By a Mathematica\textsuperscript{TM} computation we have
\[ \int_0^\infty (a + \cosh(u))^w \cdot \cosh \left( \frac{u}{3} \right) du = \frac{2^{-w}}{18w^2 - 2} \]
\[ \cdot \left[ (3 - 9w)F_1 \left( -\frac{1}{3} - w, -w, -w + \frac{2}{3}; -a + \sqrt{a^2 - 1}, \frac{1}{-a + \sqrt{a^2 - 1}} \right) \right. \]
\[ \left. -(3 + 9w)F_1 \left( \frac{1}{3} - w, -w, -w + \frac{4}{3}; -a + \sqrt{a^2 - 1}, \frac{1}{-a + \sqrt{a^2 - 1}} \right) \right]. \]

For \(|z_1|, |z_2| < 1\), we have the identity
\[ F_1(a, b_1, b_2, c; z_1, z_2) = (1 - z_1)^{-b_1}(1 - z_2)^{-b_2} \]
\[ \cdot F_1 \left( c - a, b_1, b_2, c; \frac{z_1}{z_1 - 1}, \frac{z_2}{z_2 - 1} \right). \]

It follows that
\[ F_1 \left( -\frac{1}{3} - w, -w, -w + \frac{2}{3}; -a + \sqrt{a^2 - 1}, \frac{1}{-a + \sqrt{a^2 - 1}} \right) \]
\[ = (2 + 2a)^w F_1 \left( 1, -w, -w + \frac{2}{3}; -\frac{1}{2} - \frac{1}{2} \sqrt{\frac{a - 1}{a + 1}}, \frac{1}{2} + \frac{1}{2} \sqrt{\frac{a - 1}{a + 1}} \right), \]
\[
F_1\left(\frac{1}{3} - w, -w, -w, \frac{4}{3} - w, -a + \sqrt{a^2 - 1}, \frac{1}{-a + \sqrt{a^2 - 1}}\right)
= (2 + 2a)^w F_1\left(1, -w, -w, \frac{4}{3} - w; \frac{1}{2} - \frac{1}{2} \sqrt{\frac{a - 1}{a + 1}} \pm \frac{1}{2} \sqrt{\frac{a - 1}{a + 1}}\right).
\]
Inserting the above identities in (9.11) completes the proof. ■

Émile Picard [17] proved the following integral representation of Appell’s hypergeometric function (which is valid for \(\text{Re}(c) > \text{Re}(\alpha) > 0\)):

\[
F_1(\alpha, \beta_1, \beta_2, c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(\alpha)\Gamma(c - \alpha)} \int_0^1 t^{\alpha-1}(1-t)^{c-\alpha-1}(1-z_1t)^{-\beta_1}(1-z_2t)^{-\beta_2} \, dt.
\]

Let

\[
z_1(a) := \frac{1}{2} - \frac{1}{2} \sqrt{\frac{a - 1}{a + 1}}, \quad z_2(a) := \frac{1}{2} + \frac{1}{2} \sqrt{\frac{a - 1}{a + 1}}.
\]

Note that

\[
(1 - z_1(a) \cdot t) \cdot (1 - z_2(a) \cdot t) = 1 - t + \frac{1}{4} \left(1 - \frac{a - 1}{a + 1}\right) t^2
= 1 - t = \frac{t^2}{2(a + 1)}.
\]

It follows that

\[
\Phi_1(w, a) := (-w - \frac{1}{3}) \int_0^1 (1-t)^{-w-4/3} \left(1 + \frac{t^2}{2(a + 1)}\right)^w \, dt,
\]

(9.12)

\[
\Phi_2(w, a) := (-w + \frac{1}{3}) \int_0^1 (1-t)^{-w-2/3} \left(1 + \frac{t^2}{2(a + 1)}\right)^w \, dt.
\]

The first part of Theorem 9.1 now follows from the definition of \(S_d(s)\), Lemma 9.10, and Picard’s formulae (9.12).

STEP 4: The poles of \(S_d(s)\). The only pole of \(\frac{\Gamma(2s)\Gamma(2s-2/3)}{\Gamma(2s+1/6)}\) with \(\text{Re}(s) > 0\) is a simple pole at \(s = \frac{1}{3}\). Because \(\langle P_1(*, s), \theta \theta_d \rangle\) does not have a pole at \(s = \frac{1}{3}\), there is no cancellation, so the poles of \(S_d(s)\) with \(\text{Re}(s) > 0\) are precisely a simple pole at \(s = \frac{1}{3}\) and the poles of \(\langle P_1(*, s), \theta \theta_d \rangle\) with \(\text{Re}(s) > 0\) with the same order as for that function. The computation of the residue of \(S_d(s)\) at each of those poles is then an immediate consequence of the formula. ■
10. Relating $S_d(s)$ to $L_d^\#(s)$. The function

$$S_d(s) = \frac{2^{-6s-1} \pi^{-2s+1/2} \Gamma(2s)^2}{2\pi i \Gamma(s + \frac{1}{2}) \Gamma(s)} \int_{\Re(w) = -1 - \varepsilon} \frac{\Gamma(s + \frac{1}{2} + w) \Gamma(s + w) \Gamma(-w)}{\Gamma(2s + \frac{1}{2} + w)} \cdot \sum_{\nu \in \lambda^{-3} \mathcal{O}_K} \frac{\tau(\nu) \tau(1 + \nu)}{(2|\nu| \cdot |1 + \nu|)^{-w}}
\cdot \int_0^1 (t^{-4/3} - w + t^{-2/3} - w)((a_d(\nu) + 1) \cdot t + (1 - t)^2/2)^w dt dw$$

that occurs on the geometric side of the trace formula (see Theorem 9.1) contains the coefficients $\tau(\nu) \tau(1 + \nu)$ which occur in the Dirichlet series expansions of the cubic Pell equation $L_d(s)$ and $L_d^\#(s)$. The integral in $w$ above can be evaluated by shifting the line of integration to the left and picking up residues at $w = -s - \frac{1}{2}, -s$ which leads to a proof of the following theorem.

**Theorem 10.1** (Relating $S_d(s)$ to $L_d^\#(s)$). Fix $\varepsilon > 0$ sufficiently small. Then

$$S_d(s) = \frac{2^{-7s-1} \Gamma(2s)^2}{\pi^{2s-1} \Gamma(s + \frac{1}{2})^2} \cdot L_d^\#(s) - \frac{2^{-7s+1/2} \Gamma(2s)^2}{\pi^{2s-1} \Gamma(s)^2} \cdot L_d^\#(s + \frac{1}{2})
+ \mathcal{O}(|s|^{2 \Re(s) + 3} e^{-\pi|s|}).$$

The function $L_d^\#(s)$ has a meromorphic continuation to $s \in \mathbb{C}$ with $\Re(s) > \frac{1}{3}$ and satisfies

$$|L_d^\#(s)| \ll_{\varepsilon} |s|^3$$

for $\frac{1}{3} + \varepsilon < \Re(s) < 1$ provided $|s - \rho| > \varepsilon$ for any pole $\rho \in \mathbb{C}$ of $S_d(s)$.

The poles of $L_d^\#(s)$ with $\Re(s) > \frac{1}{3}$ are precisely the poles of $S_d(s)$ (see Theorem 9.1), each having the same order as the corresponding pole of $S_d(s)$.

**Remark 10.2.** At each of the poles $\rho$ of $S_d(s)$, the residue of $L_d^\#(s)$ at that pole is

$$\text{Res}_{s=\rho} L_d^\#(s) = \frac{\pi^{2s-1} \Gamma(s + \frac{1}{2})^2}{2^{-7s-1} \Gamma(2s)^2} \cdot \text{Res}_{s=\rho} S_d(s).$$

**Proof.** The proof of Theorem 10.1 will be given in several steps.

**Step 1:** We shift the line of integration in the $w$-integral of $S_d(s)$ obtaining two residue terms and a shifted integral term.

**Proposition 10.3.** Fix $\varepsilon > 0$ sufficiently small. Let $s \in \mathbb{C}$ with $1 + \varepsilon < \Re(s) < 1 + 2\varepsilon$. Then

$$S_d(s) = R_1(s) + R_2(s) + I(s),$$

where $R_1(s)$ and $R_2(s)$ are residues at the poles of $S_d(s)$ and $I(s)$ is the integral obtained by shifting the line of integration.
where \( \mathcal{R}_1(s) \), \( \mathcal{R}_2(s) \), and \( \mathcal{I}(s) \) are holomorphic for \( 1 + \varepsilon < \Re(s) < 1 + 2\varepsilon \) and are given by

\[
\mathcal{R}_1(s) := \frac{2^{-6s-1}}{\pi^{2s-1}} \frac{\Gamma(2s)^2}{\Gamma(s + \frac{1}{2})^2} \sum_{\nu \in \lambda^{-3} \mathcal{O}_K} \frac{\tau(\nu) \overline{\tau(1 + d\nu)}}{(2|\nu| \cdot |1 + d\nu|)^s} \\
\cdot \int_0^1 (t^{-4/3+s} + t^{-2/3+s})((a_d(\nu) + 1) \cdot t + (1 - t)^2/2)^{-s} dt,
\]

\[
\mathcal{R}_2(s) := -\frac{2^{-6s}}{\pi^{2s-1}} \frac{\Gamma(2s)^2}{\Gamma(s)^2} \sum_{\nu \in \lambda^{-3} \mathcal{O}_K} \frac{\tau(\nu) \overline{\tau(1 + d\nu)}}{(2|\nu| \cdot |1 + d\nu|)^{s+1/2}} \\
\cdot \int_0^1 (t^{-5/6+s} + t^{-1/6+s})((a_d(\nu) + 1) \cdot t + (1 - t)^2/2)^{-s-1/2} dt,
\]

\[
\mathcal{I}(s) := \frac{2^{-6s-1}}{4\pi i \Gamma(s + \frac{1}{2})} \frac{\Gamma(2s)^2}{\Gamma(s + \frac{1}{2})^2} \sum_{\nu \in \lambda^{-3} \mathcal{O}_K} \frac{\tau(\nu) \overline{\tau(1 + d\nu)}}{(2|\nu| \cdot |1 + d\nu|)^{-s}} \\
\cdot \int_0^1 (t^{-4/3-w} + t^{-2/3-w})((a_d(\nu) + 1) \cdot t + (1 - t)^2/2)^w dt dw.
\]

**Proof.** Shift the line of integration in the \( w \)-integral of \( S_d(s) \) given in Theorem 9.1 to the line \( \Re(w) = -\frac{3}{2} - 3\varepsilon \). Since we are assuming that \( 1 + \varepsilon < \Re(s) < 1 + 2\varepsilon \), we will only pass the two poles of

\[
\frac{\Gamma(s + \frac{1}{2} + w) \Gamma(s + w) \Gamma(-w)}{\Gamma(2s + \frac{1}{2} + w)}
\]

at \( w = -s \) and \( w = -s - \frac{1}{2} \). The residues at these poles are given by \( \mathcal{R}_1(s), \mathcal{R}_2(s) \), and by Cauchy’s residue theorem, the original integral is the sum of the two residues plus the shifted integral \( \mathcal{I}(s) \).

Note that the sum

\[
\sum_{\nu \in \lambda^{-3} \mathcal{O}_K} \frac{\tau(\nu) \overline{\tau(1 + d\nu)}}{(2|\nu| \cdot |1 + d\nu|)^{-w}}
\]

converges absolutely for \( \Re(w) < -1 \) by Proposition 5.2. Also since \( a_d(\nu) + 1 \sim \frac{(d+1)^2}{2d} > 1 \) it follows that the \( t \)-integral in \( \mathcal{I}(s) \) above is bounded for any \( w \in \mathbb{C} \) with \( \Re(w) = -\frac{3}{2} - 3\varepsilon \). One easily concludes that \( \mathcal{R}_1(s), \mathcal{R}_2(s), \) and \( \mathcal{I}(s) \) are holomorphic functions for \( s \in \mathbb{C} \) with \( 1 + \varepsilon < \Re(s) < 1 + 2\varepsilon \). ■
Step 2: Holomorphic continuation and bounds for \( \mathcal{R}_2(s) \) and \( \mathcal{I}(s) \) in the region \( \frac{1}{2} + 4\varepsilon < \text{Re}(s) < 1 + 2\varepsilon \).

Proposition 10.4. The residue \( \mathcal{R}_2(s) \) and the integral \( \mathcal{I}(s) \) are holomorphic functions and satisfy the bounds

\[
|\mathcal{R}_2(s)| \ll |s|^{2\text{Re}(s)}e^{-\pi|s|}, \quad |\mathcal{I}(s)| \ll \varepsilon |s|^{3/2+7\varepsilon}e^{-\pi|s|},
\]

for \( s \in \mathbb{C} \) with \( \frac{1}{2} + 4\varepsilon < \text{Re}(s) < 1 + 2\varepsilon \).

Proof. By Stirling’s asymptotic formula,

\[
\left| \frac{\Gamma(2s)^2}{\Gamma(s)^2} \right| \ll |s|^{2\text{Re}(s)}e^{-\pi|s|}
\]

for \( \frac{1}{2} + 4\varepsilon < \text{Re}(s) < 1 + 2\varepsilon \). Since \( a_d(\nu) + 1 \sim \frac{(d+1)^2}{2d} > 1 \) it follows that the \( t \)-integral in \( \mathcal{R}_2(s) \) is bounded by an absolute constant for any \( s \in \mathbb{C} \) with \( \frac{1}{2} + 4\varepsilon < \text{Re}(s) < 1 + 2\varepsilon \) and any \( w \in \mathbb{C} \) with \( \text{Re}(w) = -\frac{3}{2} - 3\varepsilon \). It immediately follows that \( |\mathcal{R}_2(s)| \ll |s|^{2\text{Re}(s)}e^{-\pi|s|} \) in this region.

We have shown in Proposition [10.3] that \( \mathcal{I}(s) \) is holomorphic in the region \( 1 + \varepsilon < \text{Re}(s) < 1 + 2\varepsilon \). Note that for \( \text{Re}(w) = -\frac{3}{2} - 3\varepsilon \) the ratio of Gamma functions

\[
\frac{\Gamma(s + \frac{1}{2} + w)\Gamma(s + w)\Gamma(-w)}{\Gamma(2s + \frac{1}{2} + w)}
\]

is holomorphic in the region \( \frac{1}{2} + 4\varepsilon < \text{Re}(s) < 1 + 2\varepsilon \). We now show that the integral \( \mathcal{I}(s) \) converges absolutely in this region and, as a consequence, obtain a sharp bound for its growth in this region.

Further, for \( w = -\frac{3}{2} - 3\varepsilon + iv \) and \( s = \sigma + it \) with \( v, t \in \mathbb{R} \), Stirling’s asymptotic formula implies that for \( \frac{1}{2} + 4\varepsilon < \sigma < 1 + \varepsilon \) we have

\[
\left| \frac{\Gamma(s + \frac{1}{2} + w)\Gamma(s + w)\Gamma(-w)}{\Gamma(2s + \frac{1}{2} + w)} \right| \ll \frac{(1 + |t + v|^{2\sigma-7/2-6\varepsilon}(1 + |v|)^{1+3\varepsilon}}{(1 + |2t + v|)^{2\sigma-3/2-3\varepsilon}} \cdot e^{-\frac{\pi}{2}(2|t+v|+|v|-|2t+v|)}.
\]

Let

\[
(10.5) \quad \mathcal{I}(s) = \frac{\pi^{-2s+1}}{2^{6s+1}} \frac{\Gamma(2s)^2}{\Gamma(s + \frac{1}{2}) \Gamma(s)} \cdot \mathcal{I}_1(s).
\]

It follows that
\[ |I_1(\sigma + it)| \ll \int_{-\infty}^{\infty} \frac{(1 + |t + v|)^{2\sigma - 7/2 - 6\varepsilon}(1 + |v|)^{1+3\varepsilon}}{(1 + |2t + v|)^{2\sigma - 3/2 - 3\varepsilon}} \cdot e^{-\frac{\pi}{2}(2|t+v|+|v|-|2t+v|)} \, dv \]

\[ = \int_{-\infty}^{\infty} \frac{(1 + |u|)^{2\sigma - 7/2 - 6\varepsilon}(1 + |u - t|)^{1+3\varepsilon}}{(1 + |u + t|)^{2\sigma - 3/2 - 3\varepsilon}} \cdot e^{-\frac{\pi}{2}|u + t|-|u-t|} \, du \]

change of variables \( v \mapsto u - t \)

We now assume that \( t > 0 \); the case \( t < 0 \) can be shown by an analogous argument. To bound \( I_1(s) \) we split the interval of integration into 4 regions:

\[
\int_{-\infty}^{\infty} t \int_{0}^{t} \int_{-t}^{0} \int_{-\infty}^{0} \int_{t}^{\infty} \]

**CASE 1:** \( t \geq 0 \) and \( 0 \leq u < t \). Then

\[ \int_{0}^{t} \frac{(1 + u)^{2\sigma - 7/2 - 6\varepsilon}(1 + t - u)^{1+3\varepsilon}}{(1 + t + u)^{2\sigma - 3/2 - 3\varepsilon}} \, du \ll (1 + t)^{-2-3\varepsilon}. \]

**CASE 2:** \( t \geq 0 \) and \( t \leq u \). Then

\[ \int_{t}^{\infty} \frac{(1 + u)^{2\sigma - 7/2 - 6\varepsilon}(1 + t - u)^{1+3\varepsilon}}{(1 + t + u)^{2\sigma - 3/2 - 3\varepsilon}} e^{-\pi(u-t)} \, du \\
= \int_{t}^{t+\varepsilon} \frac{(1 + u)^{2\sigma - 7/2 - 6\varepsilon}(1 + u - t)^{1+3\varepsilon}}{(1 + t + u)^{2\sigma - 3/2 - 3\varepsilon}} e^{-\pi(u-t)} \, du + \mathcal{O}(e^{-\varepsilon}) \\
\ll (1 + t)^{-2-3\varepsilon}. \]

**CASE 3:** \( t \geq 0 \) and \( -t \leq u < 0 \). Then

\[ \int_{-t}^{0} \frac{(1 - u)^{2\sigma - 7/2 - 6\varepsilon}(1 + t - u)^{1+3\varepsilon}}{(1 + t + u)^{2\sigma - 3/2 - 3\varepsilon}} e^{2\pi u} \, du \\
= \int_{-t}^{0} \frac{(1 - u)^{2\sigma - 7/2 - 6\varepsilon}(1 + t - u)^{1+3\varepsilon}}{(1 + t + u)^{2\sigma - 3/2 - 3\varepsilon}} e^{2\pi u} \, du + \mathcal{O}(e^{-\varepsilon}) \\
\ll (1 + t)^{5/2 - 2\sigma + 7\varepsilon}. \]

**CASE 4:** \( t \geq 0 \) and \( u \leq -t \). Then

\[ \int_{-\infty}^{-t} \frac{(1 - u)^{2\sigma - 7/2 - 6\varepsilon}(1 + t - u)^{1+3\varepsilon}}{(1 + t - u)^{2\sigma - 3/2 - 3\varepsilon}} e^{\pi(u-t)} \, du \\
= \int_{-t-t}^{-t} \frac{(1 - u)^{2\sigma - 7/2 - 6\varepsilon}(1 + t - u)^{1+3\varepsilon}}{(1 + t - u)^{2\sigma - 3/2 - 3\varepsilon}} e^{\pi(u-t)} \, du + \mathcal{O}(e^{-\varepsilon}) \\
\ll (1 + t)^{-2-3\varepsilon}. \]
It immediately follows that for $\frac{1}{2} + 4\varepsilon < \text{Re}(s) < 1 + 2\varepsilon$, 

$$|I_1(s)| \ll |s|^{5/2 - 2\text{Re}(s) + 7\varepsilon}.$$  

Again, by Stirling’s formula (for $s = \sigma + it$ and $\frac{1}{2} + 4\varepsilon < \sigma < 1 + \varepsilon$), we have 

$$\left| \frac{\Gamma(2s)^2}{\Gamma(s + \frac{1}{2})^2} \right| \ll |s|^{2\sigma - 1 - \pi|t|}.$$ 

The proof of the bound for $I(s)$ given in Proposition 10.4 immediately follows from the above two bounds and (10.5). ■

**Step 3:** Meromorphic continuation of $R_1(s) + 2R_2(s)$ to $\text{Re}(s) > \frac{1}{2}$.

**Proposition 10.6.** Fix $\varepsilon > 0$. Then 

$$S_d(s) = R_1(s) + 2R_2(s) + \mathcal{O}(\left(|s|^{2\text{Re}(s)} + |s|^{3/2 + 7\varepsilon} e^{-\pi|s|}\right))$$  

for $\frac{1}{2} + 4\varepsilon < \text{Re}(s) < 1 + 2\varepsilon$.

**Proof.** This immediately follows from Proposition 10.3 (which says that $S_d(s) = R_1(s) + R_2(s) + I(s)$) and Proposition 10.4 (which gives bounds for $R_2(s)$ and $I(s)$). We already know that $S_d(s)$ has a meromorphic continuation to $\text{Re}(s) > 0$ by Theorems 8.1 and 9.1 ■

**Step 4:** Meromorphic continuation of $R_1(s) + 2R_2(s)$ to $\text{Re}(s) > \varepsilon$.

**Proposition 10.7.** Fix $\varepsilon > 0$ sufficiently small. Then 

$$S_d(s) = R_1(s) + 2R_2(s) + \mathcal{O}(|s|^{5/2 + 7\varepsilon} e^{-\pi|s|})$$  

for $4\varepsilon < \text{Re}(s) < \frac{1}{2} + 2\varepsilon$.

**Proof.** To meromorphically continue $R_1(s) + 2R_2(s)$ to $\text{Re}(s) > \varepsilon$ we first assume that $s \in \mathbb{C}$ satisfies $\frac{1}{2} + \varepsilon < \text{Re}(s) < \frac{1}{2} + 2\varepsilon$. Next, we shift the line of integration in the $w$-integral of $I(s)$ from $\text{Re}(w) = -\frac{3}{2} - 3\varepsilon$ to $\text{Re}(w) = -2 - 3\varepsilon$. In doing so we cross possible poles of $\Gamma(s + \frac{1}{2} + w)\Gamma(s + w)$ at 

$$w = -2 - \alpha\varepsilon, \quad -\frac{3}{2} - \alpha\varepsilon, \quad 1 \leq \alpha \leq 2.$$ 

Thus, the only pole that is crossed is the pole at $w = -\frac{1}{2} - s$, which has $R_2(s)$ as the residue. 

It follows that for $\frac{1}{2} + \varepsilon < \text{Re}(s) < \frac{1}{2} + 2\varepsilon$ we have $S_d(s) = R_1(s) + R_2(s) + I(s)$ where 

$$I(s) = R_2(s) + I_2(s)$$
and
\[ I_2(s) := \frac{2^{-6s} \pi^{-2s+1/2} \Gamma(2s)^2}{4\pi i \Gamma(s + \frac{1}{2}) \Gamma(s)} \int_{\text{Re}(w) = -2 - 3\epsilon} \frac{\Gamma(s + \frac{1}{2} + w) \Gamma(s + w) \Gamma(-w)}{\Gamma(2s + \frac{1}{2} + w)} \cdot \sum_{\nu \in \lambda^{-3}O_K} \frac{\tau(\nu)\tau(1 + d\nu)}{(2|\nu| \cdot |1 + d\nu|)^w} \cdot \int_0^1 (t^{4/3-w} + t^{-2/3-w})((a_d(\nu) + 1) \cdot t + (1-t)^2/2)^w \, dt \, dw. \]

Consequently,
\[ (10.8) \quad S_d(s) = R_1(s) + 2R_2(s) + I_2(s). \]

Now, for \( \text{Re}(w) = -2 - 3\epsilon \) the product of Gamma functions
\[ \Gamma(s + \frac{1}{2} + w) \Gamma(s + w) \]
has no poles for \( 4\epsilon < \text{Re}(s) < \frac{1}{2} + 2\epsilon \). This shows that \( I_2(s) \) is a holomorphic function in the region \( 4\epsilon < \text{Re}(s) < \frac{1}{2} + 2\epsilon \). It then follows (as in Step 2 above) that
\[ (10.9) \quad I_2(s) \ll \epsilon \cdot |s|^{5/2 + 7\epsilon} e^{-\pi|s|}. \]

Now, by Theorems 8.1 and 9.1, the function \( S_d(s) \) is meromorphic in the region \( \text{Re}(s) > \epsilon \). Since we also know that \( I_2(s) \) is holomorphic in this region, the proof of Proposition 10.7 immediately follows from (10.8) and (10.9). As in the proof of Proposition 10.6, we already know that \( S_d(s) \) has a meromorphic continuation to \( \text{Re}(s) > 0 \) by Theorems 8.1 and 9.1.

**Step 5:** Meromorphic continuation of \( L_d^\#(s) \) to \( \text{Re}(s) > \frac{1}{3} + \epsilon \). Recall the Picard hypergeometric function
\[ F(s, x) = \int_0^1 (t^{s-4/3} + t^{s-2/3})(x \cdot t + (t-1)^2/2)^{-s} \, dt \quad (s \in \mathbb{C}, x > 0), \]
where the integral converges absolutely for \( \text{Re}(s) > \frac{1}{3} \). For \( \text{Re}(s) > 1 \), we also recall the \( L \)-function
\[ L_d^\#(s) = \sum_{\nu \in \lambda^{-3}O_K} \frac{\tau(\nu)\tau(1 + d\nu)}{(|\nu| \cdot |1 + d\nu|)^s} \cdot F \left( s, \frac{(d+1)^2}{2d} \right) - \frac{s}{2} \sum_{\nu \in \lambda^{-3}O_K} \frac{\tau(\nu)\tau(1 + d\nu)}{(|\nu| \cdot |1 + d\nu|)^s} \left( a_d(\nu) - \frac{d^2+1}{2d} \right) \cdot F \left( s + 1, \frac{(d+1)^2}{2d} \right), \]
where
\[ a_d(\nu) - \frac{d^2+1}{2d} \ll_d |\nu(1 + d\nu)|^{-1/2}. \]
We shall now show that the function $L_d^\#(s)$ appears in the first two terms in a certain binomial expansion of the residue function $R_1(s)$ which can be written as

$$R_1(s) = G_1(s) \sum_{\nu \in \lambda^{-3}O_K} \tau(\nu)\tau(1 + dv)|\nu(1 + dv)|^{-s} \cdot F(s, a_d(\nu) + 1),$$

where

$$G_1(s) = \frac{2^{-7s-1}}{\pi^{2s-1}} \frac{\Gamma(2s)^2}{\Gamma(s + \frac{1}{2})^2}.$$ For $\Re(s) > \frac{1}{3}$, the binomial expansion of $F(s, x)$ around $x = x_0$ is given by

$$F(s, x) = \sum_{k=0}^{\infty} \binom{-s}{k} (x - x_0)^k \int_0^1 (t^{s-4/3+k} + t^{s-2/3+k}) (x_0 \cdot t + (t - 1)^2/2)^{-s-k} d t$$

$$= \sum_{k=0}^{\infty} \binom{-s}{k} (x - x_0)^k F(s + k, x_0).$$

It follows that by expanding $F(s, a_d(\nu) + 1)$ around $a_d(\nu) + 1 = \frac{(d+1)^2}{2d}$ we obtain

$$R_1(s) = G_1(s) \left[ L_d^\#(s) + \sum_{k=2}^{\infty} \binom{-s}{k} \sum_{\nu \in \lambda^{-3}O_K} \frac{\tau(\nu)\tau(1 + dv)}{|\nu(1 + dv)|^{s+1/2}} \left( a_d(\nu) - \frac{d^2 + 1}{2d} \right)^k \right] \cdot F\left(s + k, \frac{(d+1)^2}{2d}\right).$$

In a similar manner we see that $L_d^\#(s + \frac{1}{2})$ appears in the first two terms in a binomial expansion of the residue function $R_2(s)$ which can be written as

$$R_2(s) = G_2(s) \sum_{\nu \in \lambda^{-3}O_K} \frac{\tau(\nu)\tau(1 + dv)}{|\nu(1 + dv)|^{s+1/2}} F\left(s + \frac{1}{2}, a_d(\nu) + 1\right),$$

where

$$G_2(s) = \frac{-2^{-7s-1/2}}{\pi^{2s-1}} \frac{\Gamma(2s)^2}{\Gamma(s)^2}.$$ It again follows that by expanding $F\left(s + \frac{1}{2}, a_d(\nu) + 1\right)$ around $a_d(\nu) + 1 = \frac{(d+1)^2}{2d}$ we obtain

$$R_2(s) = G_2(s) \left[ L_d^\#(s + \frac{1}{2}) + \sum_{k=2}^{\infty} \binom{-s - \frac{1}{2}}{k} \sum_{\nu \in \lambda^{-3}O_K} \frac{\tau(\nu)\tau(1 + dv)}{|\nu(1 + dv)|^{s+1/2}} \left( a_d(\nu) - \frac{d^2 + 1}{2d} \right)^k \right] \cdot F\left(s + \frac{1}{2} + k, \frac{(d+1)^2}{2d}\right).$$
Since we have already proved in Proposition [10.7] that \( R_1(s) + 2R_2(s) \) has a meromorphic continuation to \( \text{Re}(s) > 0 \) and the sum

\[
\sum_{\nu \in \lambda^{-3}O_K} \frac{\tau(\nu)\tau(1+d\nu)}{(|\nu| \cdot |1+d\nu|)^s} \left( a_d(\nu) - \frac{d^2+1}{2d} \right)^k
\]

is absolutely convergent for \( \text{Re}(s) > 0 \) and \( k \geq 2 \) it then follows from the previous computations that \( L_d^*(s) \) has a meromorphic continuation to \( \text{Re}(s) > \frac{1}{3} \).

The sum (over \( k \geq 2 \)) of the binomial terms \(|s|\) grows exponentially in \(|s|\), which implies that the function

\[
\sum_{k=2}^{\infty} \binom{-s}{k} \sum_{\nu \in \lambda^{-3}O_K} \frac{\tau(\nu)\tau(1+d\nu)}{(|\nu| \cdot |1+d\nu|)^s} \left( a_d(\nu) - \frac{d^2+1}{2d} \right)^k
\]

has at most exponential growth in \(|s|\) for \( \text{Re}(s) > 0 \) as \(|s| \to \infty\). In this manner it is possible to prove that \( L_d^*(s) \) has at most exponential growth (away from poles) in the region \( \text{Re}(s) > \frac{1}{3} \). For applications, however, we would like to do better and show it has polynomial growth instead of exponential growth. Obtaining polynomial growth can be achieved by breaking the sum over \( \nu \in \lambda^{-3}O_K \) into two sums, over \(|\nu| \leq |s|\) and over \(|\nu| > |s|\). This approach will be worked out in the next step.

**Step 6: Obtaining a polynomial bound in \(|s|\) for \( L_d^*(s) \) away from poles.** As explained at the end of Step 5, in order to obtain a polynomial bound for \( L_d^*(s) \) it is necessary to break the \( \nu \)-sum in the definition of \( L_d^*(s) \) into two sums. This approach leads to the following definition.

**Definition 10.10 (The function \( L_d'^*(s) \)).** Let \( \text{Re}(s) > 1 \). Then we define

\[
L'_d(s) := \sum_{\nu \in \lambda^{-3}O_K, |\nu|>|s|} \tau(\nu)\tau(1+d\nu)|\nu(1+d\nu)|^{-s}
\]

\[
\cdot \left[ \int_0^1 \left( t^{s-4/3} + t^{s-2/3} \right) \left( \frac{(d+1)^2}{2d} t + \frac{(t-1)^2}{2} \right)^{-s} dt \right.
\]

\[
- \sum_{\nu \in \lambda^{-3}O_K, |\nu|>|s|} \tau(\nu)\tau(1+d\nu)|\nu(1+d\nu)|^{-s} \left( a_d(\nu) - \frac{d^2+1}{2d} \right)
\]

\[
\cdot \left[ \int_0^1 \left( t^{s-1/3} + t^{s+1/3} \right) \left( \frac{(d+1)^2}{2d} t + \frac{(t-1)^2}{2} \right)^{-s-1} dt \right].
\]

**Remark 10.11.** Note that this is the same as the definition of \( L_d^*(s) \) but with the sums over \( \nu \) restricted to \( \nu \in \lambda^{-3}O_K \) with \(|\nu| > |s|\).
PROPOSITION 10.12. Fix $\varepsilon > 0$. The function $L_d^\#(s) - L_d^{\#'}(s)$ has a holomorphic continuation to $\frac{1}{3} + \varepsilon < \text{Re}(s) < 1$ and in this region satisfies the bound $|L_d^\#(s) - L_d^{\#'}(s)| \ll \varepsilon |s|^3$.

Proof. For $\frac{1}{3} + \varepsilon < \text{Re}(s) < 1$ we have

$$L_d^\#(s) - L_d^{\#'}(s) = \sum_{\nu \in \lambda^{-3}O_K, |\nu| \leq |s|} \frac{\tau(\nu)}{|\nu|} \frac{\tau(1 + d\nu)|\nu(1 + d\nu)|^{-s}}{\nu(1 + d\nu)|\nu|}$$

$$\cdot \int_0^1 \left( t^{s-4/3} + t^{s-2/3} \right) \left( \frac{(d+1)^2}{2d} t + \frac{(t-1)^2}{2} \right)^{-s} dt$$

$$- s \sum_{\nu \in \lambda^{-3}O_K, |\nu| \leq |s|} \tau(\nu) \tau(1 + d\nu)|\nu(1 + d\nu)|^{-s} \left( a_d(\nu) - \frac{d^2 + 1}{2d} \right)$$

$$\cdot \int_0^1 \left( t^{s-1/3} + t^{s+1/3} \right) \left( \frac{(d+1)^2}{2d} t + \frac{(t-1)^2}{2} \right)^{-s-1} dt.$$}

Since $\frac{(d+a)^2}{2d} > 1$ and $\frac{1}{3} < \text{Re}(s)$, both of the integrals in each summand above are bounded. Furthermore, the number of terms in each sum is less than a constant times $|s|^2$ and $|\tau(\nu)\tau(1 + d\nu)| \ll |s|$ for $\nu \ll |s|$. Therefore, $|L_d^\#(s) - L_d^{\#'}(s)| \ll |s|^3$. It is also clear that $L_d^\#(s) - L_d^{\#'}(s)$ is holomorphic for $\text{Re}(s) > \frac{1}{3}$ because it is expressed as a finite sum in which each summand is holomorphic. $\blacksquare$

PROPOSITION 10.13. Fix $\varepsilon > 0$. Then for $\frac{1}{3} + \varepsilon < \text{Re}(s) < 1$ we have

$$\mathcal{R}_1(s) = \frac{2^{-7}s-1}{\pi^{2s-1}} \frac{\Gamma(2s)^2}{\Gamma(s + \frac{1}{2})^2} \left( L_d^\#(s) + O(|s|^2) \right),$$

$$\mathcal{R}_2(s) = -\frac{2^{-7}s-1/2}{\pi^{2s-1}} \frac{\Gamma(2s)^2}{\Gamma(s)^2} \left( L_d^\#(s + \frac{1}{2}) + O(|s|^2) \right).$$

Proof. Recall that

$$\mathcal{R}_1(s) = \frac{2^{-7}s-1}{\pi^{2s-1}} \frac{\Gamma(2s)^2}{\Gamma(s + \frac{1}{2})^2} \sum_{\nu \in \lambda^{-3}O_K} \tau(\nu) \tau(1 + d\nu)|\nu(1 + d\nu)|^{-s}$$

$$\cdot \int_0^1 \left( t^{s-4/3} + t^{s-2/3} \right) \left( (a_d(\nu) + 1)t + (1-t)^2/2 \right)^{-s} dt.$$}

We split the sum over $\nu \in \lambda^{-3}O_K$ as

$$\sum_{\nu \in \lambda^{-3}O_K} = \sum_{\nu \in \lambda^{-3}O_K, |\nu| \leq |s|} + \sum_{\nu \in \lambda^{-3}O_K, |\nu| > |s|}.$$
For the $|\nu| > |s|$ sum, we use the binomial theorem, yielding
\[
\sum_{k=0}^{\infty} \binom{-s}{k} \sum_{\nu \in \lambda^{-3} O_K, |\nu| > |s|} \tau(\nu)\tau(1 + d\nu)|\nu(1 + d\nu)|^{-s} \left( a_d(\nu) - \frac{d^2 + 1}{2d} \right)^k \cdot \int_0^1 (t^{s-4/3 + k} + t^{s-2/3 + k}) \left( \frac{(d + 1)^2}{2d} t + \frac{(t - 1)^2}{2} \right)^{-s-k} dt.
\]

Note that the sum of the $k = 0$ and $k = 1$ terms is precisely the definition of $L_d^\#(s)$, so that this series equals $L_d^\#(s) + T_1(s)$ where
\[
T_1(s) := \sum_{k=2}^{\infty} \binom{-s}{k} \sum_{\nu \in \lambda^{-3} O_K, |\nu| > |s|} \tau(\nu)\tau(1 + d\nu)|\nu(1 + d\nu)|^{-s} \left( a_d(\nu) - \frac{d^2 + 1}{2d} \right)^k \cdot \int_0^1 (t^{s-4/3 + k} + t^{s-2/3 + k}) \left( \frac{(d + 1)^2}{2d} t + \frac{(t - 1)^2}{2} \right)^{-s-k} dt.
\]

To bound $T_1(s)$, we claim in particular that the absolute value of the last integral is less than a constant times $c_d^{-k}$, where $c_d > 1$ is a constant that may depend on $d$. If $\frac{d}{d^2 + 1} \leq t \leq 1$, then $\frac{(d+1)^2}{2d} t + \frac{(t-1)^2}{2} \geq c_d'$, where $c_d' > 1$. For all $0 \leq t \leq 1$, $\frac{(d+1)^2}{2d} t + \frac{(t-1)^2}{2} \geq \frac{1}{2}$. Because $d \geq 2$, $0 < \frac{d}{d^2 + 1} < \frac{1}{2}$. Let $c_d = \min\left(\frac{d+1}{2d}, c_d'\right)$. Additionally, $|a_d(\nu) - \frac{d^2 + 1}{2d}| \ll |\nu|^{-1} < |s|^{-1}$, and
\[
\sum_{\nu \in \lambda^{-3} O_K, |\nu| > |s|} \tau(\nu)\tau(1 + d\nu)|\nu(1 + d\nu)|^{-s} \left( a_d(\nu) - \frac{d^2 + 1}{2d} \right)^2
\]
is bounded for $\text{Re}(s) > 0$.

It follows that
\[
T_1(s) \ll \sum_{k=2}^{\infty} \binom{|s| + k - 1}{k} (c_d|s|)^{-k+2}
\ll \sum_{k=0}^{\infty} \binom{|s| + k - 1}{k} (c_d|s|)^{-k+2} = c_d^2 |s|^2 (1 - c_d^{-1}|s|^{-1})^{-|s|}.
\]

For the $|\nu| \leq |s|$ case, we first note that there are at most $\ll |s|^2$ terms in the sum
\[
T_2(s) := \sum_{\nu \in \lambda^{-3} O_K, |\nu| \leq |s|} \tau(\nu)\tau(1 + d\nu)|\nu(1 + d\nu)|^{-s} \cdot \int_0^1 (t^{s-4/3} + t^{s-2/3}) ((a_d(\nu) + 1)t + (1 - t)^2/2)^{-s} dt.
\]
For each of those terms, $|\nu(1+dv)|^{-s}$ and the last integral are bounded. The number of summands in this case is less than a constant times $|s|^2$, so the sum is thus less than a constant times $|s|^2$.

Therefore, for $s \in \mathbb{C}$ with $\varepsilon < \Re(s) < 1$ we have

$$
R_1(s) = 2^{-7s-1} \frac{\Gamma(2s)^2}{\pi^2s-1} \frac{\Gamma(s+\frac{1}{2})^2}{
(L_d^{#}(s) + H_1(s)),
$$

where $H_1(s) := T_1(s) + T_2(s)$ and $|H_1(s)| \ll |s|^2$. An analogous argument tells us that

$$
R_2(s) = -2^{-7s-1/2} \frac{\Gamma(2s)^2}{\pi^2s-1} \frac{\Gamma(s+\frac{1}{2})^2}{
(L_d^{#}(s + \frac{1}{2}) + H_2(s)),
$$

where $|H_2(s)| \ll |s|^2$. Combining this with Proposition 10.12 completes the proof of Proposition 10.13.

**Step 7: Completion of the proof of the first two parts of Theorem 10.1.**

The first part of Theorem 10.1 given by

$$
(10.14) \quad \frac{2^{-7s-1}}{\pi^2s-1} \frac{\Gamma(2s)^2}{\Gamma(s+\frac{1}{2})^2} \cdot L_d^{#}(s) - \frac{2^{-7s+1/2}}{\pi^2s-1} \frac{\Gamma(2s)^2}{\Gamma(s)^2} \cdot L_d^{#}(s + \frac{1}{2})
$$

$$
+ O(|s|^{2\Re(s)+3}e^{-\pi|s|})
$$

for $\Re(s) > \frac{1}{3}$, follows from Propositions 10.6 and 10.13. For the second part, Theorems 8.2 and 9.1 imply that

$$
|S_d(s)| \ll \varepsilon |s|^\max(2\Re(s)+5/6,4/3)+\varepsilon e^{-\pi|s|} + \frac{\Gamma(2s)\Gamma(2s - \frac{2}{3})}{\Gamma(2s + \frac{1}{6})}
$$

$$
\ll \varepsilon |s|^\max(2\Re(s)+5/6,4/3)+\varepsilon e^{-\pi|s|}
$$

for $\varepsilon < \Re(s) < 1$ provided $|s - \rho| > \varepsilon$ for any pole $\rho \in \mathbb{C}$ of $S_d(s)$. The proof of the second part now follows from the above bound for $|S_d(S)|$ together with the first part (10.14) and the asymptotic formulae

$$
\left| \frac{\Gamma(2s)^2}{\Gamma(s)^2} \right| \sim |s|^{2\Re(s)}e^{-\pi|s|}, \quad \left| \frac{\Gamma(2s)^2}{\Gamma(s+\frac{1}{2})\Gamma(s)} \right| \sim |s|^{2\Re(s)-1/2}e^{-\pi|s|}.
$$

**Step 8: Determination of the poles of $L_d^{#}(s)$ for $\Re(s) > \frac{1}{3}$.**

The formula found in the first part of Theorem 10.1 tells us that for each pole of $S_d(s)$ with $\Re(s) > \frac{1}{3}$, either $L_d^{#}(s)$ or $L_d^{#}(s + \frac{1}{2})$ has a pole at the same location, with the corresponding order. We also know that the poles of $L_d^{#}(s)$ must have $\Re(s) \leq 1$. Thus if $S_d(s)$ has a simple pole at $s = \frac{2}{3}$, then $L_d^{#}(s)$ has a simple pole at $s = \frac{2}{3}$. Also, $L_d^{#}(s)$ has a double pole at $s = \frac{1}{2}$ and simple poles at $s = \frac{1}{2} \pm it_j, t_j \neq 0, \langle u_j, \theta_d \rangle \neq 0$. If $L_d^{#}(s)$ instead had any of the poles at $s = 1$ or $s = 1 \pm t_j$, then we would be forced either to have $S_d(s)$
have a pole at \( s = 1 \) or \( s = 1 \pm it \), or to have \( L_d^\#(s) \) have a pole at \( s = 3/2 \) or \( s = 3/2 \pm it \), both of which we know cannot happen. The computation of the residue of \( L_d^\#(s) \) at each of its poles is then an immediate consequence of the formula. ■

11. Relating \( L_d^\#(s) \) to \( L_d(s) \). We now extend the properties of \( L_d^\#(s) \) given in Theorem 10.1 to yield the following result about \( L_d(s) \).

**Theorem 11.1.** The function \( L_d(s) \) has a meromorphic continuation to the region \( \text{Re}(s) > 1/2 \) with at most a simple pole at \( s = 2/3 \) which occurs if and only if the Eisenstein contribution \( E(s) \) given in Theorem 7.4 has a pole at \( s = 2/3 \).

Fix \( \varepsilon > 0 \). Then \( L_d(s) \) has possible poles at the zeros of \( F(s, \frac{(d+1)^2}{2d}) \) with \( 1/2 < \text{Re}(s) \leq 1 \). In the region \( \text{Re}(s) > 1/2 + \varepsilon \) and \( |s - \rho| > \varepsilon \) (for any pole \( \rho \) of \( L_d(s) \)) we have the bound \( L_d(s) \ll_{d, \varepsilon} |s|^{7/2} \).

**Proof.** We first prove the following lemma, which we then use to show that the theorem is a consequence of the definition of \( L_d^\#(s) \) and its properties that were found earlier.

**Lemma 11.2.** Fix \( a > 0 \), and let \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1/3 \). Then if \( s = \sigma + it \) and \( \sigma \) is fixed with \( |t| \to \infty \), we have

\[
F(\sigma + it, a) = \left( \frac{2\pi}{|t|} \right)^{1/2} e^{-i(\log a) t} (2a^{-\sigma} + O(|t|^{-1})) a^{-1/2}.
\]

**Proof.** We may assume \( t \to \infty \). We use the method of steepest descent. Recall that

\[
F(s, a) = \int_0^1 (u^{-4/3} + u^{-2/3}) \left( a + \frac{(u - 1)^2}{2u} \right)^{-s} du.
\]

which can be written as

\[
F(\sigma + it, a) = \int_0^1 f_{a, \sigma}(u) e^{it g_a(u)} du,
\]

where

\[
f_{a, \sigma}(u) = (u^{-4/3} + u^{-2/3}) \left( a + \frac{(u - 1)^2}{2u} \right)^{-\sigma}
\]

and

\[
g_a(u) = -\log \left( a + \frac{(u - 1)^2}{2u} \right).
\]

We then compute

\[
g'_a(u) = \frac{-u^2 + 1}{u(u^2 + 2(a - 1)u + 1)}
\]
and
\[ g''_a(u) = \frac{u^4 - 4u^2 - 4(a - 1)u - 1}{u^2(u^2 + 2(a - 1)u + 1)^2}. \]

Thus there is a non-degenerate saddle point at \( u = 1 \), and if \( a \neq 2 \), there is a non-degenerate saddle point at \( u = -1 \). These are the only saddle points.

We may now apply the saddle point method; the relevant statement for our case is as follows (see [4]).

**Theorem 11.3.** Suppose that \( f(z) \) and \( S(z) \) are holomorphic functions on an open, bounded, and simply connected set \( \Omega_x \subset \mathbb{C}^n \) such that \( I_x = \Omega_x \cap \mathbb{R}^n \) is simply connected, \( \text{Re}(S(z)) \) has a single maximum \( x^0 \) in \( I_x \), and \( x^0 \) is a non-degenerate saddle point of \( S \). Then as \( \lambda \to \infty \),
\[
I(\lambda) = \left( \frac{2\pi}{\lambda} \right)^{n/2} e^{\lambda S(x^0)} (f(x^0) + O(\lambda^{-1})) \prod_{j=1}^n (-\mu_j)^{-1/2},
\]
where \( \mu_j \) are the eigenvalues of the Hessian of \( S \) and \( (-\mu_j)^{-1/2} \) are chosen so that they satisfy \( |\arg \sqrt{-\mu_j}| < \frac{\pi}{4} \).

It follows from Theorem 11.3 and the previous computations that for \( \sigma \) fixed and \( t \to \infty \), we have
\[
\mathcal{F}(\sigma + it, a) = \left( \frac{2\pi}{t} \right)^{1/2} e^{-i(\log a)t} (2a^{-\sigma} + O(t^{-1})) a^{-1/2}. \]

Recall that
\[
L^\#_d(s) = \mathcal{F} \left( s, \frac{(d + 1)^2}{2d} \right) L_d(s) - s \cdot \mathcal{F} \left( s + 1, \frac{(d + 1)^2}{2d} \right) L^*_d(s).
\]

It was shown in Theorem 10.1 that the only possible pole of \( L^\#_d(s) \) with \( \text{Re}(s) > \frac{1}{2} \) is a possible simple pole at \( s = \frac{2}{3} \) and that \( L_d(s) \) is holomorphic for \( \text{Re}(s) > 1 \), \( L^\#_d(s) \approx L_d(s + \frac{1}{2}) \) for \( \text{Re}(s) > \frac{1}{2} \), and \( \mathcal{F} \left( s, \frac{(d + 1)^2}{2d} \right) \) is holomorphic for \( \text{Re}(s) > \frac{1}{3} \). Thus if \( L^\#_d(s) \) has a simple pole at \( s = \frac{2}{3} \), then \( L_d(s) \) must have a simple pole at \( s = \frac{2}{3} \), and any other poles of \( L_d(s) \) with \( \text{Re}(s) > \frac{1}{2} \) must be at zeros of \( \mathcal{F} \left( s, \frac{(d + 1)^2}{2d} \right), \) with the order of the pole, if it occurs, being less than or equal to the order of vanishing of \( \mathcal{F} \left( s, \frac{(d + 1)^2}{2d} \right) \) at that point, since in this region \( L^\#_d(s) \) does not have any other poles and the second summand on the right side is holomorphic.

Now, for \( \text{Re}(s) > \frac{1}{2} + \varepsilon \), we have \( |s - \frac{2}{3}| > \varepsilon \), \( L^\#_d(s) \ll \varepsilon |s|^3 \). Since the second summand in the definition of \( L^\#_d(s) \) is asymptotically smaller than
the first summand, on that region we have the bound
\[ F(s, \frac{(d + 1)^2}{2d}) L_d(s) \ll \varepsilon |s|^3. \]

It follows from Lemma 11.2 that as \( t \to \infty \),
\[ \left| F(\sigma + it, \frac{(d + 1)^2}{2d}) \right| \sim c |t|^{-1/2} \]
for some constant \( c > 0 \). This immediately implies that for \( |s - \rho| > \varepsilon \) (at poles \( \rho \) of \( L_d(s) \)) we have the bound \( |L_d(s)| \ll |s|^{7/2} \).

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**References**


The cubic Pell equation L-function


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